Chapter 2
The s-Cobordism Theorem

2.1 Introduction

In this chapter we want to discuss and prove the following theorem (in the smooth category).

Theorem 2.1 (s-Cobordism Theorem) *Let M_0 be a closed connected smooth manifold with* $\dim(M_0) \geq 5$ *and fundamental group* $\pi = \pi_1(M_0)$. *Then:*

(i) *Let* $(W; M_0, f_0, M_1, f_1)$ *be a smooth h-cobordism over M_0. Then W is trivial over M_0 if and only if its Whitehead torsion* $\tau(W, M_0) \in \mathrm{Wh}(\pi)$ *vanishes;*
(ii) *For any* $x \in \mathrm{Wh}(\pi)$ *there is a smooth h-cobordism* $(W; M_0, f_0, M_1, f_1)$ *over M_0 with* $\tau(W, M_0) = x \in \mathrm{Wh}(\pi)$*;*
(iii) *The function assigning to a smooth h-cobordism* $(W; M_0, f_0, M_1, f_1)$ *over M_0 its Whitehead torsion yields a bijection from the diffeomorphism classes relative M_0 of smooth h-cobordisms over M_0 to the Whitehead group* $\mathrm{Wh}(\pi)$.

The analogous statements hold in the PL *category and in the topological category.*

Here are some explanations. In the sequel we work in the smooth category unless explicitly stated otherwise. An *n-dimensional cobordism* (sometimes also just called a bordism) $(W; M_0, f_0, M_1, f_1)$ consists of a compact n-dimensional manifold W, closed $(n-1)$-dimensional manifolds M_0 and M_1, a disjoint decomposition $\partial W = \partial_0 W \coprod \partial_1 W$ of the boundary ∂W of W, and diffeomorphisms $f_0\colon M_0 \to \partial_0 W$ and $f_1\colon M_1 \to \partial_1 W$. If we want to specify M_0, we say that W is a *cobordism over* M_0. If $\partial_0 W = M_0$, $\partial_1 W = M_1$ and f_0 and f_1 are given by the identity or if f_0 and f_1 are obvious from the context, we briefly write $(W; \partial_0 W, \partial_1 W)$. Note that the choices of the diffeomorphisms f_i do play a role, although they are often suppressed in the notation. Two cobordisms $(W; M_0, f_0, M_1, f_1)$ and $(W'; M_0, f_0', M_1', f_1')$ over M_0 are *diffeomorphic relative* M_0 if there is a diffeomorphism $F\colon W \to W'$ with $F \circ f_0 = f_0'$. We call a cobordism $(W; M_0, f_0, M_1, f_1)$ an *h-cobordism* if the inclusions $\partial_i W \to W$ for $i = 0, 1$ are homotopy equivalences. We call an h-cobordism over M_0 *trivial* if it is diffeomorphic relative M_0 to the trivial h-cobordism $(M_0 \times [0,1]; M_0 \times \{0\}, M_0 \times \{1\})$. We will discuss the Whitehead group in Sections 2.5 and 3.2.

Exercise 2.2 Let G be a finitely presented group and $n \geq 6$. Show that $\mathrm{Wh}(G)$ is trivial if every n-dimensional h-cobordism with G as fundamental group is trivial.

Exercise 2.3 Classify all two-dimensional connected h-cobordisms.

We will later see that the Whitehead group of the trivial group vanishes. Thus the s-Cobordism Theorem 2.1 implies the following theorem.

Theorem 2.4 (h-Cobordism Theorem) *Every h-cobordism over a simply connected closed smooth manifold M_0 with $\dim(M_0) \geq 5$ is trivial.*

The analogous statement holds in the PL *category and in the topological category.*

Theorem 2.5 ((Generalised) Poincaré Conjecture) *The (Generalised) Poincaré Conjecture holds for a closed topological manifold M with $\dim(M) \geq 5$, namely, if M is simply connected and its homology $H_p(M)$ is isomorphic to $H_p(S^n)$ for all $p \in \mathbb{Z}$, then M is homeomorphic to S^n.*

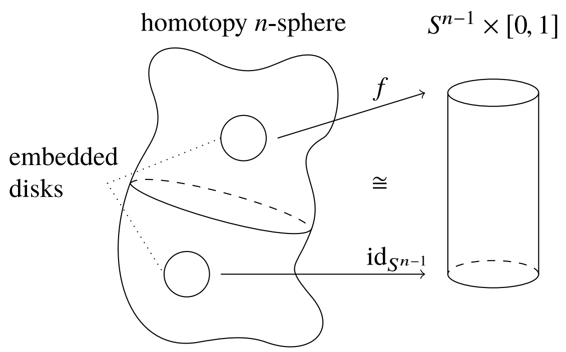
The (Generalised) Poincaré Conjecture also holds in the PL *category.*

Proof. We begin with $\dim(M) \geq 6$. As M is simply connected and $H_*(M) \cong H_*(S^n)$, one can conclude from the Hurewicz Theorem and the Whitehead Theorem, see [178, Theorem 4.32 on page 366 and Corollary 4.33 on page 367] or [431, Theorem IV.7.13 on page 181 and Theorem IV.7.17 on page 182], that there is a homotopy equivalence $f\colon M \to S^n$. Let $D_i^n \subset M$ for $i = 0, 1$ be two embedded disjoint disks. Put $W = M \setminus (\mathrm{int}(D_0^n) \coprod \mathrm{int}(D_1^n))$. Then W turns out to be a simply connected h-cobordism. By Theorem 2.4 there is a diffeomorphism $F\colon (\partial D_0^n \times [0,1], \partial D_0^n \times \{0\}, \partial D_0^n \times \{1\}) \to (W, \partial D_0^n, \partial D_1^n)$ that is the identity on $\partial D_0^n = \partial D_0^n \times \{0\}$ and induces some (unknown) diffeomorphism $f_1\colon \partial D_0^n \times \{1\} \to \partial D_1^n$. By the *Alexander trick* one can extend $f_1\colon \partial D_0^n = \partial D_0^n \times \{1\} \to \partial D_1^n$ to a homeomorphism $\overline{f_1}\colon D_0^n \to D_1^n$. Namely, any homeomorphism $f\colon S^{n-1} \to S^{n-1}$ extends to a homeomorphism $\overline{f}\colon D^n \to D^n$ by sending $t \cdot x$ for $t \in [0,1]$ and $x \in S^{n-1}$ to $t \cdot f(x)$. Now define a homeomorphism $h\colon D_0^n \times \{0\} \cup_{i_0} \partial D_0^n \times [0,1] \cup_{i_1} D_0^n \times \{1\} \to M$ for the canonical inclusions $i_k\colon \partial D_0^n \times \{k\} \to \partial D_0^n \times [0,1]$ for $k = 0, 1$ by $h|_{D_0^n \times \{0\}} = \mathrm{id}$, $h|_{\partial D_0^n \times [0,1]} = F$ and $h|_{D_0^n \times \{1\}} = \overline{f_1}$. Since the source of h is obviously homeomorphic to S^n, Theorem 2.5 follows.

In the case $\dim(M) = 5$ one uses the fact that M is the boundary of a contractible 6-dimensional manifold W and applies the s-cobordism theorem to W with an embedded disk removed. See also [439]. □

Remark 2.6 (The Poincaré Conjecture does not hold in the smooth category) Note that the proof of the Poincaré Conjecture in Theorem 2.5 works only in the topological category and PL category, but not in the smooth category. In other words, we cannot conclude the existence of a diffeomorphism $h\colon S^n \to M$. The proof in the smooth case breaks down when we apply the Alexander trick. The construction of $\overline{f}$ given by coning f yields only a PL homeomorphism $\overline{f}$ and not a diffeomorphism if we start with a diffeomorphism f. The map $\overline{f}$ is smooth outside the origin of D^n but not necessarily at the origin. We will see that not every diffeomorphism $f\colon S^{n-1} \to S^{n-1}$ can be extended to a diffeomorphism $D^n \to D^n$ and that there exist so-called *exotic spheres*, i.e., closed smooth manifolds which are homeomorphic to S^n but not diffeomorphic to S^n for some n. The classification of these exotic spheres is one of the early very important achievements of surgery theory and one motivation for its further development, see Chapter 12 and in particular Remark 12.36.

Figure 2.7 (Poincaré Conjecture)**.**



Exercise 2.8 Show that any diffeomorphism $f: S^1 \to S^1$ can be extended to a diffeomorphism $F: D^2 \to D^2$.

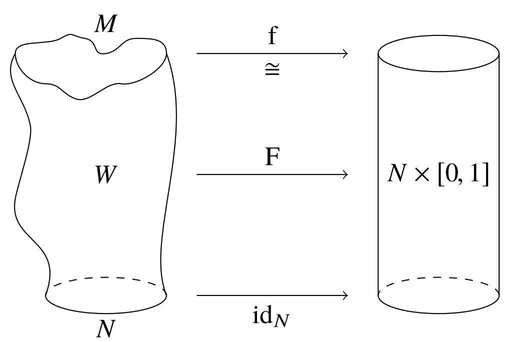
Remark 2.9 (Surgery Program) In some sense the s-Cobordism Theorem 2.1 is one of the first theorems where diffeomorphism classes of certain manifolds are determined by an algebraic invariant, namely the Whitehead torsion. Moreover, the Whitehead group $\mathrm{Wh}(\pi)$ depends only on the fundamental group $\pi = \pi_1(M_0)$ whereas the diffeomorphism classes of h-cobordisms over M_0 a priori depend on M_0 itself. The s-Cobordism Theorem 2.1 is one step in a program to decide whether two closed manifolds M and N are diffeomorphic, which is in general a very hard question. The idea is to construct an h-cobordism $(W; M, f, N, g)$ with vanishing Whitehead torsion. Then W is diffeomorphic to the trivial h-cobordism over M, which implies that M and N are diffeomorphic. So the *Surgery Program* is:

(i) Construct a homotopy equivalence $f: M \to N$;
(ii) Construct a cobordism $(W; M, N)$ and a map $(F, f, \mathrm{id}): (W; M, N) \to (N \times [0,1]; N \times \{0\}, N \times \{1\})$;
(iii) Modify W and F relative boundary by so-called *surgery* so that F becomes a homotopy equivalence and thus W becomes an h-cobordism. During these processes one should make certain that the Whitehead torsion of the resulting h-cobordism is trivial.

