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The Schoenflies Theorem after Mazur, Morse, and Brown

STEFAN BEHRENS, ALLISON N. MILLER, MATTHIAS NAGEL, AND PETER TEICHNER

We introduce some basic techniques in the study of topological manifolds by means of a discussion of the Schoenflies theorem. First we present the proof of Mazur and Morse using the Eilenberg swindle and a technique called *push-pull*. These techniques exemplify the kinds of arguments often used in the study of topological manifolds. Then we explain Brown's alternative proof of the Schoenflies theorem as an introduction to decomposition space theory, or *shrinking*.

The Schoenflies problem is a fundamental question about spheres embedded in Euclidean space. Denote the d-dimensional Euclidean space by $\mathbb{R}^d$, the closed unit disc or ball in $\mathbb{R}^d$ by D^d, and the d-dimensional sphere by S^d. We identify S^d with the boundary ∂D^{d+1}. The original Schoenflies problem can be stated as the conjecture that *for all d, every continuous embedding of S^d into $\mathbb{R}^{d+1}$ extends to a continuous embedding of D^{d+1} into $\mathbb{R}^{d+1}$.*

In 1913, the 1-dimensional case, more commonly known as the Jordan curve theorem, was proved in full generality by Carathéodory [Car13] and Osgood-Taylor [OT13] using elaborate methods from complex analysis. The 2-dimensional case was studied in the 1920s by Alexander, who first circulated an unpublished manuscript claiming a proof but soon discovered a counterexample [Ale24], which is now called the *Alexander horned sphere*, shown in Figure 3.1. Later, Alexander found that the Schoenflies conjecture holds in dimension two, given the existence of a *bicollar* [Ale30].

Definition 3.1 A continuous embedding $f\colon S^d \to \mathbb{R}^{d+1}$ has a *bicollar* if f extends to a continuous embedding $F\colon S^d \times [-1,1] \to \mathbb{R}^{d+1}$ such that F restricted to $S^d \times \{0\}$ is equal to f. We say that F is a *bicollared embedding* of S^d.

With the bicollared hypothesis added, the following became known as the Schoenflies conjecture.

Conjecture 3.2 (Schoenflies) *For all d, every bicollared embedding of S^d into $\mathbb{R}^{d+1}$ extends to a continuous embedding of D^{d+1} into $\mathbb{R}^{d+1}$.*

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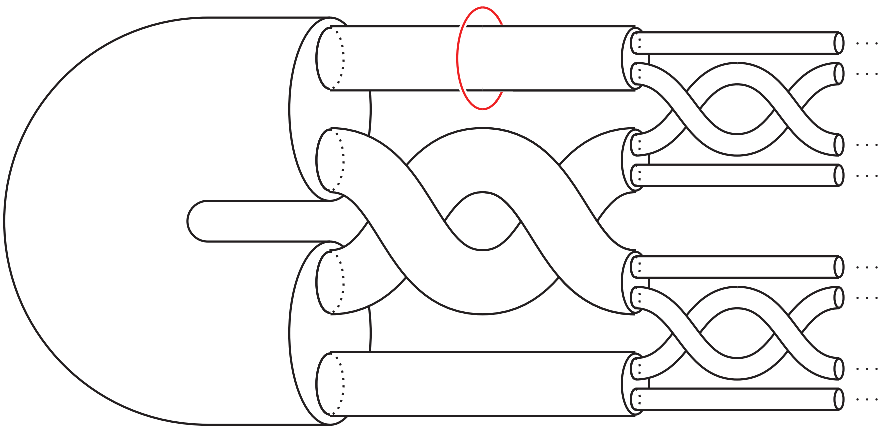


Figure 3.1 The Alexander horned sphere. Perform the indicated infinite construction, then add in a Cantor set to compactify the union of the tubes and obtain a topological embedding of a 2-sphere in $\mathbb{R}^3$. The open exterior region is not simply connected, since, for example, the red circle is not null-homotopic.

In the 30 years that followed, almost no progress was made. In the 1950s there was pervasive pessimism among manifold topologists regarding the topological category. A watershed moment came in 1959 when Mazur gave his partial proof of the Schoenflies conjecture [Maz59], which we now explain.

3.1 Mazur's Theorem

Mazur's proof uses a principle known as the *Eilenberg swindle*, which appears, for example, in the proof of the following observation in commutative algebra. Let A be any projective module over some ring. Since A is projective, it can be written as a direct summand $A \oplus B \cong F$, where F is a free module and B is some module. Then on the one hand we have

$$(A \oplus B) \oplus (A \oplus B) \oplus (A \oplus B) \oplus \cdots \cong F^\infty,$$

while on the other hand a different grouping of the summands gives

$$A \oplus (B \oplus A) \oplus (B \oplus A) \oplus (B \oplus A) \oplus \cdots \cong A \oplus F^\infty,$$

since the direct sum is associative and $B \oplus A \cong A \oplus B \cong F$. Thus, $F^\infty \cong A \oplus F^\infty$. In other words, A becomes an infinite-dimensional free module upon direct sum with an infinite-dimensional free module. That is, any projective module is *stably free* in the infinite-dimensional context.

Example 3.3 (Do knots have inverses?) The following is a standard application of the Eilenberg swindle in topology. Knots in $\mathbb{R}^3$ (or S^3) can be added by forming connected sums. We ask whether, given a knot A, there is a knot B such that the connected sum $A\#B$

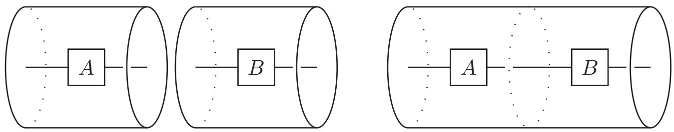


Figure 3.2 Adding knots in cylinders. The boxes denote tangles; that is, the braid closure of the strand lying within the box labelled A (respectively, B) is the knot A (respectively, B).

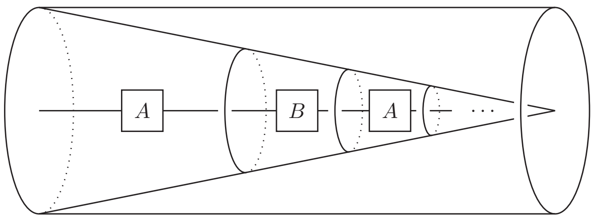


Figure 3.3 By stacking cylinders together, we construct the connected sum of infinitely many copies of $A\#B$ in a cone.

is ambiently isotopic to the trivial knot U; that is, $A\#B \cong U$. We think of knots as strands within a cylinder, and, indeed, every knot is the 'braid closure' of such a knotted strand. Then, the connected sum operation is realized by stacking cylinders next to each other, as shown in Figure 3.2. This operation is both commutative and associative. To see the commutativity, start with $A\#B$, shrink B so that it becomes very small compared to A, slide it along A to the other side, and let it grow again.

Assume that a knot A has an inverse B; that is, $A\#B \cong U$. This implies we can unknot $A\#B$ using an ambient isotopy entirely supported in the two cylinders. Now the swindle works as follows. Take the connected sum of infinitely many copies of $A\#B$ and think of the resulting knot as living in a cone, which in turn lives in a cylinder, as shown in Figure 3.3. The cone forces the summands to get progressively smaller, so they limit to a point at the tip of the cone. Then we have an ambient isotopy,

$$(A\#B)\#(A\#B)\#(A\#B)\#\cdots \cong \#_{i=1}^{\infty} U \cong U,$$

while a different grouping gives another ambient isotopy,

$$A\#(B\#A)\#(B\#A)\#(B\#A)\#\cdots \cong A\#(\#_{i=1}^{\infty} U) \cong A,$$

where we use the fact that $B\#A \cong U$ and apply infinitely many small ambient isotopies. Thus, A must be ambiently isotopic to the trivial knot. This proves that a nontrivial knot does not admit an inverse.

