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An Introduction to Manifolds

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Dedicated to the memory of Raoul Bott

Preface

It has been more than two decades since Raoul Bott and I published *Differential Forms in Algebraic Topology*. While this book has enjoyed a certain success, it does assume some familiarity with manifolds and so is not so readily accessible to the average first-year graduate student in mathematics. It has been my goal for quite some time to bridge this gap by writing an elementary introduction to manifolds assuming only one semester of abstract algebra and a year of real analysis. Moreover, given the tremendous interaction in the last twenty years between geometry and topology on the one hand and physics on the other, my intended audience includes not only budding mathematicians and advanced undergraduates, but also physicists who want a solid foundation in geometry and topology.

With so many excellent books on manifolds on the market, any author who undertakes to write another owes to the public, if not to himself, a good rationale. First and foremost is my desire to write a readable but rigorous introduction that gets the reader quickly up to speed, to the point where for example he or she can compute de Rham cohomology of simple spaces.

A second consideration stems from the self-imposed absence of point-set topology in the prerequisites. Most books laboring under the same constraint define a manifold as a subset of a Euclidean space. This has the disadvantage of making quotient manifolds, of which a projective space is a prime example, difficult to understand. My solution is to make the first four chapters of the book independent of point-set topology and to place the necessary point-set topology in an appendix. While reading the first four chapters, the student should at the same time study Appendix A to acquire the point-set topology that will be assumed starting in Chapter 5.

The book is meant to be read and studied by a novice. It is not meant to be encyclopedic. Therefore, I discuss only the irreducible minimum of manifold theory which I think every mathematician should know. I hope that the modesty of the scope allows the central ideas to emerge more clearly. In several years of teaching, I have generally been able to cover the entire book in one semester.

In order not to interrupt the flow of the exposition, certain proofs of a more routine or computational nature are left as exercises. Other exercises are scattered throughout the exposition, in their natural context. In addition to the exercises embedded in the text, there are problems at the end of each chapter. Hints and solutions to selected exercises and problems are gathered at the end of the book. I have starred the problems for which complete solutions are provided.

This book has been conceived as the first volume of a tetralogy on geometry and topology. The second volume is *Differential Forms in Algebraic Topology* cited above. I hope that Volume 3, *Differential Geometry: Connections, Curvature, and Characteristic Classes*, will soon see the light of day. Volume 4, *Elements of Equivariant Cohomology*, a long-running joint project with Raoul Bott before his passing away in 2005, should appear in a year.

This project has been ten years in gestation. During this time I have benefited from the support and hospitality of many institutions in addition to my own; more specifically, I thank the French Ministère de l'Enseignement Supérieur et de la Recherche for a senior fellowship (bourse de haut niveau), the Institut Henri Poincaré, the Institut de Mathématiques de Jussieu, and the Departments of Mathematics at the École Normale Supérieure (rue d'Ulm), the Université Paris VII, and the Université de Lille, for stays of various length. All of them have contributed in some essential way to the finished product.

I owe a debt of gratitude to my colleagues Fulton Gonzalez, Zbigniew Nitecki, and Montserrat Teixidor-i-Bigas, who tested the manuscript and provided many useful comments and corrections, to my students Cristian Gonzalez, Christopher Watson, and especially Aaron W. Brown and Jeffrey D. Carlson for their detailed errata and suggestions for improvement, to Ann Kostant of Springer and her team John Spiegelman and Elizabeth Loew for editing advice, typesetting, and manufacturing, respectively, and to Steve Schnably and Paul Gérardin for years of unwavering moral support. I thank Aaron W. Brown also for preparing the List of Symbols and the TEX files for many of the solutions. Special thanks go to George Leger for his devotion to all of my book projects and for his careful reading of many versions of the manuscripts. His encouragement, feedback, and suggestions have been invaluable to me in this book as well as in several others. Finally, I want to mention Raoul Bott whose courses on geometry and topology helped to shape my mathematical thinking and whose exemplary life is an inspiration to us all.

Medford, Massachusetts *Loring W. Tu*
June 2007

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A Brief Introduction

Undergraduate calculus progresses from differentiation and integration of functions on the real line to functions on the plane and in 3-space. Then one encounters vector-valued functions and learns about integrals on curves and surfaces. Real analysis extends differential and integral calculus from $\mathbb{R}^3$ to $\mathbb{R}^n$. This book is about the extension of the calculus of curves and surfaces to higher dimensions.

The higher-dimensional analogues of smooth curves and surfaces are called *manifolds*. The constructions and theorems of vector calculus become simpler in the more general setting of manifolds; gradient, curl, and divergence are all special cases of the exterior derivative, and the fundamental theorem for line integrals, Green's theorem, Stokes' theorem, and the divergence theorem are different manifestations of a single general Stokes' theorem for manifolds.

Higher-dimensional manifolds arise even if one is interested only in the three-dimensional space which we inhabit. For example, if we call a rotation followed by a translation an affine motion, then the set of all affine motions in $\mathbb{R}^3$ is a six-dimensional manifold. Moreover, this six-dimensional manifold is not $\mathbb{R}^6$.

We consider two manifolds to be topologically the same if there is a homeomorphism between them, that is, a bijection that is continuous in both directions. A topological invariant of a manifold is a property such as compactness that remains unchanged under a homeomorphism. Another example is the number of connected components of a manifold. Interestingly, we can use differential and integral calculus on manifolds to study the topology of manifolds. We obtain a more refined invariant called the de Rham cohomology of the manifold.

Our plan is as follows. First, we recast calculus on $\mathbb{R}^n$ in a way suitable for generalization to manifolds. We do this by giving meaning to the symbols dx, dy, and dz, so that they assume a life of their own, as *differential forms*, instead of being mere notations as in undergraduate calculus.

While it is not logically necessary to develop differential forms on $\mathbb{R}^n$ before the theory of manifolds—after all, the theory of differential forms on a manifold in Part V subsumes that on $\mathbb{R}^n$, from a pedagogical point of view it is advantageous to treat $\mathbb{R}^n$ separately first, since it is on $\mathbb{R}^n$ that the essential simplicity of differential forms and exterior differentiation becomes most apparent.

Another reason for not delving into manifolds right away is so that in a course setting the students without the background in point-set topology can read Appendix A on their own while studying the calculus of differential forms on $\mathbb{R}^n$.

Armed with the rudiments of point-set topology, we define a manifold and derive various conditions for a set to be a manifold. A central idea of calculus is the approximation of a nonlinear object by a linear object. With this in mind, we investigate the relation between a manifold and its tangent spaces. Key examples are Lie groups and their Lie algebras.

Finally we do calculus on manifolds, exploiting the interplay of analysis and topology to show on the one hand how the theorems of vector calculus generalize, and on the other hand, how the results on manifolds define new C^∞ invariants of a manifold, the de Rham cohomology groups.

The de Rham cohomology groups are in fact not merely C^∞ invariants, but also topological invariants, a consequence of the celebrated de Rham theorem that establishes an isomorphism between de Rham cohomology and singular cohomology with real coefficients. To prove this theorem would take us too far afield. Interested readers may find a proof in the sequel [3] to this book.

1

Smooth Functions on a Euclidean Space

The calculus of C^∞ functions will be our primary tool for studying higher-dimensional manifolds. For this reason, we begin with a review of C^∞ functions on $\mathbb{R}^n$.

1.1 C^∞ Versus Analytic Functions

Write the coordinates on $\mathbb{R}^n$ as $x^1, \dots, x^n$ and let $p = (p^1, \dots, p^n)$ be a point in an open set U in $\mathbb{R}^n$. In keeping with the conventions of differential geometry, the indices on coordinates are *superscripts*, not subscripts. An explanation of the rules for superscripts and subscripts is given in Section 4.7.

Definition 1.1. Let k be a nonnegative integer. A function $f\colon U \to \mathbb{R}$ is said to be C^k *at* p if its partial derivatives $\partial^j f/\partial x^{i_1} \cdots \partial x^{i_j}$ of all orders $j \le k$ exist and are continuous at p. The function $f\colon U \to \mathbb{R}$ is C^∞ *at* p if it is C^k for all $k \ge 0$; in other words, its partial derivatives of all orders

$$\frac{\partial^k f}{\partial x^{i_1} \cdots \partial x^{i_k}}$$

exist and are continuous at p. We say that f is C^k *on* U if it is C^k at every point in U. A similar definition holds for a C^∞ function on an open set U. A synonym for C^∞ is "smooth."

Example 1.2.
(i) A C^0 function on U is a continuous function on U.
(ii) Let $f\colon \mathbb{R} \to \mathbb{R}$ be $f(x) = x^{1/3}$. Then

$$f'(x) = \begin{cases} \frac{1}{3}x^{-2/3} & \text{for } x \neq 0, \\ \text{undefined} & \text{for } x = 0. \end{cases}$$

Thus the function f is C^0 but not C^1 at $x = 0$.

(iii) Let $g \colon \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \int_0^x f(t)\,dt = \int_0^x t^{1/3}\,dt = \frac{3}{4}x^{4/3}.$$

Then $g'(x) = f(x) = x^{1/3}$, so $g(x)$ is C^1 but not C^2 at $x = 0$. In the same way one can construct a function that is C^k but not C^{k+1} at a given point.

(iv) The polynomial, sine, cosine, and exponential functions on the real line are all C^∞.

The function f is *real-analytic* at p if in some neighborhood of p it is equal to its Taylor series at p:

$$\begin{aligned} f(x) = f(p) &+ \sum_i \frac{\partial f}{\partial x^i}(p)(x^i - p^i) \\ &+ \frac{1}{2!}\sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j}(p)(x^i - p^i)(x^j - p^j) + \cdots . \end{aligned}$$

A real-analytic function is necessarily C^∞, because as one learns in real analysis, a convergent power series can be differentiated term by term in its region of convergence. For example, if

$$f(x) = \sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots ,$$

then term-by-term differentiation gives

$$f'(x) = \cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots .$$

The following example shows that a C^∞ function need not be real-analytic. The idea is to construct a C^∞ function $f(x)$ on $\mathbb{R}$ whose graph, though not horizontal, is "very flat" near 0 in the sense that all of its derivatives vanish at 0.

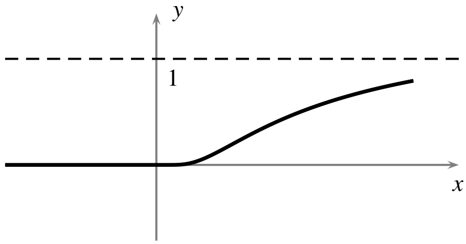


Fig. 1.1. A C^∞ function all of whose derivatives vanish at 0.

Example 1.3 (*A C^∞ function very flat at* 0). Define $f(x)$ on $\mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0; \\ 0 & \text{for } x \le 0. \end{cases}$$

(See Figure 1.1.) By induction, one can show that f is C^∞ on $\mathbb{R}$ and that the derivatives $f^{(k)}(0) = 0$ for all $k \ge 0$ (Problem 1.2).

The Taylor series of this function at the origin is identically zero in any neighborhood of the origin, since all derivatives $f^{(k)}(0) = 0$. Therefore, $f(x)$ cannot be equal to its Taylor series and $f(x)$ is not real-analytic at 0.

1.2 Taylor's Theorem with Remainder

Although a C^∞ function need not be equal to its Taylor series, there is a Taylor's theorem with remainder for C^∞ functions which is often good enough for our purposes. We prove in the lemma below the very first case when the Taylor series consists of only the constant term $f(p)$.

We say that a subset S of $\mathbb{R}^n$ is *star-shaped* with respect to a point p in S if for every x in S, the line segment from p to x lies in S (Figure 1.2).

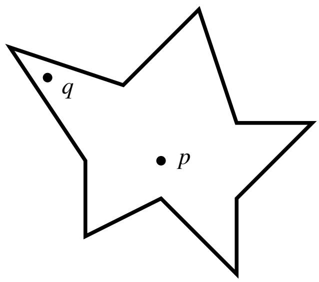


Fig. 1.2. Star-shaped with respect to p, but not with respect to q.

Lemma 1.4 (Taylor's theorem with remainder). *Let f be a C^∞ function on an open subset U of $\mathbb{R}^n$ star-shaped with respect to a point $p = (p^1, \dots, p^n)$ in U. Then there are C^∞ functions $g_1(x), \dots, g_n(x)$ on U such that*

$$f(x) = f(p) + \sum_{i=1}^{n} (x^i - p^i) g_i(x), \quad g_i(p) = \frac{\partial f}{\partial x^i}(p).$$

Proof. Since U is star-shaped with respect to p, for any x in U the line segment $p + t(x - p)$, $0 \le t \le 1$ lies in U (Figure 1.3). So $f(p + t(x - p))$ is defined for $0 \le t \le 1$.

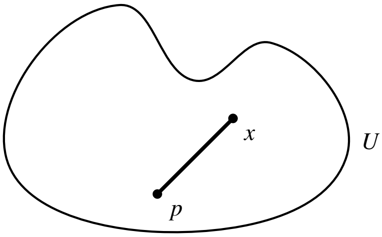


Fig. 1.3. The line segment from p to x.

By the chain rule,

$$\frac{d}{dt} f(p + t(x - p)) = \sum (x^i - p^i) \frac{\partial f}{\partial x^i}(p + t(x - p)).$$

If we integrate both sides with respect to t from 0 to 1, we get

$$f(p + t(x - p))\Big]_0^1 = \sum (x^i - p^i) \int_0^1 \frac{\partial f}{\partial x^i}(p + t(x - p))\, dt. \tag{1.1}$$

Let

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(p + t(x - p))\, dt.$$

Then $g_i(x)$ is C^∞ and (1.1) becomes

$$f(x) - f(p) = \sum (x^i - p^i) g_i(x).$$

Moreover,

$$g_i(p) = \int_0^1 \frac{\partial f}{\partial x^i}(p) dt = \frac{\partial f}{\partial x^i}(p). \qquad \square$$

In case $n = 1$ and $p = 0$, this lemma says that

$$f(x) = f(0) + x f_1(x)$$

for some C^∞ function $f_1(x)$. Applying the lemma repeatedly gives

$$f_i(x) = f_i(0) + x f_{i+1}(x),$$

where f_i, f_{i+1} are C^∞ functions. Hence,

$$\begin{aligned} f(x) &= f(0) + x(f_1(0) + x f_2(x)) \\ &= f(0) + x f_1(0) + x^2(f_2(0) + x f_3(x)) \\ &\;\;\vdots \\ &= f(0) + f_1(0)x + f_2(0)x^2 + \cdots + f_i(0)x^i + f_{i+1}(x)x^{i+1}. \end{aligned} \tag{1.2}$$

Differentiating (1.2) repeatedly and evaluating at 0, we get

$$f_k(0) = \frac{1}{k!} f^{(k)}(0), \quad k = 1, 2, \ldots, i.$$

So (1.2) is a polynomial expansion of $f(x)$ whose terms up to the last term agree with the Taylor series of $f(x)$ at 0.

Remark 1.5. Being star-shaped is not such a restrictive condition, since any open ball

$$B(p, \epsilon) = \{x \in \mathbb{R}^n \mid ||x - p|| < \epsilon\}$$

is star-shaped with respect to p. If f is a C^∞ function defined on an open set U containing p, then there is an $\epsilon > 0$ such that

$$p \in B(p, \epsilon) \subset U.$$

When its domain is restricted to $B(p, \epsilon)$, the function f is defined on a star-shaped neighborhood of p and Taylor's theorem with remainder applies.

NOTATION. It is customary to write the standard coordinates on $\mathbb{R}^2$ as x, y, and the standard coordinates on $\mathbb{R}^3$ as x, y, z.

Problems

1.1. A function that is C^2 but not C^3
Find a function $h\colon \mathbb{R} \to \mathbb{R}$ that is C^2 but not C^3 at $x = 0$.

1.2.* A C^∞ function very flat at 0
Let $f(x)$ be the function on $\mathbb{R}$ defined in Example 1.3.

(a) Show by induction that for $x > 0$ and $k \geq 0$, the kth derivative $f^{(k)}(x)$ is of the form $p_{2k}(1/x)\, e^{-1/x}$ for some polynomial $p_{2k}(y)$ of degree $2k$ in y.
(b) Prove that f is C^∞ on $\mathbb{R}$ and that $f^{(k)}(0) = 0$ for all $k \geq 0$.

1.3. A diffeomorphism of an open interval with $\mathbb{R}$
Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ be open subsets. A C^∞ map $F\colon U \to V$ is called a *diffeomorphism* if it is bijective and has a C^∞ inverse $F^{-1} : V \to U$.

(a) Show that the function $f\colon (-\pi/2, \pi/2) \to \mathbb{R}$, $f(x) = \tan x$, is a diffeomorphism.
(b) Find a linear function $h\colon (a, b) \to (-1, 1)$, thus proving that any two finite open intervals are diffeomorphic.

The composite $f \circ h\colon (a, b) \to \mathbb{R}$ is then a diffeomorphism of an open interval to $\mathbb{R}$.

1.4. A diffeomorphism of an open ball with $\mathbb{R}^n$

(a) Show that the function $h\colon (-\pi/2, \pi/2) \to [0, \infty)$,

$$h(x) = \begin{cases} e^{-1/x} \sec x & \text{for } x \in (0, \pi/2), \\ 0 & \text{for } x \leq 0, \end{cases}$$

is C^∞ on $(-\pi/2, \pi/2)$, strictly increasing on $[0, \pi/2)$, and satisfies $h^{(k)} = 0$ for all $k \geq 0$. (*Hint*: Let $f(x)$ be the function of Example 1.3 and let $g(x) = \sec x$. Then $h(x) = f(x)g(x)$. Use the properties of $f(x)$.)

(b) Define the map $F\colon B(0, \pi/2) \subset \mathbb{R}^n \to \mathbb{R}^n$ by

$$F(x) = \begin{cases} h(|x|) \dfrac{x}{|x|} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Show that $F\colon B(0, \pi/2) \to \mathbb{R}^n$ is a diffeomorphism.

1.5.* Taylor's theorem with remainder to order 2

Prove that if $f\colon \mathbb{R}^2 \to \mathbb{R}$ is C^∞, then there exist C^∞ functions f_{11}, f_{12}, f_{22} on $\mathbb{R}^2$ such that

$$\begin{aligned} f(x, y) = f(0, 0) &+ \frac{\partial f}{\partial x}(0, 0)x + \frac{\partial f}{\partial y}(0, 0)y \\ &+ x^2 f_{11}(x, y) + xy f_{12}(x, y) + y^2 f_{22}(x, y). \end{aligned}$$

1.6.* A function with a removable singularity

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^∞ function with $f(0, 0) = 0$. Define

$$g(t, u) = \begin{cases} \frac{f(t, tu)}{t} & \text{for } t \neq 0; \\ 0 & \text{for } t = 0. \end{cases}$$

Prove that $g(t, u)$ is C^∞ for $(t, u) \in \mathbb{R}^2$. (*Hint*: Apply Problem 1.5.)

1.7. Bijective C^∞ maps

Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$. Show that f is a bijective C^∞ map, but that f^{-1} is not C^∞. (In complex analysis a bijective holomorphic map $f\colon \mathbb{C} \to \mathbb{C}$ necessarily has a holomorphic inverse.)

2

Tangent Vectors in $\mathbb{R}^n$ as Derivations

In elementary calculus we normally represent a vector at a point p in $\mathbb{R}^3$ algebraically as a column of numbers

$$v = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix}$$

or geometrically as an arrow emanating from p (Figure 2.1).

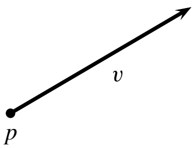


Fig. 2.1. A vector v at p.

A vector at p is tangent to a surface at p if it lies in the tangent plane at p (Figure 2.2), which is the limiting position of the secant planes through p. Intuitively, the tangent plane to a surface at p is the plane in $\mathbb{R}^3$ that just "touches" the surface at p.

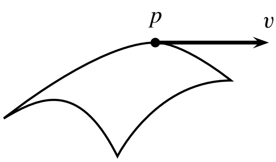


Fig. 2.2. A tangent vector v to a surface at p.

Such a definition of a tangent vector to a surface presupposes that the surface is embedded in a Euclidean space, and so would not apply to the projective plane, which does not sit inside an $\mathbb{R}^n$ in any natural way.

Our goal in this chapter is to find a characterization of a tangent vector in $\mathbb{R}^n$ that would generalize to manifolds.

2.1 The Directional Derivative

In calculus we visualize the tangent space $T_p(\mathbb{R}^n)$ at p in $\mathbb{R}^n$ as the vector space of all arrows emanating from p. By the correspondence between arrows and column vectors, this space can be identified with the vector space $\mathbb{R}^n$. To distinguish between points and vectors, we write a point in $\mathbb{R}^n$ as $p = (p^1, \dots, p^n)$ and a vector v in the tangent space $T_p(\mathbb{R}^n)$ as

$$v = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \quad \text{or} \quad \langle v^1, \dots, v^n \rangle.$$

We usually denote the standard basis for $\mathbb{R}^n$ or $T_p(\mathbb{R}^n)$ by $\{e_1, \dots, e_n\}$. Then $v = \sum v^i e_i$. We sometimes drop the parentheses and write $T_p\mathbb{R}^n$ for $T_p(\mathbb{R}^n)$. Elements of $T_p(\mathbb{R}^n)$ are called *tangent vectors* (or simply *vectors*) at p in $\mathbb{R}^n$.

The line through a point $p = (p^1, \dots, p^n)$ with direction $v = \langle v_1, \dots, v_n \rangle$ in $\mathbb{R}^n$ has parametrization

$$c(t) = (p^1 + tv^1, \dots, p^n + tv^n).$$

Its ith component $c^i(t)$ is $p^i + tv^i$. If f is C^∞ in a neighborhood of p in $\mathbb{R}^n$ and v is a tangent vector at p, the *directional derivative* of f in the direction v at p is defined to be

$$D_v f = \lim_{t \to 0} \frac{f(c(t)) - f(p)}{t} = \left.\frac{d}{dt}\right|_{t=0} f(c(t)).$$

By the chain rule,

$$D_v f = \sum_{i=1}^n \frac{dc^i}{dt}(0) \frac{\partial f}{\partial x^i}(p) = \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(p). \tag{2.1}$$

In the notation $D_v f$, it is understood that the partial derivatives are to be evaluated at p, since v is a vector at p. So $D_v f$ is a number, not a function. We write

$$D_v = \sum v^i \left.\frac{\partial}{\partial x^i}\right|_p$$

for the operator that sends a function f to the number $D_v f$. To simplify the notation we often omit the subscript p if it is clear from the context.

2.2 Germs of Functions

A *relation* on a set S is a subset R of $S \times S$. Given x, y in S, we write $x \sim y$ if and only if $(x, y) \in R$. The relation is an *equivalence relation* if it satisfies the following three properties:

(i) reflexive: $x \sim x$ for all $x \in S$.
(ii) symmetric: if $x \sim y$, then $y \sim x$.
(iii) transitive: if $x \sim y$ and $y \sim z$, then $x \sim z$.

As long as two functions agree on some neighborhood of a point p, they will have the same directional derivatives at p. This suggests that we introduce an equivalence relation on the C^∞ functions defined in some neighborhood of p. Consider the set of all pairs (f, U), where U is a neighborhood of p and $f: U \to \mathbb{R}$ is a C^∞ function. We say that (f, U) is *equivalent* to (g, V) if there is an open set $W \subset U \cap V$ containing p such that $f = g$ when restricted to W. This is clearly an equivalence relation because it is reflexive, symmetric, and transitive. The equivalence class of (f, U) is called the *germ* of f at p. We write $C_p^\infty(\mathbb{R}^n)$ or simply C_p^∞ if there is no possibility of confusion, for the set of all germs of C^∞ functions on $\mathbb{R}^n$ at p.