The advantage of this approach will be that it can be reduced to problems in homotopy theory and algebra, which can sometimes be handled by well-known techniques. In particular one will sometimes get computable obstructions for two homotopy equivalent manifolds to be diffeomorphic. Often surgery theory has proved to be very useful, when one wants to distinguish two closed manifolds that have very similar properties. The classification of homotopy spheres, see Chapter 12, is one example. Moreover, surgery techniques can be applied to problems that are of a different nature than diffeomorphism or homeomorphism classifications.

In this chapter we want to present the proof of the s-Cobordism Theorem and explain why the notion of Whitehead torsion comes in. We will encounter a typical situation in mathematics. We will consider an h-cobordism and try to prove that it is trivial. We will introduce modifications that are designed to reduce the number of handles and that we can apply to a handlebody decomposition without changing the diffeomorphism type. If we could get rid of all handles, the h-cobordism would be trivial. When attempting to cancel all handles, we run into an algebraic difficulty. A priori this difficulty could be a lack of a good idea or technique. But it will turn out to be a genuine obstruction and lead us to the definition of Whitehead torsion and the Whitehead group.

Figure 2.10 (Surgery Program)**.**



The rest of this chapter is devoted to the proof of the s-Cobordism Theorem 2.1 in the smooth category. Its proof is interesting and illuminating and it motivates the definition of Whitehead torsion. The definition of Whitehead torsion itself and the final step in the proof will appear in Chapter 3.

Guide 2.11 It is not necessary to go through the remainder of this chapter to comprehend the following chapters. It suffices to understand the statement of the s-Cobordism Theorem 2.1 and to read through Remark 2.9 about the Surgery Program.

It is even possible for the first reading to pretend that every s-cobordism is trivial and the Whitehead group $\mathrm{Wh}(G)$ is trivial, which is known to be true in the simply connected case and for many torsionfree groups G.

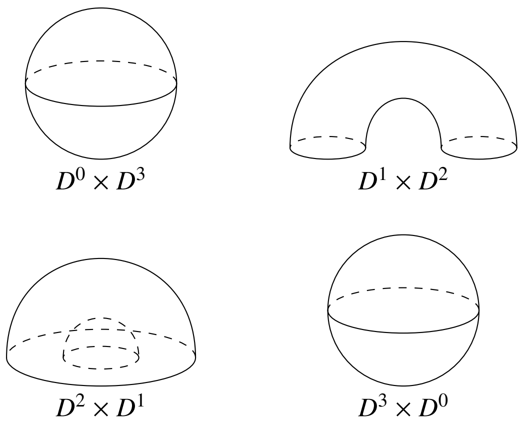
2.2 Handlebody Decompositions

In this section we explain basic facts about handles and handlebody decompositions.

Definition 2.12 (Handlebody) The *n-dimensional handle of index q* or briefly q-handle is $D^q \times D^{n-q}$. Its *core* is $D^q \times \{0\}$. The *boundary of the core* is $S^{q-1} \times \{0\}$. Its *cocore* is $\{0\} \times D^{n-q}$ and its *transverse sphere* is $\{0\} \times S^{n-q-1}$.

Consider an n-dimensional manifold M with boundary ∂M. Given an embedding $\phi^q : S^{q-1} \times D^{n-q} \hookrightarrow \partial M$, we say that the manifold $M + (\phi^q)$ defined by $M \cup_{\phi^q} D^q \times D^{n-q}$ is obtained from M by attaching a handle of index q by ϕ^q.

Figure 2.13 (Handlebody)**.**



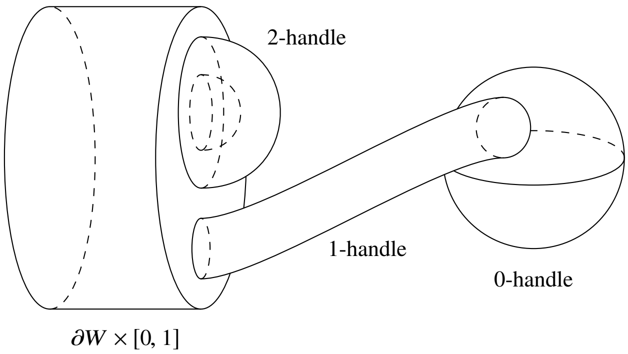
Obviously $M + (\phi^q)$ carries the structure of a topological manifold. To get a smooth structure, one has to use the technique of straightening the angle to get rid of the corners at the place where the handle is glued to M. The boundary $\partial(M + (\phi^q))$ can be described as follows. Delete from ∂M the interior of the image of ϕ^q. We obtain a manifold with boundary together with a diffeomorphism from $S^{q-1} \times S^{n-q-1}$ to its boundary induced by $\phi^q|_{S^{q-1}\times S^{n-q-1}}$. If we use this diffeomorphism to glue $D^q \times S^{n-q-1}$ to it, we obtain a closed manifold, namely, $\partial(M + (\phi^q))$.

Let W be a compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$. Then we want to construct W from $\partial_0 W \times [0, 1]$ by attaching handles as follows. Note that the following construction will not change $\partial_0 W = \partial_0 W \times \{0\}$. If $\phi^q : S^{q-1} \times D^{n-q} \hookrightarrow \partial_0 W \times \{1\}$ is an embedding, we get by attaching a handle the compact manifold $W_1 = \partial_0 W \times [0, 1] + (\phi^q)$ that is given by $\partial_0 W \times [0, 1] \cup_{\phi^q} D^q \times D^{n-q}$. Its boundary is a disjoint sum $\partial_0 W_1 \coprod \partial_1 W_1$ where $\partial_0 W_1$ is the same as $\partial_0 W$. Now we can iterate this process where we attach a handle to W_1 at $\partial_1 W_1$. Thus we obtain a compact manifold with boundary

$$W = \partial_0 W \times [0, 1] + (\phi_1^{q_1}) + (\phi_2^{q_2}) + \cdots + (\phi_r^{q_r})$$

whose boundary is the disjoint union $\partial_0 W \coprod \partial_1 W$ where $\partial_0 W$ is $\partial_0 W \times \{0\}$. We call a description of W as above a *handlebody decomposition* of W relative $\partial_0 W$.

Figure 2.14 (Handlebody decomposition)**.**



From Morse theory, see [189, Chapter 6], [298, part I], we obtain the following lemma.

Lemma 2.15 *Let W be a compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$. Then W possesses a handlebody decomposition relative $\partial_0 W$, i.e., W is up to diffeomorphism relative $\partial_0 W = \partial_0 W \times \{0\}$ of the form*

$$W = \partial_0 W \times [0,1] + (\phi_1^{q_1}) + (\phi_2^{q_2}) + \cdots + (\phi_r^{q_r}).$$

In order to show that W is diffeomorphic to $\partial_0 W \times [0,1]$ relative $\partial_0 W = \partial_0 W \times \{0\}$, we must get rid of the handles. For this purpose we have to find modifications of the handlebody decomposition that reduce the number of handles without changing the diffeomorphism type of W relative $\partial_0 W$.

Lemma 2.16 (Isotopy Lemma) *Let W be an n-dimensional compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$. Given isotopic embeddings $\phi^q, \psi^q \colon S^{q-1} \times D^{n-q} \hookrightarrow \partial_1 W$, there is a diffeomorphism $W + (\phi^q) \to W + (\psi^q)$ relative $\partial_0 W$.*

Proof. Let $i \colon S^{q-1} \times D^{n-q} \times [0,1] \to \partial_1 W$ be an isotopy from ϕ^q to ψ^q. Then one can find a diffeotopy $H \colon W \times [0,1] \to W$ with $H_0 = \mathrm{id}_W$ such that the composition of H with $\phi^q \times \mathrm{id}_{[0,1]}$ is i and H is stationary on $\partial_0 W$, see [189, Theorem 1.3 in

Chapter 8 on page 184]. Thus $H_1 : W \to W$ is a diffeomorphism relative $\partial_0 W$ and satisfies $H_1 \circ \phi^q = \psi^q$. It induces a diffeomorphism $W + (\phi^q) \to W + (\psi^q)$ relative $\partial_0 W$. □

Lemma 2.17 (Diffeomorphism Lemma) *Let W resp. W' be a compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$ resp. $\partial_0 W' \coprod \partial_1 W'$. Let $F : W \to W'$ be a diffeomorphism that induces a diffeomorphism $f_0 : \partial_0 W \to \partial_0 W'$. Let $\phi^q : S^{q-1} \times D^{n-q} \hookrightarrow \partial_1 W$ be an embedding.*

Then there is an embedding $\overline{\phi}^q : S^{q-1} \times D^{n-q} \hookrightarrow \partial_1 W'$ and a diffeomorphism $F' : W + (\phi^q) \to W' + (\overline{\phi}^q)$ that induces f_0 on $\partial_0 W$.