The above proof has the drawback that it loses category; that is, we may have started with smooth or piecewise linear knots but the conclusion holds only in the topological category, since the ambient isotopy we constructed may not be smooth or piecewise linear at the cone point: we obtain a homeomorphism of S^3 sending A to the unknot rather than a diffeomorphism. Other proofs of the non-cancellation of knots, such as the proof using additivity of the Seifert genus, do not have this drawback.

Mazur used the Eilenberg swindle to give a proof of the Schoenflies theorem, with a hypothesis about a *standard spot*.

Definition 3.4 Let $i\colon S^d \times [-1,1] \to \mathbb{R}^{d+1}$ be a bicollared embedding with a point $p \in S^d$ such that $i(p,0) = 0$. Write $\mathbb{R}^{d+1}$ as $\mathbb{R}^d \times \mathbb{R}$ and then the function i as (i_1, i_2), where $i_1\colon S^d \times [-1,1] \to \mathbb{R}^d$ and $i_2\colon S^d \times [-1,1] \to \mathbb{R}$.

We say that $(p,0) \in S^d \times [-1,1]$ is a *standard spot* of i if there is a standard d-dimensional disc $D^d \subseteq S^d$ around p such that

(a) the function i maps $D^d \times \{0\}$ to a standard round disc in $\mathbb{R}^d \times \{0\}$,

(b) for each $q \in D^d$, the interval $\{q\} \times [-1,1]$ in $S^d \times [-1,1]$ is mapped by i such that $i(q,t) = (i_1(q,0),t)$.

Note that, in particular, the closure of the complement of D^d in S^d is also a standard disc. Morally, this definition means that i is 'as standard as possible' around p.

Theorem 3.5 (Mazur [Maz59]) *Let $d \geq 1$ and let $i\colon S^d \times [-1,1] \to \mathbb{R}^{d+1}$ be a bicollared embedding with a standard spot. Then i extends to a continuous embedding of D^{d+1}.*

Proof Let $i\colon S^d \times [-1,1] \to \mathbb{R}^{d+1}$ be the given bicollared embedding with a standard spot $(p,0)$. By passing to the one-point compactification of $\mathbb{R}^{d+1}$, we can consider i to be an embedding $S^d \times [-1,1] \hookrightarrow S^{d+1}$. Let $D^d \subseteq S^d$ be the disc in the definition of the standard spot. Cut out the image $i(D^d \times [-\frac{1}{2},\frac{1}{2}]) = i(D^d) \times [-\frac{1}{2},\frac{1}{2}]$ around $i(p,0)$. By definition, we have removed a standard ball from S^{d+1}, so the closure of the complement is also a standard ball (in particular, this does not assume the Schoenflies theorem).

Next, we claim that the space $\overline{S^{d+1} \smallsetminus i(S^d \times [-\frac{1}{2},\frac{1}{2}])}$ has two components, as indicated in Figure 3.4 in the case $d = 1$. To see this, let $X := i(S^d \times [-\frac{1}{2},\frac{1}{2}])$ and $Y := \overline{S^{d+1} \smallsetminus X}$. Then $S^{d+1} = X \cup_{i(S^d \times \{-\frac{1}{2},\frac{1}{2}\})} Y$, and so the Mayer–Vietoris sequence yields

$$H_1(S^{d+1}) \to H_0\big(S^d \times \{-\tfrac{1}{2},\tfrac{1}{2}\}\big) \to H_0(X) \oplus H_0(Y) \to H_0(S^{d+1}) \to 0.$$

To apply the Mayer–Vietoris sequence, we use that $i(S^d \times \{\pm\frac{1}{2}\})$ sits inside a larger collar, so is itself bicollared. Since $d \geq 1$, we have that $H_1(S^{d+1}) = 0$ and $\mathbb{Z} \cong H_0(S^{d+1}) \cong H_0(S^d) \cong H_0(X)$. We compute that $H_0(Y) \cong \mathbb{Z}^2$, so Y has two connected components, as claimed. We call these two pieces A^+ and A^-, where A^- is the piece contained in $\mathbb{R}^{d+1} \subseteq S^{d+1}$.

We also see from the existence of the standard spot that the boundary of $A^\pm$ is a d-dimensional sphere that is decomposed into two standard d-dimensional discs $P^\pm$ and $Q^\pm$, as shown in Figure 3.4, where $P^+ = i(D^d \times \{-\frac{1}{2}\})$, $P^- = i(D^d \times \{\frac{1}{2}\})$ and $Q^\pm$ are the closures of the complementary regions.

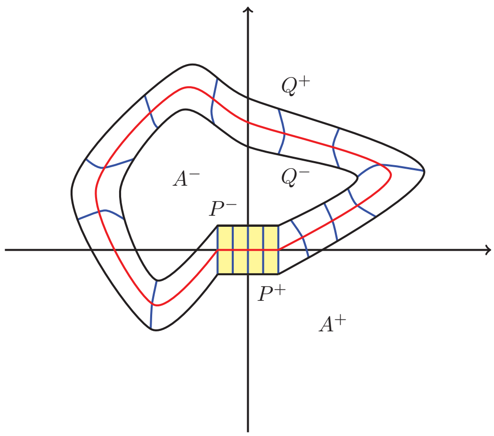


Figure 3.4 Mazur's partial proof of the Schoenflies conjecture. Red denotes the image $i(S^d)$. The standard spot is shown around the origin. Let $A^{\pm}$ denote the connected components of the complement of $i(S^d \times [-\frac{1}{2}, \frac{1}{2}])$. The boundary $i(S^d \times \{-\frac{1}{2}\})$ is decomposed into $P^+ \cup Q^+$ and $i(S^d \times \{\frac{1}{2}\})$ is decomposed into $P^- \cup Q^-$.

Consider the space

$$A^- \cup_Q A^+ := A^- \cup_{Q^-} i((\overline{S^d \setminus D^d}) \times [-\tfrac{1}{2}, \tfrac{1}{2}]) \cup_{Q^+} A^+.$$

By definition, this is the closure of the complement in S^{d+1} of $i(D^d \times [-\frac{1}{2}, \frac{1}{2}])$, and we have already established that it is homeomorphic to D^{d+1}. Next we show that the space

$$A^- \cup_P A^+ := A^- \cup_{P^-} i(D^d \times [-\tfrac{1}{2}, \tfrac{1}{2}]) \cup_{P^+} A^+$$

is also homeomorphic to D^{d+1}. To see this, note that $P^{\pm}$ and $Q^{\pm}$ are ambiently isotopic in $\partial A^{\pm}$, via some ambient isotopy

$$F^{\pm} \colon \partial A^{\pm} \times [0,1] \to \partial A^{\pm},$$

with $F^{\pm}|_{\partial A^{\pm} \times \{0\}} = \mathrm{Id}$, since $P^{\pm}$ and $Q^{\pm}$ may be considered to be the standard northern and southern hemispheres of the d-dimensional sphere $\partial A^{\pm}$. By construction, $\partial A^{\pm}$ is collared. Thus there is an embedding, $\partial A^{\pm} \times [0,1] \to A^{\pm}$, with $\partial A^{\pm} \times \{1\}$ mapped homeomorphically to $\partial A^{\pm}$. Then we have the following homeomorphism obtained by inserting the ambient isotopies into the boundary collars.

$$\begin{array}{ccccccccc}
A^- & \cup_{P^-} & \partial A^- \times [0,1] & \cup_{P^-} & i(D^d \times [-\frac{1}{2}, \frac{1}{2}]) & \cup_{P^+} & \partial A^+ \times [0,1] & \cup_{P^+} & A^+ \\
\downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle F^- \times \mathrm{Id}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle F^+ \times \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id}} \\
A^- & \cup_{P^-} & \partial A^- \times [0,1] & \cup_{Q^-} & i((\overline{S^d \setminus D^d}) \times [-\frac{1}{2}, \frac{1}{2}]) & \cup_{Q^+} & \partial A^+ \times [0,1] & \cup_{P^+} & A^+
\end{array}$$

The middle map is obtained using the abstract homeomorphism $D^d \cong \overline{S^d \smallsetminus D^d}$. The diagram shows that $A^- \cup_P A^+ \cong A^- \cup_Q A^+ \cong D^{d+1}$, as desired.