Example 2.1. The functions

$$f(x) = \frac{1}{1-x}$$

with domain $\mathbb{R} - \{1\}$ and

$$g(x) = 1 + x + x^2 + x^3 + \cdots$$

with domain the open interval $(-1, 1)$ have the same germ at any point p in the open interval $(-1, 1)$.

An *algebra* over a field K is a vector space A over K with a multiplication map

$$\mu\colon A \times A \to A,$$

usually written $\mu(a, b) = a \times b$, such that for all $a, b, c \in A$ and $r \in K$,

(i) (associativity) $(a \times b) \times c = a \times (b \times c)$,
(ii) (distributivity) $(a + b) \times c = a \times c + b \times c$ and $a \times (b + c) = a \times b + a \times c$,
(iii) (homogeneity) $r(a \times b) = (ra) \times b = a \times (rb)$.

Equivalently, an algebra over a field K is a ring A which is also a vector space over K such that the ring multiplication satisfies the homogeneity condition (iii). Thus, an algebra has three operations: the addition and multiplication of a ring and the scalar multiplication of a vector space. Usually we omit the multiplication sign and write ab instead of $a \times b$.

Addition and multiplication of functions induce corresponding operations on C_p^∞, making it into an algebra over $\mathbb{R}$ (Problem 2.2).

2.3 Derivations at a Point

A map $L\colon V \to W$ between vector spaces over a field K is called a *linear map* or a *linear operator* if for any $r \in K$ and $u, v \in V$,

(i) $L(u+v) = L(u) + L(v)$;
(ii) $L(rv) = rL(v)$.

To emphasize the fact that the scalars are in the field K, such a map is also said to be *K-linear*.

For each tangent vector v at a point p in $\mathbb{R}^n$, the directional derivative at p gives a map of real vector spaces

$$D_v\colon C_p^\infty \to \mathbb{R}.$$

By (2.1), D_v is $\mathbb{R}$-linear and satisfies the Leibniz rule

$$D_v(fg) = (D_v f)g(p) + f(p)D_v g, \tag{2.2}$$

essentially because the partial derivatives $\partial/\partial x^i|_p$ have these properties.

In general, any linear map $D\colon C_p^\infty \to \mathbb{R}$ satisfying the Leibniz rule (2.2) is called a *derivation at p* or a *point-derivation* of C_p^∞. Denote the set of all derivations at p by $\mathcal{D}_p(\mathbb{R}^n)$. This set is in fact a real vector space, since the sum of two derivations at p and a scalar multiple of a derivation at p are again derivations at p (Problem 2.3).

Thus far, we know that directional derivatives at p are all derivations at p, so there is a map

$$\begin{aligned} \phi\colon T_p(\mathbb{R}^n) &\to \mathcal{D}_p(\mathbb{R}^n), \\ v \mapsto D_v &= \sum v^i \left.\frac{\partial}{\partial x^i}\right|_p . \end{aligned} \tag{2.3}$$

Since D_v is clearly linear in v, the map ϕ is a linear operator of vector spaces.

Lemma 2.2. *If D is a point-derivation of C_p^∞, then $D(c) = 0$ for any constant function c.*

Proof. As we do not know if every derivation at p is a directional derivative, we need to prove this lemma using only the defining properties of a derivation at p.

By $\mathbb{R}$-linearity, $D(c) = cD(1)$. So it suffices to prove that $D(1) = 0$. By the Leibniz rule

$$D(1) = D(1 \times 1) = D(1) \times 1 + 1 \times D(1) = 2D(1).$$

Subtracting $D(1)$ from both sides gives $0 = D(1)$. □

Theorem 2.3. *The linear map $\phi\colon T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$ defined in* (2.3) *is an isomorphism of vector spaces.*

Proof. To prove injectivity, suppose $D_v = 0$ for $v \in T_p(\mathbb{R}^n)$. Applying D_v to the coordinate function x^j gives

$$0 = D_v(x^j) = \sum_i v^i \left.\frac{\partial}{\partial x^i}\right|_p x^j = \sum_i v^i \delta_i^j = v^j.$$

Hence, $v = 0$ and ϕ is injective.

To prove surjectivity, let D be a derivation of at p and let (f, V) be a representative of a germ in C_p^∞. Making V smaller if necessary, we may assume that V is an open ball, hence star-shaped. By Taylor's theorem with remainder (Lemma 1.4) there are C^∞ functions $g_i(x)$ in a neighborhood of p such that

$$f(x) = f(p) + \sum (x^i - p^i) g_i(x), \quad g_i(p) = \frac{\partial f}{\partial x^i}(p).$$

Applying D to both sides and noting that $D(f(p)) = 0$ and $D(p^i) = 0$ by Lemma 2.2, we get by the Leibniz rule

$$\begin{aligned} Df(x) &= \sum (Dx^i) g_i(p) + \sum (p^i - p^i) Dg_i(x) \\ &= \sum (Dx^i) \frac{\partial f}{\partial x^i}(p). \end{aligned}$$

This proves that $D = D_v$ for $v = \langle Dx^1, \ldots, Dx^n \rangle$. $\square$

This theorem shows that one may identify the tangent vectors at p with the derivations at p. Under the identification $T_p(\mathbb{R}^n) \simeq \mathcal{D}_p(\mathbb{R}^n)$, the standard basis $\{e_1, \ldots, e_n\}$ for $T_p(\mathbb{R}^n)$ corresponds to the set $\{\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p\}$ of partial derivatives. From now on, we will make this identification and write a tangent vector $v = \langle v^1, \ldots, v^n \rangle = \sum v^i e_i$ as

$$v = \sum v^i \left.\frac{\partial}{\partial x^i}\right|_p.$$

The vector space $\mathcal{D}_p(\mathbb{R}^n)$ of derivations at p, although not as geometric as arrows, turns out to be more suitable for generalization to manifolds.

2.4 Vector Fields

A *vector field* X on an open subset U of $\mathbb{R}^n$ is a function that assigns to each point p in U a tangent vector X_p in $T_p(\mathbb{R}^n)$. Since $T_p(\mathbb{R}^n)$ has basis $\{\partial/\partial x^i|_p\}$, the vector X_p is a linear combination

$$X_p = \sum a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p, \quad p \in U.$$

We say that the vector field X is C^∞ on U if the coefficient functions a^i are all C^∞ on U.

Example 2.4. On $\mathbb{R}^2 - \{\mathbf{0}\}$, let $p = (x, y)$. Then

$$X = \frac{-y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}$$

is the vector field of Figure 2.3.

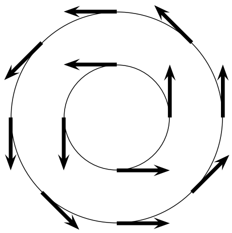


Fig. 2.3. A vector field on $\mathbb{R}^2 - \{\mathbf{0}\}$.

One can identify vector fields on U with column vectors of C^∞ functions on U:

$$X = \sum a^i \frac{\partial}{\partial x^i} \quad \longleftrightarrow \quad \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix}.$$

The ring of C^∞ functions on U is commonly denoted $C^\infty(U)$ or $\mathcal{F}(U)$. Since one can multiply a C^∞ vector field by a C^∞ function and still get a C^∞ vector field, the set of all C^∞ vector fields on U, denoted $\mathfrak{X}(U)$, is not only a vector space over $\mathbb{R}$, but also a *module* over the ring $C^\infty(U)$. We recall the definition of a module.

Definition 2.5. If R is a commutative ring with identity, then an R-*module* is a set A with two operations, addition and scalar multiplication, such that

(1) under addition, A is an abelian group;
(2) for $r, s \in R$ and $a, b \in A$,
  (i) (closure) $ra \in A$;
  (ii) (identity) if 1 is the multiplicative identity in R, then $1a = a$;
  (iii) (associativity) $(rs)a = r(sa)$;
  (iv) (distributivity) $(r+s)a = ra + sa$, $r(a+b) = ra + rb$.

If R is a field, then an R-module is precisely a vector space over R. In this sense, a module generalizes a vector space by allowing scalars in a ring rather than a field.

2.5 Vector Fields as Derivations

If X is a C^∞ vector field on an open subset U of $\mathbb{R}^n$ and f is a C^∞ function on U, we define a new function Xf on U by

$$(Xf)(p) = X_p f \quad \text{for any } p \in U.$$

Writing $X = \sum a^i \partial/\partial x^i$, we get

$$(Xf)(p) = \sum a^i(p) \frac{\partial f}{\partial x^i}(p),$$

or

$$Xf = \sum a^i \frac{\partial f}{\partial x^i},$$

which shows that Xf is a C^∞ function on U. Thus, a C^∞ vector field X gives rise to an $\mathbb{R}$-linear map

$$\begin{aligned} C^\infty(U) &\to C^\infty(U) \\ f &\mapsto Xf. \end{aligned}$$

Proposition 2.6 (Leibniz rule for a vector field). *If X is a C^∞ vector field and f and g are C^∞ functions on an open subset U of $\mathbb{R}^n$, then $X(fg)$ satisfies the product rule (Leibniz rule):*

$$X(fg) = (Xf)g + fXg.$$

Proof. At each point $p \in U$, the vector X_p satisfies the Leibniz rule:

$$X_p(fg) = (X_p f)g(p) + f(p)X_p g.$$

As p varies over U, this becomes an equality of functions:

$$X(fg) = (Xf)g + fXg. \qquad \square$$

If A is an algebra over a field K, a *derivation* of A is a K-linear map $D\colon A \to A$ such that

$$D(ab) = (Da)b + aDb \quad \text{for all } a, b \in A.$$

The set of all derivations of A is closed under addition and scalar multiplication and forms a vector space, denoted $\mathrm{Der}(A)$. As noted above, a C^∞ vector field on an open set U gives rise to a derivation of the algebra $C^\infty(U)$. We therefore have a map

$$\begin{aligned} \varphi\colon \mathfrak{X}(U) &\to \mathrm{Der}(C^\infty(U)), \\ X &\mapsto (f \mapsto Xf). \end{aligned}$$

Just as the tangent vectors at a point p can be identified with the point-derivations of C_p^∞, so the vector fields on an open set U can be identified with the derivations of the algebra $C^\infty(U)$, i.e., the map φ is an isomorphism of vector spaces. The injectivity of φ is easy to establish, but the surjectivity of φ takes some work (see Problem 19.11).

Note that a derivation at p is not a derivation of the algebra C_p^∞. A derivation at p is a map from C_p^∞ to $\mathbb{R}$, while a derivation of the algebra C_p^∞ is a map from C_p^∞ to C_p^∞.

Problems

2.1. Vector fields
Let X be the vector field $x\,\partial/\partial x + y\,\partial/\partial y$ and $f(x, y, z)$ the function $x^2 + y^2 + z^2$ on $\mathbb{R}^3$. Compute Xf.

2.2. Algebra structure on C_p^∞
Define carefully addition, multiplication, and scalar multiplication in C_p^∞. Prove that addition in C_p^∞ is commutative.

2.3. Vector space structure on derivations at a point
Let D and D' be derivations at p in $\mathbb{R}^n$, and $c \in \mathbb{R}$. Prove that

(a) the sum $D + D'$ is a derivation at p.
(b) the scalar multiple cD is a derivation at p.

2.4. Product of derivations
Let A be an algebra over a field K. If D_1 and D_2 are derivations of A, show that $D_1 \circ D_2$ is not necessarily a derivation (it is if D_1 or $D_2 = 0$), but $D_1 \circ D_2 - D_2 \circ D_1$ is always a derivation of A.

3

Alternating k-Linear Functions

This chapter is purely algebraic. Its purpose is to develop the properties of alternating k-linear functions on a vector space for later application to the tangent space at a point of a manifold.

3.1 Dual Space

If V and W are real vector spaces, we denote by $\mathrm{Hom}(V, W)$ the vector space of all linear maps $f: V \to W$. Define the *dual space* V^* to be the vector space of all real-valued linear functions on V:

$$V^* = \mathrm{Hom}(V, \mathbb{R}).$$

The elements of V^* are called *covectors* or 1*-covectors* on V.

In the rest of this section, assume V to be a *finite-dimensional* vector space. Let $\{e_1, \ldots, e_n\}$ be a basis for V. Then every v in V is uniquely a linear combination $v = \sum v^i e_i$ with $v^i \in \mathbb{R}$. Let $\alpha^i : V \to \mathbb{R}$ be the linear function that picks out the ith coordinate, $\alpha^i(v) = v^i$. Note that α^i is characterized by

$$\alpha^i(e_j) = \delta^i_j = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

Proposition 3.1. *The functions $\alpha^1, \ldots, \alpha^n$ form a basis for V^*.*

Proof. We first prove that $\alpha^1, \ldots, \alpha^n$ span V^*. If $f \in V^*$ and $v = \sum v^i e_i$ in V, then

$$f(v) = \sum v^i f(e_i) = \sum f(e_i)\alpha^i(v).$$

Hence,

$$f = \sum f(e_i)\alpha^i,$$

which shows that $\alpha^1, \ldots, \alpha^n$ span V^*.

To show linear independence, suppose $\sum c_i \alpha^i = 0$ for some $c_i \in \mathbb{R}$. Applying both sides to the vector e_j gives

$$0 = \sum c_i \alpha^i(e_j) = \sum c_i \delta^i_j = c_j, \quad j = 1, \dots, n.$$

Hence, $\alpha^1, \dots, \alpha^n$ are linearly independent. □

This basis $\{\alpha^1, \dots, \alpha^n\}$ for V^* is said to be *dual* to the basis $\{e_1, \dots, e_n\}$ for V.

Corollary 3.2. *The dual space V^* of a finite-dimensional vector space V has the same dimension as V.*

Example 3.3 (*Coordinate functions*). With respect to a basis $e_1, \dots, e_n$ for a vector space V, every $v \in V$ can be written uniquely as a linear combination $v = \sum b^i(v)e_i$, where $b^i(v) \in \mathbb{R}$. Let $\alpha^1, \dots, \alpha^n$ be the basis of V^* dual to $e_1, \dots, e_n$. Then

$$\alpha^i(v) = \alpha^i\left(\sum_j b^j(v)e_j\right) = \sum_j b^j(v)\alpha^i(e_j) = \sum_j b^j(v)\delta^i_j = b^i(v).$$

Thus, the set of coordinate functions $b^1, \dots, b^n$ with respect to the basis $e_1, \dots, e_n$ is precisely the dual basis to $e_1, \dots, e_n$.

3.2 Permutations

Fix a positive integer k. A *permutation* of the set $A = \{1, \dots, k\}$ is a bijection $\sigma\colon A \to A$. The product $\tau\sigma$ of two permutations τ and σ of A is the composition $\tau \circ \sigma\colon A \to A$, in that order; first apply σ, then τ. The *cyclic permutation* $(a_1\, a_2\, \cdots\, a_r)$ is the permutation σ such that $\sigma(a_1) = a_2, \sigma(a_2) = a_3, \dots, \sigma(a_{r-1}) = (a_r), \sigma(a_r) = a_1$, and such that σ fixes all the other elements of A. The cyclic permutation $(a_1\, a_2\, \cdots\, a_r)$ is also called a *cycle of length r* or an *r-cycle*. A *transposition* is a cycle of the form $(a\ b)$ that interchanges a and b, leaving all other elements of A fixed.

A permutation $\sigma\colon A \to A$ can be described in two ways: as a matrix

$$\begin{bmatrix} 1 & 2 & \cdots & k \\ \sigma(1) & \sigma(2) & \cdots & \sigma(k) \end{bmatrix}$$

or as a product of disjoint cycles $(a_1 \cdots a_{r_1})(b_1 \cdots b_{r_2}) \cdots$.

Example 3.4. Suppose the permuation $\sigma\colon \{1,2,3,4,5\} \to \{1,2,3,4,5\}$ maps 1, 2, 3, 4, 5 to 2, 4, 5, 1, 3 in that order. Then

$$\sigma = \begin{bmatrix} 1\,2\,3\,4\,5 \\ 2\,4\,5\,1\,3 \end{bmatrix} = (1\ 2\ 4)(3\ 5).$$

Let S_k be the group of all permutations of the set $\{1, \dots, k\}$. A permutation is *even* or *odd* depending on whether it is the product of an even or an odd number of transpositions. From the theory of permutations we know that this is a well-defined concept: an even permutation can never be written as the product of an odd number of transpositions and vice versa. The *sign* of a permutation σ, denoted $\operatorname{sgn}(\sigma)$ or $\operatorname{sgn}\sigma$, is defined to be $+1$ or -1 depending on whether the permutation is even or odd. Clearly, the sign of a permutation satisfies

$$\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$$

for $\sigma, \tau \in S_k$.

Example 3.5. The decomposition

$$(1\ 2\ 3\ 4\ 5) = (1\ 5)(1\ 4)(1\ 3)(1\ 2)$$

shows that the 5-cycle $(1\ 2\ 3\ 4\ 5)$ is an even permutation.

More generally, the decomposition

$$(a_1\ a_2\ \cdots\ a_r) = (a_1\ a_r)(a_1\ a_{r-1}) \cdots (a_1\ a_3)(a_1\ a_2)$$

shows that an r-cycle is an even permutation if and only if r is odd, and an odd permutation if and only if r is even. Thus one way to compute the sign of a permutation is to decompose it into a product of cycles and to count the number of cycles of even length. For example, the permutation σ in Example 3.4 is odd because $(1\ 2\ 4)$ is even and $(3\ 5)$ is odd.

An *inversion* in a permutation σ is an ordered pair $(\sigma(i), \sigma(j))$ such that $i < j$ but $\sigma(i) > \sigma(j)$. Thus, the permutation σ in Example 3.4 has five inversions: $(2, 1)$, $(4, 1)$, $(5, 1)$, $(4, 3)$, and $(5, 3)$.

A second way to compute the sign of a permutation is to count the number of inversions as in the following proposition.

Proposition 3.6. *A permutation is even if and only if it has an even number of inversions.*

Proof. By multiplying σ by a number of transpositions, we can obtain the identity. This can be achieved in k steps.

(1) First, look for the number 1 among $\sigma(1), \sigma(2), \dots, \sigma(k)$. Every number preceding 1 in this list gives rise to an inversion. Suppose $1 = \sigma(i)$. Then $(\sigma(1), 1), \dots, (\sigma(i-1), 1)$ are inversions of σ. Now move 1 to the beginning of the list across the $i-1$ elements $\sigma(1), \dots, \sigma(i-1)$. This requires $i-1$ transpositions. Note that the number of transpositions is the number of inversions ending in 1.

(2) Next look for the number 2 in the list: $1, \sigma(1), \dots, \sigma(i-1), \sigma(i+1), \dots, \sigma(k)$. Every number other than 1 preceding 2 in this list gives rise to an inversion $(\sigma(m), 2)$. Suppose there are i_2 such numbers. Then there are i_2 inversions ending in 2. In moving 2 to its natural position $1, 2, \sigma(1), \sigma(2), \dots$, we need to move it across i_2 numbers. This requires i_2 transpositions.

Repeating this procedure, we see that for each $j = 1, \dots, k$, the number of transpositions required to move j to its natural position is the same as the number of inversions ending in j. In the end we achieve the ordered list $1, 2, \dots, k$ from $\sigma(1), \sigma(2), \dots, \sigma(k)$ by multiplying σ by as many transpositions as the total number of inversions in σ. Therefore, $\operatorname{sgn}(\sigma) = (-1)^{\#\text{ inversions in } \sigma}$. $\square$

3.3 Multilinear Functions

Denote by $V^k = V \times \cdots \times V$ the Cartesian product of k copies of a real vector space V. A function $f\colon V^k \to \mathbb{R}$ is *k-linear* if it is linear in each of its k arguments

$$f(\dots, av + bw, \dots) = af(\dots, v, \dots) + bf(\dots, w, \dots)$$

for $a, b \in \mathbb{R}$ and $v, w \in V$. Instead of 2-linear and 3-linear, it is customary to say "bilinear" and "trilinear." A k-linear function on V is also called a *k-tensor* on V. We will denote the vector space of all k-tensors on V by $L_k(V)$. If f is a k-tensor on V, we also call k the *degree* of f.

Example 3.7. The dot product $f(v, w) = v \cdot w$ on $\mathbb{R}^n$ is bilinear:

$$v \cdot w = \sum v^i w^i,$$

where $v = \sum v^i e_i$ and $w = \sum w^i e_i$.

Example 3.8. The determinant $f(v_1, \dots, v_n) = \det[v_1 \cdots v_n]$, viewed as a function of the n column vectors $v_1, \dots, v_n$ in $\mathbb{R}^n$, is n-linear.

Definition 3.9. A k-linear function $f\colon V^k \to \mathbb{R}$ is *symmetric* if

$$f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = f(v_1, \dots, v_k)$$

for all permutations $\sigma \in S_k$; it is *alternating* if

$$f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = (\operatorname{sgn} \sigma) f(v_1, \dots, v_k)$$

for all $\sigma \in S_k$.

Example 3.10.

(i) The dot product $f(v, w) = v \cdot w$ on $\mathbb{R}^n$ is symmetric.
(ii) The determinant $f(v_1, \dots, v_n) = \det[v_1 \cdots v_n]$ on $\mathbb{R}^n$ is alternating.

We are especially interested in the space $A_k(V)$ of all alternating k-linear functions on a vector space V for $k > 0$. These are also called *alternating k-tensors*, *k-covectors*, or *multicovectors* on V. For $k = 0$, we define a 0-covector to be a constant so that $A_0(V)$ is the vector space $\mathbb{R}$. A 1-covector is simply a covector.

3.4 Permutation Action on k-Linear Functions

If f is a k-linear function on a vector space V and σ is a permutation in S_k, we define a new k-linear function σf by

$$(\sigma f)(v_1, \dots, v_k) = f(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

Thus, f is symmetric if and only if $\sigma f = f$ for all $\sigma \in S_k$ and f is alternating if and only if $\sigma f = (\operatorname{sgn} \sigma) f$ for all $\sigma \in S_k$.

When there is only one argument, the permutation group S_1 is the identity group and a 1-linear function is both symmetric and alternating. In particular,

$$A_1(V) = L_1(V) = V^*.$$

Lemma 3.11. *If $\sigma, \tau \in S_k$ and f is a k-linear function on V, then $\tau(\sigma f) = (\tau\sigma) f$.*

Proof. For $v_1, \dots, v_k \in V$,

$$\begin{aligned} \tau(\sigma f)(v_1, \dots, v_k) &= (\sigma f)(v_{\tau(1)}, \dots, v_{\tau(k)}) \\ &= (\sigma f)(w_1, \dots, w_k) \quad (\text{letting } w_i = v_{\tau(i)}) \\ &= f(w_{\sigma(1)}, \dots, w_{\sigma(k)}) \\ &= f(v_{\tau(\sigma(1))}, \dots, v_{\tau(\sigma(k))}) = f(v_{(\tau\sigma)(1)}, \dots, v_{(\tau\sigma)(k)}) \\ &= (\tau\sigma) f(v_1, \dots, v_k). \end{aligned}$$

□

In general, if G is a group and X is a set, a map

$$\begin{aligned} G \times X &\to X \\ (\sigma, x) &\mapsto \sigma \cdot x \end{aligned}$$

is called a *left action* of G on X if

(i) $1 \cdot x = x$ where 1 is the identity in G and x is any element in X, and
(ii) $\tau \cdot (\sigma \cdot x) = (\tau\sigma) \cdot x$ for all $\tau, \sigma \in G$, $x \in X$.

In this terminology, we have defined a left action of the permutation group S_k on the space $L_k(V)$ of k-linear functions on V. Note that each permutation acts as a linear function on the vector space $L_k(V)$ since σf is $\mathbb{R}$-linear in f.

A *right action* of G on X is defined similarly; it is a map $X \times G \to X$ such that

(i) $x \cdot 1 = x$,
(ii) $(x \cdot \sigma) \cdot \tau = x \cdot (\sigma\tau)$

for all $\sigma, \tau \in G$ and $x \in X$.