Proof. Put $\overline{\phi}^q = F \circ \phi^q$. □

Lemma 2.18 *Let W be an n-dimensional compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$. Suppose that $V = W + (\psi^r) + (\phi^q)$ for $q \leq r$. Then V is diffeomorphic relative $\partial_0 W$ to $V' = W + (\overline{\phi}^q) + (\psi^r)$ for appropriate $\overline{\phi}^q$.*

Proof. By transversality and the assumption $(q-1) + (n-1-r) < n-1$, we can show that the embedding $\phi^q|_{S^{q-1}\times\{0\}} : S^{q-1} \times \{0\} \hookrightarrow \partial_1(W + (\psi^r))$ is isotopic to an embedding that does not meet the transverse sphere of the handle (ψ^r) attached by ψ^r [189, Theorem 2.3 in Chapter 3 on page 78]. This isotopy can be embedded in a diffeotopy on $\partial_1(W + (\psi^r))$. Thus the embedding $\phi^q : S^{q-1} \times D^{n-q} \hookrightarrow \partial_1(W + (\psi^r))$ is isotopic to an embedding whose restriction to $S^{q-1} \times \{0\}$ does not meet the transverse sphere of the handle (ψ^r). Since we can isotope an embedding $S^{q-1} \times D^{n-q} \hookrightarrow W + (\psi^r)$ so that its image becomes arbitrary close to the image of $S^{q-1} \times \{0\}$, we can isotope $\phi^q : S^{q-1} \times D^{n-q} \hookrightarrow \partial_1(W + (\psi^r))$ to an embedding that does not meet a closed neighbourhood $U \subset \partial_1(W + (\psi^r))$ of the transverse sphere of the handle (ψ^r). There is an obvious diffeotopy on $\partial_1(W + (\psi^r))$ that is stationary on the transverse sphere of (ψ^r) and moves any point on $\partial_1(W + (\psi^r))$, which belongs to the handle (ψ^r) but not to U, to a point outside the handle (ψ^r). Thus we can find an isotopy of ϕ^q to an embedding $\overline{\phi}^p$ that does not meet the handle (ψ^r) at all. Obviously $W + (\psi^r) + (\overline{\phi}^q)$ and $W + (\overline{\phi}^q) + (\psi^r)$ agree. By the Isotopy Lemma 2.16 there is a diffeomorphism relative $\partial_0 W$ from $W + (\psi^r) + (\overline{\phi}^q)$ to $W + (\psi^r) + (\phi^q)$. □

Example 2.19 (Cancelling handles) Here is a standard situation where attaching first a q-handle and then a $(q+1)$-handle does not change the diffeomorphism type of an n-dimensional compact manifold W with the disjoint union $\partial_0 W \coprod \partial_1 W$ as boundary ∂W. Let $0 \leq q \leq n-1$. Consider an embedding

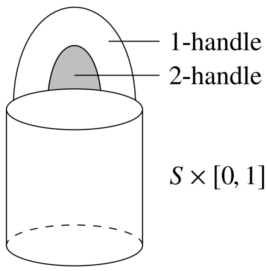
$$\mu : S^{q-1} \times D^{n-q} \cup_{S^{q-1} \times S_+^{n-1-q}} D^q \times S_+^{n-1-q} \hookrightarrow \partial_1 W$$

where S_+^{n-1-q} is the upper hemisphere in $S^{n-1-q} = \partial D^{n-q}$. Note that the source of μ is diffeomorphic to D^{n-1}. Let $\phi^q : S^{q-1} \times D^{n-q} \to \partial_1 W$ be its restriction to $S^{q-1} \times D^{n-q}$. Let $\phi_+^{q+1} : S_+^q \times S_+^{n-q-1} \hookrightarrow \partial_1(W + (\phi^q))$ be the embedding given by

$$S_+^q \times S_+^{n-q-1} = D^q \times S_+^{n-q-1} \subset D^q \times S^{n-q-1} = \partial(\phi^q) \subset \partial_1(W + (\phi^q)).$$

It does not meet the interior of W. Let $\phi_-^{q+1}\colon S_-^q \times S_+^{n-1-q} \hookrightarrow \partial_1(W + (\phi^q))$ be the embedding obtained from μ by restriction to $S_-^q \times S_+^{n-1-q} = D^q \times S_+^{n-1-q}$. Then ϕ_-^{q+1} and ϕ_+^{q+1} fit together to yield an embedding $\psi^{q+1}\colon S^q \times D^{n-q-1} = S_-^q \times S_+^{n-q-1} \cup_{S^{q-1}\times S_+^{n-q-1}} S_+^q \times S_+^{n-q-1} \hookrightarrow \partial_1(W+(\phi^q))$. Then it is not difficult to check that $W+(\phi^q)+(\psi^{q+1})$ is diffeomorphic relative $\partial_0 W$ to W since up to diffeomorphism $W+(\phi^q)+(\psi^{q+1})$ is obtained from W by taking the boundary connected sum of W and D^n along the embedding μ of $D^{n-1} = S_+^{n-1} = S^{q-1} \times D^{n-q} \cup_{S^{q-1}\times S_+^{n-1-q}} D^q \times S_+^{n-1-q}$ into $\partial_1 W$.

Figure 2.20 (Handle cancellation)**.**



This cancellation of two handles of consecutive index can be generalised as follows.

Lemma 2.21 (Cancellation Lemma) *Let W be an n-dimensional compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \coprod \partial_1 W$. Let $\phi^q\colon S^{q-1} \times D^{n-q} \hookrightarrow \partial_1 W$ be an embedding. Let $\psi^{q+1}\colon S^q \times D^{n-1-q} \hookrightarrow \partial_1(W+(\phi^q))$ be an embedding. Suppose that $\psi^{q+1}(S^q \times \{0\})$ is transversal to the transverse sphere of the handle (ϕ^q) and meets the transverse sphere in exactly one point.*

Then there is a diffeomorphism relative $\partial_0 W$ from W to $W + (\phi^q) + (\psi^{q+1})$.

Proof. Given any neighbourhood $U \subset \partial(\phi^q)$ of the transverse sphere of (ϕ^q), there is an obvious diffeotopy on $\partial_1(W + (\phi^q))$ that is stationary on the transverse sphere of (ϕ^q) and moves any point on $\partial_1(W + (\phi^q))$, which belongs to the handle (ϕ^q) but not to U, to a point outside the handle (ϕ^q). Thus we can achieve that ψ^{q+1} maps the lower hemisphere $S_-^q \times \{0\}$ to points outside (ϕ^q) and on the upper hemisphere $S_+^q \times \{0\}$ it is given by the obvious inclusion $D^q \times \{x\} \hookrightarrow D^q \times D^{n-q} = (\phi^q)$ for some $x \in S^{n-q-1}$ and the obvious identification of $S_+^q \times \{0\}$ with $D^q \times \{x\}$. Now it is not hard to construct a diffeomorphism relative $\partial_0 W$ from $W + (\phi^q) + (\psi^{q+1})$ to W modelling the standard situation of Example 2.19. □

The Cancellation Lemma 2.21 will be our only tool to reduce the number of handles. Note that one can never get rid of one handle alone, there must be at least two handles involved simultaneously. The reason for this is that the Euler characteristic $\chi(W, \partial_0 W)$ is independent of the handle decomposition and can be computed by $\sum_{q \geq 0} (-1)^q \cdot p_q$, where p_q is the number of q-handles, see Section 2.3.

We call an embedding $S^q \times D^{n-q} \hookrightarrow M$ for $q < n$ into an n-dimensional manifold *trivial* if it can be written as the composition of an embedding $D^n \hookrightarrow M$ and a fixed standard embedding $S^q \times D^{n-q} \hookrightarrow D^n$. We call an embedding $S^q \hookrightarrow M$ for $q < n$ *trivial* if it can be extended to a trivial embedding $S^q \times D^{n-q} \hookrightarrow M$. We conclude from the Cancellation Lemma 2.21 the following result.

Lemma 2.22 *Let $\phi^q : S^{q-1} \times D^{n-q} \hookrightarrow \partial_1 W$ be a trivial embedding. Then there is an embedding $\phi^{q+1} : S^q \times D^{n-1-q} \hookrightarrow \partial_1 (W + (\phi^q))$ such that W and $W + (\phi^q) + (\phi^{q+1})$ are diffeomorphic relative $\partial_0 W$.*

Consider a compact n-dimensional manifold W whose boundary is the disjoint union $\partial_0 W \coprod \partial_1 W$. In view of Lemma 2.15 and Lemma 2.18 we can write it as

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_0} (\phi_i^0) + \sum_{i=1}^{p_1} (\phi_i^1) + \cdots + \sum_{i=1}^{p_n} (\phi_i^n) \qquad (2.23)$$

where $\cong$ means diffeomorphic relative $\partial_0 W$.

Notation 2.24 Put for $-1 \leq q \leq n$

$$\begin{aligned} W_q &:= \partial_0 W \times [0,1] + \sum_{i=1}^{p_0} (\phi_i^0) + \sum_{i=1}^{p_1} (\phi_i^1) + \cdots + \sum_{i=1}^{p_q} (\phi_i^q); \\ \partial_1 W_q &:= \partial W_q - \partial_0 W \times \{0\}; \\ \partial_1^\circ W_q &:= \partial_1 W_q - \coprod_{i=1}^{p_{q+1}} \phi_i^{q+1} (S^q \times \operatorname{int}(D^{n-1-q})). \end{aligned}$$

Note for the sequel that $\partial_1^\circ W_q \subset \partial_1 W_{q+1}$.

Lemma 2.25 (Elimination Lemma) *Fix an integer q with $1 \leq q \leq n-3$. Suppose that $p_j = 0$ for $j < q$, i.e., W looks like*

$$W = \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) + \cdots + \sum_{i=1}^{p_n} (\phi_i^n).$$

Fix an integer i_0 with $1 \leq i_0 \leq p_q$. Suppose that there is an embedding $\psi^{q+1} : S^q \times D^{n-1-q} \hookrightarrow \partial_1^\circ W_q$ with the following properties:

(i) *The restriction $\psi^{q+1}|_{S^q \times \{0\}}$ is isotopic in $\partial_1 W_q$ to an embedding $\psi_1^{q+1} : S^q \times \{0\} \hookrightarrow \partial_1 W_q$ that meets the transverse sphere of the handle $(\phi_{i_0}^q)$ transversally and in exactly one point and is disjoint from the transverse sphere of ϕ_i^q for $i \neq i_0$;*

(ii) *The restriction $\psi^{q+1}|_{S^q \times \{0\}}$ is isotopic in $\partial_1 W_{q+1}$ to a trivial embedding $\psi_2^{q+1} : S^q \times \{0\} \hookrightarrow \partial_1^\circ W_{q+1}$.*

Then W is diffeomorphic relative $\partial_0 W$ to a manifold of the shape

$$\partial_0 W \times [0,1] + \sum_{\substack{i=1,2,\ldots,p_q,\\ i \neq i_0}} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\overline{\phi}_i^{q+1}) + (\psi^{q+2}) + \sum_{i=1}^{p_{q+2}} (\overline{\phi}_i^{q+2}) + \cdots + \sum_{i=1}^{p_n} (\overline{\phi}_i^{n}).$$