We are now ready for the Eilenberg swindle. We have the following sequence of homeomorphisms:

$$
\begin{aligned}
D^{d+1} &\cong \left(A^- \cup_Q A^+\right) \cup_{P^-} i\left(D^d \times \left[-\frac{1}{2}, \frac{1}{2}\right]\right) \cup_{P^+} \left(A^- \cup_Q A^+\right) \cup_{P^-} \cdots \cup \{\infty\} \\
&\cong A^- \cup_{Q^-} i\left(\left(\overline{S^d \smallsetminus D^d}\right) \times \left[-\frac{1}{2}, \frac{1}{2}\right]\right) \cup_{Q^+} \left(A^+ \cup_P A^-\right) \cup_{Q^-} \cdots \cup \{\infty\} \\
&\cong A^- \cup_{Q^-} i\left(\left(\overline{S^d \smallsetminus D^d}\right) \times \left[-\frac{1}{2}, \frac{1}{2}\right]\right) \cup_{Q^+} D^{d+1} \cup_{Q^-} D^{d+1} \cup \cdots \cup \{\infty\} \\
&\cong A^- \cup_{Q^-} i\left(\left(\overline{S^d \smallsetminus D^d}\right) \times \left[-\frac{1}{2}, \frac{1}{2}\right]\right) \cup_{Q^+} D^{d+1} \\
&\cong A^-.
\end{aligned}
$$

For the first homeomorphism above, we are using the fact that $A^- \cup_Q A^+ \cong D^{d+1}$ and that gluing infinitely many balls in pairs along balls of one lower dimension in their boundaries and then taking the one-point compactification gives another ball. The second step is the Eilenberg swindle, where the rebracketing occurs. The third step uses that $A^+ \cup_P A^- \cong D^{d+1}$, as shown above. Then we again use the fact that a compactified infinite sequence of balls glued together along balls of one lower dimension is homeomorphic to a ball. The last homeomorphism is easier: the boundary connected sum of finitely many balls is homeomorphic to a ball, and since ∂A^- is collared, boundary connected sum with a ball along part of its boundary is trivial. We have now shown that the space A^- is homeomorphic to D^{d+1}. Note that the closure of the component of the complement of $i(S^d \times \{0\})$ in $\mathbb{R}^{d+1} \subseteq S^{d+1}$ is A^- equipped with a boundary collar and thus is also homeomorphic to D^{d+1}. The proof is then completed by the Alexander trick, which provides an extension of a given homeomorphism $S^d \to S^d$ to a homeomorphism $D^{d+1} \to D^{d+1}$. □

Note that we could reverse the rôles of A^- and A^+ in the proof above to conclude that A^+ is also a ball. Thus, we have shown that given a bicollared embedding $i\colon S^d \hookrightarrow S^{d+1}$ with a standard spot, both of the connected components of the complement $S^{d+1} \smallsetminus i(S^d)$ have closures homeomorphic to D^{d+1}. Next we show that the standard spot is not required.

3.2 Morse's Theorem

Mazur's work generated a lot of interest in the problem of removing the standard spot hypothesis. This was solved in 1960 in a paper by Morse [Mor60] using a technique called *push-pull*. We introduce it by proving a theorem that uses the technique.

Theorem 3.6 (Application of push-pull, Brown (unpublished), see [EK71, p. 85; Sie68, p. 535; Sie70b, Corollary 5.4]) *Let X and Y be compact metric spaces. If $X \times \mathbb{R}$ is homeomorphic to $Y \times \mathbb{R}$, then $X \times S^1$ is homeomorphic to $Y \times S^1$.*

Sketch of proof Let $h\colon X \times \mathbb{R} \to Y \times \mathbb{R}$ be a homeomorphism. The key point in this argument will be that $Y \times \mathbb{R}$ has two product structures, the intrinsic one and the one induced from $X \times \mathbb{R}$ via h.

Let X_t denote $X \times \{t\}$ for $t \in \mathbb{R}$ and let $X_{[t,u]}$ denote $X \times [t,u]$ for $[t,u] \subseteq \mathbb{R}$. Similarly, let Y_s denote $Y \times \{s\}$ for $s \in \mathbb{R}$ and let $Y_{[r,s]}$ denote $Y \times [r,s]$ for $[r,s] \subseteq \mathbb{R}$. By compactness of X and Y, there exist $a < c < e$ and $b < d$ such that

(1) $Y_a, Y_c, Y_e, h(X_b)$, and $h(X_d)$ are pairwise disjoint in $Y \times \mathbb{R}$,

(2) $h(X_b) \subseteq Y_{[a,c]}$,

(3) $Y_c \subseteq h(X_{[b,d]})$,

(4) $h(X_d) \subseteq Y_{[c,e]}$,

as illustrated in the leftmost panel in Figure 3.5. This may be achieved by first fixing a, and then choosing as follows.

- Choose b so that (1) is satisfied for a and b.
- Choose $c > a$ so that (1) and (2) are satisfied for a, b, and c.
- Choose $d > b$ so that (1) and (3) are satisfied for a, b, c, and d.
- Choose $e > c$ so that (1) and (4) are satisfied.

Now we construct a self-homeomorphism χ of $Y \times \mathbb{R}$ as the composition

$$\chi = C^{-1} \circ P_Y \circ P_X \circ C,$$

where the steps are illustrated in Figure 3.5. The maps P_X and P_Y will constitute the actual pushing and pulling, while C, which we might call *cold storage*, makes sure that nothing is pushed or pulled unless it is supposed to be.

The maps are obtained as follows:

- The map C rescales the intrinsic $\mathbb{R}$-coordinate of $Y \times \mathbb{R}$ such that $C(Y_{[a,c]})$ lies below $h(X_b)$ and leaves $h(X_d)$ untouched. We require C to be the identity on $Y_{[c+\varepsilon,\infty)}$ and $Y_{(-\infty,a]}$, for ε small enough so that $Y_{c+\varepsilon} \subsetneq h(X_{[b,d]})$.
- The map P_X pushes $h(X_d)$ down to $h(X_b)$ along the $\mathbb{R}$-coordinate induced by h—that is, the image of the product structure of $X \times \mathbb{R}$—without moving $C(Y_{[a,c]})$.
- The map P_Y pulls $h(X_b) = (P_X \circ C \circ h)(X_d)$ up along the intrinsic $\mathbb{R}$-coordinate of $Y \times \mathbb{R}$ so that it lies above the support of C^{-1}, again without moving $C(Y_{[a,c]})$. This can be done in such a way that P_Y is supported below Y_e.

The map χ is the identity outside of $Y_{[a,e]}$. Observe that χ leaves $h(X_b)$ untouched and that $\chi(h(X_d))$ appears as a translate of $h(X_b)$ in the intrinsic $\mathbb{R}$-coordinate.