Remark 3.12. In some books the notation for σf is f^σ. In that notation, $(f^\sigma)^\tau = f^{\tau\sigma}$, not $f^{\sigma\tau}$.

3.5 The Symmetrizing and Alternating Operators

Given any k-linear function f on a vector space V, there is a way to make a symmetric k-linear function Sf from it:

$$(Sf)(v_1, \ldots, v_k) = \sum_{\sigma \in S_k} f(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

or, in our new shorthand,

$$Sf = \sum_{\sigma \in S_k} \sigma f.$$

Similarly, there is a way to make an alternating k-linear function from f. Define

$$Af = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\sigma f.$$

Proposition 3.13.

(i) *The k-linear function Sf is symmetric.*

(ii) *The k-linear function Af is alternating.*

Proof. We prove (ii) only and leave (i) as an exercise. If $\tau \in S_k$,

$$\begin{aligned}
\tau(Af) &= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\tau(\sigma f) \\
&= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)(\tau\sigma) f \qquad \text{(Lemma 3.11)} \\
&= (\operatorname{sgn} \tau) \sum_{\sigma \in S_k} (\operatorname{sgn} \tau\sigma)(\tau\sigma) f \\
&= (\operatorname{sgn} \tau) Af,
\end{aligned}$$

since as σ runs through all permutations in S_k, so does $\tau\sigma$. □

Exercise 3.14 (Symmetrizing operator). Show that the k-linear function Sf is symmetric.

Lemma 3.15. *If f is an alternating k-linear function on a vector space V, then $Af = (k!)f$.*

Proof.

$$Af = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\sigma f = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)(\operatorname{sgn} \sigma) f = (k!) f.$$ □

Exercise 3.16 (The alternating operator). If f is a 3-linear function on a vector space V, what is $(Af)(v_1, v_2, v_3)$, where $v_1, v_2, v_3 \in V$?

3.6 The Tensor Product

Let f be a k-linear function and g an ℓ-linear function on a vector space V. Their *tensor product* is the $(k+\ell)$-linear function $f \otimes g$ defined by

$$(f \otimes g)(v_1, \dots, v_{k+\ell}) = f(v_1, \dots, v_k)g(v_{k+1}, \dots, v_{k+\ell}).$$

Example 3.17 (*Euclidean inner product*). Let $e_1, \dots, e_n$ be the standard basis for $\mathbb{R}^n$ and let $\alpha^1, \dots, \alpha^n$ be its dual basis. The Euclidean inner product on $\mathbb{R}^n$ is the bilinear function $\langle\ ,\ \rangle \colon \mathbb{R}^n \times \mathbb{R}^n \to R$ defined by

$$\langle v, w \rangle = \sum v^i w^i$$

for $v = \sum v^i e_i$ and $w = \sum w^i e_i$. We can express $\langle\ ,\ \rangle$ in terms of the tensor product:

$$\begin{aligned} \langle v, w \rangle &= \sum_i v^i w^i = \sum_i \alpha^i(v)\alpha^i(w) \\ &= \sum_i (\alpha^i \otimes \alpha^i)(v, w). \end{aligned}$$

Hence, $\langle\ ,\ \rangle = \sum_i \alpha^i \otimes \alpha^i$. This notation is often used in differential geometry to describe an inner product on a vector space.

Exercise 3.18 (Associativity of the tensor product). Check that the tensor product of multilinear functions is associative: if f, g, and h are multilinear functions on V, then

$$(f \otimes g) \otimes h = f \otimes (g \otimes h).$$

3.7 The Wedge Product

If two multilinear functions f and g on a vector space V are alternating, then we would like to have a product that is alternating as well. This motivates the definition of the *wedge product*: for $f \in A_k(V)$ and $g \in A_\ell(V)$,

$$f \wedge g = \frac{1}{k!\ell!} A(f \otimes g); \tag{3.1}$$

or explicitly,

$$\begin{aligned} &(f \wedge g)(v_1, \dots, v_{k+\ell}) \\ &\quad = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}). \end{aligned} \tag{3.2}$$

By Proposition 3.13, $f \wedge g$ is alternating.

When $k = 0$, the element $f \in A_0(V)$ is simply a constant c. In this case, the wedge product $c \wedge g$ is scalar multiplication, since the right-hand side of (3.2) is

$$\frac{1}{\ell!}\sum_{\sigma\in S_\ell}(\operatorname{sgn}\sigma)cg(v_{\sigma(1)},\dots,v_{\sigma(\ell)}) = cg(v_1,\dots,v_\ell).$$

Thus $c\wedge g = cg$ for $c\in\mathbb{R}$ and $g\in A_\ell(V)$.

The coefficient $1/(k!\ell!)$ in the definition of the wedge product compensates for repetitions in the sum: for every permutation $\sigma\in S_{k+\ell}$, there are $k!$ permutations τ in S_k that permute the first k arguments $v_{\sigma(1)},\dots,v_{\sigma(k)}$ and leave the arguments of g alone; for all τ in S_k, the resulting permutations $\sigma\tau$ in $S_{k+\ell}$ contribute the same term to the sum since

$$\begin{aligned}(\operatorname{sgn}\sigma\tau)f(v_{\sigma\tau(1)},\dots,v_{\sigma\tau(k)}) &= (\operatorname{sgn}\sigma\tau)(\operatorname{sgn}\tau)f(v_{\sigma(1)},\dots,v_{\sigma(k)})\\ &= (\operatorname{sgn}\sigma)f(v_{\sigma(1)},\dots,v_{\sigma(k)}),\end{aligned}$$

where the first equality follows from the fact that $(\tau(1),\dots,\tau(k))$ is a permutation of $(1,\dots,k)$. So we divide by $k!$ to get rid of the $k!$ repeating terms in the sum coming from the permutations of the k arguments of f; similarly, we divide by $\ell!$ on account of the ℓ arguments of g.

Example 3.19. For $f\in A_2(V)$ and $g\in A_1(V)$,

$$\begin{aligned}A(f\otimes g)(v_1,v_2,v_3) &= f(v_1,v_2)g(v_3) - f(v_1,v_3)g(v_2) + f(v_2,v_3)g(v_1)\\ &\quad - f(v_2,v_1)g(v_3) + f(v_3,v_1)g(v_2) - f(v_3,v_2)g(v_1).\end{aligned}$$

Among these 6 terms, there are three pairs of equal terms:

$$f(v_1,v_2)g(v_3) = -f(v_2,v_1)g(v_3), \quad \text{and so on.}$$

Therefore, after dividing by 2,

$$(f\wedge g)(v_1,v_2,v_3) = f(v_1,v_2)g(v_3) - f(v_1,v_3)g(v_2) + f(v_2,v_3)g(v_1).$$

One way to avoid redundancies in the definition of $f\wedge g$ is to stipulate that in the sum (3.2), $\sigma(1),\dots,\sigma(k)$ be in ascending order and $\sigma(k+1),\dots,\sigma(k+\ell)$ also be in ascending order. We call a permutation $\sigma\in S_{k+\ell}$ a (k,ℓ)-*shuffle* if

$$\sigma(1)<\cdots<\sigma(k) \quad\text{and}\quad \sigma(k+1)<\cdots<\sigma(k+\ell).$$

Then one may rewrite (3.2) as

$$\begin{aligned}&(f\wedge g)(v_1,\dots,v_{k+\ell})\\ &\qquad = \sum_{\substack{(k,\ell)\text{-shuffles}\\ \sigma}}(\operatorname{sgn}\sigma)f(v_{\sigma(1)},\dots,v_{\sigma(k)})g(v_{\sigma(k+1)},\dots,v_{\sigma(k+\ell)}).\end{aligned} \tag{3.3}$$

Written this way, the definition of $(f\wedge g)(v_1,\dots,v_{k+\ell})$ is a sum of $\binom{k+\ell}{k}$ terms, instead of $(k+\ell)!$ terms.

Example 3.20 (*Wedge product of two covectors*). If f and g are covectors on a vector space V and $v_1,v_2\in V$, then by (3.3)

$$(f\wedge g)(v_1,v_2) = f(v_1)g(v_2) - f(v_2)g(v_1).$$

Exercise 3.21 (Wedge product of two 2-covectors). For $f,g\in A_2(V)$, write out the definition of $f\wedge g$ using $(2,2)$-shuffles.

3.8 Anticommutativity of the Wedge Product

It follows directly from the definition of the wedge product (3.2) that $f \wedge g$ is bilinear in f and in g.

Proposition 3.22. *The wedge product is anticommutative: if $f \in A_k(V)$ and $g \in A_\ell(V)$, then*

$$f \wedge g = (-1)^{k\ell} g \wedge f.$$

Proof. Define $\tau \in S_{k+\ell}$ to be the permutation

$$\tau = \begin{bmatrix} 1 & \cdots & \ell & \ell+1 & \cdots & \ell+k \\ k+1 & \cdots & k+\ell & 1 & \cdots & k \end{bmatrix}.$$

This means that

$$\tau(1) = k+1, \ldots, \tau(\ell) = k+\ell, \tau(\ell+1) = 1, \ldots, \tau(\ell+k) = k.$$

Then

$$\begin{aligned} \sigma(1) &= \sigma\tau(\ell+1), \ldots, \sigma(k) = \sigma\tau(\ell+k), \\ \sigma(k+1) &= \sigma\tau(1), \ldots, \sigma(k+\ell) = \sigma\tau(\ell). \end{aligned}$$

For any $v_1, \ldots, v_{k+\ell} \in V$,

$$\begin{aligned} &A(f \otimes g)(v_1, \ldots, v_{k+\ell}) \\ &\quad= \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn}\sigma) f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}) \\ &\quad= \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn}\sigma) f(v_{\sigma\tau(\ell+1)}, \ldots, v_{\sigma\tau(\ell+k)}) g(v_{\sigma\tau(1)}, \ldots, v_{\sigma\tau(\ell)}) \\ &\quad= (\operatorname{sgn}\tau) \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn}\sigma\tau) g(v_{\sigma\tau(1)}, \ldots, v_{\sigma\tau(\ell)}) f(v_{\sigma\tau(\ell+1)}, \ldots, v_{\sigma\tau(\ell+k)}) \\ &\quad= (\operatorname{sgn}\tau) A(g \otimes f)(v_1, \ldots, v_{k+\ell}). \end{aligned}$$

The last equality follows from the fact that as σ runs through all permutations in $S_{k+\ell}$, so does $\sigma\tau$.

We have proved

$$A(f \otimes g) = (\operatorname{sgn}\tau) A(g \otimes f).$$

Dividing by $k!\ell!$ gives

$$f \wedge g = (\operatorname{sgn}\tau) g \wedge f.$$

Exercise 3.23 (The sign of a permutation). Show that $\operatorname{sgn}\tau = (-1)^{k\ell}$. □

Corollary 3.24. *If f is a k-covector on V and k is odd, then $f \wedge f = 0$.*

Proof. By anticommutativity,

$$\begin{aligned} f \wedge f &= (-1)^{k^2} f \wedge f \\ &= -f \wedge f, \end{aligned}$$

since k is odd. Hence, $2f \wedge f = 0$. Dividing by 2 gives $f \wedge f = 0$. □

3.9 Associativity of the Wedge Product

If f is a k-covector and g is an ℓ-covector, we have defined their wedge product to be the $(k+\ell)$-covector

$$f \wedge g = \frac{1}{k!\ell!} A(f \otimes g).$$

To prove the associativity of the wedge product, we will follow Godbillon [7] by first proving the following lemma on the alternating operator A.

Lemma 3.25. *Suppose f is a k-linear function and g an ℓ-linear function on a vector space V. Then*

(i) $A(A(f) \otimes g) = k!A(f \otimes g)$*, and*
(ii) $A(f \otimes A(g)) = \ell!A(f \otimes g)$.

Proof. (i) By definition,

$$A(A(f) \otimes g) = \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma)\sigma \left(\sum_{\tau \in S_k} (\operatorname{sgn} \tau)(\tau f) \otimes g \right).$$

We can view $\tau \in S_k$ as a permutation in $S_{k+\ell}$ such that

$$\tau(i) = i \quad \text{for } i = k+1, \dots, k+\ell.$$

For such a τ,

$$(\tau f) \otimes g = \tau(f \otimes g).$$

Hence,

$$A(A(f) \otimes g) = \sum_{\sigma \in S_{k+\ell}} \sum_{\tau \in S_k} (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau)(\sigma\tau)(f \otimes g).$$

Let $\mu = \sigma\tau \in S_{k+\ell}$. For each $\mu \in S_{k+\ell}$, there are $k!$ ways to write $\mu = \sigma\tau$ with $\sigma \in S_{k+\ell}$ and $\tau \in S_k$, because each $\tau \in S_k$ determines a unique σ by the formula $\sigma = \mu\tau^{-1}$. So the double sum above can be rewritten as

$$\begin{aligned} A(A(f) \otimes g) &= k! \sum_{\mu \in S_{k+\ell}} (\operatorname{sgn} \mu)\mu(f \otimes g) \\ &= k!A(f \otimes g). \end{aligned}$$

The equality in (ii) is proved in the same way. $\square$

Proposition 3.26 (Associativity of the wedge product). *Let V be a real vector space and f, g, h alternating multilinear functions on V of degrees k, ℓ, m, respectively. Then*

$$(f \wedge g) \wedge h = f \wedge (g \wedge h).$$

Proof. By the definition of the wedge product,

$$\begin{aligned}(f \wedge g) \wedge h &= \frac{1}{(k+\ell)!m!} A((f \wedge g) \otimes h) \\ &= \frac{1}{(k+\ell)!m!} \frac{1}{k!\ell!} A(A(f \otimes g) \otimes h) \\ &= \frac{(k+\ell)!}{(k+\ell)!m!k!\ell!} A((f \otimes g) \otimes h) \quad \text{(by Lemma 3.25(i))} \\ &= \frac{1}{k!\ell!m!} A((f \otimes g) \otimes h).\end{aligned}$$

Similarly,

$$\begin{aligned}f \wedge (g \wedge h) &= \frac{1}{k!(\ell+m)!} A\left(f \otimes \frac{1}{\ell!m!} A(g \otimes h)\right) \\ &= \frac{1}{k!\ell!m!} A(f \otimes (g \otimes h)).\end{aligned}$$

Since the tensor product is associative, we conclude that

$$(f \wedge g) \wedge h = f \wedge (g \wedge h). \qquad \square$$

By associativity, we can omit the parentheses in a multiple wedge product such as $(f \wedge g) \wedge h$ and write simply $f \wedge g \wedge h$.

Corollary 3.27. *Under the hypotheses of the proposition,*

$$f \wedge g \wedge h = \frac{1}{k!\ell!m!} A(f \otimes g \otimes h).$$

This corollary easily generalizes to an arbitrary number of factors: if $f_i \in A_{d_i}(V)$, then

$$f_1 \wedge \cdots \wedge f_r = \frac{1}{(d_1)! \ldots (d_r)!} A(f_1 \otimes \cdots \otimes f_r). \tag{3.4}$$

In particular, we have the following proposition. We use the notation $[b^i_j]$ to denote the matrix whose (i, j)-entry is b^i_j.

Proposition 3.28 (Wedge product of 1-covectors). *If $\alpha^1, \ldots, \alpha^k$ are linear functions on a vector space V and $v_1, \ldots, v_k \in V$, then*

$$(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \ldots, v_k) = \det[\alpha^i(v_j)].$$

Proof. By (3.4),

$$\begin{aligned}(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \ldots, v_k) &= A(\alpha^1 \otimes \cdots \otimes \alpha^k)(v_1, \ldots, v_k) \\ &= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \alpha^1(v_{\sigma(1)}) \cdots \alpha^k(v_{\sigma(k)}) \\ &= \det[\alpha^i(v_j)].\end{aligned}$$

$\square$

3.10 A Basis for k-Covectors

Let $e_1, \dots, e_n$ be a basis for a real vector space V, and let $\alpha^1, \dots, \alpha^n$ be the dual basis for V^*. Introduce the multi-index notation

$$I = (i_1, \dots, i_k)$$

and write e_I for $(e_{i_1}, \dots, e_{i_k})$ and α^I for $\alpha^{i_1} \wedge \dots \wedge \alpha^{i_k}$.

A k-linear function f on V is completely determined by its values on all k-tuples $(e_{i_1}, \dots, e_{i_k})$. If f is alternating, then it is completely determined by its values on $(e_{i_1}, \dots, e_{i_k})$ with $1 \le i_1 < \dots < i_k \le n$; that is, it suffices to consider e_I with I in ascending order. Suppose I, J are ascending multi-indices of length k. By Proposition 3.28,

$$\alpha^I(e_J) = \begin{cases} 1 & \text{if } I = J; \\ 0 & \text{if } I \neq J. \end{cases}$$

Proposition 3.29. *The alternating k-linear functions α^I, $I = (i_1 < \dots < i_k)$, form a basis for the space $A_k(V)$ of alternating k-linear functions on V.*

Proof. First, we show linear independence. Suppose $\sum c_I \alpha^I = 0$, $c_I \in \mathbb{R}$, and I runs over ascending multi-indices of length k. Applying both sides to e_J, $J = (j_1 < \dots < j_k)$, we get

$$0 = \sum c_I \alpha^I(e_J) = c_J,$$

since among all ascending multi-indices I of length k there is only one equal to J. This proves that the α^I are linearly independent.

To show that the α^I span $A_k(V)$, let $f \in A_k(V)$. We claim that

$$f = \sum f(e_I)\alpha^I,$$

where I runs over all ascending multi-indices of length k. Let $g = \sum f(e_I)\alpha^I$. By k-linearity and the alternating property, if two k-covectors agree on all e_J, $J = (j_1 < \dots < j_k)$, then they are equal. But

$$g(e_J) = \sum f(e_I)\alpha^I(e_J) = \sum f(e_I)\delta^I_J = f(e_J).$$

Therefore, $f = g = \sum f(e_I)\alpha^I$. □

Corollary 3.30. *If the vector space V has dimension n, then the vector space $A_k(V)$ of k-covectors on V has dimension $\binom{n}{k}$.*

Proof. An ascending multi-index $I = (i_1 < \dots < i_k)$ is obtained by choosing a subset of k numbers from $1, \dots, n$. This can be done in $\binom{n}{k}$ ways. □

Corollary 3.31. *If $k > \dim V$, then $A_k(V) = 0$.*

Proof. In $\alpha^{i_1} \wedge \dots \wedge \alpha^{i_k}$, at least two of the factors must be the same, say α. Because α is a 1-covector, $\alpha \wedge \alpha = 0$ by Corollary 3.24, so $\alpha^{i_1} \wedge \dots \wedge \alpha^{i_k} = 0$. □

Problems

3.1. Tensor product of covectors
Let $e_1, \dots, e_n$ be a basis for a vector space V and let $\alpha^1, \dots, \alpha^n$ be its dual basis for V^*. Suppose $[g_{ij}] \in \mathbb{R}^{n\times n}$ is an $n \times n$ matrix. Define a bilinear function $f\colon V \times V \to \mathbb{R}$ by

$$f(v, w) = \sum g_{ij} v^i w^j$$

for $v = \sum v^i e_i$ and $w = \sum w^j e_j$ in V. Describe f in terms of the tensor product of α^i and α^j.

3.2. Hyperplanes

(a) Let V be a vector space of dimension n and $f\colon V \to \mathbb{R}$ a nonzero linear functional. Show that $\dim \ker f = n-1$. A linear subspace of V of dimension $n-1$ is called a *hyperplane* in V.
(b) Show that a nonzero linear functional on a vector space V is determined up to a constant by its kernel, a hyperplane in V. In other words, if f and $g\colon V \to \mathbb{R}$ are nonzero linear functionals and $\ker f = \ker g$, then $g = cf$ for some constant $c \in \mathbb{R}$.

3.3. A basis for k-tensors
Let V be a vector space of dimension n with basis $e_1, \dots, e_n$. Let $\alpha^1, \dots, \alpha^n$ be the dual basis for V^*. Show that a basis for the space $L_k(V)$ of k-linear functions on V is $\{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$ for all multi-indices $(i_1, \dots, i_k)$. In particular, this shows that $\dim L_k(V) = n^k$.

3.4. Alternating k-tensors
Let ω be a k-tensor on a vector space V. Prove that ω is alternating if and only if ω changes sign whenever two successive arguments are interchanged:

$$\omega(\dots, v_{i+1}, v_i, \dots) = -\omega(\dots, v_i, v_{i+1}, \dots)$$

for $i = 1, \dots, k-1$.

3.5. Alternating k-tensors
Let ω be a k-tensor on a vector space V. Prove that ω is alternating if and only if $\omega(v_1, \dots, v_k) = 0$ whenever two of the vectors $v_1, \dots, v_k$ are equal.

3.6. Wedge product and scalars
Let V be a vector space. For $a, b \in \mathbb{R}$, $f \in A_k(V)$ and $g \in A_\ell(V)$, show that $af \wedge bg = (ab)\, f \wedge g$.

3.7. Transformation of a wedge product of covectors
Suppose two sets of covectors on a vector space V, $\omega^1, \dots, \omega^k$ and $\tau^1, \dots, \tau^k$, are related by

$$\omega^i = \sum_{j=1}^{k} a^i_j \tau^j, \quad i = 1, \dots, k,$$

for a $k \times k$ matrix $A = [a^i_j]$. Show that

$$\omega^1 \wedge \cdots \wedge \omega^k = (\det A)\, \tau^1 \wedge \cdots \wedge \tau^k.$$

3.8. Transformation rule for a k-covector
Let ω be a k-covector on a vector space V. Suppose two sets of vectors $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ in V are related by

$$u_j = \sum_{j=1}^{i} a^i_j v_i, \quad j = 1, \ldots, k,$$

for a $k \times k$ matrix $A = [a^i_j]$. Show that

$$\omega(u_1, \ldots, u_k) = (\det A)\omega(v_1, \ldots, v_k).$$

3.9.* Linear independence of covectors
Let $\alpha^1, \ldots, \alpha^k$ be 1-covectors on a vector space V. Show that $\alpha^1 \wedge \cdots \wedge \alpha^k \neq 0$ if and only if $\alpha^1, \ldots, \alpha^k$ are linearly independent in the dual space V^*.

3.10.* Exterior multiplication
Let α be a nonzero 1-covector and ω a k-covector on a finite-dimensional vector space V. Show that $\alpha \wedge \omega = 0$ if and only if $\omega = \alpha \wedge \tau$ for some $(k-1)$-covector τ on V.

3.11. Pullback of a k-covector
For any linear map $L\colon V \to W$ of vector spaces and any positive integer k, there is a *pullback map* $L^*\colon A_k(W) \to A_k(V)$ defined by

$$L^*(f)(v_1, \ldots, v_k) = f(L(v_1), \ldots, L(v)_k)$$

for all $v_1, \ldots, v_k \in V$. Show that if $L\colon V \to V$ is a linear operator of a vector space V of dimension n, then $L^*\colon A_n(V) \to A_n(V)$ is multiplication by the determinant of L.

4

Differential Forms on $\mathbb{R}^n$

In this chapter we apply the multilinear algebra of Chapter 3 to define differential forms on an open subset of $\mathbb{R}^n$. Differential forms provide a way to unify the main theorems of vector calculus in $\mathbb{R}^3$.

4.1 Differential 1-Forms and the Differential of a Function

The *cotangent space* to $\mathbb{R}^n$ at p, denoted by $T_p^*(\mathbb{R}^n)$ or $T_p^*\mathbb{R}^n$, is defined to be the dual space $(T_p\mathbb{R}^n)^*$ of the tangent space $T_p(\mathbb{R}^n)$. Thus, an element of the cotangent space $T_p^*(\mathbb{R}^n)$ is a covector or a linear functional on the tangent space $T_p(\mathbb{R}^n)$. In parallel with the definition of a vector field, a *covector field* or a *differential* 1*-form* ω on an open subset U of $\mathbb{R}^n$ is a function that assigns to each point p in U a covector $\omega_p \in T_p^*(\mathbb{R}^n)$. We call a differential 1-form a 1-form for short.