Proof. Since $\psi^{q+1}|_{S^q \times \{0\}}$ is isotopic to ψ_1^{q+1} and ψ_2^{q+1} is trivial, we can extend ψ_1^{q+1} and ψ_2^{q+1} to embeddings denoted in the same way $\psi_1^{q+1}\colon S^q \times D^{n-q-1} \hookrightarrow \partial_1 W_q$ and $\psi_2^{q+1}\colon S^q \times D^{n-1-q} \hookrightarrow \partial_1^\circ W_{q+1}$ with the following properties: ψ^{q+1} is isotopic to ψ_1^{q+1} in $\partial_1 W_q$, ψ_1^{q+1} does not meet the transverse spheres of the handles (ϕ_i^q) for $i \neq i_0$, $\psi_1^{q+1}|_{S^q \times \{0\}}$ meets the transverse sphere of the handle $(\phi_{i_0}^q)$ transversally and in exactly one point, ψ^{q+1} is isotopic to ψ_2^{q+1} within $\partial_1 W_{q+1}$, and ψ_2^{q+1} is trivial, see [189, Theorem 1.5 in Chapter 8 on page 180]. Because of the Diffeomorphism Lemma 2.17 we can assume without loss of generality that there are no handles of index $\geq q+2$, i.e., $p_{q+2} = p_{q+3} = \cdots = p_n = 0$. It suffices to show for appropriate embeddings $\overline{\phi}_i^{q+1}$ and ψ^{q+2} that

$$\begin{aligned}
&\partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) \\
&\qquad \cong \partial_0 W \times [0,1] + \sum_{i=1,2,\ldots,p_q, i \neq i_0} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\overline{\phi}_i^{q+1}) + (\psi^{q+2})
\end{aligned}$$

where $\cong$ means diffeomorphic relative $\partial_0 W$. Because of Lemma 2.22 there is an embedding ψ^{q+2} satisfying

$$\begin{aligned}
&\partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) \\
&\qquad \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) + (\psi_2^{q+1}) + (\psi^{q+2}).
\end{aligned}$$

We conclude from the Isotopy Lemma 2.16 and the Diffeomorphism Lemma 2.17 for appropriate embeddings ψ_k^{q+2} for $k = 1, 2$

$$\begin{aligned}
&\partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) + (\psi_2^{q+1}) + (\psi^{q+2}) \\
&\cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) + (\psi^{q+1}) + (\psi_1^{q+2}) \\
&\cong \partial_0 W \times [0,1] + \sum_{i=1,2,\ldots,p_q, i \neq i_0}^{p_q} (\phi_i^q) + (\phi_{i_0}^q) + (\psi^{q+1}) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) + (\psi_2^{q+2}).
\end{aligned}$$

We get from the Diffeomorphism Lemma 2.17 and the Cancellation Lemma 2.21 for appropriate embeddings $\overline{\phi}_i^{q+1}$ and ψ_3^{q+2}

$$\partial_0 W \times [0,1] + \sum_{i=1,2,\ldots,p_q, i\neq i_0}^{p_q} (\phi_i^q) + (\phi_{i_0}^q) + (\psi^{q+1}) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) + (\psi_2^{q+2})$$
$$\cong \partial_0 W \times [0,1] + \sum_{i=1,2,\ldots,p_q, i\neq i_0}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\overline{\phi}_i^{q+1}) + (\psi_3^{q+2}).$$

This finishes the proof of the Elimination Lemma 2.25. □

2.3 Handlebody Decompositions and CW-Structures

Next we explain how we can associate to a handlebody decomposition (2.23) a CW-pair $(X, \partial_0 W)$ such that there is a bijective correspondence between the q-handles of the handlebody decomposition and the q-cells of $(X, \partial_0 W)$. The key ingredient is that the projection $(D^q \times D^{n-q}, S^{q-1} \times D^{n-q}) \to (D^q, S^{q-1})$ is a homotopy equivalence and actually, as we will explain later, a simple homotopy equivalence.

Recall that a (relative) CW-complex (X, A) consists of a pair of topological spaces (X, A) together with a filtration

$$X_{-1} = A \subset X_0 \subset X_1 \subset \ldots \subset X_q \subset X_{q+1} \subset \ldots \subset \cup_{q \geq 0} X_q = X$$

such that for any $q \geq 0$ there exists a pushout of spaces

$$\begin{array}{ccc} \coprod_{i \in I_q} S^{q-1} & \xrightarrow{\coprod_{i \in I_q} \phi_i^q} & X_{q-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_q} D^q & \xrightarrow{\coprod_{i \in I_q} \Phi_i^q} & X_q \end{array}$$

and X carries the colimit topology with respect to this filtration. The map ϕ_i^q is called the *attaching map* and the map (Φ_i^q, ϕ_i^q) is called the *characteristic map* of the q-cell belonging to $i \in I_q$. The pushouts above are not part of the structure, only their existence is required. Only the filtration $\{X_q \mid q \geq -1\}$ is part of the structure. The path components of $X_q - X_{q-1}$ are called the *open cells*. The open cells coincide with the sets $\Phi_i^q(D^q - S^{q-1})$. The closure of an open cell $\Phi_i^q(D^q - S^{q-1})$ is called a *closed cell* and turns out to be $\Phi_i^q(D^q)$.

Suppose that X is connected with fundamental group π. Let $p\colon \widetilde{X} \to X$ be the *universal covering* of X, i.e., a covering with simply connected total space. Put $\widetilde{X_q} = p^{-1}(X_q)$ and $\widetilde{A} = p^{-1}(A)$. Then $(\widetilde{X}, \widetilde{A})$ inherits a CW-structure from (X, A) by the filtration $\{\widetilde{X_q} \mid q \geq -1\}$. The *cellular $\mathbb{Z}\pi$-chain complex* $C_*(\widetilde{X}, \widetilde{A})$ has as q-th

$\mathbb{Z}\pi$-chain module the singular homology $H_q(\widetilde{X_q}, \widetilde{X_{q-1}})$ with $\mathbb{Z}$-coefficients and the π-action coming from the deck transformations. The q-th differential d_q is given by the composition

$$H_q(\widetilde{X_q}, \widetilde{X_{q-1}}) \xrightarrow{\partial_q} H_{q-1}(\widetilde{X_{q-1}}) \xrightarrow{i_q} H_{q-1}(\widetilde{X_{q-1}}, \widetilde{X_{q-2}})$$

where ∂_q is the boundary operator of the long exact sequence of the pair $(\widetilde{X_q}, \widetilde{X_{q-1}})$ and i_q is induced by the inclusion. If we choose for each $i \in I_q$ a lift $(\widetilde{\Phi^q_i}, \widetilde{\phi^q_i})\colon (D^q, S^{q-1}) \to (\widetilde{X_q}, \widetilde{X_{q-1}})$ of the characteristic map (Φ^q_i, ϕ^q_i), we obtain a $\mathbb{Z}\pi$-basis $\{b_i \mid i \in I_q\}$ for $C_q(\widetilde{X}, \widetilde{A})$ if we define b_i as the image of a generator in $H_q(D^q, S^{q-1}) \cong \mathbb{Z}$ under the map $H_q(\widetilde{\Phi^q_i}, \widetilde{\phi^q_i})\colon H_q(D^q, S^{q-1}) \to H_q(\widetilde{X_q}, \widetilde{X_{q-1}}) = C_q(\widetilde{X}, \widetilde{A})$. We call $\{b_i \mid i \in I_q\}$ *the cellular basis*. Note that we have made several choices in defining the cellular basis. We call two $\mathbb{Z}\pi$-bases $\{\alpha_j \mid j \in J\}$ and $\{\beta_k \mid k \in K\}$ for $C_q(\widetilde{X}, \widetilde{A})$ *equivalent* if there is a bijection $\phi\colon J \to K$ and elements $\epsilon_j \in \{\pm 1\}$ and $\gamma_j \in \pi$ for $j \in J$ such that $\epsilon_j \cdot \gamma_j \cdot \alpha_j = \beta_{\phi(j)}$. The equivalence class of the basis $\{b_i \mid i \in I_q\}$ constructed above only depends on the CW-structure on (X, A) and is independent of all further choices such as (Φ^q_i, ϕ^q_i), its lift $(\widetilde{\Phi^q_i}, \widetilde{\phi^q_i})$, and the generator of $H_q(D^q, S^{q-1})$.

Now suppose we are given a handlebody decomposition (2.23). We want to define a finite n-dimensional relative CW-complex $(X, \partial_0 W)$ and a homotopy equivalence

$$(f, \mathrm{id})\colon (W, \partial_0 W) \xrightarrow{\simeq} (X, \partial_0 W). \tag{2.26}$$

For this purpose we construct by induction over $q = -1, 0, 1, \ldots, n$ a sequence of spaces $X_{-1} = \partial_0 W \subset X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_n$ together with homotopy equivalences $f_q\colon W_q \to X_q$ such that $f_q|_{W_{q-1}} = f_{q-1}$ and $(X, \partial_0 W)$ is a CW-complex with respect to the filtration $\{X_q \mid q = -1, 0, 1, \ldots, n\}$. Then f will be f_n. The induction beginning with $f_1\colon W_{-1} = \partial_0 W \times [0,1] \to X_{-1} = \partial_0 W$ is given by the projection. The induction step from $(q-1)$ to q is done as follows. We attach for each handle (ϕ^q_i) for $i = 1, 2, \ldots, p_q$ a cell D^q to X_{q-1} by the attaching map $f_{q-1} \circ \phi^q_i|_{S^{q-1} \times \{0\}}$. In other words, we define X_q by the pushout

$$\begin{array}{ccc}
\coprod_{i=1}^{p_q} S^{q-1} & \xrightarrow{\coprod_{i=1}^{p_q} f_{q-1} \circ \phi^q_i|_{S^{q-1}\times\{0\}}} & X_{q-1} \\
\downarrow & & \downarrow \\
\coprod_{i=1}^{p_q} D^q & \longrightarrow & X_q.
\end{array}$$

Recall that W_q is the pushout

$$\begin{tikzcd}
\coprod_{i=1}^{p_q} S^{q-1} \times D^{n-q} \arrow[r, "\coprod_{i=1}^{p_q} \phi_i^q"] \arrow[d] & W_{q-1} \arrow[d] \\
\coprod_{i=1}^{p_q} D^q \times D^{n-q} \arrow[r] & W_q.
\end{tikzcd}$$

Define a space Y_q by the pushout

$$\begin{tikzcd}
\coprod_{i=1}^{p_q} S^{q-1} \arrow[r, "\coprod_{i=1}^{p_q} \phi_i^q|_{S^{q-1}\times\{0\}}"] \arrow[d] & W_{q-1} \arrow[d] \\
\coprod_{i=1}^{p_q} D^q \arrow[r] & Y_q.
\end{tikzcd}$$

Define $(g_q, f_{q-1})\colon (Y_q, W_{q-1}) \to (X_q, X_{q-1})$ by the pushout property applied to homotopy equivalences given by $f_{q-1}\colon W_{q-1} \to X_{q-1}$ and the identity maps on S^{q-1} and D^q. Define $(h_q, \mathrm{id})\colon (Y_q, W_{q-1}) \to (W_q, W_{q-1})$ by the pushout property applied to homotopy equivalences given by the obvious inclusions $S^{q-1} \to S^{q-1} \times D^{n-q}$ and $D^q \to D^q \times D^{n-q}$ and the identity on W_{q-1}. The resulting maps are homotopy equivalences of pairs since the left vertical arrows in the three pushouts above are cofibrations, see [60, page 249]. Choose a homotopy inverse $(h_q^{-1}, \mathrm{id})\colon (W_q, W_{q-1}) \to (Y_q, W_{q-1})$. Define f_q by the composite $g_q \circ h_q^{-1}$.