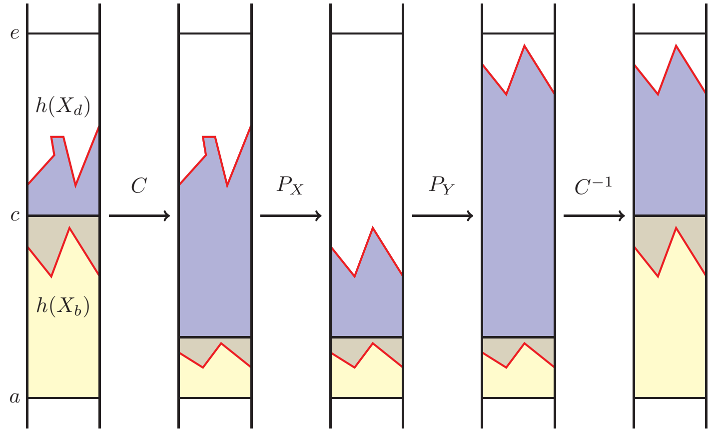


Figure 3.5 The push-pull construction. Each panel depicts the space $Y \times \mathbb{R}$. The blue and yellow regions denote $h(X_{[b,d]})$ and $Y_{[a,c]}$, respectively. Note that the regions overlap.

The embedding $H := \chi \circ h\colon X \times [b,d] \to Y \times \mathbb{R}$ now descends to a homeomorphism $X \times S^1 \to Y \times S^1$. That such a continuous map exists is straightforward to see. It takes some work to verify that it is a bijection; we leave this for the reader. The closed map lemma (Lemma 3.23) then shows that the inverse is also continuous. □

Remark 3.7 The converse of Theorem 3.6 is not true in general. There exist compact manifolds X and Y such that $X \times S^1$ and $Y \times S^1$ are diffeomorphic but $X \times \mathbb{R}$ and $Y \times \mathbb{R}$ are not even homotopy equivalent [Cha65, Theorem 3.9].

Remark 3.8 The compactness hypothesis of Theorem 3.6 is necessary. That is, there exist examples of noncompact metric spaces X and Y such that $X \times \mathbb{R}$ and $Y \times \mathbb{R}$ are homeomorphic but $X \times S^1$ and $Y \times S^1$ are not, as follows. Let $\Sigma_{g,n}$ denote the compact, orientable surface with genus g and n boundary components. Note that $\Sigma_{g,1} \times [0,1]$ is homeomorphic to $\Sigma_{0,2g+1} \times [0,1]$. Indeed, both are obtained from D^3 by attaching $2g$ orientable 1-handles, and there is an essentially unique way to attach orientable 3-dimensional 1-handles to D^3. Let X and Y be the interiors of $\Sigma_{g,1}$ and $\Sigma_{0,2g+1}$, respectively. Then $X \times \mathbb{R}$ and $Y \times \mathbb{R}$ are homeomorphic (indeed, diffeomorphic), since they are the interiors of the homeomorphic spaces $\Sigma_{g,1} \times [0,1]$ and $\Sigma_{0,2g+1} \times [0,1]$, respectively. However, the end of $X \times S^1$ is homotopy equivalent to a torus, but the set of ends of $Y \times S^1$ is homotopy equivalent to the disjoint union of $2g+1$ copies of tori. Therefore $X \times S^1$ is not homeomorphic to $Y \times S^1$.

Morse used the technique of push-pull to prove the following theorem.

Theorem 3.9 (Morse [Mor60]) *For all d, every bicollared embedding $S^d \times [-1,1] \to \mathbb{R}^{d+1}$ has a standard spot after applying a self-homeomorphism of $\mathbb{R}^{d+1}$.*

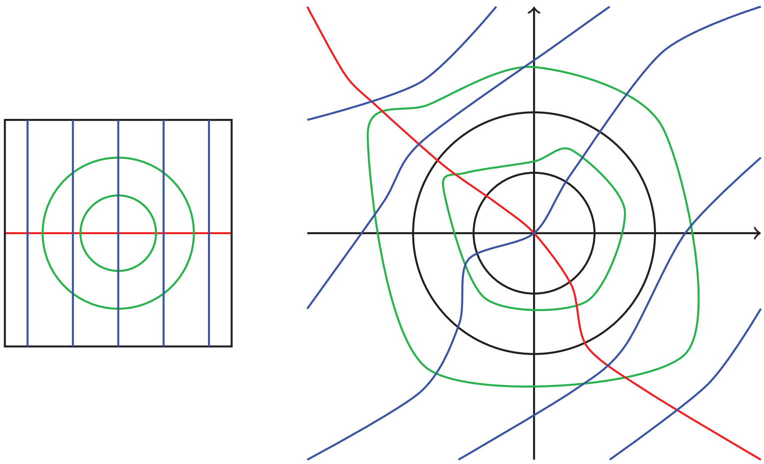


Figure 3.6 Creating a standard spot. Left: The disc D^d appears as the central horizontal red segment. The blue vertical lines show the induced coordinate system on $D^d \times [-1,1]$. A collection of standard round spheres are indicated in green. Right: We show the image of $D^d \times [-1,1]$ in $\mathbb{R}^{d+1}$. The proof of Theorem 3.9 compares the intrinsic round spheres (black) with the induced round spheres (green).

Sketch of proof Consider a bicollared embedding, $i\colon S^d \times [-1,1] \to \mathbb{R}^{d+1}$, and fix a point, $p \in S^d$. Up to translation, we can assume that $i(p,0) = 0$. Choose local coordinates on a standard disc $D^d \subseteq S^d$ containing p, which yields an induced local coordinate system on $i(D^d \times [-1,1]) \subseteq \mathbb{R}^{d+1}$, as shown in Figure 3.6.

In this new local coordinate system on $\mathbb{R}^{d+1}$, the embedded sphere has a standard spot, so it remains to extend it to a global coordinate system. We achieve this by using a push-pull argument. The idea is to compare the standard polar coordinate system in $\mathbb{R}^{d+1}$ with the one induced by i. Again, by compactness we can find interlaced pairs of standard d-dimensional spheres in $\mathbb{R}^{d+1}$ and homeomorphically mapped spheres, as indicated in Figure 3.6. Then, by using push-pull, we can find an isotopy that transforms one of the homeomorphically mapped spheres into a translate of the other homeomorphically mapped sphere along the standard radial coordinate and preserves a neighbourhood of the origin. □

Combining the results of Mazur and Morse, we immediately deduce the following theorem.

Theorem 3.10 (Schoenflies theorem) *Let $d \geq 1$. Every bicollared embedding of S^d into $\mathbb{R}^{d+1}$ extends to a continuous embedding of D^{d+1}.*

As a historical note, by the time Morse had augmented Mazur's argument with his theorem, Brown had already given an independent and complete proof of the Schoenflies theorem, which we will discuss shortly.

We observe that the utility of the push-pull technique is in gaining control over a homeomorphism in one linear direction. As we will see, a major technical problem when working with topological manifolds is to gain control of a homeomorphism in many directions simultaneously. Results in this direction culminated in Kirby's work on the torus trick [Kir69].

We end this section by stating some more applications of push-pull to topological manifolds.

Theorem 3.11 (**[Bro62, Theorem 3]**) *A locally bicollared codimension one embedding in any topological manifold is globally bicollared.*

Theorem 3.12 (**[Bro62, Theorem 2]**) *The boundary of every topological manifold is collared.*

Theorem 3.13 (**[Arm70, Theorem 2]**) *Any two locally flat collars for either a codimension one submanifold or for the boundary of a topological manifold are ambiently isotopic to one another.*

3.3 Shrinking Cellular Sets

At the end of this chapter, we will give Brown's alternative proof of the Schoenflies theorem. In this section, we set the stage by introducing certain elementary notions from *decomposition space theory*, a field of ideas that will be central to the proof of the disc embedding theorem. In this section we follow [Bro60] and [Dav07].