From any C^∞ function $f\colon U \to \mathbb{R}$, we can construct a 1-form df, called the *differential* of f, as follows. For $p \in U$ and $X_p \in T_pU$, define

$$(df)_p(X_p) = X_p f.$$

Let $x^1, \dots, x^n$ be the standard coordinates on $\mathbb{R}^n$. We saw in Section 2.3 that the set $\{\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p\}$ is a basis for the tangent space $T_p(\mathbb{R}^n)$.

Proposition 4.1. *If $x^1, \dots, x^n$ are the standard coordinates on $\mathbb{R}^n$, then at each point $p \in \mathbb{R}^n$, $\{(dx^1)_p, \dots, (dx^n)_p\}$ is the basis for the cotangent space $T_p^*(\mathbb{R}^n)$ dual to the basis $\{\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p\}$ for the tangent space $T_p(\mathbb{R}^n)$.*

Proof. By definition,

$$(dx^i)_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \left.\frac{\partial}{\partial x^j}\right|_p x^i = \delta^i_j. \qquad \square$$

If ω is a 1-form on an open subset U of $\mathbb{R}^n$, then by Proposition 4.1, at each point p in U there is a linear combination

$$\omega_p = \sum a_i(p)\,(dx^i)_p,$$

for some $a_i(p) \in \mathbb{R}$. As p varies over U, the coefficients a_i become functions on U, and we may write $\omega = \sum a_i\,dx^i$. The covector field ω is said to be C^∞ *on* U if the coefficient functions a_i are all C^∞ on U.

If x, y, and z are the coordinates on $\mathbb{R}^3$, then dx, dy, and dz are 1-forms on $\mathbb{R}^3$. In this way, we give independent meaning to what was merely a notation in elementary calculus.

Proposition 4.2 (The differential in terms of coordinates). *If $f\colon U \to \mathbb{R}$ is a C^∞ function on an open set U in $\mathbb{R}^n$, then*

$$df = \sum \frac{\partial f}{\partial x^i}\,dx^i. \tag{4.1}$$

Proof. By Proposition 4.1, at each point p in U,

$$(df)_p = \sum a_i(p)\,(dx^i)_p \tag{4.2}$$

for some constants $a_i(p)$ depending on p. Thus, $df = \sum a_i\,dx^i$ for some functions a_i on U. To evaluate a_j, apply both sides of (4.2) to the coordinate vector field $\partial/\partial x^j$:

$$df\left(\frac{\partial}{\partial x^j}\right) = \sum_i a_i\,dx^i\left(\frac{\partial}{\partial x^j}\right) = \sum_i a_i\delta^i_j = a_j.$$

On the other hand, by the definition of the differential,

$$df\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j}. \qquad \square$$

Equation (4.1) shows that if f is a C^∞ function, then the 1-form df is also C^∞.

Example 4.3. Differential 1-forms occur naturally, even if one is interested only in tangent vectors. Every tangent vector $X_p \in T_p(\mathbb{R}^n)$ is a linear combination of the standard basis vectors:

$$X_p = \sum_i b^i(X_p)\,\frac{\partial}{\partial x^i}\bigg|_p.$$

In Example 3.3 we saw that at each point $p \in \mathbb{R}^n$, we have $b^i(X_p) = (dx^i)_p(X_p)$. Hence, the coefficient b^i of a vector with respect to the standard basis $\partial/\partial x^1, \dots, \partial/\partial x^n$ is none other than the dual form dx^i on $\mathbb{R}^n$.

4.2 Differential k-Forms

More generally, a *differential form ω of degree k* or a *k-form* on an open subset U of $\mathbb{R}^n$ is a function that assigns to each point p in U an alternating k-linear function on the tangent space $T_p(\mathbb{R}^n)$, i.e., $\omega_p \in A_k(T_p\mathbb{R}^n)$. Since $A_1(T_p\mathbb{R}^n) = T_p^*(\mathbb{R}^n)$, the definition of a k-form generalizes that of a 1-form in the preceding section.

By Proposition 3.29, a basis for $A_k(T_p\mathbb{R}^n)$ is

$$dx_p^I = dx_p^{i_1} \wedge \cdots \wedge dx_p^{i_k}, \quad 1 \le i_1 < \cdots < i_k \le n.$$

Therefore, at each point p in U, ω_p is a linear combination

$$\omega_p = \sum a_I(p)\, dx_p^I, \quad 1 \le i_1 < \cdots < i_k \le n,$$

and a k-form ω on U is a linear combination

$$\omega = \sum a_I\, dx^I,$$

with function coefficients $a_I \colon U \to \mathbb{R}$. We say that a k-form ω is C^∞ on U if all the coefficients a_I are C^∞ functions on U.

Denote by $\Omega^k(U)$ the vector space of C^∞ k-forms on U. A 0-form on U assigns to each point p in U an element of $A_0(T_p\mathbb{R}^n) = \mathbb{R}$. Thus, a 0-form on U is simply a function on U, and $\Omega^0(U) = C^\infty(U)$.

Since one can multiply C^∞ k-forms by C^∞ functions, the set $\Omega^k(U)$ of C^∞ k-forms on U is both a vector space over $\mathbb{R}$ and a module over $C^\infty(U)$. With the wedge product as multiplication, the direct sum $\Omega^*(U) = \bigoplus_{k=0}^n \Omega^k(U)$ becomes an algebra over $\mathbb{R}$ as well as a module over $C^\infty(U)$. As an algebra, it is anticommutative and associative.

Remark 4.4. There are no differential forms of degree $> n$ on an open subset of $\mathbb{R}^n$, other than the zero differential form. This is because if $\deg dx^I > n$, then in the expression dx^I at least two of the 1-forms dx^{i_α} must be the same, forcing $dx^I = 0$.

Example 4.5. Let x, y, z be the coordinates on $\mathbb{R}^3$. The C^∞ 1-forms on $\mathbb{R}^3$ are

$$a(x, y, z)\, dx + b(x, y, z)\, dy + c(x, y, z)\, dz,$$

where a, b, c range over all C^∞ functions on $\mathbb{R}^3$. The C^∞ 2-forms are

$$a(x, y, z)\, dy \wedge dz + b(x, y, z)\, dx \wedge dz + c(x, y, z)\, dx \wedge dy$$

and the C^∞ 3-forms are

$$a(x, y, z)\, dx \wedge dy \wedge dz.$$

Exercise 4.6 (A basis for 3-covectors). Let x^1, x^2, x^3, x^4 be the coordinates on $\mathbb{R}^4$ and p a point in $\mathbb{R}^4$. Write down a basis for the vector space $A_3(T_p(\mathbb{R}^4))$.

4.3 Differential Forms as Multilinear Functions on Vector Fields

If ω is a C^∞ 1-form and X is a C^∞ vector field on an open set U in $\mathbb{R}^n$, we define a function $\omega(X)$ on U by the formula

$$\omega(X)_p = \omega_p(X_p), \quad p \in U.$$

Written out in coordinates,

$$\omega = \sum a_i\, dx^i, \qquad X = \sum b^j \frac{\partial}{\partial x^j},$$

so

$$\omega(X) = \left(\sum a_i\, dx^i\right)\left(\sum b^j \frac{\partial}{\partial x^j}\right) = \sum a_i b_i,$$

which shows that $\omega(X)$ is C^∞ on U. Thus, a C^∞ 1-form on U gives rise to a map: $\mathfrak{X}(U) \to C^\infty(U)$.

This function is actually linear over the ring $C^\infty(U)$ since if $f \in C^\infty(U)$, then $\omega(fX) = f\omega(X)$. Let $\mathcal{F}(U) = C^\infty(U)$. In this notation, a 1-form on U gives rise to an $\mathcal{F}(U)$-linear map: $\mathfrak{X}(U) \to \mathcal{F}(U)$.

Similarly, a k-form on U gives rise to a k-linear map over $\mathcal{F}(U)$:

$$\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U) (k \text{ times}) \to \mathcal{F}(U).$$

Exercise 4.7 (Wedge product of a 2-form with a 1-form). Let ω be a 2-form and τ a 1-form on $\mathbb{R}^3$. If X, Y, Z are vector fields on M, then

$$\omega \wedge \tau(X, Y, Z) = \underline{\quad ? \quad}.$$

4.4 The Exterior Derivative

To define the *exterior derivative* of a C^∞ k-form on an open subset U of $\mathbb{R}^n$, we first define it on 0-forms: the exterior derivative of a C^∞ function $f \in C^\infty(U)$ is its differential

$$df = \sum \frac{\partial f}{\partial x^i}\, dx^i \in \Omega^1(U).$$

Definition 4.8. If $\omega = \sum_I a_I\, dx^I \in \Omega^k(U)$, then

$$d\omega = \sum_I da_I \wedge dx^I = \sum_I \left(\sum_j \frac{\partial a_I}{\partial x^j}\, dx^j \right) \wedge dx^I \in \Omega^{k+1}(U).$$

Example 4.9. Let ω be the 1-form $f\,dx + g\,dy$ on $\mathbb{R}^2$, where f and g are C^∞ functions on $\mathbb{R}^2$. To simplify the notation, write $f_x = \partial f/\partial x$, $f_y = \partial f/\partial y$. Then

$$\begin{aligned} d\omega &= df \wedge dx + dg \wedge dy \\ &= (f_x\, dx + f_y\, dy) \wedge dx + (g_x\, dx + g_y\, dy) \wedge dy \\ &= (g_x - f_y)\, dx \wedge dy. \end{aligned}$$

In this computation $dy \wedge dx = -dx \wedge dy$ and $dx \wedge dx = 0$ by the anticommutativity property of the wedge product (Proposition 3.22 and Corollary 3.24).

An algebra A over a field K is said to be *graded* if it can be written as a direct sum $A = \bigoplus_{k=0}^{\infty} A^k$ of vector spaces over K so that the multiplication map sends $A^k \times A^\ell$ to $A^{k+\ell}$. The notation $A = \bigoplus_{k=0}^{\infty} A^k$ means that each element of A is uniquely a *finite sum*

$$a = a_{i_1} + \cdots + a_{i_m},$$

where $a_{i_j} \in A^{i_j}$.

Example 4.10. The polynomial algebra $A = \mathbb{R}[x, y]$ is graded by the degree: A^k consists of all homogeneous polynomials of degree k in the variables x and y.

Example 4.11. The algebra $\Omega^*(U)$ of C^∞ differential forms on U is also graded by the degree.

Definition 4.12. Let $A = \oplus_{k=0}^{\infty} A^k$ be a graded algebra over a field K. An *antiderivation* of the graded algebra A is a K-linear map $D\colon A \to A$ such that for $\omega \in A^k$ and $\tau \in A^\ell$,

$$D(\omega\tau) = (D\omega)\tau + (-1)^k \omega D\tau. \tag{4.3}$$

If the antiderivation sends A^k to A^{k+m}, then we say that it is an antiderivation of *degree* m. (The degree m could be negative.)

Proposition 4.13.
(i) *The exterior differentiation* $d\colon \Omega^*(U) \to \Omega^*(U)$ *is an antiderivation of degree* 1:

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau.$$

(ii) $d^2 = 0$.
(iii) *If* $f \in C^\infty(U)$ *and* $X \in \mathfrak{X}(U)$, *then* $(df)(X) = Xf$.

Proof.
(i) Since both sides of (4.3) are linear in ω and in τ, it suffices to check the equality for $\omega = f\, dx^I$ and $\tau = g\, dx^J$. Then

$$\begin{aligned} d(\omega \wedge \tau) &= d(fg\, dx^I \wedge dx^J) \\ &= \sum \frac{\partial (fg)}{\partial x^i}\, dx^i \wedge dx^I \wedge dx^J \\ &= \sum \frac{\partial f}{\partial x^i}\, dx^i \wedge dx^I \wedge g\, dx^J + \sum f \frac{\partial g}{\partial x^i}\, dx^i \wedge dx^I \wedge dx^J. \end{aligned}$$

In the second sum, moving the 1-form $(\partial g/\partial x^i)\,dx^i$ across the k-form dx^I results in the sign $(-1)^k$ by anticommutativity. Hence,

$$\begin{aligned} d(\omega\wedge\tau) &= d\omega\wedge\tau + (-1)^k \sum f\,dx^I \wedge \frac{\partial g}{\partial x^i}\,dx^i \wedge dx^J \\ &= d\omega\wedge\tau + (-1)^k \omega\wedge d\tau. \end{aligned}$$

(ii) Again, by the $\mathbb{R}$-linearity of d, it suffices to show that $d^2\omega = 0$ for $\omega = f\,dx^I$. We compute:

$$\begin{aligned} d^2(f\,dx^I) &= d\left(\sum \frac{\partial f}{\partial x^i}\,dx^i\wedge dx^I\right) \\ &= \sum \frac{\partial^2 f}{\partial x^j \partial x^i}\,dx^j\wedge dx^i \wedge dx^I. \end{aligned}$$

In this sum if $i = j$, then $dx^j \wedge dx^i = 0$; if $i \neq j$, then $\partial^2 f/\partial x^i \partial x^j$ is symmetric in i and j, but $dx^j \wedge dx^i$ is alternating in i and j, so the terms with $i \neq j$ pair up and cancel out. For example,

$$\begin{aligned} &\frac{\partial^2 f}{\partial x^1 \partial x^2}\,dx^1\wedge dx^2 + \frac{\partial^2 f}{\partial x^2 \partial x^1}\,dx^2\wedge dx^1 \\ &\quad = \frac{\partial^2 f}{\partial x^1 \partial x^2}\,dx^1\wedge dx^2 + \frac{\partial^2 f}{\partial x^1 \partial x^2}(-dx^1\wedge dx^2) = 0. \end{aligned}$$

(iii) Let $X = \sum a^i \partial/\partial x^i$. Then

$$\begin{aligned} (df)(X) &= \left(\sum \frac{\partial f}{\partial x^j}\,dx^j\right)\left(\sum a^i \frac{\partial}{\partial x^i}\right) \\ &= \sum \frac{\partial f}{\partial x^i} a^i = Xf. \end{aligned}$$ □

Proposition 4.14 (Characterization of the exterior derivative). *The three properties of Proposition 4.13 characterize uniquely exterior differentiation on an open set U in $\mathbb{R}^n$; that is, if* (i) *$D\colon \Omega^*(U) \to \Omega^*(U)$ is an antiderivation of degree* 1 *such that* (ii) *$D^2 = 0$ and* (iii) *for $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, $(Df)(X) = Xf$, then $D = d$.*

Proof. Since every k-form on U is a sum of terms such as $f\,dx^{i_1}\wedge\cdots\wedge dx^{i_k}$, by linearity it suffices to show that $D = d$ on a k-form of this type. Applying the three properties, we get

$$\begin{aligned} &D(f\,dx^{i_1}\wedge\cdots\wedge dx^{i_k}) \\ &\quad = D(f\,Dx^{i_1}\wedge\cdots\wedge Dx^{i_k}) && \text{(by (iii), } Dx^i = dx^i\text{)} \\ &\quad = Df \wedge (Dx^{i_1}\wedge\cdots\wedge Dx^{i_k}) && \text{(by (i) and (ii), since } D^2 = 0\text{)} \\ &\quad = df \wedge (dx^{i_1}\wedge\cdots\wedge dx^{i_k}) && \text{(by (iii) again)} \\ &\quad = d(f\,dx^{i_1}\wedge\cdots\wedge dx^{i_k}). \end{aligned}$$

Hence, $D = d$ on $\Omega^*(U)$. □

4.5 Closed Forms and Exact Forms

A k-form ω on U is *closed* if $d\omega = 0$; it is *exact* if there is a $(k-1)$-form τ such that $\omega = d\tau$ on U. Since $d^2 = 0$, every exact form is closed. In the next section we will discuss the meaning of closed and exact forms in the context of vector calculus on $\mathbb{R}^3$.

Exercise 4.15 (A closed 1-form on the punctured plane). Define a 1-form ω on $\mathbb{R}^2 - \{\mathbf{0}\}$ by

$$\omega = \frac{1}{x^2 + y^2}(-y\,dx + x\,dy).$$

Show that ω is closed.

A collection of vector spaces $\{V^k\}_{k=0}^{\infty}$ with linear maps $d_k\colon V^k \to V^{k+1}$ such that $d_{k+1} \circ d_k = 0$ is called a *differential complex* or a *cochain complex*. For any open subset U of $\mathbb{R}^n$, the exterior derivative d makes the vector space $\Omega^*(U)$ of C^∞ forms on U into a cochain complex, called the *de Rham complex* of U:

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \to \cdots .$$

The closed forms are precisely the elements of the kernel of d and the exact forms are the elements of the image of d.

4.6 Applications to Vector Calculus

The theory of differential forms unifies many theorems in vector calculus on $\mathbb{R}^3$. We summarize here some results from vector calculus and then show how they fit into the framework of differential forms.

A vector-valued function on $\mathbb{R}^3$ is the same as a vector field. Recall the three operators on scalar- and vector-valued functions on $\mathbb{R}^3$:

$$\{\text{scalar func.}\} \xrightarrow{\text{grad}} \{\text{vector func.}\} \xrightarrow{\text{curl}} \{\text{vector func.}\} \xrightarrow{\text{div}} \{\text{scalar func.}\}$$

$$\begin{aligned}
\operatorname{grad} f &= \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}, \\
\operatorname{curl} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} &= \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} R_y - Q_z \\ -(R_x - P_z) \\ Q_x - P_y \end{bmatrix}, \\
\operatorname{div} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} &= P_x + Q_y + R_z.
\end{aligned}$$

Proposition A. $\operatorname{curl}(\operatorname{grad} f) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Proposition B. $\operatorname{div}\left(\operatorname{curl}\begin{bmatrix} P \\ Q \\ R \end{bmatrix}\right) = 0$.

Proposition C. *On $\mathbb{R}^3$, a vector field F is the gradient of some scalar function f if and only if* $\operatorname{curl} F = 0$.

Since every 1-form on $\mathbb{R}^3$ is a linear combination with function coefficients of dx, dy, and dz, we can identify 1-forms with vector fields on $\mathbb{R}^3$ via

$$P\,dx + Q\,dy + R\,dz \longleftrightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix}.$$

Similarly, the 2-forms on $\mathbb{R}^3$ can also be identified with vector fields on $\mathbb{R}^3$:

$$P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy \longleftrightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix}.$$

In terms of these identifications, the exterior derivative of a 0-form f is

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz \longleftrightarrow \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial x \end{bmatrix} = \operatorname{grad} f;$$

the exterior derivative of a 1-form is

$$\begin{aligned} &d(P\,dx + Q\,dy + R\,dz) \\ &\quad = (R_y - Q_z)\,dy \wedge dz - (R_x - P_z)\,dz \wedge dx + (Q_x - P_y)\,dx \wedge dy, \end{aligned} \tag{4.4}$$

which corresponds to

$$\operatorname{curl}\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} R_y - Q_z \\ -(R_x - P_z) \\ Q_x - P_y \end{bmatrix};$$

the exterior derivative of a 2-form is

$$\begin{aligned} &d(P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy) \\ &\quad = (P_x + Q_y + R_z)\,dx \wedge dy \wedge dz, \end{aligned} \tag{4.5}$$

which corresponds to

$$\operatorname{div}\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = P_x + Q_y + R_z.$$

Thus, after appropriate identifications, the exterior derivatives d on 0-forms, 1-forms, and 2-forms are simply the three operators grad, curl, and div. In summary, on an open subset U of $\mathbb{R}^3$, there are identifications

$$\begin{array}{ccccccc}
\Omega^0(U) & \xrightarrow{d} & \Omega^1(U) & \xrightarrow{d} & \Omega^2(U) & \xrightarrow{d} & \Omega^3(U) \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\
C^\infty(U) & \xrightarrow[\text{grad}]{} & \mathfrak{X}(U) & \xrightarrow[\text{curl}]{} & \mathfrak{X}(U) & \xrightarrow[\text{div}]{} & C^\infty(U).
\end{array}$$

Propositions A and B express the property $d^2 = 0$ of the exterior derivative.

A vector field $\langle P, Q, R\rangle$ on $\mathbb{R}^3$ is the gradient of a C^∞ function f if and only if the corresponding 1-form $P\,dx + Q\,dy + R\,dz$ is df. Proposition C expresses the fact that a 1-form on $\mathbb{R}^3$ is exact if and only if it is closed.

On the other hand, Proposition C need not be true on a region other than $\mathbb{R}^3$, as the following well-known example from calculus shows.

Example 4.16. If $U = \mathbb{R}^3 - \{z\text{-axis}\}$, and $\mathbf{F}$ is the vector field

$$\mathbf{F} = \left\langle \frac{-y}{x^2+y^2},\ \frac{x}{x^2+y^2},\ 0 \right\rangle$$

on $\mathbb{R}^3$, then curl $\mathbf{F} = \mathbf{0}$, but $\mathbf{F}$ is not the gradient of any C^∞ function on U. The reason is that if $\mathbf{F}$ were the gradient of a C^∞ function f on U, then by the fundamental theorem for line integrals, the line integral

$$\int_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

over any closed curve C would be zero. However, on the unit circle C, with $x = \cos t$ and $y = \sin t$ for $0 \le t \le 2\pi$, this integral is

$$\int_C -y\,dx + x\,dy = \int_0^{2\pi} -(\sin t)\,d(\cos t) + (\cos t)\,d(\sin t) = 2\pi.$$

In terms of differential forms, the 1-form

$$\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

is closed but not exact on U.

It turns out that whether Proposition C is true for a region U depends only on the topology of U. One measure of the failure of a closed k-form to be exact is the quotient vector space

$$H^k(U) := \frac{\{\text{closed } k\text{-forms on } U\}}{\{\text{exact } k\text{-forms on } U\}},$$

called the kth *de Rham cohomology* of U.

The generalization of Proposition C to any differential form on $\mathbb{R}^n$ is called the *Poincaré lemma*: for $k \geq 1$, every closed k-form on $\mathbb{R}^n$ is exact. This is of course equivalent to the vanishing of the kth de Rham cohomology $H^k(\mathbb{R}^n)$ for $k \geq 1$. We will prove it in Chapter 26.

4.7 Convention on Subscripts and Superscripts

In differential geometry it is customary to index vector fields with subscripts $e_1, \ldots, e_n$, and differential forms with superscripts $\omega^1, \ldots, \omega^n$. Being 0-forms, coordinate functions take superscripts, $x^1, \ldots, x^n$. Their differentials, being 1-forms, should also have superscripts, and indeed they do, $dx^1, \ldots, dx^n$. Coordinate vector fields $\partial/\partial x^1, \ldots, \partial/\partial x^n$ are considered to have subscripts because the i in $\partial/\partial x^i$, although a superscript for x^i, is in the lower half of the fraction.

Coefficient functions can have superscripts or subscripts depending on whether they are the coefficient functions of a vector field or of a differential form. For a vector field $X = \sum a^i e_i$, the coefficient functions a^i have superscripts; the idea is that the superscript in a^i "cancels out" the subscript in e_i. For the same reason, the coefficient functions b_j in a differential form $\omega = \sum b_j \, dx^j$ have subscripts.

The beauty of this convention is that there is a "conservation of indices" on the two sides of the equality sign. For example, with $e_i = \partial/\partial x^i$,

$$\omega(X) = \left(\sum b_j \, dx^j\right)\left(\sum a^i \frac{\partial}{\partial x^i}\right) = \sum b_i a^i;$$

after cancellation of superscripts and subscripts, both sides of the equality sign have zero net index. As another example, if $X = \sum a^i \, \partial/\partial x^i$, then

$$a^i = (dx^i)(X).$$

Here both sides have a net superscript i. This convention is a useful mnemonic aid in some of the transformation formulas of differential geometry.