In particular we see that the inclusions $W_q \to W$ are q-connected since the inclusion of the q-skeleton $X_q \to X$ is q-connected for a CW-complex X.

Denote by $p\colon \widetilde{W} \to W$ the universal covering with $\pi = \pi_1(W)$ as group of deck transformations. Let $\widetilde{W_q}$ be the preimage of W_q under p. Note that this is the universal covering for $q \geq 2$ since each inclusion $W_q \to W$ induces an isomorphism on the fundamental groups. Define the *handlebody $\mathbb{Z}\pi$-chain complex* $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ to be the $\mathbb{Z}\pi$-chain complex whose q-th chain group is $H_q(\widetilde{W_q}, \widetilde{W_{q-1}})$ and whose q-th differential is given by the composition

$$H_q(\widetilde{W_q}, \widetilde{W_{q-1}}) \xrightarrow{\partial_q} H_{q-1}(\widetilde{W_{q-1}}) \xrightarrow{i_q} H_{q-1}(\widetilde{W_{q-1}}, \widetilde{W_{q-2}})$$

where ∂_q is the boundary operator of the long homology sequence associated to the pair $(\widetilde{W_q}, \widetilde{W_{q-1}})$ and i_q is induced by the inclusion. The map $(f, \mathrm{id})\colon (W, \partial_0 W) \xrightarrow{\simeq} (X, \partial_0 W)$ of (2.26) induces an isomorphism of $\mathbb{Z}\pi$-chain complexes

$$C_*(\widetilde{f}, \mathrm{id}_{\widetilde{\partial_0 W}})\colon C_*(\widetilde{W}, \widetilde{\partial_0 W}) \xrightarrow{\cong} C_*(\widetilde{X}, \widetilde{\partial_0 W}). \tag{2.27}$$

Each handle (ϕ_i^q) determines an element

$$[\phi_i^q] \in C_q(\widetilde{W}, \widetilde{\partial_0 W}) \tag{2.28}$$

after choosing a lift $(\widetilde{\Phi_i^q}, \widetilde{\phi_i^q})\colon (D^q \times D^{n-q}, S^{q-1} \times D^{n-q}) \to (\widetilde{W_q}, \widetilde{W_{q-1}})$ of its characteristic map $(\Phi_i^q, \phi_i^q)\colon (D^q \times D^{n-q}, S^{q-1} \times D^{n-q}) \to (W_q, W_{q-1})$, namely, the image under the map $H_q(\widetilde{\Phi_i^q}, \widetilde{\phi_i^q})$ of the preferred generator in $H_q(D^q \times D^{n-q}, S^{q-1} \times D^{n-q}) \cong H_0(\{*\}) = \mathbb{Z}$. This element is only well defined up to multiplication by an element $\gamma \in \pi$. The elements $\{[\phi_i^q] \mid i = 1, 2, \ldots, p_q\}$ form a $\mathbb{Z}\pi$-basis for $C_q(\widetilde{W}, \widetilde{\partial_0 W})$. Its image under the isomorphism (2.27) is a cellular $\mathbb{Z}\pi$-basis.

If W has no handles of index ≤ 1, i.e., $p_0 = p_1 = 0$, one can express $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ also in terms of homotopy groups as follows. Fix a base point $z \in \partial_0 W$ and a lift $\widetilde{z} \in \widetilde{\partial_0 W}$. All homotopy groups are taken with respect to these base points. Let $\pi_*(W_*, W_{*-1})$ be the $\mathbb{Z}\pi$-chain complex whose q-th $\mathbb{Z}\pi$-module is $\pi_q(W_q, W_{q-1})$ for $q \geq 2$ and zero for $q \leq 1$ and whose q-th differential is given by the composition

$$\pi_q(W_q, W_{q-1}) \xrightarrow{\partial_q} \pi_{q-1}(W_{q-1}) \xrightarrow{\pi_{q-1}(i)} \pi_{q-1}(W_{q-1}, W_{q-2}).$$

The $\mathbb{Z}\pi$-action comes from the canonical $\pi_1(A)$-action on the group $\pi_q(Y, A)$ and the $\pi_1(Y)$ action on $\pi_q(Y)$, see [178, Section 4.A] or [431, Theorem I.3.1 on page 164], and the identification $\pi_1(W_{q-1}) \to \pi_1(W) = \pi$ coming from the inclusions $W_{q-1} \to W$. Note that $\pi_q(Y, A)$ is abelian for any pair of spaces (Y, A) for $q \geq 3$ and is abelian also for $q = 2$ if A is simply connected or empty. For $q \geq 2$ the Hurewicz homomorphism is an isomorphism, see [178, Theorem 4.32 on page 366] or [431, Corollary IV.7.11 on page 181], $\pi_q(\widetilde{W_q}, \widetilde{W_{q-1}}) \to H_q(\widetilde{W_q}, \widetilde{W_{q-1}})$ and the projection $p\colon \widetilde{W} \to W$ induces isomorphisms $\pi_q(\widetilde{W_q}, \widetilde{W_{q-1}}) \to \pi_q(W_q, W_{q-1})$. Thus we obtain an isomorphism of $\mathbb{Z}\pi$-chain complexes

$$C_*(\widetilde{W}, \widetilde{\partial_0 W}) \xrightarrow{\cong} \pi_*(W_*, W_{*-1}). \tag{2.29}$$

Fix a path w_i in W from a point in the transverse sphere of (ϕ_i^q) to the base point z. Then the handle (ϕ_i^q) determines an element

$$[\phi_i^q] \in \pi_q(W_q, W_{q-1}). \tag{2.30}$$

It is represented by the obvious map $(D^q \times \{0\}, S^{q-1} \times \{0\}) \to (W_q, W_{q-1})$ together with w_i. It agrees with the element $[\phi_i^q] \in C_q(\widetilde{W}, \widetilde{\partial_0 W})$ defined in (2.28) under the isomorphism (2.29) if we use the lift of the characteristic map determined by the path w_i.

Exercise 2.31 Let M be a closed odd-dimensional manifold. Show that for any handlebody decomposition the number of handles of odd index is equal to the number of handles of even index.

Exercise 2.32 Let M be a closed connected manifold that possesses a handlebody decomposition without handles of index one. Show that M is simply connected.

Exercise 2.33 Show that a handlebody decomposition for $W = S^n \times S^n$ must have at least one 0-handle, two n-handles and one $2n$-handle. Describe one.

2.4 Reducing the Handlebody Decomposition

In the next step we want to get rid of the handles of index zero and one in the handlebody decomposition (2.23).

Lemma 2.34 *Let W be an n-dimensional manifold for $n \geq 6$ whose boundary is the disjoint union $\partial W = \partial_0 W \coprod \partial_1 W$. Then the following statements are equivalent:*

(i) *The inclusion $\partial_0 W \to W$ is 1-connected;*
(ii) *We can find a diffeomorphism relative $\partial_0 W$*

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_2} (\phi_i^2) + \sum_{i=1}^{p_3} (\overline{\phi}_i^3) + \cdots + \sum_{i=1}^{p_n} (\overline{\phi}_i^n).$$

Proof. (ii) $\Rightarrow$ (i) has already been proved in Section 2.3. It remains to conclude (ii) provided that (i) holds.

We first get rid of all 0-handles in the handlebody decomposition (2.23). It suffices to give a procedure to reduce the number of handles of index 0 by one. As the inclusion $\partial_0 W \to W$ is 1-connected, the inclusion $\partial_0 W \to W_1$ induces a bijection on the set of path components. Given any index i_0, there must be an index i_1 such that the core of the handle $\phi_{i_1}^1$ is a path connecting a point in $\partial_0 W \times \{1\}$ with a point in $(\phi_{i_0}^0)$. We conclude from the Diffeomorphism Lemma 2.17 and the Cancellation Lemma 2.21 that (ϕ_{i_0}) and $(\phi_{i_1}^1)$ cancel one another, i.e., we have

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1,2,\ldots,p_0, i \neq i_0} (\phi_i^0) + \sum_{i=1,2,\ldots,p_1, i \neq i_1} (\phi_i^1) + \cdots + \sum_{i=1}^{p_n} (\phi_i^n).$$

Hence we can assume $p_0 = 0$ in (2.23).