Definition 3.14 Let M^d be a d-dimensional manifold. A subset $X \subseteq M^d$ is said to be *cellular* if it is the intersection of countably many nested closed balls in M^d; that is, if there exist embedded, closed d-dimensional balls $B_i \subseteq M^d$, $i \geq 1$, with $B_i \cong D^d$, such that $B_{i+1} \subseteq \operatorname{Int} B_i$ and $X = \bigcap_{i=1}^{\infty} B_i$.

Figure 3.7 illustrates that the letter

$$\mathsf{X} := \{(x, y) \in D^2 \mid x^2 = y^2, |x| \leq 1/2\}$$

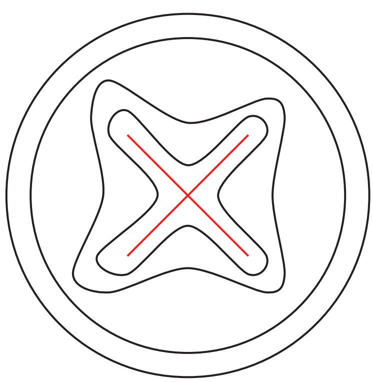


Figure 3.7 A cellular set (red) in D^2. The boundaries of embedded discs (black) converging to the cellular set are shown.

is a cellular subset of D^2. Most of the cellular sets in this section will be denoted by the symbol X. We begin with some elementary properties of cellular sets.

Proposition 3.15 *Every cellular subset X of a manifold M is closed and compact.*

Proof Let $\{B_i\}$ be the nested balls, as in Definition 3.14. Then $X = \bigcap_{i=1}^{\infty} B_i$ is closed as an intersection of closed sets. Further, X is compact, since it is a closed subset of the compact space B_1. □

Proposition 3.16 *Let X be a cellular set in a d-dimensional manifold M, and let U be an open set with $X \subseteq U$. Then there exist embedded, closed d-dimensional balls $B_i \subseteq U$, $i \geq 1$, with $B_i \cong D^d$ and $B_{i+1} \subseteq \operatorname{Int} B_i$, for all i such that $X = \bigcap_{i=1}^{\infty} B_i$.*

Proof By definition, there exist embedded, closed d-dimensional balls $B_i \subseteq M, i \geq 1$, with $B_i \cong D^d$, such that $B_{i+1} \subseteq \operatorname{Int} B_i$ and $X = \bigcap_{i=1}^{\infty} B_i$. It suffices to show that there exists a j such that $B_j \subseteq U$. Suppose not. Then for all i, $B_i \cap (M \setminus U)$ is nonempty. Choose a point $x_i \in B_i \cap (M \setminus U) \subseteq B_1$ for each i. Since B_1 is sequentially compact, the sequence $\{x_i\}$ has a convergent subsequence $\{x_{i_k}\}$, converging to some $x \in B_1$. We assert that $x \in \bigcap B_i$. To see this, note that $x_{i_k} \in B_{i_k}$ for all k. Fix ℓ. Then for $k \geq \ell$, $x_{i_k} \in B_{i_k} \subseteq B_{i_\ell}$. Since B_{i_ℓ} is closed, $x \in B_{i_\ell}$. Thus $x \in B_{i_\ell}$ for all ℓ, so $x \in \bigcap B_i$, as asserted. We therefore have that $x \in \bigcap B_i = X \subseteq U$ and $\{x_{i_k}\}$ is contained in the closed set $M \setminus U$, which is a contradiction, since closed sets contain their limit points. □

Remark 3.17 Note that cellularity is not an intrinsic property of a space X but rather depends on its specific embedding within the ambient space. For example, there exist noncellular embeddings of a closed arc in S^3 [Edw80].

The key property of cellular sets is that they can be *shrunk* by homeomorphisms, as seen in the following proposition.

Proposition 3.18 *Let X be a cellular set in a d-dimensional manifold M and let U be an open set with $X \subseteq U$. For every $\varepsilon > 0$, there exists a homeomorphism $h_\varepsilon \colon M \to M$ such that h_ε is the identity outside U and $\operatorname{diam} h_\varepsilon(X) < \varepsilon$.*

Proof By Proposition 3.16, there is a closed ball B in U such that $X \subseteq \operatorname{Int} B$. Since X is closed by Proposition 3.15, there exists a collar N of ∂B disjoint from X. Find a ball D in $B \setminus N$ such that $\operatorname{diam} D < \varepsilon$. Now pick a homeomorphism $s_\varepsilon \colon B \to B$ which is the identity on the boundary ∂B and maps the complement of the collar N into D. Consequently, $s_\varepsilon(X) \subseteq D$ and therefore $\operatorname{diam} s_\varepsilon(X) < \varepsilon$. The map h_ε is obtained by extending s_ε to all of M by the identity map. □

For a homeomorphism h_ε as in the statement above, we say that $h_\varepsilon \colon M \to M$ *shrinks X in U to diameter less than ε.*

Our eventual goal is to use decomposition space theory, specifically the idea of shrinking, to approximate certain functions by homeomorphisms. Next we define precisely what this means.

Let X and Y be compact metric spaces. Recall that the uniform metric is defined by setting $d(f,g) = \sup_{x \in X} d_Y(f(x), g(x))$ for functions $f, g \colon X \to Y$. We denote the metric space of continuous functions from X to Y, equipped with the uniform metric, by $\mathcal{C}(X,Y)$. This is known to be a complete metric space [Mun00, Theorems 43.6 and 45.1]. Observe that the metric space $\mathcal{C}(X,X)$ contains the subspace $\mathcal{C}_A(X,X)$ of functions $f\colon X \to X$ with $f|_A = \mathrm{Id}_A$, for any subset $A \subseteq X$. This is a closed set in $\mathcal{C}(X,X)$ and thus is itself a complete metric space under the induced metric. The following definition formalizes the notion of approximating functions by homeomorphisms.

Definition 3.19 Let X and Y be compact metric spaces and let $f\colon X \to Y$ be a surjective continuous map. The map f is said to be *approximable by homeomorphisms* if there is a sequence of homeomorphisms $\{h_n\colon X \to Y\}_{n=1}^{\infty}$ that converges to f in $\mathcal{C}(X,Y)$.

In particular, a necessary condition for f to be approximable by homeomorphisms is that X and Y are homeomorphic. In applications, frequently we will not know that X and Y are homeomorphic until we have shown that a map $f\colon X \to Y$ is approximable by homeomorphisms.

We will often wish to approximate quotient maps by homeomorphisms. This will only be meaningful when the quotient spaces are metric spaces. We record the following fact for use in this chapter. This will later be subsumed by Corollary 4.13.

Given a surjective map $f\colon X \to Y$ between topological spaces, we say that a subset $C \subseteq X$ is *saturated* (with respect to f) if, whenever $f^{-1}(y)$ intersects C, for some $y \in Y$ we have $f^{-1}(y) \subseteq C$, or in other words, the set C is a union of fibres of f.

