LECTURE 8.

§ Top SCHOENFLIES THEOREM

From now on: consider topological manifolds.

Recall: Alexander's Horned Sphere is a counterexample to Top Schoenflies Conjecture. It is an example of a wild topological embedding

$:=$ homeomorphism onto the image.

def. Let $f: Y \hookrightarrow N$ be a top. emb. We say that f is flat at $x \in Y$ (or $f(x) \in N$) if there is a chart $(U \subseteq N,\ \varphi: U \to \mathbb{R}^n$ or $\mathbb{R}^n_+)$ such that one of the following holds:

$$\varphi(U \cap f(Y)) = \begin{cases} \mathbb{R}^m \times \{\vec{0}\} \subseteq \mathbb{R}^n & x \in \operatorname{int} Y,\ f(x) \in \operatorname{int} N \\ \mathbb{R}^m_+ \times \{\vec{0}\} \subseteq \mathbb{R}^n & x \in \partial Y,\ f(x) \in \operatorname{int} N \\ \mathbb{R}^m_+ \times \{\vec{0}\} \subseteq \mathbb{R}^n_+ & x \in \partial Y,\ f(x) \in \partial N \end{cases}$$

We say f is locally flat if it is flat at every $x \in Y$. Otherwise, f is wild. By a submanifold of a top. manifold we will mean the image of a loc. flat emb.

Bicollar THM. Any locally flat two-sided embedding $f: Y \xhookrightarrow{\text{lf}} N$ with Y closed and of codimension $\dim N - \dim Y = 1$ extends to a bicollared embedding $\hat{f}: Y \times [-1,1] \xhookrightarrow{\text{top}} N$ s.t. $\hat{f}|_{Y \times \{0\}} = f$.

proof. Collar THM. Boundary of a top. manifold admits a collar.

Assuming this, two-sidedness by definition means that $\exists\, U \subseteq N,\ Y \subseteq U$ s.t. U connected and $U \setminus Y$ has two components, say U' and U''. Then $U' \cup Y$ and $U'' \cup Y$ are manifolds with boundary (since each point admits an appropriate chart). Therefore, they both have collars which we can glue to a bicollar. $\square$

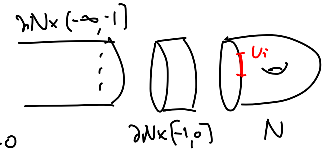
proof of the Collar Thm (due to Connelly):

We need to prove that $\exists\ \partial N \times [0,1] \hookrightarrow N$ s.t. $\partial N \times \{0\} \xrightarrow{\text{id}} \partial N$.

Consider

$$N_{\geq -\infty} := \partial N \times (-\infty, 0] \cup_\varphi N \quad \text{where } \varphi: \partial N \times \{0\} \xrightarrow{\text{id}} \partial N$$

and $N_{\geq s} := \partial N \times [s, 0] \cup_\varphi N$.



It suffices to show

$$N_{\geq -1} \approx N = N_{\geq 0}$$

Let us cover ∂N with $\{U_1, \dots, U_k\}$ s.t. $\exists\, \phi^i: U_i \times [0,1] \hookrightarrow N$. "local collars"

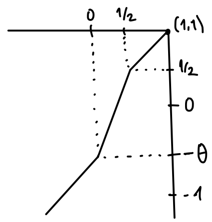
We can extend ϕ^i to $\phi^i_{\geq -\infty}: U_i \times (-\infty, 1] \hookrightarrow N_{\geq -\infty}$.

Let $\{\rho_i : U_i \to [0,1]\}_{1 \leq i \leq k}$ be a partition of unity subordinate to $\{U_i\}$.

For any $0 \leq \theta \leq 1$ let

$$\tau_\theta : (-\infty, 1] \xrightarrow{\approx} (-\infty, 1] \qquad t \longmapsto \begin{cases} t & 1/2 \leq t \leq 1 \\ (1+2\theta)t - \theta & 0 \leq t \leq 1/2 \\ t - \theta & -\infty < t \leq 0 \end{cases}$$

Note: $\tau_0 = \mathrm{Id}$.