Next we want to get rid of the 1-handles assuming that the inclusion $\partial_0 W \to W$ is 1-connected. It suffices to give a procedure to reduce the number of handles of index 1 by one. We want to do this by constructing an embedding $\psi^2 \colon S^1 \times D^{n-2} \hookrightarrow \partial_1^\circ W_1$ that satisfies the two conditions of the Elimination Lemma 2.25 and then applying the Elimination Lemma 2.25. Consider the embedding $\psi_+^2 \colon S_+^1 = D^1 = D^1 \times \{x\} \hookrightarrow D^1 \times D^{n-1} = (\phi_1^1)$ for some fixed $x \in S^{n-2} = \partial D^{n-1}$. The inclusion $\partial_1^\circ W_0 \to \partial_1 W_0 = \partial_0 W \times \{1\}$ induces an isomorphism on the fundamental group since $\partial_1^\circ W_0$ is obtained from $\partial_1 W_0 = \partial_0 W \times \{1\}$ by removing the interior of a finite number of embedded $(n-1)$-dimensional disks. Since by assumption the inclusion $\partial_0 W \to W$ is 1-connected, the inclusion $\partial_1^\circ W_0 \to W$ induces an epimorphism on the fundamental groups. Therefore we can find an embedding $\psi_-^2 \colon S_-^1 \hookrightarrow \partial_1^\circ W_0$ with $\psi_-^2|_{S^0} = \psi_+^2|_{S^0}$ such that the map $\psi_0^2 \colon S^1 = S_+^1 \cup_{S^0} S_-^1 \to \partial_1 W_1$ given by $\psi_+^2 \cup \psi_-^2$ is nullhomotopic in W. One can isotope the attaching maps $\phi_i^2 \colon S^1 \times D^{n-2} \to \partial_1 W_1$ of the 2-handles (ϕ_i^2) so that they do not meet the image of ψ_0^2 because the sum of the dimension of the source of ψ_0^2 and of $S^1 \times \{0\} \subset S^1 \times D^{n-2}$ is less than the dimension $(n-1)$ of $\partial_1 W_1$ and one can always shrink inside D^{n-2}. Thus we can assume without loss of generality by the Isotopy Lemma 2.16 and the Diffeomorphism Lemma 2.17 that the image of ψ_0^2 lies in $\partial_1^\circ W_1$. The inclusion $\partial_1 W_2 \to W$ is 2-connected. Hence ψ_0^2 is nullhomotopic in $\partial_1 W_2$. Let $h \colon D^2 \to \partial_1 W_2$ be a nullhomotopy for ψ_0^2. Since $2 \cdot \dim(D^2) < \dim(\partial_1 W_2)$, we can change h relative to S^1 into an embedding. (Here we need for the first time the assumption $n \geq 6$.) Since D^2 is contractible, the normal bundle of h and thus of $\psi_0^2 = \psi_+^2 \cup \psi_-^2$ are trivial. Therefore we can extend ψ_0^2 to an embedding $\psi^2 \colon S^1 \times D^{n-1} \hookrightarrow \partial_1^\circ W_1$ that is isotopic to a trivial embedding in $\partial_1 W_2$, meets the transverse sphere of the handle (ϕ_1^1) transversally and in exactly one point, and does not meet the transverse spheres of the handles (ϕ_i^1) for $2 \leq i \leq p_1$. Now Lemma 2.34 follows from the Elimination Lemma 2.25. □

Now consider an h-cobordism $(W; \partial_0 W, \partial_1 W)$. Because of Lemma 2.34 we can write it as

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_2} (\phi_i^2) + \sum_{i=1}^{p_3} (\phi_i^3) + \cdots .$$

Lemma 2.35 (Homology Lemma) *Suppose $n \geq 6$. Fix $2 \leq q \leq n-3$ and $i_0 \in \{1, 2, \ldots, p_q\}$. Let $f \colon S^q \hookrightarrow \partial_1 W_q$ be an embedding. Then the following statements are equivalent:*

(i) *The embedding f is isotopic to an embedding $g \colon S^q \hookrightarrow \partial_1 W_q$ such that g meets the transverse sphere of $(\phi_{i_0}^q)$ transversally and in exactly one point and is disjoint from transverse spheres of the handles (ϕ_i^q) for $i \neq i_0$;*

(ii) *Let $\widetilde{f} \colon S^q \to \widetilde{W_q}$ be a lift of f under $p|_{\widetilde{W_q}} \colon \widetilde{W_q} \to W_q$. Let $[\widetilde{f}]$ be the image of the class represented by $\widetilde{f}$ under the obvious composition*

$$\pi_q(\widetilde{W_q}) \to \pi_q(\widetilde{W_q}, \widetilde{W_{q-1}}) \to H_q(\widetilde{W_q}, \widetilde{W_{q-1}}) = C_q(\widetilde{W}).$$

Then there is a $\gamma \in \pi$ with

$$[\widetilde{f}] = \pm \gamma \cdot [\phi_{i_0}^q].$$

Proof. (i) $\Rightarrow$ (ii) We can isotope f so that $f|_{S_+^q} \colon S_+^q \to \partial_1 W_q$ looks like the canonical embedding $S_+^q = D^q \times \{x\} \hookrightarrow D^q \times S^{n-1-q} = \partial(\phi_{i_0}^q)$ for some $x \in S^{n-1-q}$ and $f(S_-^q)$ does not meet any of the handles (ϕ_i^q) for $i = 1, 2, \ldots, p_q$. One easily checks that then (ii) is true.

(ii) $\Rightarrow$ (i) We can isotope f so that it is transverse to the transverse spheres of the handles (ϕ_i^q) for $i = 1, 2, \ldots, p_q$. Since the sum of the dimension of the source of f and of the dimension of the transverse spheres is the dimension of $\partial_1 W_q$, the intersection of the image of f with the transverse sphere of the handle (ϕ_i^q) consists of finitely many points $x_{i,1}, x_{i,2}, \ldots, x_{i,r_i}$ for $i = 1, 2, \ldots, p_q$. Fix a base point $y \in S^q$.

It yields a base point $z = f(y) \in W$. Fix for each handle (ϕ_i^q) a path w_i in W from a point in its transverse sphere to z. Let $u_{i,j}$ be a path in S^q with the property that $u_{i,j}(0) = y$ and $f(u_{i,j}(1)) = x_{i,j}$ for $1 \leq j \leq r_i$ and $1 \leq i \leq p_q$. Let $v_{i,j}$ be any path in the transverse sphere of (ϕ_i^q) from $x_{i,j}$ to $w_i(0)$. Then the composite $f(u_{i,j}) * v_{i,j} * w_i$ is a loop in W with base point z and thus represents an element denoted by $\gamma_{i,j}$ in $\pi = \pi_1(W, z)$. It is independent of the choice of $u_{i,j}$ and $v_{i,j}$ since S^q and the transverse sphere of each handle (ϕ_i^q) are simply connected. The tangent space $T_{x_{i,j}} \partial_1 W_q$ is the tangent space of the handle (ϕ_i^q) at $x_{i,j}$ and is the direct sum of $T_{f^{-1}(x_{i,j})} S^q$ and the tangent space of the transverse sphere $\{0\} \times S^{n-1-q}$ of the handle (ϕ_i^q) at $x_{i,j}$. All these three tangent spaces come with preferred orientations. We define elements $\epsilon_{i,j} \in \{\pm 1\}$ by requiring that it is 1 if these orientations fit together and -1 otherwise. Now one easily checks that

$$[\widetilde{f}] = \sum_{i=1}^{p_q} \sum_{j=1}^{r_i} \epsilon_{i,j} \cdot \gamma_{i,j} \cdot [\phi_i^q]$$

where $[\phi_i^q]$ is the element associated to the handle (ϕ_i^q) after the choice of the path w_i, see (2.28) and (2.30). We have by assumption $[\widetilde{f}] = \pm\gamma \cdot [\phi_{i_0}^q]$ for some $\gamma \in \pi$. We want to isotope f so that f does not meet the transverse spheres of the handles (ϕ_i^q) for $i \neq i_0$ and does meet the transverse sphere of $(\phi_{i_0}^q)$ transversally and in exactly one point. Therefore it suffices to show in the case that the number $\sum_{i=1}^{p_q} r_i$ of all intersection points of f with the transverse spheres of the handles (ϕ_i^q) for $i = 1, 2, \ldots, p_i$ is bigger than one that we can change f by an isotopy so that this number becomes smaller. We have

$$\pm\gamma \cdot [\phi_{i_0}^q] = \sum_{i=1}^{p_q} \sum_{j=1}^{r_i} \epsilon_{i,j} \cdot \gamma_{i,j} \cdot [\phi_i^q].$$

Recall that the elements $[\phi_i^q]$ for $i = 1, 2, \ldots, p_q$ form a $\mathbb{Z}\pi$-basis. Hence we can find an index $i \in \{1, 2, \ldots, p_q\}$ and two different indices j_1, j_2 in $\{1, 2, \ldots, r_i\}$ such that the composite of the paths $f(u_{i,j_1}) * v_{i,j_1} * v_{i,j_2}^- * f(u_{i,j_2}^-)$ is nullhomotopic in W and hence in $\partial_1 W_q$ and the signs ϵ_{i,j_1} and ϵ_{i,j_2} are different. Now by the Whitney trick, see [301, Theorem 6.6 on page 71], [435], we can change f by an isotopy so that the two intersection points x_{i,j_1} and x_{i,j_2} disappear, the other intersection points of f with transverse spheres of the handles (ϕ_i^q) for $i \in \{1, 2, \ldots, p_q\}$ remain and no further intersection points are introduced. For the application of the Whitney trick we need the assumption $n - 1 \geq 5$ and when $q = 2$ or $q = n - 3$ an additional assumption on fundamental groups. One of the referees pointed out that we did not address the second assumption. In our situation it turns out that this requirement is always fulfilled, a subtlety explained in the book by Scorpan [373, pages 51-53] in dimensions $(n - 1) \geq 5$. (What happens for $(n - 1) = 4$ is discussed in [373, pages 57-58].) This finishes the proof of the Homology Lemma 2.35. $\square$

Lemma 2.36 (Modification Lemma) *Let $f\colon S^q \hookrightarrow \partial_1^\circ W_q$ be an embedding and let $x_j \in \mathbb{Z}\pi$ be elements for $j = 1, 2 \dots, p_{q+1}$. Then there is an embedding $g\colon S^q \hookrightarrow \partial_1^\circ W_q$ with the following properties:*

(i) *f and g are isotopic in $\partial_1 W_{q+1}$;*
(ii) *For a given lift $\widetilde{f}\colon S^q \to \widetilde{W_q}$ of f one can find a lift $\widetilde{g}\colon S^q \to \widetilde{W_q}$ of g such that we get in $C_q(\widetilde{W})$*

$$[\widetilde{g}] = [\widetilde{f}] + \sum_{j=1}^{p_{q+1}} x_j \cdot d_{q+1}[\phi_j^{q+1}]$$

where d_{q+1} is the $(q+1)$-th differential in $C_*(\widetilde{W}, \widetilde{\partial_0 W})$.