Proposition 3.20 *Let M be a compact d-dimensional manifold, possibly with nonempty boundary. Let $X \subseteq \operatorname{Int} M$ be a cellular set. Then the quotient M/X is a compact metric space.*

Proof Fix some metric on M, inducing its topology. The quotient M/X is compact, since M is compact. We show that M/X is Hausdorff. Let x denote the image of X in M/X. Choose $y, z \in M/X$ with $y \neq z$. Consider the quotient map $\pi\colon M \to M/X$. The restriction of π to the saturated open set $M \setminus X$ is an open, continuous bijection and thus a homeomorphism. If $y, z \neq x$, then $\pi^{-1}(y)$ and $\pi^{-1}(z)$ are distinct points in $M \setminus X$ with disjoint open neighbourhoods in $M \setminus X$ which are mapped by π to disjoint open neighbourhoods in M/X. Moreover, since M is a metric space and X is closed, we can find disjoint open neighbourhoods of X and $\pi^{-1}(y)$ in M. These are saturated open sets and are thus mapped to (disjoint) open sets in M/X separating x and y. Therefore, M/X is Hausdorff. This finishes the proof, since the continuous image of a compact metric space in a Hausdorff space is metrizable [Wil70, Corollary 23.2, p. 166]. □

A simple class of quotient maps consists of those where a unique point in the codomain has more than one point in its preimage. In other words, such a map is many to one on this pre-image but one to one everywhere else. The following terminology will be helpful in describing such maps.

Definition 3.21 Let $f\colon X \to Y$ be a map between topological spaces. The set $f^{-1}(y)$, where $y \in Y$, is called an *inverse set* of f if $|f^{-1}(y)| > 1$.

The following proposition shows that crushing a cellular set to a point does not change the homeomorphism type of a manifold.

Proposition 3.22 *Let M be a compact d-dimensional manifold, possibly with nonempty boundary. Let $X \subseteq \operatorname{Int} M$ be a cellular set. Then the quotient map $\pi\colon M \to M/X$ is approximable by homeomorphisms. In particular, the quotient space M/X is homeomorphic to M.*

Before giving the proof, we recall the following elementary lemma, since we will use it frequently.

Lemma 3.23 (Closed map lemma) *Every continuous map from a compact space A to a Hausdorff space B sends closed subsets of A to closed subsets of B.*

Proof Let $U \subseteq A$ be closed. Then U is compact as a closed set in a compact space. Continuous maps preserve compactness, so $f(U)$ is compact. Finally, a compact subset of a Hausdorff space is closed, so $f(U) \subseteq B$ is closed. □

Proof of Proposition 3.22 Fix metrics on M and M/X, using Proposition 3.20. Most of the proof will consist of building a surjective, continuous function $f\colon M \to M$ which has X as its unique inverse set.

Since X is cellular, there is a family $\{B_i\}$ of closed balls with $B_0 \subseteq \operatorname{Int} M$, $B_{i+1} \subseteq \operatorname{Int} B_i$ for all i, and $\bigcap_i B_i = X$. We inductively define a family of homeomorphisms $f_i\colon M \to M$, starting with $f_0 = \mathrm{Id}_M$. Assume that f_i is already defined for some i. From the proof of Proposition 3.18, there is a homeomorphism $h_i\colon M \to M$ shrinking $f_i(B_{i+1})$ in $f_i(B_i)$ to diameter less than $\frac{1}{i+1}$ that restricts to the identity outside $\operatorname{Int} f_i(B_i)$. Define $f_{i+1} = h_i \circ f_i$. Note that $\operatorname{diam} f_i(B_i) < \frac{1}{i}$ for all i, by construction.

Next we will show that the sequence $\{f_i\}$ in the complete metric space $\mathcal{C}_{\partial M}(M,M)$ is Cauchy. Fix integers $m > n$. Note that $f_m = f_n$ outside B_n. For every point $x \in B_n$, we have that $f_m(x), f_n(x) \in f_n(B_n)$, and, as $\operatorname{diam} f_n(B_n) < \frac{1}{n}$, we get $d(f_m(x), f_n(x)) < \frac{1}{n}$. This implies that $d(f_n, f_m) < \frac{1}{n}$ in $\mathcal{C}_{\partial M}(M,M)$ and so $\{f_i\}$ is a Cauchy sequence. We define f to be the limit of the sequence $\{f_i\}$, which exists since $\mathcal{C}_{\partial M}(M,M)$ is complete. By construction, if $x \notin B_i$, then $f(x) = f_i(x)$.

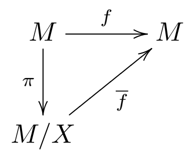
Next, we show that f has the correct inverse sets. Let $z \in M$ be such that $z \notin B_i$ for some i, and let $x \in X$. Then

$$d(f(z), f(x)) = d(f_i(z), f(x)) \geq d(f_i(z), f_i(B_{i+1})) > 0.$$

Above, in the penultimate inequality, we use the fact that $f(x) \in f_i(B_{i+1})$. In the final inequality, we use that $B_{i+1} \subseteq \operatorname{Int} B_i$, so for every $z \notin B_i$, $d_X(z, B_{i+1}) > 0$. The inequality follows, since f_i is a homeomorphism on $M \smallsetminus \operatorname{Int} B_{i+1}$. Thus $f(X)$ is disjoint from $f(M \smallsetminus X)$. Additionally, note that $\operatorname{diam} f_i(X) < \frac{1}{i}$ for all i, and thus $f(X)$ consists of a single point, y. As a result, $f^{-1}(y) = X$.

Next we show that $f^{-1}(z)$ for $z \neq y$ consists of precisely one element. Note that $f^{-1}(z) \subseteq M \smallsetminus X$. Thus $f^{-1}(z) = (f|_{M \smallsetminus X})^{-1}(z)$, and it suffices to show that $f|_{M \smallsetminus X}$ is injective. Given any two points $p, q \in M \smallsetminus X$, there exists some i such that $p, q \notin B_i$. Then, $f(p) = f_i(p)$ and $f(q) = f_i(q)$. Since each f_i is a homeomorphism and therefore injective, this completes the proof that X is the unique inverse set of f.

Finally we are ready to investigate the quotient map $\pi\colon M \to M/X$ directly. Note that the surjective map f descends to a map $M/X \to M$ via the quotient map, and we obtain a bijective continuous function $\overline{f}\colon M/X \to M$.



By the closed map lemma (Lemma 3.23), $\overline{f}$ is a homeomorphism. Note that $f = \overline{f} \circ \pi$.

Given the sequence of homeomorphisms $\{f_i \colon M \to M\}$ converging to f, consider the functions

$$\{\overline{f}^{-1} \circ f_i \colon M \to M/X\}.$$

These are homeomorphisms, since $\overline{f}^{-1}$ and f_i are. The sequence $\{\pi_i := \overline{f}^{-1} \circ f_i\}$ converges to

$$\overline{f}^{-1} \circ f = \overline{f}^{-1} \circ \overline{f} \circ \pi = \pi,$$

as desired, since $\overline{f}^{-1}$ is uniformly continuous by Theorem 3.25 below. □

There is no need to restrict ourselves to the case of a single cellular set. We will show next that any finite collection of cellular sets in a manifold can be crushed to individual points (one per cellular set) while preserving the homeomorphism type of the manifold. We will need the following proposition.

Proposition 3.24 *Let $f \colon X \to Y$ and $g \colon Y \to Z$ be maps between compact metric spaces that are approximable by homeomorphisms. Then $g \circ f \colon X \to Z$ is also approximable by homeomorphisms.*

We will use the next elementary result from analysis [Rud76, Theorem 4.19] in the proof of the proposition, but also many times in the future, so we record it here.

Theorem 3.25 (Heine–Cantor theorem) *Let Y be a metric space, X be a compact metric space, and $f \colon X \to Y$ be a continuous function. Then f is uniformly continuous.*

Proof of Proposition 3.24 Let $\varepsilon > 0$. The Heine–Cantor theorem implies that, since Y is compact and g is continuous, the function $g \colon Y \to Z$ is uniformly continuous. Thus there is a $\delta > 0$ such that $d_Z(g(y), g(y')) < \frac{\varepsilon}{2}$ whenever $y, y' \in Y$ are such that $d_Y(y, y') < \delta$.