Problems

4.1. A 1-form on $\mathbb{R}^3$

Let ω be the 1-form $z \, dx - dz$ and X be the vector field $y \, \partial/\partial x + x \, \partial/\partial y$ on $\mathbb{R}^3$. Compute $\omega(X)$ and $d\omega$.

4.2. A 2-form on $\mathbb{R}^3$

At each point $p \in \mathbb{R}^3$, define a bilinear function ω_p on $T_p(\mathbb{R}^3)$ by

$$\omega_p(\mathbf{a}, \mathbf{b}) = \omega_p\left(\begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix}, \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix}\right) = p^3 \det \begin{bmatrix} a^1 & b^1 \\ a^2 & b^2 \end{bmatrix},$$

for tangent vectors $\mathbf{a}, \mathbf{b} \in T_p(\mathbb{R}^3)$, where p^3 is the third component of $p = (p^1, p^2, p^3)$. Since ω_p is an alternating bilinear function on $T_p(\mathbb{R}^3)$, ω is a 2-form on $\mathbb{R}^3$. Write ω in terms of the standard basis $dx^i \wedge dx^j$ at each point.

4.3. Exterior calculus

Suppose the standard coordinates on $\mathbb{R}^2$ are called r and θ (this $\mathbb{R}^2$ is the (r, θ)-plane, not the (x, y)-plane). If $x = r\cos\theta$ and $y = r\sin\theta$, calculate dx, dy, and $dx \wedge dy$ in terms of dr and $d\theta$.

4.4. Exterior calculus

Suppose the standard coordinates on $\mathbb{R}^3$ are called ρ, ϕ, and θ. If $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$, calculate dx, dy, dz, and $dx \wedge dy \wedge dz$ in terms of $d\rho$, $d\phi$, and $d\theta$.

4.5. Wedge product

Let α be a 1-form and β a 2-form on $\mathbb{R}^3$. Then

$$\begin{aligned}\alpha &= a_1\,dx^1 + a_2\,dx^2 + a_3\,dx^3,\\ \beta &= b_1\,dx^2 \wedge dx^3 + b_2\,dx^3 \wedge dx^1 + b_3\,dx^1 \wedge dx^2.\end{aligned}$$

Compute $\alpha \wedge \beta$.

4.6. Wedge product and cross product

To a 1-covector $\alpha = a_1\,dx + a_2\,dy + a_3\,dz$ on $\mathbb{R}^3$ we associate the vector $\mathbf{v}_\alpha = \langle a_1, a_2, a_3\rangle$ in $\mathbb{R}^3$; to a 2-covector $\gamma = c_1\,dy \wedge dz + c_2\,dz \wedge dx + c_3\,dx \wedge dy$ on $\mathbb{R}^3$, we associate the vector $\mathbf{v}_\gamma = \langle c_1, c_2, c_3\rangle$. Show that under this correspondence, the wedge product of 1-covectors corresponds to the cross product of vectors in $\mathbb{R}^3$: if $\alpha = a_1\,dx + a_2\,dy + a_3\,dz$ and $\beta = b_1\,dx + b_2\,dy + b_3\,dz$, then $\mathbf{v}_{\alpha\wedge\beta} = \mathbf{v}_\alpha \times \mathbf{v}_\beta$.

4.7.* Interior multiplication

If ω is a k-covector on a vector space V and $v \in V$, the *interior multiplication* or *contraction* of ω with v is the $(k-1)$-covector $\iota_v\omega$ defined by

$$(\iota_v\omega)(v_2, \dots, v_k) = \omega(v, v_2, \dots, v_k)$$

for all $v_2, \dots, v_k \in V$. If $\alpha^1, \dots, \alpha^k$ are 1-covectors on V, prove that

$$\iota_v(\alpha^1 \wedge \cdots \wedge \alpha^k) = \sum_{i=1}^{k} (-1)^{i+1}\alpha^i(v)\alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k,$$

where the caret $\widehat{\ }$ over α^i means that α^i is omitted from the wedge product. (*Hint*: Use the determinant formula for the wedge product of 1-covectors (Proposition 3.28).)

4.8.* Interior multiplication

Keeping the same notation as in the preceding problem, prove that

(a) $\iota_v \circ \iota_v = 0$;

(b) for $\omega \in A_k(V)$ and $\tau \in A_\ell(V)$,

$$\iota_v(\omega \wedge \tau) = (\iota_v \omega) \wedge \tau + (-1)^k \omega \wedge \iota_v \tau.$$

Thus, ι_v is an antiderivation of degree -1 whose square is zero. (*Hint for* (b): By the linearity of ι_v, we may assume that ω and τ are products of 1-covectors. Apply Problem 4.7.)

4.9. Commutator of derivations and antiderivations

Let $A = \oplus_{k=0}^{\infty} A^k$ be a graded algebra over a field K. A *superderivation of A of degree m* is a K-linear map $D\colon A \to A$ such that $D(A^k) \subset (A^{k+m})$ and for all $a \in A^k$ and $b \in A^\ell$,

$$D(ab) = (Da)b + (-1)^{km} a(Db).$$

If D_1 and D_2 are two superderivations of A of respective degrees m_1 and m_2, define their *commutator* to be

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{m_1 m_2} D_2 \circ D_1.$$

Show that $[D_1, D_2]$ is a superderivation of degree $m_1 + m_2$. (A superderivation is said to be *even* or *odd* depending on the parity of its degree. An even superderivation is a derivation; an odd superderivation is an antiderivation.)

5

Manifolds

Intuitively, a manifold is a generalization of curves and surfaces to arbitrary dimension. While there are many different kinds of manifolds—topological manifolds, C^k-manifolds, analytic manifolds, and complex manifolds, in this book we are concerned mainly with smooth manifolds.

5.1 Topological Manifolds

We first recall a few definitions from point-set topology. For more details, see Appendix A. A topological space is *second countable* if it has a countable basis. A *neighborhood* of a point p in a topological space M is any open set containing p. An *open cover* of M is a collection $\{U_\alpha\}_{\alpha\in A}$ of open sets in M whose union $\bigcup_{\alpha\in A} U_\alpha$ is M.

Definition 5.1. A topological space M is *locally Euclidean of dimension n* if every point p in M has a neighborhood U such that there is a homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$. We call the pair $(U, \phi\colon U \to \mathbb{R}^n)$ a *chart*, U a *coordinate neighborhood* or a *coordinate open set*, and ϕ a *coordinate map* or a *coordinate system* on U. We say that a chart (U, ϕ) is *centered* at $p \in U$ if $\phi(p) = 0$. A *chart* (U, ϕ) *about* p simply means that (U, ϕ) is a chart and $p \in U$.

Definition 5.2. A *topological manifold of dimension n* is a Hausdorff, second countable, locally Euclidean space of dimension n.

For the dimension to be well defined, we need to know that for $n \neq m$ an open subset of $\mathbb{R}^n$ is not homeomorphic to an open subset of $\mathbb{R}^m$. This is indeed true, but is not easy to prove (see [4] for a discussion and further references). We will not pursue this point as we are mainly interested in *smooth* manifolds, for which the analogous result is easy to prove (Corollary 8.8). Of course, if a topological manifold has several connected components, it is possible for each component to have a different dimension.

Example 5.3. The Euclidean space $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, 1_{\mathbb{R}^n})$, where $1_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. It is the prime example of a topological manifold. Every open subset of $\mathbb{R}^n$ is also a topological manifold, with chart $(U, 1_U)$.

Recall that the Hausdorff condition and second countability are "hereditary properties"; that is, they are inherited by subspaces: a subspace of a Hausdorff space is Hausdorff (Proposition A.23) and a subspace of a second countable space is second countable (Proposition A.19). So any subspace of $\mathbb{R}^n$ is automatically Hausdorff and second countable.

Example 5.4 (*The cusp*). The graph of $y = x^{2/3}$ in $\mathbb{R}^2$ is a topological manifold (Figure 5.1(a)). By virtue of being a subspace of $\mathbb{R}^2$, it is Hausdorff and second countable. It is locally Euclidean, because it is homeomorphic to $\mathbb{R}$ via $(x, x^{2/3}) \mapsto x$.

(a) Cusp (b) Cross

Fig. 5.1.

Example 5.5 (*The cross*). Show that the cross in $\mathbb{R}^2$ in Figure 5.1 with the subspace topology is not locally Euclidean at p, and so cannot be a topological manifold.

Solution. If a space is locally Euclidean of dimension n at p, then p has a neighborhood U homeomorphic to an open ball $B := B(0, \epsilon) \subset \mathbb{R}^n$ with p mapping to 0. The homeomorphism: $U \to B$ restricts to a homeomorphism: $U - \{p\} \to B - \{0\}$. Now $B - \{0\}$ is either connected if $n \geq 2$ or has two connected components if $n = 1$. Since $U - \{p\}$ has four connected components, there can be no homeomorphism from $U - \{p\}$ to $B - \{0\}$. This contradiction proves that the cross is not locally Euclidean at p. □

5.2 Compatible Charts

Definition 5.6. Two charts $(U, \phi\colon U \to \mathbb{R}^n)$, $(V, \psi\colon V \to \mathbb{R}^n)$ of a topological manifold are C^∞-*compatible* if the two maps

$$\phi \circ \psi^{-1}\colon \psi(U \cap V) \to \phi(U \cap V), \quad \psi \circ \phi^{-1}\colon \phi(U \cap V) \to \psi(U \cap V)$$

are C^∞ (Figure 5.2). These two maps are called the *transition functions* between the charts. If $U \cap V$ is empty, then the two charts are automatically C^∞-compatible. To simplify the notation, we will sometimes write $U_{\alpha\beta}$ for $U_\alpha \cap U_\beta$ and $U_{\alpha\beta\gamma}$ for $U_\alpha \cap U_\beta \cap U_\gamma$.

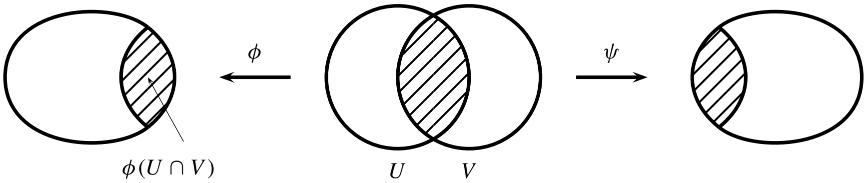


Fig. 5.2. The transition function $\psi \circ \phi^{-1}$ is defined on $\phi(U \cap V)$.

Since we are interested only in C^∞-compatible charts, we often omit to mention C^∞ and speak simply of compatible charts.

Definition 5.7. A C^∞ *atlas* or simply an *atlas* on a locally Euclidean space M is a collection $\{(U_\alpha, \phi_\alpha)\}$ of C^∞-compatible charts that cover M, i.e., such that $M = \bigcup_\alpha U_\alpha$.

Although the C^∞ compatibility of charts is clearly reflexive and symmetric, it is not transitive. The reason is as follows. Suppose (U_1, ϕ_1) is C^∞-compatible with (U_2, ϕ_2), and (U_2, ϕ_2) is C^∞-compatible with (U_3, ϕ_3). Note that the three coordinate functions are simultaneously defined only on the triple intersection U_{123}. Thus, the composite

$$\phi_3 \circ \phi_1^{-1} = (\phi_3 \circ \phi_2^{-1}) \circ (\phi_2 \circ \phi_1^{-1})$$

is C^∞ but only on $\phi_1(U_{123})$, not necessarily on $\phi_1(U_{13})$ (Figure 5.3). A priori we know nothing about $\phi_3 \circ \phi_1^{-1}$ on $\phi_1(U_{13} - U_{123})$ and so we cannot conclude that (U_1, ϕ_1) and (U_3, ϕ_3) are C^∞-compatible.

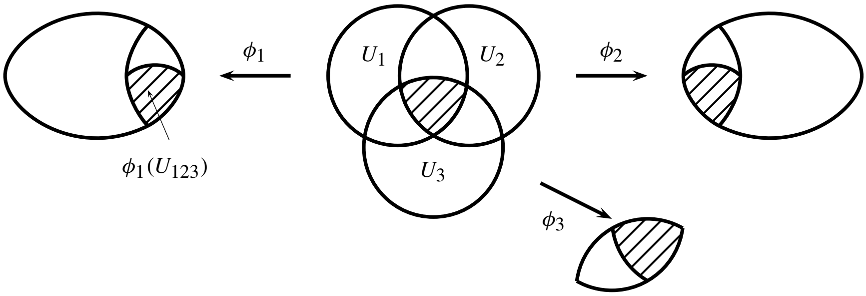


Fig. 5.3. $\phi_3 \circ \phi_1^{-1}$ is C^∞ on $\phi_1(U_{123})$.

We say that a chart (V, ψ) is *compatible with an atlas* $\{(U_\alpha, \phi_\alpha)\}$ if it is compatible with all the charts (U_α, ϕ_α) of the atlas.

Lemma 5.8. *Let* $\{(U_\alpha, \phi_\alpha)\}$ *be an atlas on a locally Euclidean space. If two charts* (V, ψ) *and* (W, σ) *are both compatible with the atlas* $\{(U_\alpha, \phi_\alpha)\}$*, then they are compatible with each other.*

Proof. (See Figure 5.4.) Let $p \in V \cap W$. We need to show that $\sigma \circ \psi^{-1}$ is C^∞ at $\psi(p)$. Since $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for M, $p \in U_\alpha$ for some α. Then p is in the triple intersection $V \cap W \cap U_\alpha$.

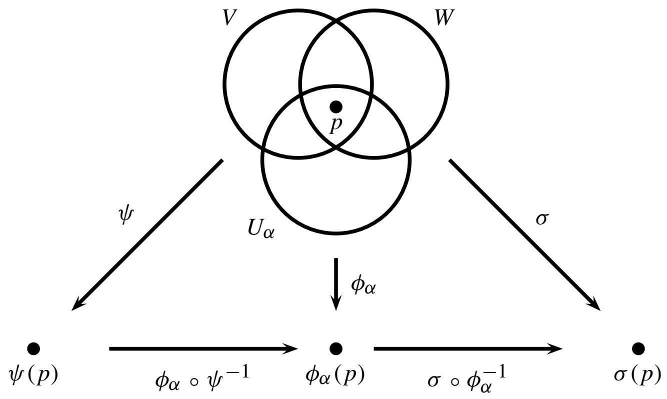


Fig. 5.4. Two charts (V, ψ), (W, σ) compatible with an atlas.

By the remark above, $\sigma \circ \psi^{-1} = (\sigma \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \psi^{-1})$ is C^∞ on $\psi(V \cap W \cap U_\alpha)$, hence at $\psi(p)$. Since p is an arbitrary point of $V \cap W$, this proves that $\sigma \circ \psi^{-1}$ is C^∞ on $\psi(V \cap W)$. Similarly, $\psi \circ \sigma^{-1}$ is C^∞ on $\sigma(V \cap W)$. □

5.3 Smooth Manifolds

An atlas $\mathfrak{A}$ on a locally Euclidean space is said to be *maximal* if it is not contained in a larger atlas; in other words, if $\mathfrak{M}$ is any other atlas containing $\mathfrak{A}$, then $\mathfrak{M} = \mathfrak{A}$.

Definition 5.9. A *smooth* or C^∞ *manifold* is a topological manifold M together with a maximal atlas. The maximal atlas is also called a *differentiable structure* on M. A manifold is said to have dimension n if all of its connected components have dimension n. A manifold of dimension n is also called an n*-manifold*.

In Corollary 8.8 we will prove that if an open set $U \subset \mathbb{R}^n$ is diffeomorphic to an open set $V \subset \mathbb{R}^m$, then $n = m$. As a consequence, the dimension of a manifold at a point is well defined.

In practice, to check that a topological manifold M is a smooth manifold, it is not necessary to exhibit a maximal atlas. The existence of *any* atlas on M will do, because of the following proposition.

Proposition 5.10. *Any atlas* $\mathfrak{A} = \{(U_\alpha, \phi_\alpha)\}$ *on a locally Euclidean space is contained in a unique maximal atlas.*

Proof. Adjoin to the atlas $\mathfrak{A}$ all charts (V_i, ψ_i) that are compatible with $\mathfrak{A}$. By Proposition 5.8 the charts (V_i, ψ_i) are compatible with one another. So the enlarged collection of charts is an atlas. Any chart compatible with the new atlas must be compatible with the original atlas $\mathfrak{A}$ and so by construction belongs to the new atlas. This proves that the new atlas is maximal.

Let $\mathfrak{M}$ be the maximal atlas containing $\mathfrak{A}$ that we have just constructed. If $\mathfrak{M}'$ is another maximal atlas containing $\mathfrak{A}$, then all the charts in $\mathfrak{M}'$ are compatible with $\mathfrak{A}$ and so by construction must belong to $\mathfrak{M}$. This proves that $\mathfrak{M}' \subset \mathfrak{M}$. Since both are maximal, $\mathfrak{M}' = \mathfrak{M}$. Therefore, the maximal atlas containing $\mathfrak{A}$ is unique. □

In summary, to show that a topological space M is a C^∞ manifold, it suffices to check:

(i) M is Hausdorff and second countable,
(ii) M has a C^∞ atlas (not necessarily maximal).

From now on by a *manifold* we will mean a C^∞ manifold. We use the words *smooth* and C^∞ interchangeably.

5.4 Examples of Smooth Manifolds

Example 5.11. The Euclidean space $\mathbb{R}^n$ is a smooth manifold with a single chart $(\mathbb{R}^n, r^1, \ldots, r^n)$, where $r^1, \ldots, r^n$ are the standard coordinates on $\mathbb{R}^n$.

Example 5.12. Any open subset V of a manifold M is also a manifold. If $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for M, then $\{(U_\alpha \cap V, \phi_\alpha|_{U_\alpha \cap V}\}$ is an atlas for V, where $\phi_\alpha|_{U_\alpha \cap V} : U_\alpha \cap V \to \mathbb{R}^n$ denotes the restriction of ϕ_α to the subset $U_\alpha \cap V$.

Example 5.13 (*The graph of a smooth function*). For U an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ a C^∞ function, the *graph* of f is defined to be the subspace

$$\Gamma(f) = \{(x, f(x)) \in U \times \mathbb{R}^m\}.$$

The two maps

$$\phi\colon \Gamma(f) \to U, \qquad (x, f(x)) \mapsto x,$$

and

$$1 \times f : U \to \Gamma(f), \qquad x \mapsto (x, f(x))$$

are continuous and inverse to each other, and so are homeomorphisms. The graph $\Gamma(f)$ of a C^∞ function $f : U \to \mathbb{R}^m$ has an atlas with a single chart $(\Gamma(f), \phi)$, and is therefore a C^∞ manifold. This shows that many of the familiar surfaces of calculus, for example an elliptic paraboloid or a hyperbolic paraboloid, are manifolds.

Example 5.14. For any two positive integers m and n let $\mathbb{R}^{m\times n}$ be the vector space of all $m \times n$ matrices. Since $\mathbb{R}^{m\times n}$ is isomorphic to $\mathbb{R}^{mn}$, we give it the topology of $\mathbb{R}^{mn}$. The *general linear group* $\mathrm{GL}(n, \mathbb{R})$ is by definition

$$\mathrm{GL}(n, \mathbb{R}) := \{A \in \mathbb{R}^{n \times n} \mid \det A \neq 0\} = \det^{-1}(\mathbb{R} - \{0\}).$$

Since the determinant function

$$\det : \mathbb{R}^{n \times n} \to \mathbb{R}$$

is continuous, $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n \times n} \simeq \mathbb{R}^{n^2}$ and is therefore a manifold.

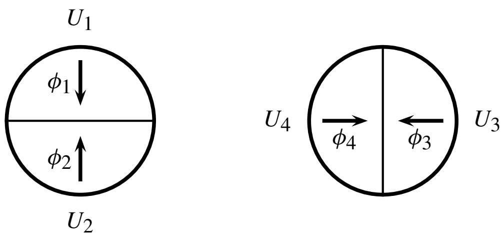


Fig. 5.5. Charts on the unit circle.

Example 5.15 (*The unit circle in the plane*). The equation $x^2 + y^2 = 1$ defines the unit circle S^1 in $\mathbb{R}^2$. We can cover the unit circle by four open sets: the upper and lower semicircles U_1, U_2, and the right and left semicircles U_3, U_4. On U_1 and U_2, the coordinate function x is a homeomorphism onto the open interval $(-1, 1)$ in the x-axis. Thus, $\phi_i(x, y) = x$ for $i = 1, 2$. Similarly, on U_3 and U_4, y is a homeomorphism onto the open interval $(-1, 1)$ in the y-axis (Figure 5.5).

It is easy to check that on every nonempty pairwise intersection $U_\alpha \cap U_\beta$, $\phi_\beta \circ \phi_\alpha^{-1}$ is C^∞. For example, on $U_1 \cap U_3$,

$$\phi_3 \circ \phi_1^{-1}(x) = \phi_3(x, \sqrt{1 - x^2}) = \sqrt{1 - x^2},$$

which is C^∞. On $U_2 \cap U_4$,

$$\phi_4 \circ \phi_2^{-1}(x) = \phi_4(x, -\sqrt{1 - x^2}) = -\sqrt{1 - x^2},$$

which is also C^∞. Thus, $\{(U_i, \phi_i)\}_{i=1}^4$ is an atlas on S^1. By Proposition 5.10, this atlas is contained in a unique maximal atlas. Hence, the unit circle is a manifold.

Example 5.16 (*The product manifold*). If M and N are C^∞ manifolds, then $M \times N$ with its product topology is Hausdorff and second countable (Corollary A.25 and Proposition A.26). To show that $M \times N$ is a manifold, it remains to exhibit an atlas on it.

Proposition 5.17 (An atlas for a product manifold). *If* $\{(U_\alpha, \phi_\alpha)\}$ *and* $\{(V_i, \psi_i)\}$ *are atlases for* M *and* N*, respectively, then*

$$\{(U_\alpha \times V_i, \phi_\alpha \times \psi_i : U_\alpha \times V_i \to \mathbb{R}^{m+n})\}$$

is an atlas on $M \times N$*. Therefore, if* M *and* N *are manifolds, then so is* $M \times N$*.*

Proof. Problem 5.4. □

Example 5.18. It follows from Proposition 5.17 that the infinite cylinder $S^1 \times \mathbb{R}$ and the torus $S^1 \times S^1$ are manifolds (Figure 5.6).

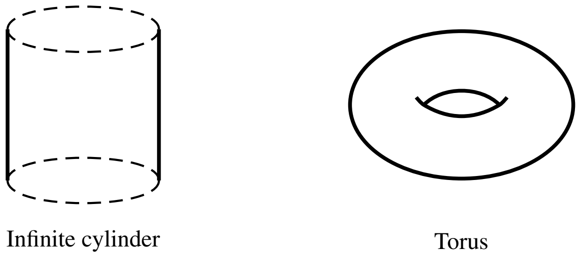


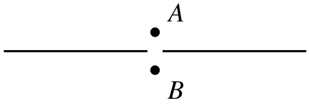
Fig. 5.6.

Since $M \times N \times P = (M \times N) \times P$ is the successive product of pairs of spaces, if M, N and P are manifolds, then so is $M \times N \times P$. Thus, the n-dimensional torus, $S^1 \times \cdots \times S^1$ (n times), is a manifold.

Problems

5.1. The real line with two origins

Let A and B be two points not on the real line $\mathbb{R}$. Consider the set $S = (\mathbb{R} - \{0\}) \cup \{A, B\}$.



For any two positive real numbers c, d, define

$$I_A(-c, d) = (-c, 0) \cup \{A\} \cup (0, d)$$

and similarly for $I_B(-c, d)$, with B instead of A. Define a topology on S as follows: On $(\mathbb{R} - \{0\})$, use the subspace topology inherited from $\mathbb{R}$, with open intervals as a basis. A basis at A is the set $\{I_A(-c, d) \mid c, d > 0\}$; similarly, a basis at B is $\{I_B(-c, d) \mid c, d > 0\}$.