Proof. Any element in $\mathbb{Z}\pi$ can be written as a sum of elements of the form $\pm\gamma$ for $\gamma \in \pi$. Hence it suffices to prove for a fixed number $j \in \{1, 2 \dots, p_q\}$, a fixed element $\gamma \in \pi$, and a fixed sign $\epsilon \in \{\pm 1\}$ that one can find an embedding $g\colon S^q \hookrightarrow \partial_1^\circ W_q$ that is isotopic to f in $\partial_1 W_{q+1}$ and satisfies for an appropriate lifting $\widetilde{g}$

$$[\widetilde{g}] = [\widetilde{f}] + \epsilon \cdot \gamma \cdot d_{q+1}[\phi_j^{q+1}].$$

Consider the embedding $t_j\colon S^q = S^q \times \{z\} \subset S^q \times S^{n-2-q} \hookrightarrow \partial(\phi_j^{q+1}) \subset \partial_1 W_q$ for some point $z \in S^{n-2-q} = \partial D^{n-1-q}$. It is in $\partial_1 W_{q+1}$ isotopic to a trivial embedding. Choose a path w in $\partial_1^\circ W_q$ connecting a point in the image of f with a point in the image of t_j. Without loss of generality we can arrange w to be an embedding. Moreover, we can thicken $w\colon [0,1] \to \partial_1^\circ W_q$ to an embedding $\overline{w}\colon [0,1] \times D^q \hookrightarrow \partial_1^\circ W_q$ such that $\overline{w}(\{0\} \times D^q)$ and $\overline{w}(\{1\} \times D^q)$ are embedded q-dimensional disks in the images of f and t_j and $\overline{w}((0,1) \times D^q)$ does not meet the images of f and t_j. Now one can form a new embedding, the connected sum $g := f \sharp_w t_j\colon S^q \to \partial_1^\circ W_q$. It is essentially given by restriction of f and t_j to the part of S^q which is not mapped under f and t_j to the interior of the disks $\overline{w}(\{0\} \times D^q)$, $\overline{w}(\{1\} \times D^q)$, and $\overline{w}|_{[0,1] \times S^{q-1}}$. Since t_j is isotopic to a trivial embedding in $\partial_1 W_{q+1}$, the embedding g is isotopic in $\partial_1 W_{q+1}$ to f. Recall that we have fixed a lifting $\widetilde{f}$ of f. This determines a unique lifting of $\widetilde{g}$, namely, we require that $\widetilde{f}$ and $\widetilde{g}$ coincide on those points where f and g already coincide. For an appropriate element $\gamma' \in \pi$ one gets $[\widetilde{g}] = [\widetilde{f}] + \gamma' \cdot d_{q+1}([\phi_j^{q+1}])$ since $t_j\colon S^q \to \partial_1 W_q \subset W_q$ is homotopic to $\phi_j^{q+1}|_{S^q \times \{0\}}\colon S^q \times \{0\} = S^q \to W_q$ in W_q. We can change the path w by composing it with a loop representing $\gamma \cdot (\gamma')^{-1} \in \pi$. Then we get for the new embedding g that

$$[\widetilde{g}] = [\widetilde{f}] + \gamma \cdot d_{q+1}([\phi_j^{q+1}]).$$

If we compose t_j with a diffeomorphism $S^q \to S^q$ of degree -1, we still get an embedding g that is isotopic to f in $\partial_1 W_{q+1}$ and satisfies

$$[\widetilde{g}] = [\widetilde{f}] - \gamma \cdot d_{q+1}([\phi_j^{q+1}]).$$

This finishes the proof of the Modification Lemma 2.36. □

Lemma 2.37 (Normal Form Lemma) *Let $(W; \partial_0 W, \partial_1 W)$ be a compact h-cobordism of dimension $n \geq 6$. Let q be an integer with $2 \leq q \leq n-3$.*

Then there is a handlebody decomposition that has only handles of index q and $(q+1)$, i.e., there is a diffeomorphism relative $\partial_0 W$

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}).$$

Proof. In the first step we show that we can arrange $W_{-1} = W_{q-1}$, i.e., $p_r = 0$ for $r \leq q-1$. We do this by induction over q. The induction beginning with $q = 2$ has already been carried out in Lemma 2.34. In the induction step from q to $(q+1)$ we must explain how we can decrease the number of q-handles, provided that there are no handles of index $< q$. In order to get rid of the handle (ϕ_1^q), we want to attach a new $(q+1)$-handle and a new $(q+2)$-handle such that (ϕ_1^q) and the new $(q+1)$-handle cancel and the new $(q+1)$-handle and the new $(q+2)$-handle cancel each other. The effect will be that the number of q-handles is decreased by one at the cost of increasing the number of $(q+2)$-handles by one.

Fix a trivial embedding $\overline{\psi}^{q+1} \colon S^q \times D^{n-1-q} \hookrightarrow \partial_1^\circ W_q$. As the inclusion $\partial_0 W \to W$ is a homotopy equivalence, $H_p(\widetilde{W}, \widetilde{\partial_0 W}) = 0$ for all $p \geq 0$. Since the p-th homology of $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ is $H_p(\widetilde{W}, \widetilde{\partial_0 W}) = 0$, the $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ is acyclic. Since $C_{q-1}(\widetilde{W}, \widetilde{\partial_0 W})$ is trivial, the q-th differential of $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ is zero and hence the $(q+1)$-th differential d_{q+1} is surjective. We can choose elements $x_j \in \mathbb{Z}\pi$ such that

$$[\phi_1^q] = \sum_{i=1}^{p_{q+1}} x_j \cdot d_{q+1}([\phi_i^{q+1}]).$$

Since $\alpha := \overline{\psi}^{q+1}|_{S^q \times \{0\}} \to \partial_1^\circ W_q$ is nullhomotopic, $[\widetilde{\alpha}] = 0$ in $H_q(\widetilde{W_q}, \widetilde{W_{q-1}})$. We conclude from the Modification Lemma 2.36 that we can find an embedding $\psi^{q+1} \colon S^q \times D^{n-1-q} \hookrightarrow \partial_1^\circ W_q$ such that $\beta := \psi^{q+1}|_{S^q \times \{0\}}$ is isotopic in $\partial_1 W_{q+1}$ to α and we get

$$[\widetilde{\beta}] = [\widetilde{\alpha}] + \sum_{i=1}^{p_{q+1}} x_j \cdot d_{q+1}([\phi_i^{q+1}]) = [\phi_1^q].$$

Because of the Homology Lemma 2.35 the embedding $\beta = \psi^q|_{S^q \times \{0\}}$ is isotopic in $\partial_1 W_q$ to an embedding $\gamma \colon S^q \hookrightarrow \partial_1 W_q$ that meets the transverse sphere of (ϕ_1^q) transversally and in exactly one point and is disjoint from the transverse spheres of all other handles of index q. By construction ψ^{q+1} is isotopic in $\partial_1 W_{q+1}$ to the trivial embedding $\overline{\psi}^{q+1}$. Now we can apply the Elimination Lemma 2.25. This finishes the proof that we can arrange $W_{-1} = W_{q-1}$.

Next we explain the *dual handlebody decomposition*. Suppose that W is obtained from $\partial_0 W \times [0,1]$ by attaching one q-handle (ϕ^q), i.e., $W = \partial_0 W \times [0,1] + (\phi^q)$. Then we can interchange the role of $\partial_0 W$ and $\partial_1 W$ and try to build W from $\partial_1 W$ by handles. It turns out that W can be written as

$$W = \partial_1 W \times [0,1] + (\psi^{n-q}) \tag{2.38}$$

by the following argument.

Let M be the manifold with boundary $S^{q-1} \times S^{n-1-q}$ obtained from $\partial_0 W$ by removing the interior of $\phi^q(S^{q-1} \times D^{n-q})$. We get

$$\begin{aligned} W &\cong \partial_0 W \times [0,1] \cup_{S^{q-1}\times D^{n-q}} D^q \times D^{n-q} \\ &= M \times [0,1] \cup_{S^{q-1}\times S^{n-1-q}\times[0,1]} \\ &\qquad \left(S^{q-1} \times D^{n-q} \times [0,1] \cup_{S^{q-1}\times D^{n-q}\times\{1\}} D^q \times D^{n-q}\right). \end{aligned}$$

Inside $S^{q-1} \times D^{n-q} \times [0,1] \cup_{S^{q-1}\times D^{n-q}\times\{1\}} D^q \times D^{n-q}$ we have the following submanifolds

$$\begin{aligned} X &:= S^{q-1} \times 1/2 \cdot D^{n-q} \times [0,1] \cup_{S^{q-1}\times 1/2\cdot D^{n-q}\times\{1\}} D^q \times 1/2 \cdot D^{n-q}; \\ Y &:= S^{q-1} \times 1/2 \cdot S^{n-1-q} \times [0,1] \cup_{S^{q-1}\times 1/2\cdot S^{n-1-q}\times\{1\}} D^q \times 1/2 \cdot S^{n-1-q}. \end{aligned}$$

The pair (X,Y) is diffeomorphic to $(D^q \times D^{n-q}, D^q \times S^{n-1-q})$, i.e., it is a handle of index $(n-q)$. Let N be obtained from W by removing the interior of X. Then W is obtained from N by adding an $(n-q)$-handle, the so-called *dual handle*. One easily checks that N is diffeomorphic to $\partial_1 W \times [0,1]$ relative $\partial_1 W \times \{1\}$. Thus (2.38) follows.

Suppose that W is relatively $\partial_0 W$ of the shape

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_0}(\phi_i^0) + \sum_{i=1}^{p_1}(\phi_i^1) + \cdots + \sum_{i=1}^{p_n}(\phi_i^n).$$

Then we can conclude inductively using the Diffeomorphism Lemma 2.17 and (2.38) that W is diffeomorphic relative to $\partial_1 W$ to

$$W \cong \partial_1 W \times [0,1] + \sum_{i=1}^{p_n}(\overline{\phi}_i^0) + \sum_{i=1}^{p_{n-1}}(\overline{\phi}_i^1) + \cdots + \sum_{i=1}^{p_0}(\overline{\phi}_i^n). \tag{2.39}$$

This corresponds to replacing a Morse function f by $-f$. The effect is that the number of q-handles now becomes the number of $(n-q)$-handles.