Then for $1 \leq i \leq k$ define

$$\psi^i: U_i \times (-\infty, 1] \xrightarrow{\approx} U_i \times (-\infty, 1]$$
$$(x,t) \longmapsto (x, \tau_{\rho_i(x)}(t)).$$

We have:

$\psi^i = \mathrm{Id}$ outside $\operatorname{supp}(\rho_i) \times (-\infty, 1/2]$

so $\phi^i_{\geq\infty} \circ \psi^i$ extends to a homeomorphism $\hat{\psi}^i$ of $N_{\geq -\infty}$.

Then $\hat{\psi}^1 \circ \hat{\psi}^2 \circ \dots \circ \hat{\psi}^k : N_{\geq -\infty} \circlearrowleft \approx$ restricts to $\underset{\displaystyle N}{\overset{\displaystyle N_{\geq 0}}{\|}} \xrightarrow{\approx} N_{\geq -1}$. $\square$

If ∂N not compact have loc. finite $\{\rho_i\}$, so still have $\psi^1 \circ \dots$ defined.

THM [Mazur 1959] If a bicollared embedding $S^d \times [-1,1] \xhookrightarrow{F} S^{d+1}$ has a standard spot, then the closures of both connected components of $S^{d+1} \setminus F(S^d \times \{0\})$ are homeomorphic to $\mathbb{D}^{d+1}$.

+ Note: this complement has 2 components by the Mayer–Vietoris sequence:

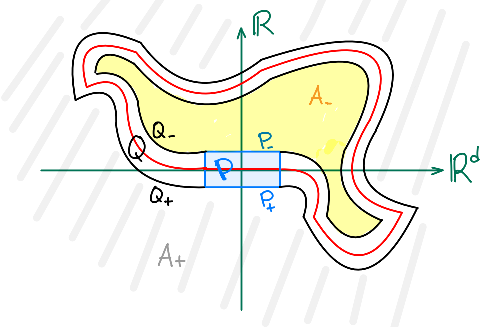
$$\underbrace{H_1(S^{d+1})}_{=0} \longrightarrow \underbrace{H_0\left(F(S^d \times \{-\tfrac{1}{2},\tfrac{1}{2}\})\right)}_{=\mathbb{Z}^2} \longrightarrow \underbrace{H_0\left(F(S^d \times [-\tfrac{1}{2},\tfrac{1}{2}])\right)}_{=\mathbb{Z}} \oplus H_0\left(S^{d+1} \setminus F(S^d \times [-\tfrac{1}{2},\tfrac{1}{2}])\right) \twoheadrightarrow \underbrace{H_0(S^{d+1})}_{=\mathbb{Z}} +$$

note: for MV we use that this is bicollared

def. $p \in S^d$ is a standard spot for F if $F(p,0) = 0 \in \mathbb{R}^{d+1} \subseteq S^{d+1}$ and there exists $\mathbb{D}^d \hookrightarrow S^d$ s.t. 1° $0 \mapsto p$

2° $F(\mathbb{D}^d \times \{0\}) = \text{standard } \mathbb{D}^d \subseteq \mathbb{R}^d \times \{0\} \subseteq \mathbb{R}^{d+1} \subseteq S^{d+1}$

3° $F(q,t) = (F(q,0), t)$



Note: $S^d \times \{0\} \setminus F(\mathbb{D}^d \times \{0\})$ is a standard disc.

proof. Denote $P := F(\mathbb{D}^d \times [-\tfrac{1}{2},\tfrac{1}{2}])$ and $P_\pm := F(\mathbb{D}^d \times \{\pm\tfrac{1}{2}\})$

and $Q := F(\overline{(S^d \setminus \mathbb{D}^d)} \times [-\tfrac{1}{2},\tfrac{1}{2}])$ and $Q_\pm := F(\overline{(S^d \setminus \mathbb{D}^d)} \times \{\pm\tfrac{1}{2}\})$

and $A_\pm$ the connected comp. of $S^{d+1} \setminus F(S^d \times \{0\})$ containing $P_\pm \cup Q_\pm$.