(a) Prove that the map $h\colon I_A(-c, d) \to (-c, d)$ defined by

$$h(x) = x \quad \text{for } x \in (-c, 0) \cup (0, d),$$
$$h(A) = 0,$$

is a homeomorphism.

(b) Show that S is locally Euclidean and second countable, but not Hausdorff.

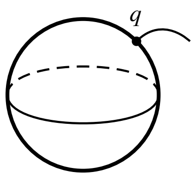


Fig. 5.7. Sphere with a hair.

5.2. Sphere with a hair

Prove that the sphere with a hair in $\mathbb{R}^3$ (Figure 5.7) is not locally Euclidean at q. Hence it cannot be a topological manifold. (*Hint*: Mimic Example 5.5.)

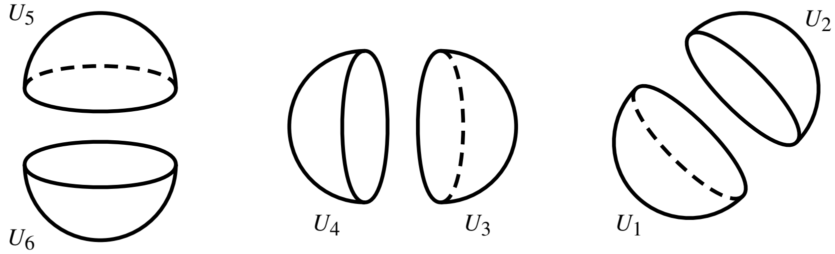


Fig. 5.8. Charts on the unit sphere.

5.3. Charts on the sphere

Let S^2 be the unit sphere

$$x^2 + y^2 + z^2 = 1$$

in $\mathbb{R}^3$. Define in S^2 the six charts corresponding to the six hemispheres—the front, rear, right, left, upper, and lower hemispheres (Figure 5.8):

$$\begin{aligned}
U_1 &= \{(x, y, z) \in S^2 \mid x > 0\}, & \phi_1(x, y, z) &= (y, z),\\
U_2 &= \{(x, y, z) \in S^2 \mid x < 0\}, & \phi_2(x, y, z) &= (y, z),\\
U_3 &= \{(x, y, z) \in S^2 \mid y > 0\}, & \phi_3(x, y, z) &= (x, z),\\
U_4 &= \{(x, y, z) \in S^2 \mid y < 0\}, & \phi_4(x, y, z) &= (x, z),\\
U_5 &= \{(x, y, z) \in S^2 \mid z > 0\}, & \phi_5(x, y, z) &= (x, y),\\
U_6 &= \{(x, y, z) \in S^2 \mid z < 0\}, & \phi_6(x, y, z) &= (x, y).
\end{aligned}$$

Describe the domain $\phi_4(U_{14})$ of $\phi_1 \circ \phi_4^{-1}$ and show that $\phi_1 \circ \phi_4^{-1}$ is C^∞ on $\phi_4(U_{14})$. Do the same for $\phi_6 \circ \phi_1^{-1}$.

5.4. An atlas for a product manifold
Prove Proposition 5.17.

6

Smooth Maps on a Manifold

Using coordinate charts we can transfer the notion of differentiability from $\mathbb{R}^m$ to a smooth manifold M.

6.1 Smooth Functions and Maps

Definition 6.1. Let M be a smooth manifold of dimension n. A function $f: M \to \mathbb{R}$ is said to be C^∞ or *smooth at a point* p in M if there is a chart (U, ϕ) containing p in the atlas of M such that $f \circ \phi^{-1}$, which is defined on the open subset $\phi(U)$ of $\mathbb{R}^n$, is C^∞ at $\phi(p)$ (see Figure 6.1).

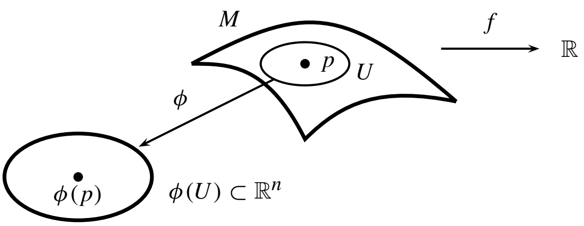


Fig. 6.1. Checking that a function f is C^∞ at p by pulling back to $\mathbb{R}^n$.

This definition is independent of the chart (U, ϕ), for if (V, ψ) is any other chart in the atlas containing p, then on $\psi(U \cap V)$

$$f \circ \psi^{-1} = (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1}),$$

which is C^∞ at $\psi(p)$ (see Figure 6.2). The function f is said to be C^∞ on M if it is C^∞ at every point of M.

We emphasize again that by manifolds we always mean C^∞ manifolds and that we use the terms "C^∞" and "smooth" interchangeably.

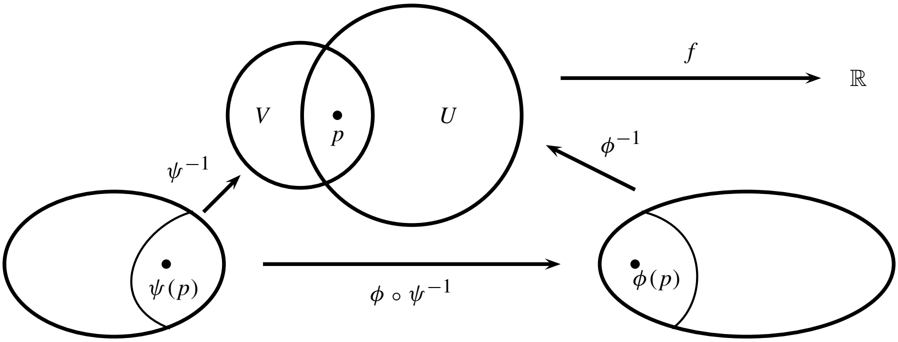


Fig. 6.2. Checking that a function f is C^∞ at p via two charts.

Notation. We generally denote a manifold by M and its dimension by n. However, in speaking of two manifolds simultaneously, as in a map $f\colon N \to M$, the dimension of N will be n and the dimension of M will be m.

Definition 6.2. Let $F\colon N \to M$ be a map and h a function on M. The *pullback* of h by F, denoted F^*h, is the composite function $h \circ F$.

In this terminology, a function f on M is C^∞ on a chart (U, ϕ) if its pullback by ϕ^{-1} is C^∞ on the subset $\phi(U)$ of a Euclidean space.

Definition 6.3. Let N and M be manifolds of dimension n and m, respectively. A map $F\colon N \to M$ is C^∞ *at a point* p in N if there is a chart (V, ψ) in M containing $F(p)$ and a chart (U, ϕ) in N containing p such that the composition $\psi \circ F \circ \phi^{-1}$, a map from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^m$, is C^∞ at $\phi(p)$ (see Figure 6.3). By the continuity of F, one can always choose U small enough so that $F(U) \subset V$.

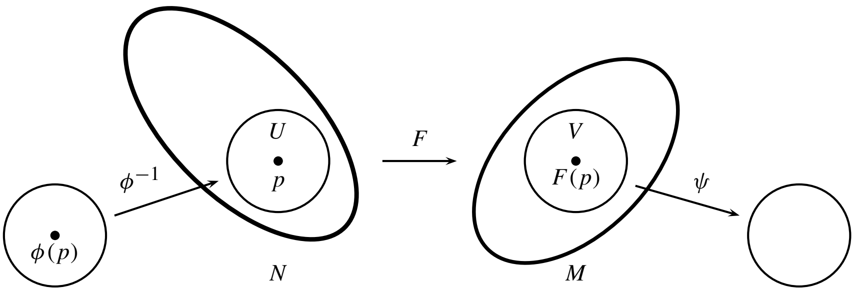


Fig. 6.3. Checking that a map $F\colon N \to M$ is C^∞ at p.

This definition of a map $F\colon N \to M$ being C^∞ at p in N is in fact independent of the choice of charts (see Problem 6.1).

Definition 6.4. The map $F: N \to M$ is said to be C^∞ if it is C^∞ at every point of N. It is a *diffeomorphism* if it is bijective and both F and its inverse F^{-1} are C^∞.

Example 6.5. If (U, F) is a chart in the atlas of a manifold M of dimension n, then F is C^∞, because with $\psi: \mathbb{R}^n \to \mathbb{R}^n$ being the identity map, $\psi \circ F \circ F^{-1}$ is C^∞. The inverse map $F^{-1}: F(U) \to U$ is also C^∞ because in Definition 6.3 one can take $\psi = F$ and $\phi = 1_{F(U)}$. Then $\psi \circ F^{-1} \circ 1_{F(U)} = 1_{F(U)}$ is C^∞.

Proposition 6.6 (Composite of C^∞ maps). *If $F: N \to M$ and $G: M \to P$ are C^∞ maps of manifolds, then the composite $G \circ F: N \to P$ is C^∞.*

Proof. Problem 6.2. □

Proposition 6.7. *Let U be an open subset of a manifold M. If $F : U \to \mathbb{R}^n$ is a diffeomorphism onto its image, then (U, F) is a chart in the atlas of M.*

Proof. For any chart (U_α, ϕ_α) in the atlas of M, both $F \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ F^{-1}$ are C^∞. Hence, (U, F) is compatible with the atlas. By the maximality of the atlas of M, the chart (U, F) is in the atlas. □

Now that we know what it means for a map between manifolds to be C^∞, we can define a Lie group.

Definition 6.8. A *Lie group* is a C^∞ manifold G having a group structure such that the multiplication map

$$\mu: G \times G \to G$$

and the inverse map

$$\iota: G \to G, \quad \iota(x) = x^{-1},$$

are both C^∞.

Similarly, a *topological group* is a topological space having a group structure such that the multiplication and inverse maps are both continuous. Note that a topological group is required to be a topological space, not a topological manifold.

Example 6.9.

(i) The Euclidean space $\mathbb{R}^n$ is a Lie group under addition.
(ii) The set $\mathbb{C}^\times$ of nonzero complex numbers is a Lie group under multiplication.
(iii) The unit circle S^1 in $\mathbb{C}^\times$ is a Lie group under multiplication.

In Chapter 15 we will study a few less obvious examples of Lie groups.

6.2 Partial Derivatives

Let (U, ϕ) be a chart and f a C^∞ function on a manifold M of dimension n. As a function into $\mathbb{R}^n$, ϕ has n components $x^1, \dots, x^n$. This means if $r^1, \dots, r^n$ are the standard coordinates on $\mathbb{R}^n$, then $x^i = r^i \circ \phi$. For $p \in U$, we define the *partial derivative* $\partial f/\partial x^i$ at p to be

$$\left.\frac{\partial}{\partial x^i}\right|_p f := \frac{\partial f}{\partial x^i}(p) = \frac{\partial (f \circ \phi^{-1})}{\partial r^i}(\phi(p)).$$

This equation may be rewritten in the form

$$\frac{\partial f}{\partial x^i}(\phi^{-1}(\phi(p))) = \frac{\partial (f \circ \phi^{-1})}{\partial r^i}(\phi(p)).$$

Thus, as functions on $\phi(U)$,

$$\frac{\partial f}{\partial x^i} \circ \phi^{-1} = \frac{\partial (f \circ \phi^{-1})}{\partial r^i}.$$

The partial derivative $\partial f/\partial x^i$ is C^∞ on U because its pullback $(\partial f/\partial x^i) \circ \phi^{-1}$ is C^∞ on $\phi(U)$.

In the next proposition we see that partial derivatives on a manifold behave as they should.

Proposition 6.10. *Suppose $(U, x^1, \dots, x^n)$ is a chart on a manifold. Then $\partial x^i/\partial x^j = \delta^i_j$.*

Proof. At a point $p \in U$, by the definition of $\partial/\partial x^j|_p$,

$$\begin{aligned}\frac{\partial x^i}{\partial x^j}(p) &= \left.\frac{\partial}{\partial r^j}\right|_{\phi(p)} x^i \circ \phi^{-1} \\ &= \left.\frac{\partial}{\partial r^j}\right|_{\phi(p)} (r^i \circ \phi) \circ \phi^{-1} = \left.\frac{\partial}{\partial r^j}\right|_{\phi(p)} r^i = \delta^i_j.\end{aligned}$$

□

6.3 The Inverse Function Theorem

Let U be an open subset of $\mathbb{R}^n$. Suppose

$$f = (f^1, \dots, f^n) \colon U \to \mathbb{R}^n$$

is a diffeomorphism of U onto some open subset of $\mathbb{R}^n$. Then the chart (U, f) is compatible with the standard chart $(\mathbb{R}^n, r^1, \dots, r^n)$. Therefore, (U, f) is an element of the maximal atlas that determines the differentiable structure of $\mathbb{R}^n$. In other words, any diffeomorphism of an open subset U of $\mathbb{R}^n$ may be thought of as a coordinate system on U.

More generally, given n smooth functions $f^1, \dots, f^n$ in a neighborhood of a point p in a manifold of dimension n, one would like to know if they form a coordinate system, possibly on a smaller neighborhood of p. The inverse function theorem provides an answer.

Definition 6.11. Let U be an open subset of $\mathbb{R}^n$. A smooth map $f = (f^1, \dots, f^n) : U \to \mathbb{R}^n$ is *locally invertible* at $p \in U$ if f has a smooth inverse in some neighborhood of p. The matrix of partial derivatives $[\partial f^i / \partial r^j]$ is called the *Jacobian matrix* of f, and its determinant $\det[\partial f^i / \partial r^j]$ the *Jacobian determinant* of f. The Jacobian determinant is also written as $\partial(f^1, \dots, f^n)/\partial(r^1, \dots, r^n)$.

We will use the phrase "near p" to mean "in a neighborhood of p."

Theorem 6.12 (Inverse function theorem for $\mathbb{R}^n$). *Let $f \colon W \to \mathbb{R}^n$ be a C^∞ map defined on an open subset W of $\mathbb{R}^n$. For any point p in W, the map f is locally invertible at p if and only if the Jacobian determinant* $\det[\partial f^i / \partial r^j(p)]$ *is not zero.*

This theorem is usually proved in a course on real analysis, for example, in [17].

Theorem 6.13 (Inverse function theorem for a manifold). *Let M be an n-dimensional manifold, p a point in M, and $f = (f^1, \dots, f^n) \colon W \to \mathbb{R}^n$ a C^∞ map defined on a neighborhood W of p. Suppose that relative to some chart $(U, \phi) = (U, x^1, \dots, x^n)$ containing p, the Jacobian determinant* $\det[\partial f^i / \partial x^j(p)] \neq 0$. *Then there is a neighborhood V of p on which f is a diffeomorphism onto its image. Moreover, (V, f) is a chart in the differentiable structure of M.* (*See Figure* 6.4.)

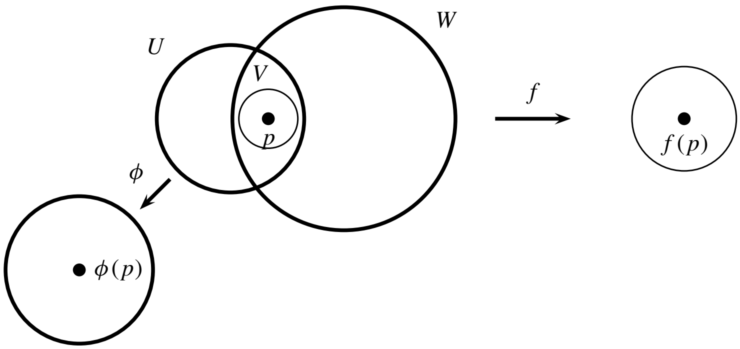


Fig. 6.4. The map f is locally invertible at p.

Proof. By definition,

$$\frac{\partial f^i}{\partial x^j}(p) = \frac{\partial(f^i \circ \phi^{-1})}{\partial r^j}(\phi(p)).$$

By the inverse function theorem for $\mathbb{R}^n$, the map $f \circ \phi^{-1}$ is locally invertible at $\phi(p)$. So $f = (f \circ \phi^{-1}) \circ \phi$ is a diffeomorphism in some neighborhood V of p. By Proposition 6.7, (V, f) is a chart in the atlas of M. $\square$

Problems

6.1. Smoothness of a map at a point
Suppose $F: N \to M$ is C^∞ at $p \in N$. Show that if (U', ϕ') is any chart containing p in the atlas of N and (V', ψ') is any chart containing $F(p)$ in the atlas of M, then $\psi' \circ F \circ (\phi')^{-1}$ is C^∞ at $\phi'(p)$.

6.2. Composition of smooth maps
Prove Proposition 6.6.

6.3. Differentiable structures on $\mathbb{R}$
Let $\mathbb{R}$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \phi = \mathrm{id}: \mathbb{R} \to \mathbb{R})$, and let $\mathbb{R}'$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \psi: \mathbb{R} \to \mathbb{R})$, where $\psi(x) = x^{1/3}$.

(a) Show that these two differentiable structures are distinct.
(b) Show that there is a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}'$. (*Hint*: The identity map is not the desired diffeomorphism; in fact, the identity map is not smooth.)

6.4.* Coordinate maps are C^∞
Show that if (U, ϕ) is a chart in the atlas of a manifold, then ϕ is C^∞.

6.5. Smooth functions
Show that $f: M \to \mathbb{R}$ is C^∞ if and only if for every chart (U, ϕ) in the atlas of M, the function $f \circ \phi^{-1}$ is C^∞ on $\phi(U)$.

6.6. Smooth maps
Show that a map $f: M \to N$ of manifolds is C^∞ if and only if for every chart (U, ϕ) in the atlas of M and (V, ψ) in the atlas of N, the composite $\psi \circ f \circ \phi^{-1}$ is C^∞ on $\phi(f^{-1}(V) \cap U)$.

6.7. General linear group
Show that the general linear group $\mathrm{GL}(n, \mathbb{R})$ defined in Example 5.14 is a Lie group under matrix multiplication.

6.8. Group of automorphisms of a vector space
Let V be a finite-dimensional vector space over $\mathbb{R}$, and $\mathrm{GL}(V)$ the group of all linear isomorphisms of V itself. A basis $e_1, \dots, e_n$ for V induces a bijection

$$\begin{aligned} \mathrm{GL}(n, \mathbb{R}) &\to \mathrm{GL}(V), \\ [a^i_j] &\mapsto \Big(e_j \mapsto \sum_i a^i_j e_i\Big), \end{aligned}$$

making $\mathrm{GL}(V)$ into a C^∞ manifold, which we denote temporarily by $\mathrm{GL}(V)_e$. If $\mathrm{GL}(V)_u$ is the manifold structure induced from another basis $u_1, \dots, u_n$ for V, show that $\mathrm{GL}(V)_e$ is diffeomorphic to $\mathrm{GL}(V)_u$.

7

Quotients

Gluing the edges of a malleable square is one way to create new surfaces. For example, gluing together the top and bottom edges of a square gives a cylinder; gluing together the boundaries of the cylinder with matching orientations gives a torus (Figure 7.1). This gluing process is called an *identification* or a *quotient construction*.

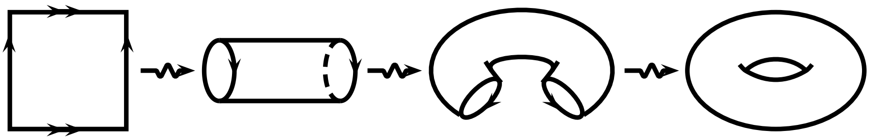


Fig. 7.1. Gluing the edges of a malleable square.

Even if the original space is a manifold, a quotient space is often not a manifold. The main results of this chapter give conditions under which the quotient space remains second countable and Hausdorff. We then study the real projective space as an example of a quotient manifold.

7.1 The Quotient Topology

Suppose $\sim$ is an equivalence relation on the set S. The *equivalence class* $[x]$ of x is the set of all elements in S equivalent to x. An equivalence relation on S partitions S into disjoint subsets consisting of equivalence classes. We denote the set of equivalence classes by $S/\sim$ and call this set the *quotient* of S by the equivalence relation $\sim$. There is a natural *projection map* $\pi: S \to S/\sim$ that sends $x \in S$ to its equivalence class $[x]$.

We call a set U in $S/\sim$ *open* if and only if $\pi^{-1}(U)$ is open in S. Clearly, both the empty set $\varnothing$ and the entire quotient $S/\sim$ are open. Since

$$\pi^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha \pi^{-1}(U_\alpha),$$

and

$$\pi^{-1}\left(\bigcap_i U_i\right) = \bigcap_i \pi^{-1}(U_i),$$

the collection of open sets in $S/\!\sim$ is closed under arbitrary union and finite intersection, and is therefore a topology. It is called the *quotient topology* on $S/\!\sim$. With this topology, $S/\!\sim$ is called the *quotient space* of S by the equivalence relation $\sim$. With the quotient topology on $S/\!\sim$, the projection map $\pi: S \to S/\!\sim$ is automatically continuous, because the inverse image of an open set in $S/\!\sim$ is by definition open in S. However, π need not be an open map, as Example 7.7 shows.

7.2 Continuity of a Map on a Quotient

Let $\sim$ be an equivalence relation on the topological space S and give $S/\!\sim$ the quotient topology. Suppose a function $f: S \to Y$ from S to another topological space Y is constant on each equivalence class. Then it induces a map $\bar{f}: S/\!\sim\, \to Y$ by

$$\bar{f}([p]) = f(p) \quad \text{for } p \in S.$$

In other words, there is a commutative diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ S/\!\sim \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Proposition 7.1. *The induced map $\bar{f}: S/\!\sim\, \to Y$ is continuous if and only if the map $f: S \to Y$ is continuous.*

Proof.
($\Rightarrow$) If $\bar{f}$ is continuous, then as the composite $\bar{f} \circ \pi$ of continuous functions, f is also continuous.

($\Leftarrow$) Suppose f is continuous. Let V be open in Y. Then $f^{-1}(V) = \pi^{-1}(\bar{f}^{-1}(V))$ is open in S. By the definition of quotient topology, $\bar{f}^{-1}(V)$ is open in $S/\!\sim$. Hence, $\bar{f}: S/\!\sim\, \to Y$ is continuous. □

This proposition gives a useful criterion for checking if a function $\bar{f}$ on a quotient space $S/\!\sim$ is continuous: simply lift the function $\bar{f}$ to $f := \bar{f} \circ \pi$ on S and check the continuity of the lifted map f on S. For an example of this, see Example 7.2 and Proposition 7.3.

7.3 Identification of a Subset to a Point

If A is a subspace of a topological space S, we define a relation $\sim$ on S by declaring

$$x \sim x \quad \text{for all } x \in S$$

(so that the relation would be reflexive) and

$$x \sim y \quad \text{for all } x, y \in A.$$

This is an equivalence relation on S. We say that the quotient space $S/\sim$ is obtained from S by *identifying A to a point*.

Example 7.2. Let I be the unit interval $[0, 1]$ and $I/\sim$ the quotient space obtained from I by identifying the two points $\{0, 1\}$ to a point. Denote by S^1 the unit circle in the complex plane. The function $f: I \to S^1$, $f(x) = \exp(2\pi i x)$ assumes the same value at 0 and 1 (Figure 7.2), and so induces a function

$$\bar{f}: I/\sim \to S^1.$$

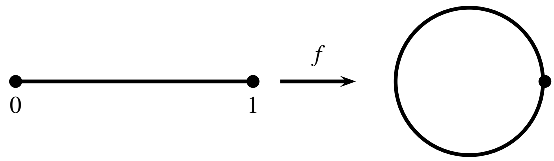


Fig. 7.2. The unit circle as a quotient space of the unit interval.