Recall that for two functions $f, f' \colon X \to Y$, the uniform metric is defined by $d(f, f') := \sup_{x \in X} d_Y(f(x), f'(x))$. Similarly, for functions $g, g' \colon Y \to Z$ we have $d(g, g') := \sup_{y \in Y} d_Z(g(y), g'(y))$. Let $\{f_n \colon X \to Y\}$ be a sequence of homeomorphisms converging to f and let $\{g_m \colon Y \to Z\}$ be a sequence of homeomorphisms converging to g. That is, there exists $N > 0$ such that $d(f, f_n) < \delta$ whenever $n \geq N$, and similarly there exists $M > 0$ such that $d(g, g_m) < \frac{\varepsilon}{2}$ whenever $m \geq M$.

Let L be the maximum of M and N. Then, for every $x \in X$ and every $n \geq L$, $d_Y(f(x), f_n(x)) < \delta$. By the uniform continuity property,

$$d_Z(g(f(x)), g(f_n(x))) < \tfrac{\varepsilon}{2}.$$

Also, for every $x \in X$ and for every $n \geq L$, we have

$$d_Z(g(f_n(x)), g_n(f_n(x))) < \tfrac{\varepsilon}{2}.$$

Thus for every $x \in X$ and for every $n \geq L$, we have

$$\begin{aligned} d_Z(g(f(x)), g_n(f_n(x))) &\leq d_Z(g(f(x)), g(f_n(x))) + d_Z(g(f_n(x)), g_n(f_n(x))) \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, $d(g \circ f, g_n \circ f_n) < \varepsilon$ for every $n \geq L$, and so $g_n \circ f_n \to g \circ f$. Since $g_n \circ f_n$ is a homeomorphism for every n, this proves that $g \circ f$ is approximable by homeomorphisms. □

Proposition 3.26 *Let M be a compact d-dimensional manifold, possibly with nonempty boundary. Let $\{X_1, \ldots, X_n\}$ be a finite collection of pairwise disjoint cellular sets in $\operatorname{Int} M$. Then the quotient map $\pi\colon M \to M/\{X_1, \ldots, X_n\}$ is approximable by homeomorphisms. In particular, the quotient $M/\{X_1, \ldots, X_n\}$ is homeomorphic to M.*

We point out that the space $M/\{X_1, \ldots, X_n\}$ is the quotient where the collection of cellular subsets $X_1, \ldots, X_n$ is crushed to n distinct points rather than $M/\bigcup\{X_1, \ldots, X_n\}$, where all the X_i are identified to a single point.

Proof of Proposition 3.26 We give a proof by induction. For the case $n = 1$, see Proposition 3.22. Suppose that the quotient map on M crushing any given pairwise disjoint collection of $n-1$ cellular sets in $\operatorname{Int} M$ is approximable by homeomorphisms, for some $n \geq 2$. The quotient map $\pi\colon M \to M/\{X_1, \ldots, X_n\}$ factors as the composition

$$M \to M/X_1 \to M/\{X_1, \ldots, X_n\}.$$

The first quotient map is approximable by homeomorphisms by Proposition 3.22. The second map is approximable by homeomorphisms by the inductive hypothesis. Here we are using the fact that $\{X_2, \ldots, X_n\}$ is mapped to a pairwise disjoint collection of cellular sets in M/X_1 by the quotient map. This follows from Proposition 3.16 and the fact that the quotient map $M \to M/X_1$ is a homeomorphism when restricted to $M \setminus X_1$. Compositions of maps between compact metric spaces which are approximable by homeomorphisms are themselves approximable by homeomorphisms, according to Proposition 3.24. This completes the proof that π is approximable by homeomorphisms. □

For the application to the Schoenflies theorem, we will need the following proposition.

Proposition 3.27 *Let $f\colon S^d \to S^d$ be a continuous surjection for some d with exactly two inverse sets, A and B. Then each of A and B is cellular.*

To prove Proposition 3.27, we will need the following lemma.

Lemma 3.28 *Let $f\colon D^d \to S^d$ be a continuous function such that $X \subseteq \operatorname{Int} D^d$ is the only inverse set of f and $f(\operatorname{Int} D^d)$ is open in S^d. Then X is cellular in D^d.*

Proof Let D denote the domain D^d. Let $X = f^{-1}(y)$ for some $y \in S^d$. Since $X \subseteq \operatorname{Int} D$ and we know that $f(\operatorname{Int} D)$ is open in S^d, there is an $\varepsilon > 0$ such that the standard disc $B_\varepsilon(y)$ of radius ε around y in S^d is contained in $f(\operatorname{Int} D)$. Next, choose some $z \in S^d$ not in $\operatorname{Im}(D)$. In particular, we have $z \neq y$. Let V be a standard open ball neighbourhood of z in S^d such that $S^d \smallsetminus V$ is a d-dimensional disc B with $B_\varepsilon(y) \subseteq f(D) \subseteq S^d \smallsetminus V = B$. For every $n \in \mathbb{N}$, choose some homeomorphism $s_{\varepsilon/2^n} : S^d \to S^d$ that restricts to the identity on the small disc $B_{\varepsilon/2^{n+1}}(y)$ and squeezes the rest of B into $B_{\varepsilon/2^n}(y)$. In a ball containing B, this could, for instance, be defined radially and then extended to all of S^d, stretching out V so as to cover the complement of $B_{\varepsilon/2^n}(y)$. Using this function, we now define a map $\sigma_{\varepsilon/2^n} : D \to D$ by setting

$$\sigma_{\varepsilon/2^n}(x) = \begin{cases} x & \text{if } x \in X \\ f^{-1} \circ s_{\varepsilon/2^n} \circ f(x) & \text{if } x \notin X. \end{cases}$$

Here, f^{-1} may be used, because f is injective on $D \smallsetminus X$ and $s_{\varepsilon/2^n} \circ f$ does not map x to $y = f(X)$ as long as $x \in D \smallsetminus X$. By the closed map lemma (Lemma 3.23), we see that $f\colon D \to S^d$ is a closed map. The restriction $f\colon D \smallsetminus X \to S^d \smallsetminus \{y\}$ is also closed as a restriction of a closed map to a saturated set. As a consequence, the composition

$$f^{-1} \circ s_{\varepsilon/2^n} \circ f|_{D \smallsetminus X} \colon D \smallsetminus X \to D \smallsetminus X$$

is continuous. Define $U := f^{-1}(B_{\varepsilon/2^{n+1}}(y)) \supseteq X$. Then, by construction, $\sigma_{\varepsilon/2^n}|_U = \operatorname{Id}_U$. We deduce that $\sigma_{\varepsilon/2^n}$ is continuous, since both $f^{-1} \circ s_{\varepsilon/2^n} \circ f|_{D \smallsetminus X}$ and $\sigma_{\varepsilon/2^n}|_U$ are continuous. Furthermore, the map $\sigma_{\varepsilon/2^n}$ is injective, because the maps $\sigma_{\varepsilon/2^n}|_{D \smallsetminus X}$ and $\sigma_{\varepsilon/2^n}|_X$ are injective and have disjoint images. As a result, the image $\operatorname{Im}\sigma_{\varepsilon/2^n} \subseteq D$ is Hausdorff, and by the closed map lemma (Lemma 3.23) the map $\sigma_{\varepsilon/2^n} : D \to \operatorname{Im}\sigma_{\varepsilon/2^n}$ is a closed map. Thus, the inverse $\sigma^{-1}_{\varepsilon/2^n} : \operatorname{Im}\sigma_{\varepsilon/2^n} \to D$ is continuous and $\sigma_{\varepsilon/2^n}$ is an embedding. Therefore, $\sigma_{\varepsilon/2^n}(D)$ is homeomorphic to a ball for every n. To finish the proof, observe that the balls $\mathcal{B}_n := \sigma_{\varepsilon/2^n}(D) \subseteq D$, for $n = 1, 2, \ldots$, exhibit X as a cellular set. In particular, note that $\mathcal{B}_{n+1} \subseteq \operatorname{Int}\mathcal{B}_n$, since $\sigma_{\varepsilon/2^n}(\partial D)$ lies in $f^{-1}(B_{\varepsilon/2^n})$ but not in $f^{-1}(B_{\varepsilon/2^{n+1}})$. □