Note: $P \approx Q \approx \mathbb{D}^{d+1}$ as images of $\mathbb{D}^{d+1} \approx \mathbb{D}^d \times [-1,1] \approx \overline{(S^d \setminus \mathbb{D}^d)} \times [-1,1]$ under embeddings

and $\overline{S^{d+1} \setminus P} \approx \mathbb{D}^{d+1}$ since P is standardly embedded, by the definition of a standard spot.

Then:

$$\begin{tikzcd} A_- \cup_{Q_-} Q \cup_{Q_+} A_+ = \overline{S^{d+1} \setminus P} \approx \mathbb{D}^{d+1} \\ A_- \cup_{P_-} P \cup_{P_+} A_+ \arrow[u, "\mathrm{id}" near start, "\approx h"', "\mathrm{id}" near end] \end{tikzcd}$$

can be extended into a homeomorphism.

More precisely:

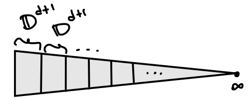
$$\begin{tikzcd}[column sep=small] A_- \arrow[r, phantom, "\cup_{\partial A_-}"] & \partial A_- \times [0,1] \arrow[r, phantom, "\cup_{Q_-}"] & Q \arrow[r, phantom, "\cup_{Q_+}"] & \partial A_+ \times [0,1] \arrow[r, phantom, "\cup_{\partial A_-}"] & A_+ \\ A_- \arrow[u, "\mathrm{id}"] \arrow[r, phantom, "\cup_{\partial A_-}"] & \partial A_- \times [0,1] \arrow[u, "H_-"'] \arrow[r, phantom, "\cup_{P_-}"] & P \arrow[u, "\approx h"'] \arrow[r, phantom, "\cup_{P_+}"] & \partial A_+ \times [0,1] \arrow[u, "H_+"'] \arrow[r, phantom, "\cup_{\partial A_-}"] & A_+ \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

where $H_\pm : \partial A_\pm \times [0,1] \longrightarrow \partial A_\pm$ is an ambient isotopy such that $H_\pm(x,0) = x$ and $H_\pm(\cdot,1)|_{P_\pm} : P_\pm \xrightarrow{\approx} Q_\pm$.

This exists since $\underset{\mathbb{D}^d}{\underset{\wr\wr}{P_\pm}} \subseteq \underset{S^d}{\underset{\wr\wr}{\partial A_\pm}}$ standard, so also $Q_\pm \approx \mathbb{D}^d$ and by Palais any 2 discs amb. isotopic.

Thus: $A_- \cup_{P_-} P \cup_{P_+} A_+ \approx \mathbb{D}^{d+1}$ as well.

Finally, the swindle:



$\approx \mathbb{D}^{d+1}$

$$A_- \approx A_- \cup_{Q_-} \overbrace{Q \cup_{Q_+} \mathbb{D}^{d+1}}^{\text{this is } \natural\, \mathbb{D}^{n+1}} \qquad \text{since } \natural\, \mathbb{D}^{d+1} \text{ does nothing}$$

$$\approx A_- \cup_{Q_-} Q \cup_{Q_+} (A_+ \cup_{P_+} P \cup_{P_-} A_-) \cup_{Q_-} Q \cup_{Q_+} (A_+ \cup_{P_+} P \cup_{P_-} A_-) \ldots \cup \{\infty\}$$

$$\approx \underbrace{(A_- \cup_{Q_-} Q \cup_{Q_+} A_+)}_{\mathbb{D}^{d+1}} \cup_{P_+} P \cup_{P_-} \underbrace{(A_- \cup_{Q_-} Q \cup_{Q_+} A_+)} \cup_{P_+} P \cup_{P_-} \cdots \cup \{\infty\}$$

$$\approx \mathbb{D}^{n+1} \underbrace{\cup_{P_+} P \cup_{P_-} \mathbb{D}^{n+1}}_{\natural\, \mathbb{D}^{n+1}} \underbrace{\cup_{P_+} P \cup_{P_-} \mathbb{D}^{n+1}}_{\natural\, \mathbb{D}^{n+1}} \cup \ldots \cup \{\infty\}$$

$$\approx \mathbb{D}^{n+1}.$$

□