Proposition 7.3. *The function $\bar{f}: I/\sim \to S^1$ is a homeomorphism.*

Proof. Since f is continuous, $\bar{f}$ is also continuous by Proposition 7.1. Clearly, $\bar{f}$ is a bijection. As the continuous image of the compact set I, the quotient $I/\sim$ is compact. Thus, $\bar{f}$ is a continuous bijection from the compact space $I/\sim$ to the Hausdorff space S^1. By Proposition A.39, $\bar{f}$ is a homeomorphism. $\square$

7.4 A Necessary Condition for a Hausdorff Quotient

The quotient construction does not in general preserve the Hausdorff property or second countability. Indeed, since every singleton set in a Hausdorff space is closed, if $\pi : S \to S/\sim$ is the projection and the quotient $S/\sim$ is Hausdorff, then for any $p \in S$, its image $\{\pi(p)\}$ is closed in $S/\sim$. By the continuity of π, the inverse image $\pi^{-1}(\{\pi(p)\}) = [p]$ is closed in S. This gives a necessary condition for a quotient space to be Hausdorff.

Proposition 7.4. *If the quotient space $S/\sim$ is Hausdorff, then the equivalence class $[p]$ of any point p in S is closed in S.*

Example 7.5. Define an equivalence relation $\sim$ on $\mathbb{R}$ by identifying the open interval $(0, \infty)$ to a point. Then the quotient space $\mathbb{R}/\sim$ is not Hausdorff because the point corresponding to the equivalence class $(0, \infty)$ is not closed.

7.5 Open Equivalence Relations

In this section we follow the treatment of Boothby [2] and derive conditions under which a quotient space is Hausdorff or second countable. Recall that a map $f : X \to Y$ of topological spaces is *open* if the image of any open set under f is open.

Definition 7.6. An equivalence relation $\sim$ on a topological space S is said to be *open* if the projection map $\pi : S \to S/\sim$ is open.

In other words, the equivalence relation $\sim$ on S is open if and only if for every open set U in S, the set

$$\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x]$$

of all points equivalent to some point of U is open.

Example 7.7. The projection map to a quotient space is in general not open. For example, let $\sim$ be the equivalence relation on the real line $\mathbb{R}$ that identifies the two points 1 and -1, and $\pi : \mathbb{R} \to \mathbb{R}/\sim$ the projection map.

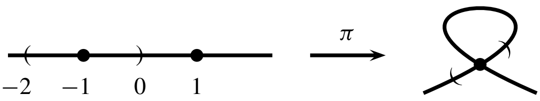


Fig. 7.3. A projection map that is not open.

Let V be the open interval $(-2, 0)$. Then

$$\pi^{-1}(\pi(V)) = (-2, 0) \cup \{-1\},$$

which is not open in $\mathbb{R}$ (Figure 7.3). Thus, $\pi(V)$ is not open in the quotient space. In this example the projection map $\pi : \mathbb{R} \to \mathbb{R}/\sim$ is not an open map.

Given an equivalence relation $\sim$ on S, we let R be the subset of $S \times S$ that defines the relation:

$$R = \{(x, y) \in S \times S \mid x \sim y\}.$$

We call R the *graph* of the equivalence relation $\sim$.

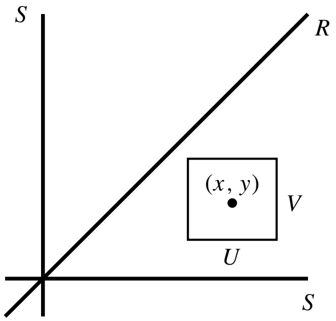


Fig. 7.4. The graph R of an equivalence relation.

Theorem 7.8. *Suppose $\sim$ is an open equivalence relation on S. Then the quotient space $S/\!\sim$ is Hausdorff if and only if the graph R of the equivalence relation is closed in $S \times S$.*

Proof.
($\Rightarrow$) Suppose $S/\!\sim$ is Hausdorff. We will show that $S \times S - R$ is an open set. Let $(x, y) \in S \times S - R$. Then $x \nsim y$. So $[x] \neq [y]$ in $S/\!\sim$. Since $S/\!\sim$ is Hausdorff, there are disjoint open sets $\tilde{U}$, $\tilde{V}$ in $S/\!\sim$ with $[x] \in \tilde{U}$ and $[y] \in \tilde{V}$. Since $\tilde{U}$ and $\tilde{V}$ are disjoint, no element in $U := \pi^{-1}(\tilde{U})$ is equivalent to an element of $V := \pi^{-1}(\tilde{V})$. This means $U \times V$ is open and disjoint from R in $S \times S$. So

$$(x, y) \in U \times V \subset S \times S - R,$$

which proves that $S \times S - R$ is open in $S \times S$.

($\Leftarrow$) Suppose R is closed in $S \times S$ and $[x] \neq [y]$ in $S/\!\sim$. Then $x \nsim y$. Thus, $(x, y) \in S \times S - R$. Since $S \times S - R$ is open, there is a basic open set $U \times V$ containing (x, y) and contained in $S \times S - R$ (Figure 7.4). Thus, no element of U is equivalent to an element of V, so $\pi(U)$ and $\pi(V)$ are disjoint in $S/\!\sim$. Since $\pi : S \to S/\!\sim$ is an open map, $\pi(U)$ and $\pi(V)$ are open in $S/\!\sim$. Moreover, $[x] \in \pi(U)$ and $[y] \in \pi(V)$. This proves that $S/\!\sim$ is Hausdorff. □

Theorem 7.9. *Let $\sim$ be an open equivalence relation on a space S with projection $\pi : S \to S/\!\sim$. If $\mathcal{B} = \{B_\alpha\}$ is a basis for S, then its image $\{\pi(B_\alpha)\}$ under π is a basis for $S/\!\sim$.*

Proof. Since π is an open map, $\{\pi(B_\alpha)\}$ is a collection of open sets in $S/\!\sim$. Let W be an open set in $S/\!\sim$ and $[x] \in W$, $x \in S$. Then $x \in \pi^{-1}(W)$. Since $\pi^{-1}(W)$ is open, there is a basic open set $B_\alpha \in \mathcal{B}$ such that

$$x \in B_\alpha \subset \pi^{-1}(W).$$

Then

$$[x] = \pi(x) \in \pi(B_\alpha) \subset W,$$

which proves that $\{\pi(B_\alpha)\}$ is a basis for $S/\!\sim$. □

Corollary 7.10. *If $\sim$ is an open equivalence relation on a second countable space S, then the quotient space $S/\!\sim$ is second countable.*

7.6 The Real Projective Space

Define an equivalence relation on $\mathbb{R}^{n+1} - \{0\}$ by

$$x \sim y \quad \text{iff } y = tx \text{ for some nonzero real number } t,$$

where $x, y \in \mathbb{R}^{n+1} - \{0\}$. The *real projective space* $\mathbb{R}P^n$ is the quotient space of $\mathbb{R}^{n+1} - \{0\}$ by this equivalence relation. We denote the equivalence class of a point $(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} - \{0\}$ by $[a_0, \ldots, a_n]$ and let $\pi : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}P^n$ be the projection. We call $[a_0, \ldots, a_n]$ the *homogeneous coordinates* on $\mathbb{R}P^n$.

Geometrically two nonzero points in $\mathbb{R}^{n+1}$ are equivalent if and only if they lie on the same line through the origin. So $\mathbb{R}P^n$ can be interpreted as the set of all lines through the origin in $\mathbb{R}^{n+1}$.

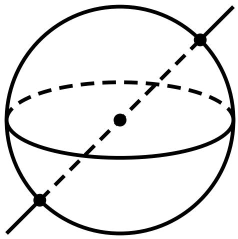


Fig. 7.5. The real projective plane $\mathbb{R}P^2$ as the set of lines through 0 in $\mathbb{R}^3$.

Each line through the origin in $\mathbb{R}^{n+1}$ meets the unit sphere S^n in a pair of antipodal points, and conversely, a pair of antipodal points on S^n determines a unique line through the origin (Figure 7.5). This suggests that we define an equivalence relation $\sim$ on S^n by identifying the antipodal points:

$$x \sim y \quad \text{iff} \quad x = \pm y, \qquad x, y \in S^n.$$

We then have a bijection $\mathbb{R}P^n \leftrightarrow S^n/\!\sim$.

Exercise 7.11 (Real projective space as a quotient of a sphere).* Prove that the map $f\colon \mathbb{R}^{n+1} - \{0\} \to S^n$ given by

$$f(x) = \frac{x}{|x|}$$

induces a homeomorphism $\bar{f}\colon \mathbb{R}P^n \to S^n/\!\sim$. (*Hint*: Find an inverse map

$$\bar{g}\colon S^n/\!\sim \to \mathbb{R}P^n$$

and show that both $\bar{f}$ and $\bar{g}$ are continuous.)

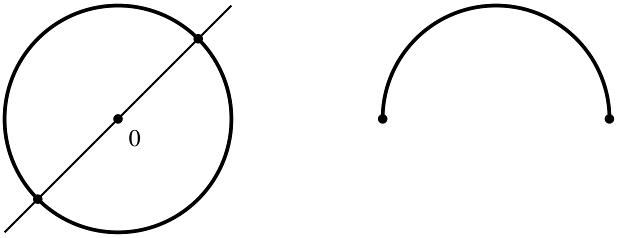


Fig. 7.6. The real projective line $\mathbb{R}P^1$ as the set of lines through 0 in $\mathbb{R}^2$.

Example 7.12. *The real projective line* $\mathbb{R}P^1$.

Each line through the origin in $\mathbb{R}^2$ meets the unit circle in a pair of antipodal points. By Exercise 7.11, $\mathbb{R}P^1$ is homeomorphic to the quotient $S^1/{\sim}$, which is in turn homeomorphic to the closed upper semicircle with the two endpoints identified (Figure 7.6). Thus, $\mathbb{R}P^1$ is homeomorphic to S^1.

Example 7.13. *The real projective plane* $\mathbb{R}P^2$. By Exercise 7.11, there is a homeomorphism

$$\mathbb{R}P^2 \simeq S^2/\{\text{antipodal points}\} = S^2/{\sim}.$$

For points not on the equator, each pair of antipodal points contains a unique point in the upper hemisphere. Thus, there is a bijection between $S^2/{\sim}$ and the quotient of the closed upper hemisphere in which each pair of antipodal points on the equator are identified. It is not difficult to show that this bijection is a homeomorphism.

Let H^2 be the closed upper hemisphere

$$H^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ z \geq 0\}$$

and let D^2 be the closed unit disk

$$D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

These two spaces are homeomorphic to each other via the continuous map

$$\varphi\colon H^2 \to D^2$$
$$\varphi(x, y, z) = (x, y)$$

and its inverse

$$\psi\colon D^2 \to H^2$$
$$\psi(x, y) = \left(x, y, \sqrt{1 - x^2 - y^2}\right).$$

On H^2, define an equivalence relation $\sim$ by identifying the antipodal points on the equator:

$$(x, y, 0) \sim (-x, -y, 0), \quad x^2 + y^2 = 1.$$

On D^2, define an equivalence relation $\sim$ by identifying the antipodal points on the boundary circle:

$$(x, y) \sim (-x, -y), \quad x^2 + y^2 = 1.$$

Then φ and ψ induce homeomorphisms

$$\bar{\varphi}\colon H^2/\!\sim \to D^2/\!\sim, \quad \bar{\psi}\colon D^2/\!\sim \to H^2/\!\sim.$$

In summary, there is a sequence of homeomorphisms

$$\mathbb{R}P^2 \simeq S^2/\!\sim \simeq H^2/\!\sim \simeq D^2/\!\sim$$

that identifies the real projective plane as the quotient of the closed disk D^2 with the antipodal points on its boundary identified. This may be the best way to picture $\mathbb{R}P^2$ (Figure 7.7).

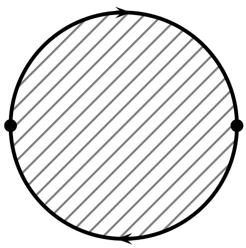


Fig. 7.7. The real projective plane as the quotient of a disk.

The real projective plane $\mathbb{R}P^2$ cannot be embedded as a submanifold of $\mathbb{R}^3$. However, if we allow self-intersection, then we can map $\mathbb{R}P^2$ into $\mathbb{R}^3$ as a cross-cap (Figure 7.8). This map is not one-to-one.

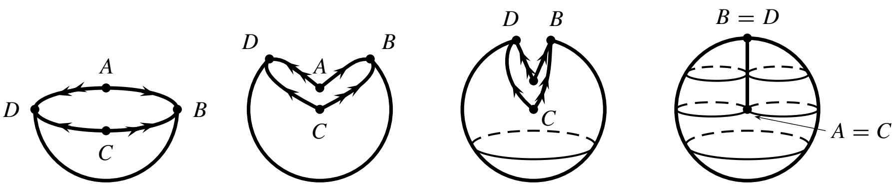


Fig. 7.8. The real projective plane immersed as a cross-cap in $\mathbb{R}^3$.

Proposition 7.14. *The equivalence relation $\sim$ on $\mathbb{R}^{n+1} - \{0\}$ in the definition of $\mathbb{R}P^n$ is an open equivalence relation.*

Proof. For an open set $U \subset \mathbb{R}^{n+1} - \{0\}$, the image $\pi(U)$ is open in $\mathbb{R}P^n$ if and only if $\pi^{-1}(\pi(U))$ is open in $\mathbb{R}^{n+1} - \{0\}$. But $\pi^{-1}(\pi(U))$ consists of all points equivalent to some points of U; that is,

$$\pi^{-1}(\pi(U)) = \bigcup_{t \in \mathbb{R}^\times} tU.$$

Since multiplication by $t \in \mathbb{R}^\times$ is a homeomorphism of $\mathbb{R}^{n+1} - \{0\}$, the set tU is open for any t. Therefore, their union $\pi^{-1}(\pi(U))$ is also open. □

Corollary 7.15. *The real projective space $\mathbb{R}P^n$ is second countable.*

Proof. Apply Corollary 7.10. □

Proposition 7.16. *The real projective space $\mathbb{R}P^n$ is Hausdorff.*

Proof. Let $S = \mathbb{R}^{n+1} - \{0\}$ and consider the set

$$R = \{(x, y) \in S \times S \mid y = tx \text{ for some } t \in \mathbb{R}^\times\}.$$

If we write x and y as column vectors, then $[x\ y]$ is an $(n+1) \times 2$ matrix, and R may be characterized as the set of matrices $[x\ y]$ in $S \times S$ of rank ≤ 1. By a standard fact from linear algebra, this is equivalent to the vanishing of all 2×2 minors of $[x\ y]$ (see Problem B.1). As the zero set of finitely many polynomials, R is a closed subset of $S \times S$. Since $\sim$ is an open equivalence relation on S, and R is closed in $S \times S$, by Theorem 7.8 the quotient $S/\sim\ \simeq \mathbb{R}P^n$ is Hausdorff. □

7.7 The Standard C^∞ Atlas on a Real Projective Space

Let $[a_0, \ldots, a_n]$ be the homogeneous coordinates on the projective space $\mathbb{R}P^n$. Although a_0 is not a well-defined function on $\mathbb{R}P^n$, the condition $a_0 \neq 0$ is independent of the choice of a representative for $[a_0, \ldots, a_n]$. Hence, the condition $a_0 \neq 0$ makes sense on $\mathbb{R}P^n$, and we may define

$$U_0 := \{[a_0, \ldots, a_n] \in \mathbb{R}P^n \mid a_0 \neq 0\}.$$

Similarly, for each $i = 1, \ldots, n$, let

$$U_i := \{[a_0, \ldots, a_n] \in \mathbb{R}P^n \mid a_i \neq 0\}.$$

Define

$$\phi_0 \colon U_0 \to \mathbb{R}^n$$

by

$$[a_0, \ldots, a_n] \mapsto \left(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}\right).$$

This map has a continuous inverse

$$(b_1, \dots, b_n) \mapsto [1, b_1, \dots, b_n]$$

and is therefore a homeomorphism. Similarly, there are homeomorphisms for each $i = 1, \dots, n$:

$$\begin{aligned} \phi_i \colon U_i &\to \mathbb{R}^n \\ [a_0, \dots, a_n] &\mapsto \left(\frac{a_0}{a_i}, \dots, \widehat{\frac{a_i}{a_i}}, \dots, \frac{a_n}{a_i} \right), \end{aligned}$$

where the caret sign $\widehat{\ }$ over a_i/a_i means that that entry is to be omitted. This proves that $\mathbb{R}P^n$ is locally Euclidean with the (U_i, ϕ_i) as charts.

On the intersection $U_0 \cap U_1$, $a_0 \neq 0$ and $a_1 \neq 0$, and there are two coordinates systems

$$\begin{tikzcd}[column sep=small] & {[a_0, a_1, a_2, \dots, a_n]} \arrow[dl, "\phi_0"'] \arrow[dr, "\phi_1"] & \\ \left(\dfrac{a_1}{a_0}, \dfrac{a_2}{a_0}, \dots, \dfrac{a_n}{a_0} \right) & & \left(\dfrac{a_0}{a_1}, \dfrac{a_2}{a_1}, \dots, \dfrac{a_n}{a_1} \right). \end{tikzcd}$$

Let us call the coordinate functions on U_0, $x_1, \dots, x_n$, and the coordinate functions on U_1, $y_1, \dots, y_n$. On U_0,

$$x_i = \frac{a_i}{a_0}, \quad i = 1, \dots, n,$$

and on U_1,

$$y_1 = \frac{a_0}{a_1}, \ y_2 = \frac{a_2}{a_1}, \ \dots, \ y_n = \frac{a_n}{a_1}.$$

Then on $U_0 \cap U_1$,

$$y_1 = \frac{1}{x_1}, y_2 = \frac{x_2}{x_1}, y_3 = \frac{x_3}{x_1}, \dots, y_n = \frac{x_n}{x_1};$$

that is,

$$\phi_1 \circ \phi_0^{-1}(x) = \left(\frac{1}{x_1}, \frac{x_2}{x_1}, \frac{x_3}{x_1}, \dots, \frac{x_n}{x_1} \right).$$

This is a C^∞ function because $x_1 \neq 0$ on $\phi_0(U_0 \cap U_1)$. On any other $U_i \cap U_j$ an analogous formula holds. Therefore, the collection $\{(U_i, \phi_i)\}_{i=0,\dots,n}$ is a C^∞ atlas for $\mathbb{R}P^n$, called the *standard atlas*. This concludes the proof that $\mathbb{R}P^n$ is a C^∞ manifold.

Problems

7.1.* Quotient space by a group
Suppose a left action of a topological group G on a topological space S is continuous; this simply means that the map $G \times S \to S$ describing the action is continuous. Define two point x, y of S to be equivalent if there is a $g \in G$ such that $y = gx$. Let $G \backslash S$ be the quotient space. Prove that the projection map $\pi : S \to G \backslash S$ is an open map.

7.2. The Grassmannian $G(k, n)$
The Grassmannian $G(k, n)$ is the set of all k-planes through the origin in $\mathbb{R}^n$. Such a k-plane is a linear subspace of dimension k of $\mathbb{R}^n$ and has a basis consisting of k linearly independent vectors $a_1, \dots, a_k$ in $\mathbb{R}^n$. It is therefore completely specified by an $n \times k$ matrix $A = [a_1 \cdots a_k]$ of rank k, where the *rank* of a matrix A, denoted by $\operatorname{rk} A$, is defined to be the number of linearly independent columns of A. This matrix is called a *matrix representative* of the k-plane. (For properties of the rank, see the problems in Appendix B.)

Two bases $a_1, \dots, a_k$ and $b_1, \dots, b_k$ determine the same k-plane if there is a change of basis matrix $g = [g_{ij}] \in \mathrm{GL}(k, \mathbb{R})$ such that

$$b_j = \sum_i a_i g_{ij}, \quad 1 \le i, j \le k.$$

In matrix notation, $B = Ag$.

Let $F(k, n)$ be the set of all $n \times k$ matrices of rank k, topologized as a subspace of $\mathbb{R}^{n \times k}$, and $\sim$ the equivalence relation:

$$A \sim B \quad \text{iff there is a matrix } g \in \mathrm{GL}(k, \mathbb{R}) \text{ such that } B = Ag.$$

In the notation of Problem B.3, $F(k, n)$ is the set $D_{\max}$ in $\mathbb{R}^{n \times k}$ and is therefore an open subset. There is a bijection between $G(k, n)$ and the quotient space $F(k, n)/\sim$. We give the Grassmannian $G(k, n)$ the quotient topology on $F(k, n)/\sim$.

(a) Show that $\sim$ is an open equivalence relation. (*Hint*: Mimic the proof of Proposition 7.14.)
(b) Prove that the Grassmannian $G(k, n)$ is second countable. (*Hint*: Mimic the proof of Corollary 7.15.)
(c) Let $S = F(k, n)$. Prove that the graph R in $S \times S$ of the equivalence relation $\sim$ is closed. (*Hint*: Two matrices $A = [a_1 \cdots a_k]$ and $B = [b_1 \cdots b_k]$ in $F(k, n)$ are equivalent iff every column of B is a linear combination of the columns of A iff $\operatorname{rk}[A B] \le k$ iff all $(k+1) \times (k+1)$ minors of $[A B]$ are zero.)
(d) Prove that the Grassmannian $G(k, n)$ is Hausdorff. (*Hint*: Mimic the proof of Proposition 7.16.)

Next we want to find a C^∞ atlas on the Grassmannian $G(k, n)$. For simplicity, we specialize to $G(2, 4)$. For any 4×2 matrix A, let A_{ij} be the 2×2 submatrix consisting of its ith row and jth row. Define

$$V_{ij} = \{A \in F(2, 4) \mid A_{ij} \text{ is nonsingular}\}.$$

Because the complement of V_{ij} in $F(2, 4)$ is defined by the vanishing of det A_{ij}, we conclude that V_{ij} is an open subset of $F(2, 4)$.

(e) Prove that if $A \in V_{ij}$, then $Ag \in V_{ij}$ for any nonsingular matrix $g \in \mathrm{GL}(2, \mathbb{R})$.

Define $U_{ij} = V_{ij}/\sim$. Since $\sim$ is an open equivalence relation, $U_{ij} = V_{ij}/\sim$ is an open subset of $G(2, 4)$.

For $A \in V_{12}$,

$$A \sim AA_{12}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} I \\ A_{34}A_{12}^{-1} \end{bmatrix}.$$

This shows that the matrix representatives of a 2-plane in U_{12} have a canonical form B in which B_{12} is the identity matrix.

(e) Show that the map $\tilde{\phi}_{12} \colon V_{12} \to \mathbb{R}^{2\times 2}$,

$$\tilde{\phi}_{12}(A) = A_{34}A_{12}^{-1},$$

induces a homeomorphism $\phi_{12} \colon U_{12} \to \mathbb{R}^{2\times 2}$.

(f) Define similarly a homeomorphism $\phi_{ij} \colon U_{ij} \to \mathbb{R}^{2\times 2}$. Compute $\phi_{12} \circ \phi_{23}^{-1}$, and show that it is C^∞.

(g) Show that $\{U_{ij} \mid 1 \le i < j \le 4\}$ is an open over of $G(2, 4)$ and that $G(2, 4)$ is a smooth manifold.

Similar consideration shows that $F(k, n)$ has an open cover $\{V_I\}$, where I is an ascending multi-index

$$1 \le i_1 < \dots i_k \le n.$$

For $A \in F(k, n)$, let A_I be the $k \times k$ submatrix of A consisting of i_1th, …, i_kth rows of A. Define

$$V_I = \{A \in G(k, n) \mid \det A_I \neq 0\}.$$

Next define $\tilde{\phi}_I \colon V_I \to \mathbb{R}^{(n-k)\times k}$ by

$$\tilde{\phi}_I(A) = (AA_I^{-1})_{I'},$$

where $(\ )_{I'}$ denotes the $(n - k) \times k$ submatrix obtained from the complement I' of the multi-index I. Let $U_I = V_I/\sim$. Then $\tilde{\phi}$ induces a homeomorphism $\phi \colon U_I \to \mathbb{R}^{(n-k)\times k}$. It is not difficult to show that $\{(U_I, \phi_I)\}$ is a C^∞ atlas for $G(k, n)$. Therefore the Grassmannian $G(k, n)$ is a C^∞ manifold of dimension $k(n - k)$.