Now applying the first step to the dual handlebody decomposition for q replaced by $(n-q-1)$ and then considering the dual handlebody decomposition of the result finishes the proof of the Normal Form Lemma 2.37. □

2.5 Handlebody Decompositions and Whitehead Groups

Let $(W, \partial_0 W, \partial_1 W)$ be an n-dimensional compact h-cobordism for $n \geq 6$. By the Normal Form Lemma 2.37 we can fix a handlebody decomposition for some fixed number $2 \leq q \leq n-3$

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}).$$

Recall that the $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ is acyclic. Hence the only non-trivial differential $d_{q+1} \colon H_{q+1}(\widetilde{W_{q+1}}, \widetilde{W_q}) \to H_q(\widetilde{W_q}, \widetilde{W_{q-1}})$ is bijective. Recall that $\{[\phi_i^{q+1}] \mid i = 1, 2 \ldots, p_{q+1}\}$ is a $\mathbb{Z}\pi$-basis for $H_{q+1}(\widetilde{W_{q+1}}, \widetilde{W_q})$ and $\{[\phi_i^q] \mid i = 1, 2 \ldots, p_q\}$ is a $\mathbb{Z}\pi$-basis for $H_q(\widetilde{W_q}, \widetilde{W_{q-1}})$. In particular $p_q = p_{q+1}$. The matrix A, which describes the differential d_{q+1} with respect to these bases, is an invertible (p_q, p_q)-matrix over $\mathbb{Z}\pi$. Since we are working with left modules, d_{q+1} sends an element $x \in \mathbb{Z}G^n$ to $x \cdot A \in \mathbb{Z}G^n$, or equivalently, $d_{q+1}([\phi_i^{q+1}]) = \sum_{j=1}^n a_{i,j} [\phi_j^q]$.

Next we define an abelian group $\mathrm{Wh}(\pi)$ as follows. It is the set of equivalence classes of invertible matrices of arbitrary size with entries in $\mathbb{Z}\pi$ where we call an invertible (m, m)-matrix A and an invertible (n, n)-matrix B over $\mathbb{Z}\pi$ equivalent if we can pass from A to B by a sequence of the following operations:

(i) B is obtained from A by adding the k-th row multiplied by x from the left to the l-th row for $x \in \mathbb{Z}\pi$ and $k \neq l$;
(ii) B is obtained by taking the direct sum of A and the $(1,1)$-matrix $I_1 = (1)$, i.e., B looks like the block matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$;
(iii) A is the direct sum of B and I_1. This is the inverse operation to (ii);
(iv) B is obtained from A by multiplying the i-th row from the left by a trivial unit, i.e., by an element of the shape $\pm\gamma$ for $\gamma \in \pi$;
(v) B is obtained from A by interchanging two rows or two columns.

The group structure is given on representatives A and B as follows. By taking the direct sum $A \oplus I_m$ and $B \oplus I_n$ with the identity matrices I_m and I_n of size m and n for appropriate m and n one can arrange that $A \oplus I_m$ and $B \oplus I_n$ are invertible matrices of the same size and can be multiplied. Define $[A] \cdot [B]$ by $[(A \oplus I_m) \cdot (B \oplus I_n)]$. The zero element $0 \in \mathrm{Wh}(\pi)$ is represented by I_n for any positive integer n. The inverse of $[A]$ is given by $[A^{-1}]$. We will show later in Lemma 3.8 that the multiplication is well defined and yields an abelian group $\mathrm{Wh}(\pi)$.

Exercise 2.40 Show that the Whitehead group of the trivial group vanishes.

Exercise 2.41 Let $t \in \mathbb{Z}/5$ be a generator. Consider the $(1,1)$-matrix given as $(1 - t - t^{-1})$ over $\mathbb{Z}[\mathbb{Z}/5]$. Show that it represents a non-trivial element in $\mathrm{Wh}(\mathbb{Z}/5)$.

Lemma 2.42 (i) *Let $(W, \partial_0 W, \partial_1 W)$ be an n-dimensional compact h-cobordism for $n \geq 6$ and A be the matrix defined above. If $[A] = 0$ in $\mathrm{Wh}(\pi)$, then the h-cobordism W is trivial relative $\partial_0 W$.*
(ii) *Consider an element $u \in \mathrm{Wh}(\pi)$, a closed manifold M of dimension $n - 1 \geq 5$ with fundamental group π and an integer q with $2 \leq q \leq n - 3$. Then we can find an h-cobordism of the shape*

$$W = M \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1})$$

such that $[A] = u$.

Proof. (i) Let B be a matrix which is obtained from A by applying one of the operations (i), (ii), (iii), (iv), and (v). It suffices to show that we can modify the given handlebody decomposition in normal form of W with associated matrix A such that we get a new handlebody decomposition in normal form whose associated matrix is B.

We begin with (i). Consider $W' = \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{j=1, j \neq l}^{p_{q+1}} (\phi_j^{q+1})$. Note that we get from W' our h-cobordism W if we attach the handle (ϕ_l^{q+1}). By the Modification Lemma 2.36 we can find an embedding $\overline{\phi}_l^{q+1} : S^q \times D^{n-1-q} \hookrightarrow \partial_1 W'$ such that $\overline{\phi}_l^{q+1}$ is isotopic to ϕ_l^{q+1} and we get

$$\left[\widetilde{\overline{\phi}_l^{q+1}|_{S^q \times \{0\}}}\right] = \left[\widetilde{\phi_l^{q+1}|_{S^q \times \{0\}}}\right] + x \cdot d_{q+1}([\phi_k^{q+1}]).$$

If we attach to W' the handle $(\overline{\phi}_l^{q+1})$, the result is diffeomorphic to W relative $\partial_0 W$ by the Isotopy Lemma 2.16. One easily checks that the associated invertible matrix B is obtained from A by adding the k-th row multiplied by x from the left to the l-th row.

The claim for the operations (ii) and (iii) follows from the Cancellation Lemma 2.21 and the Homology Lemma 2.35. The claim for the operation (iv) follows from the observation that we can replace the attaching map of a handle $\phi^q : S^q \times D^{n-1-q} \hookrightarrow \partial_1 W_q$ by its composition with $f \times \mathrm{id}$ for some diffeomorphism $f : S^q \to S^q$ of degree -1 and that the base element $[\phi_i^q]$ can also be changed to $\gamma \cdot [\phi_i^q]$ by choosing a different lift along $\widetilde{W_q} \to W_q$. Operation (v) can be realised by interchanging the numeration of the q-handles and $(q+1)$-handles.

(ii) Fix an invertible matrix $A = (a_{i,j}) \in \mathrm{GL}(n, \mathbb{Z}\pi)$. Choose trivial pairwise disjoint embeddings $\phi_i^2 : S^1 \times D^{n-2} \hookrightarrow M_0 \times \{1\}$. Consider

$$W_2 = M_0 \times [0,1] + (\phi_1^2) + (\phi_2^2) + \cdots + (\phi_n^2).$$

As the embeddings ϕ_i^2 are trivial, there are embeddings $\phi_i^3 : S^2 \times D^{n-3} \hookrightarrow \partial_1 W_2$ and lifts $\widetilde{\phi_i^3} : S^2 \times D^{n-3} \to \widetilde{\partial_1 W_2}$ such that in $\pi_2(\widetilde{W_2}, \widetilde{\partial_0 W})$

$$[\widetilde{\phi_i^3}|_{S^2 \times \{0\}}] = \sum_{j=1}^{n} a_{i,j} \cdot [\phi_j^2].$$

Put $W = W_2 + (\phi_1^3) + (\phi_2^3) + \cdots + (\phi_n^3)$. One easily checks that W is an h-cobordism over M_0 with a handlebody decomposition realising the matrix A. This finishes the proof Lemma 2.42. □

If π is trivial, then $\mathrm{Wh}(\pi)$ is trivial. Hence Lemma 2.42 (i) implies already the h-Cobordism Theorem 2.4.

Remark 2.43 (Strategy to finish the proof of the s-Cobordism Theorem 2.1) As soon as we have shown that $[A] \in \mathrm{Wh}(\pi)$ agrees with the Whitehead torsion $\tau(W, M_0)$ of the h-cobordism W over M_0 and that this invariant depends only on the diffeomorphism type of W relative M_0, the s-Cobordism Theorem 2.1 (i) will follow.

Obviously Lemma 2.42 (ii) implies the s-Cobordism Theorem 2.1 (ii). We will later see that assertion (iii) of the s-Cobordism Theorem 2.1 follows from assertions (i) and (ii) if we have more information about the Whitehead torsion, namely the sum and the composition formulas. All this will be carried out in Section 3.3.

2.6 Notes

The h-Cobordism Theorem 2.4 is due to Smale [379]. The s-Cobordism Theorem 2.1 is due to Barden, Mazur, and Stallings, see [20, 291]. In the PL category proofs can be found in [367, 6.19 on page 88]. Its topological version follows from Kirby and Siebenmann [219, Conclusion 7.4 on page 320]. More information about the s-Cobordism Theorem can be found for instance in [215], [301], [367, page 87-90]. The s-Cobordism Theorem is known to be false for $\dim(M_0) = 4$ in general, by the work of Donaldson [135], but it is true for $n = \dim(M_0) = 4$ for good fundamental groups in the topological category by results of Quinn and Freedman [37, 157, 158, 159]. Counterexamples in the case $\dim(M_0) = 3$ are constructed by Matsumoto and Siebenmann [288] and Cappell and Shaneson [82] where the relevant 4-dimensional s-cobordism is a topological manifold. It is not known whether one can choose the s-cobordism to be smooth in these counterexamples. It follows from Kwasik and Schultz [236] and Perelman's proof of the Thurston Geometrisation Conjecture, see [222, 311], that every h-cobordism between two orientable closed 3-manifolds is an s-cobordism. The Poincaré Conjecture, see Theorem 2.5, is known in all dimensions where dimension 3 is due to the work of Perelman, see [222, 310, 311, 331, 332, 333], and dimension 4 is due to Freedman, see [37, 157, 158, 159]. The first proof of the Poincaré Conjecture in the topological category in dimension ≥ 5 was given by Newman [319] using engulfing theory. The smooth version of the Poincaré Conjecture holds in dimensions ≤ 3, is open in dimension 4, and is discussed in dimensions ≥ 5 in Remark 12.36.