Proof of Proposition 3.27 In an attempt to reduce confusion, let S and T denote the two copies of S^d. That is, we have a function $f\colon S \to T$. We show that B is cellular. Let $a := f(A)$ and $b := f(B)$. Since A and B are precisely the two inverse sets of f, we know that they are closed and disjoint. Thus, there exists some standard open ball $U \subseteq S$ disjoint from $A \cup B$ such that if $D := S \smallsetminus U$, then D is a standard closed d-dimensional ball and $A \cup B \subseteq \operatorname{Int} D$.

Then we claim that $f(\operatorname{Int} D)$ is open in T. Note that f is a closed map by the closed map lemma (Lemma 3.23). Thus $f(\overline{U})$ is closed. But then $f(\operatorname{Int} D) = T \smallsetminus f(\overline{U})$ is open as claimed.

Then, since $a, b \in f(\operatorname{Int} D)$ are distinct, there exists some open set $V \subseteq f(\operatorname{Int} D)$ with $a \in V$ and $b \notin V$. Choose a homeomorphism $h\colon T \to T$, taking $f(D)$ to V bijectively and fixing some smaller neighbourhood $W \subsetneq V$ of a. Recall that $D \subseteq S$. Define a map $\psi\colon D \to S$, as follows.

$$\psi(x) = \begin{cases} x & \text{if } x \in f^{-1}(W) \\ f^{-1} \circ h \circ f(x) & \text{if } x \in D \smallsetminus A. \end{cases}$$

The function above is well defined and continuous, since f is injective away from A and B and $f^{-1} \circ h \circ f(x) = x$ on $f^{-1}(W) \smallsetminus A$.

We also check that $B \subseteq \operatorname{Int} D$ is the only inverse set of ψ. This follows, since h maps $f(D)$ into V and f is injective away from A and B. To finish, we need to show that $\psi(\operatorname{Int} D)$ is open in S. We check by hand that $\psi(\operatorname{Int} D) = f^{-1} \circ h \circ f(\operatorname{Int} D)$. We know from before that $f(\operatorname{Int} D)$ is open in T. Then $f^{-1} \circ h \circ f(\operatorname{Int} D)$ is open, since h is a homeomorphism and f is continuous. Now apply Lemma 3.28 to the map $\psi\colon D \to S$ to conclude that B is cellular. A similar proof shows that A is cellular. □

3.4 Brown's Proof of the Schoenflies Theorem

After our lengthy interlude in the previous section, we return to the Schoenflies theorem, which we restate below in an equivalent form.

Theorem 3.29 (Schoenflies theorem [Bro60]) *Let $i\colon S^{d-1} \hookrightarrow S^d$ be a continuous embedding admitting a bicollar. Then the closure of each component of $S^d \smallsetminus i(S^{d-1})$ is homeomorphic to D^d.*

Proof By the bicollar hypothesis, there exists $J\colon S^{d-1} \times [-1,1] \to S^d$ such that $J|_{S^{d-1}\times\{0\}}$ equals i. From elementary homology computations, as in the proof of Theorem 3.5, it follows that the complement of the image of J in S^d has exactly two connected components. Denote their closures by A and B, where A meets $J(S^{d-1} \times \{1\})$.

Observe that the quotient space $S^d/\{A,B\}$ is homeomorphic to S^d due to the existence of the bicollar. In other words, $S^d/\{A,B\}$ can be identified with the (unreduced) suspension of S^{d-1}, which is homeomorphic to S^d, as indicated in Figure 3.8. Thus we have the composition

$$f\colon S^d \xrightarrow{\pi} S^d/\{A,B\} \xrightarrow{\cong} S^d,$$

where $\cong$ denotes homeomorphism and π denotes the quotient map. This map $f\colon S^d \to S^d$ has exactly two inverse sets, namely A and B, and is surjective. By Proposition 3.27, each of A and B is cellular.

Let D denote the closed northern hemisphere of S^d, thought of as a subset of the codomain S^d. By definition, D is a copy of the d-dimensional ball. Let $U := A \cup (J(S^{d-1} \times (0,1]))$; that is, U is the component of $S^d \smallsetminus i(S^{d-1})$ containing A. Then we have the restriction $f|_{\overline{U}}\colon \overline{U} \to D$, whose unique inverse set is A, which is cellular in S^d and thus in $\overline{U}$ by Proposition 3.16.

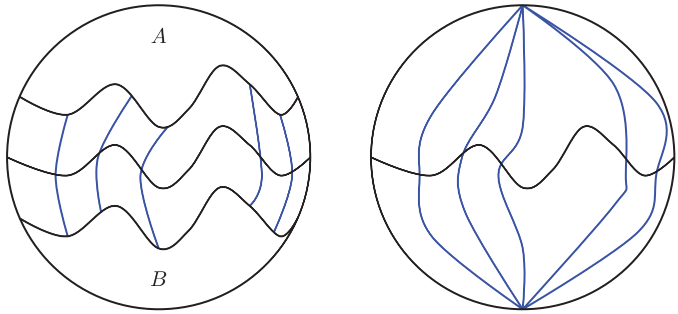


Figure 3.8 Brown's proof of the Schoenflies theorem. Going from left to right, the regions labelled A and B are collapsed to a point each, stretching out a neighbourhood of the equator in the process.

Our goal is to apply Proposition 3.22 to $\overline{U}$. To do so, we first need to show that $\overline{U}$ is a manifold (with boundary). The only possible failure could be near the boundary. As a subspace of S^d, $\overline{U}$ is already Hausdorff and second countable, so we only need to show that it is locally Euclidean. Let $E := J(S^{d-1} \times [0, \frac{1}{2}])$. Note that $\overline{U} = U \cup E$. Moreover, f restricts to a continuous bijection from E to some collar of ∂D. This collar of D is closed by the closed map lemma (Lemma 3.23). Therefore $f|_E$ is a homeomorphism, so $\overline{U}$ is a manifold, as needed.

Then we have the following diagram:

$$\begin{array}{ccc} \overline{U} & \xrightarrow{f} & D \\ {\scriptstyle\pi}\downarrow & \nearrow{\scriptstyle\overline{f}} & \\ \overline{U}/A. & & \end{array}$$

The map f is constant on the fibres of the quotient map $\pi\colon \overline{U} \to \overline{U}/A$ and thus descends to the map $\overline{f}$, which is a homeomorphism by the closed map lemma (Lemma 3.23). Next, since A is cellular, the map π is approximable by homeomorphisms by Proposition 3.22. Let $\widetilde{\pi}\colon \overline{U} \to \overline{U}/A$ be any such approximating homeomorphism. Then $\overline{f} \circ \widetilde{\pi}\colon \overline{U} \to D$ is a homeomorphism. It follows that $\overline{U}$ is homeomorphic to the d-dimensional ball $D \cong D^d$, as claimed. □