7.3.* The real projective space

Show that the real projective space $\mathbb{R}P^n$ is compact. (*Hint*: Use Exercise 7.11.)

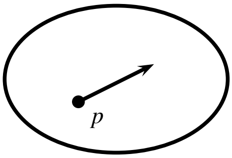
8

The Tangent Space

8.1 The Tangent Space at a Point

In Chapter 2 we saw that for any point p in an open set U in $\mathbb{R}^n$ there are two equivalent ways to define a tangent vector at p:

(i) as an arrow (Figure 8.1), represented by a column vector;



$$\begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix}$$

Fig. 8.1. A tangent vector in $\mathbb{R}^n$ as an arrow and as a column vector.

(ii) as a point-derivation of C_p^∞, the algebra of germs of C^∞ functions at p.

Both definitions generalize to a manifold. In the arrow approach, one defines a tangent vector at p in a manifold M by first choosing a chart (U, ϕ) at p and then decreeing a tangent vector at p to be an arrow at $\phi(p)$ in $\phi(U)$. This approach, while more visual, is complicated to work with, since a different chart (V, ψ) at p would give rise to a different set of tangent vectors at p and one would have to decide how to identify the arrows at $\phi(p)$ in U with the arrows at $\psi(p)$ in $\psi(V)$.

The cleanest and most intrinsic definition of a tangent vector at p in M is as a point-derivation and this is the approach we shall adopt.

Just as for $\mathbb{R}^n$, we define a *germ* of a C^∞ function at p in M to be an equivalence class of C^∞ functions defined in a neighborhood of p in M, two such functions being equivalent if they agree on some, possibly smaller, neighborhood of p. The set of germs of C^∞ real-valued functions at p in M is denoted $C_p^\infty(M)$. The addition and multiplication of functions make $C_p^\infty(M)$ into a ring; with scalar multiplication by real numbers, $C_p^\infty(M)$ becomes an algebra over $\mathbb{R}$.

Generalizing a derivation at a point in $\mathbb{R}^n$, we define a *derivation at a point in a manifold M*, or a *point-derivation of $C_p^\infty(M)$*, to be a linear map $D\colon C_p^\infty(M) \to \mathbb{R}$ such that

$$D(fg) = (Df)g(p) + f(p)Dg.$$

Definition 8.1. A *tangent vector* at a point p in a manifold M is a derivation at p.

As for $\mathbb{R}^n$, the tangent vectors at p form a vector space $T_p(M)$, called the *tangent space of M at p*. We also write T_pM instead of $T_p(M)$.

Remark 8.2 (*Tangent space to an open subset*). If U is an open set containing p in M, then the algebra $C_p^\infty(U)$ of germs of C^∞ functions in U at p is the same as $C_p^\infty(M)$. Hence, $T_pU = T_pM$.

Given a coordinate neighborhood $(U, \phi) = (U, x^1, \ldots, x^n)$ about a point p in a manifold M, we recall the definition of the partial derivatives $\partial/\partial x^i$ first introduced in Chapter 6. Let $r^1, \ldots, r^n$ be the standard coordinates on $\mathbb{R}^n$. Then

$$x^i = r^i \circ \phi\colon U \to \mathbb{R}.$$

If f is a smooth function in a neighborhood of p, we set

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} f \circ \phi^{-1} \in \mathbb{R}.$$

It is easily checked that $\partial/\partial x^i|_p$ satisfies the derivation property and so is a tangent vector at p.

To simplify the notation, we will often write $\partial/\partial x^i$ instead $\partial/\partial x^i|_p$ if it is understood at which point the tangent vector is located.

8.2 The Differential of a Map

Let $F\colon N \to M$ be a C^∞ map between two manifolds. At each point $p \in N$, the map F induces a linear map of tangent spaces, called its *differential at p*,

$$F_*\colon T_pN \to T_{F(p)}M$$

as follows. If $X_p \in T_pN$, then $F_*(X_p)$ is the tangent vector in $T_{F(p)}M$ defined by

$$(F_*(X_p))f = X_p(f \circ F) \in \mathbb{R} \quad \text{for } f \in C_{F(p)}^\infty(M). \tag{8.1}$$

Exercise 8.3 (The differential of a map). Check that $F_*(X_p)$ is a derivation at $F(p)$ and that $F_*\colon T_pN \to T_{F(p)}M$ is a linear map.

To make the dependence on p explicit we sometimes write $F_{*,p}$ instead of F_*.

Example 8.4. Suppose $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is smooth and p is a point in $\mathbb{R}^n$. Let $x^1, \dots, x^n$ be the coordinates on $\mathbb{R}^n$ and $y^1, \dots, y^m$ the coordinates on $\mathbb{R}^m$. Then the tangent vectors $\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p$ form a basis for the tangent space $T_p(\mathbb{R}^n)$ and $\partial/\partial y^1|_{F(p)}, \dots, \partial/\partial y^m|_{F(p)}$ form a basis for the tangent space $T_{F(p)}(\mathbb{R}^m)$. The linear map $F_*\colon T_p(\mathbb{R}^n) \to T_{F(p)}(\mathbb{R}^m)$ is described by a matrix $[a^i_j]$ relative to these two bases:

$$F_*\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \sum a^k_j \left.\frac{\partial}{\partial y^k}\right|_{F(p)}, \quad a^k_j \in \mathbb{R}. \tag{8.2}$$

Let $F^i = y^i \circ F$ be the ith component of F. We can find a^i_j by evaluating both sides of (8.2) on y^i:

$$\begin{aligned} \text{RHS} &= \sum a^k_j \left.\frac{\partial}{\partial y^k}\right|_{F(p)} y^i = \sum a^k_j \delta^i_k = a^i_j, \\ \text{LHS} &= F_*\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) y^i = \left.\frac{\partial}{\partial x^j}\right|_p y^i \circ F = \frac{\partial F^i}{\partial x^j}(p). \end{aligned}$$

So the matrix of F_* relative to the bases $\{\partial/\partial x^j|_p\}$ and $\{\partial/\partial y^i|_{F(p)}\}$ is $[\partial F^i/\partial x^j(p)]$. This is precisely the Jacobian matrix of the derivative of F at p. Thus, the differential of a map between manifolds generalizes the derivative of a map between Euclidean spaces.

8.3 The Chain Rule

Let $F\colon N \to M$ and $G\colon M \to P$ be smooth maps of manifolds, and $p \in N$. The differentials of F at p and G at $F(p)$ are linear maps:

$$T_pN \xrightarrow{F_{*,p}} T_{F(p)}M \xrightarrow{G_{*,F(p)}} T_{G(F(p))}P.$$

Theorem 8.5 (The chain rule). *If $F\colon N \to M$ and $G\colon M \to P$ are smooth maps of manifolds and $p \in N$, then*

$$(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}.$$

Proof. Let $X_p \in T_pN$ and f a smooth function at $G(F(p))$ in P. Then

$$((G \circ F)_* X_p) f = X_p(f \circ G \circ F)$$

and

$$\begin{aligned} ((G_* \circ F_*)X_p) f &= (G_*(F_* X_p)) f \\ &= (F_* X_p)(f \circ G) \\ &= X_p(f \circ G \circ F). \end{aligned}$$

□

Example 8.13 shows that when written out in terms of matrices, the chain rule of Theorem 8.5 assumes a more familiar form as a sum of products of derivatives of functions.

Remark 8.6. The differential of the identity map $1_M : M \to M$ at any point p in M is the identity map

$$1_{T_pM} : T_pM \to T_pM,$$

because

$$((1_M)_* X_p) f = X_p(f \circ 1_M) = X_p f,$$

for any $X_p \in T_pM$ and $f \in C_p^\infty(M)$.

Corollary 8.7. *If $F : N \to M$ is a diffeomorphism of manifolds and $p \in N$, then $F_* : T_pN \to T_{F(p)}M$ is an isomorphism of vector spaces.*

Proof. To say that F is a diffeomorphism means that it has a differentiable inverse $G : M \to N$ such that $G \circ F = 1_N$ and $F \circ G = 1_M$. By the chain rule,

$$(G \circ F)_* = G_* \circ F_* = (1_N)_* = 1_{T_pN},$$
$$(F \circ G)_* = F_* \circ G_* = (1_M)_* = 1_{T_{F(p)}M}.$$

Hence, F_* and G_* are isomorphisms. □

Corollary 8.8 (Invariance of dimension). *If an open set $U \subset \mathbb{R}^n$ is diffeomorphic to an open set $V \subset \mathbb{R}^m$, then $n = m$.*

Proof. Let $F : U \to V$ be a diffeomorphism and let $p \in U$. By Corollary 8.7, $F_{*,p} : T_pU \to T_{F(p)}V$ is an isomorphism of vector spaces. Since $T_pU \simeq \mathbb{R}^n$ and $T_{F(p)} \simeq \mathbb{R}^m$, we must have that $n = m$. □

8.4 Bases for the Tangent Space at a Point

If $(U, \phi) = (U, x^1, \dots, x^n)$ is a coordinate neighborhood about a point p in a manifold M and $r^1, \dots, r^n$ are the standard coordinates on $\mathbb{R}^n$, we defined earlier the partial derivatives

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} f \circ \phi^{-1} \in \mathbb{R}. \tag{8.3}$$

Since $\phi : U \to \mathbb{R}^n$ is a diffeomorphism onto its image, by Corollary 8.7 the differential

$$\phi_* : T_pM \to T_{\phi(p)}(\mathbb{R}^n)$$

is a vector space isomorphism.

Proposition 8.9. *Let $(U, \phi) = (U, x^1, \dots, x^n)$ be a chart about a point p in a manifold M. Then*

$$\phi_* \left(\left.\frac{\partial}{\partial x^i}\right|_p \right) = \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)}.$$

Proof. For any $f \in C^\infty_{\phi(p)}(\mathbb{R}^n)$,

$$\begin{aligned}\phi_*\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) f &= \left.\frac{\partial}{\partial x^i}\right|_p f \circ \phi && \text{(definition of } \phi_*) \\ &= \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} f \circ \phi \circ \phi^{-1} && \text{(definition of } \partial/\partial x^i|_p) \\ &= \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} f. \end{aligned}$$ □

Proposition 8.10. *If $(U, \phi) = (U, x^1, \ldots, x^n)$ is a chart containing p, then the tangent space $T_p M$ has basis*

$$\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p.$$

Proof. An isomorphism of vector spaces carries a basis to a basis. By Proposition 8.9 the isomorphism $\phi_* \colon T_p M \to T_{\phi(p)}(\mathbb{R}^n)$ maps $\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p$ to $\partial/\partial r^1|_{\phi(p)}, \ldots, \partial/\partial r^n|_{\phi(p)}$, which is a basis for the tangent space $T_{\phi(p)}(\mathbb{R}^n)$. Therefore, $\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p$ is a basis for $T_p M$. □

Proposition 8.11 (Transition matrix for coordinate vectors). *Suppose $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^n)$ are two coordinate charts on a manifold M. Then*

$$\frac{\partial}{\partial x^j} = \sum_i \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i}$$

on $U \cap V$.

Proof. At each point $p \in U \cap V$, the sets $\{\partial/\partial x^j|_p\}$ and $\{\partial/\partial y^i|_p\}$ are both bases for the tangent space $T_p M$, so there is a matrix $[a^i_j(p)]$ of real numbers such that on $U \cap V$

$$\frac{\partial}{\partial x^j} = \sum_k a^k_j \frac{\partial}{\partial y^k}.$$

Applying both sides of the equation to y^i, we get

$$\begin{aligned}\frac{\partial y^i}{\partial x^j} &= \sum_k a^k_j \frac{\partial y^i}{\partial y^k} \\ &= \sum_k a^k_j \delta^i_k \quad \text{(by Proposition 6.10)} \\ &= a^i_j. \end{aligned}$$ □

8.5 Local Expression for the Differential

Given a smooth map of manifolds, $F\colon N \to M$, and $p \in N$, we choose charts $(U, x^1, \dots, x^n)$ about p in N and $(V, y^1, \dots, y^m)$ about $F(p)$ in M. We can now find a local expression for the differential $F_{*,p}\colon T_pN \to T_{F(p)}M$ as in Example 8.4.

By Proposition 8.10, $\{\partial/\partial x^j|_p\}_{j=1}^n$ is a basis for T_pN and $\{\partial/\partial y^i|_{F(p)}\}_{i=1}^m$ is a basis for $T_{F(p)}M$. Therefore, $F_* = F_{*,p}$ is completely determined by the numbers a^i_j such that

$$F_*\left(\frac{\partial}{\partial x^j}\bigg|_p\right) = \sum_{k=1}^m a^k_j \frac{\partial}{\partial y^k}\bigg|_{F(p)}, \qquad j = 1, \dots, n.$$

Evaluating both sides on y^i, we find that

$$a^i_j = \left(\sum_{k=1}^m a^k_j \frac{\partial}{\partial y^k}\bigg|_{F(p)}\right) y^i = F_*\left(\frac{\partial}{\partial x^j}\bigg|_p\right) y^i = \frac{\partial}{\partial x^j}\bigg|_p y^i \circ F = \frac{\partial F^i}{\partial x^j}(p).$$

We state this result as a proposition.

Proposition 8.12. *Given a smooth map $F\colon N \to M$ of manifolds and a point $p \in N$, let $(U, x^1, \dots, x^n)$ and $(V, y^1, \dots, y^m)$ be coordinate charts about p in N and $F(p)$ in M, respectively. Relative to the bases $\{\partial/\partial x^j|_p\}$ for $T_p(N)$ and $\{\partial/\partial y^i|_{F(p)}\}$ for $T_{F(p)}(M)$, the differential $F_{*,p}\colon T_p(N) \to T_{F(p)}(M)$ is represented by the matrix $[\partial F^i/\partial x^j(p)]$, where $F^i = y^i \circ F$ is the ith component of F.*

This proposition is in the spirit of the "arrow" approach to tangent vectors. Here each tangent vector in $T_p(N)$ is represented by a column vector relative to the basis $\{\partial/\partial x^j|_p\}$ and the differential $F_{*,p}$ is represented by a matrix.

Example 8.13 (*The chain rule in calculus notation*). Suppose $w = G(x, y, z)$ is a C^∞ function: $\mathbb{R}^3 \to \mathbb{R}$ and $(x, y, z) = F(t)$ is a C^∞ function: $\mathbb{R} \to \mathbb{R}^3$. Under composition,

$$w = (G \circ F)(t) = G(x(t), y(t), z(t))$$

becomes a C^∞ function of $t \in \mathbb{R}$. The differentials F_*, G_*, and $(G \circ F)_*$ are represented by the matrices

$$\begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix}, \qquad \begin{bmatrix} \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{bmatrix}, \quad \text{and} \quad \frac{dw}{dt},$$

respectively. In terms of matrices, the chain rule $(G \circ F)_* = G_* \circ F_*$ is equivalent to

$$\begin{aligned} \frac{dw}{dt} &= \begin{bmatrix} \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{bmatrix} \begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix} \\ &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}. \end{aligned}$$

This is the usual form of the chain rule taught in calculus.

8.6 Curves in a Manifold

A *smooth curve* in a manifold M is by definition a smooth map $c\colon (a, b) \to M$ from some open interval (a, b) into M. Usually we assume $0 \in (a, b)$ and say that c is a *curve starting at* p if $c(0) = p$. The *velocity vector* $c'(t)$ of the curve c at time $t \in (a, b)$ is defined to be

$$\frac{dc}{dt}(t) := c'(t) := c_*\left(\left.\frac{d}{dt}\right|_t\right) \in T_{c(t)}(M).$$

We also say that $c'(t)$ is the velocity of c at the point $c(t)$.

Example 8.14. Define $c\colon \mathbb{R} \to \mathbb{R}^2$ by

$$c(t) = (t^2, t^3).$$

(See Figure 8.2.)

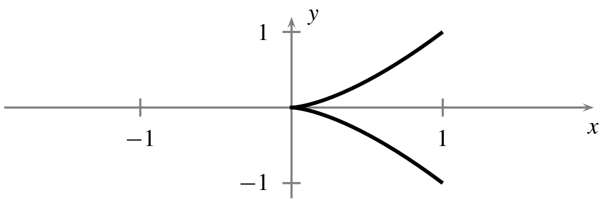


Fig. 8.2. A cuspidal cubic.

Then $c'(t)$ is a linear combination of $\partial/\partial x$ and $\partial/\partial y$ at $c(t)$:

$$c'(t) = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}.$$

To compute a, we evaluate both sides on x:

$$\begin{aligned} a &= \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)x = c'(t)x \\ &= c_*\left(\frac{d}{dt}\right)x = \frac{d}{dt}(x \circ c) = \frac{d}{dt}t^2 = 2t. \end{aligned}$$

Similarly,

$$\begin{aligned} b &= \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)y = c'(t)y \\ &= c_*\left(\frac{d}{dt}\right)y = \frac{d}{dt}(y \circ c) = \frac{d}{dt}t^3 = 3t^2. \end{aligned}$$

Thus,

$$c'(t) = 2t\frac{\partial}{\partial x} + 3t^2\frac{\partial}{\partial y}.$$

In terms of the basis $\{\partial/\partial x|_{c(t)}, \partial/\partial y|_{c(t)}\}$ for $T_{c(t)}(\mathbb{R}^2)$,

$$c'(t) = \begin{bmatrix} 2t \\ 3t^2 \end{bmatrix}.$$

More generally, as in this example, to compute the velocity vector of a curve c in $\mathbb{R}^n$, one can simply differentiate the components of c. This shows that our definition of the velocity vector of a curve agrees with the usual definition in vector calculus.

Proposition 8.15 (Velocity of a curve in local coordinates). *Let* $c\colon (a,b) \to M$ *be a curve, and let* $(U, x^1, \dots, x^n)$ *be a coordinate chart about* $c(t)$*. Write* $c^i = x^i \circ c$ *for the* i*th component of* c *in the chart. Then* $c'(t)$ *is given by*

$$c'(t) = \sum_{j=1}^{n} (c^i)'(t)\,\frac{\partial}{\partial x^i}\bigg|_{c(t)}.$$

Thus, relative to the basis $\{\partial/\partial x^i|_p\}$ *for* $T_{c(t)}(M)$*, the velocity* $c'(t)$ *is represented by the column vector*

$$\begin{bmatrix} (c^1)'(t) \\ \vdots \\ (c^n)'(t) \end{bmatrix}.$$

Proof. Problem 8.4. □

Every curve c at p in a manifold M gives rise to a tangent vector $c'(0)$ in $T_p(M)$. Conversely, one can show that every tangent vector $X_p \in T_p(M)$ is the velocity vector of some curve at p, as follows.

Proposition 8.16 (Existence of a curve with a given initial vector). *For any point* p *in a manifold* M *and any tangent vector* $X_p \in T_pM$*, there is a smooth curve* $c\colon (-\epsilon, \epsilon) \to M$ *for some* $\epsilon > 0$ *such that* $c(0) = p$ *and* $c'(0) = X_p$*.*

Proof. Let $(U, \phi) = (U, x^1, \dots, x^n)$ be a chart centered at p, i.e., $\phi(p) = (0, \dots, 0)$. Suppose $X_p = \sum a^i \partial/\partial x^i|_p$ at p. Let $r^1, \dots, r^n$ be the standard coordinates on $\mathbb{R}^n$. Then $x^i = r^i \circ \phi$. To find a curve c at p with $c'(0) = X_p$, start with a curve α in $\mathbb{R}^n$ with $\alpha(0) = 0$ and $\alpha'(0) = \sum a^i \partial/\partial r^i|_0$. We then map α to M via ϕ^{-1} (Figure 8.3). The simplest such α is

$$\alpha(t) = (a^1 t, \dots, a^n t), \quad t \in (-\epsilon, \epsilon),$$

where ϵ is sufficiently small so that $\alpha(t)$ lies in $\phi(U)$. Define $c = \phi^{-1} \circ \alpha\colon (-\epsilon, \epsilon) \to M$. We leave it as an exercise to show that $c(0) = p$ and $c'(0) = X_p$. □

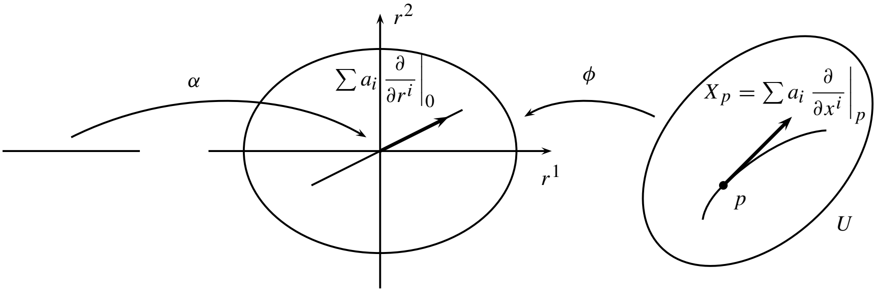


Fig. 8.3. Existence of a curve through a point with a given initial vector.

8.7 Computing the Differential Using Curves

We have introduced two ways of computing the differential of a smooth map, in terms of derivations at a point (Equation (8.1)) and local coordinates (Proposition 8.12). The next proposition gives still another way of computing the differential $F_{*,p}$, this time using curves.

Proposition 8.17. *Let $F\colon N \to M$ be a smooth map of manifolds, $p \in N$, and $X_p \in T_pN$. If c is a curve starting at p in N with velocity X_p at p, then*

$$F_{*,p}(X_p) = \frac{d}{dt}\bigg|_0 F \circ c(t);$$

In other words, $F_{*,p}(X_p)$ is the velocity vector of the image curve $F \circ c$ at $F(p)$.

Proof. By hypothesis, $c(0) = p$ and $c'(0) = X_p$. Then

$$\begin{aligned} F_{*,p}(X_p) &= F_{*,p}(c'(0)) \\ &= F_{*,p} \circ c_{*,0}\left(\frac{d}{dt}\bigg|_0\right) \\ &= (F \circ c)_{*,0}\left(\frac{d}{dt}\bigg|_0\right) \qquad \text{(by the chain rule, Theorem 8.5)} \\ &= \frac{d}{dt}\bigg|_0 F \circ c(t). \end{aligned}$$

□

Example 8.18 (*Differential of left multiplication*). If g is a matrix in the general linear group $\mathrm{GL}(n,\mathbb{R})$, let $\ell_g\colon \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ be left multiplication by g; thus, $\ell_g(B) = gB$ for any $B \in \mathrm{GL}(n,\mathbb{R})$. Since $\mathrm{GL}(n,\mathbb{R})$ is an open subset of the vector space $\mathbb{R}^{n\times n}$, the tangent space $T_g(\mathrm{GL}(n,\mathbb{R}))$ can be identified with $\mathbb{R}^{n\times n}$. Show that with this identification the differential $(\ell_g)_{*,I}\colon T_I(\mathrm{GL}(n,\mathbb{R})) \to T_g(\mathrm{GL}(n,\mathbb{R}))$ is also left multiplication by g.

Solution. Let $X \in T_I(\mathrm{GL}(n,\mathbb{R})) = \mathbb{R}^{n\times n}$. To compute $(\ell_g)_{*,I}(X)$, choose a curve $c(t)$ in $\mathrm{GL}(n,\mathbb{R})$ with $c(0) = I$ and $c'(0) = X$. By Proposition 8.17,

$$(\ell_g)_{*,I}(X) = \left.\frac{d}{dt}\right|_{t=0} \ell_g(c(t)) = \left.\frac{d}{dt}\right|_{t=0} gc(t) = gc'(0) = gX. \qquad \square$$

8.8 Rank, Critical and Regular Points

The *rank* of a linear transformation $L\colon V \to W$ between finite-dimensional vector spaces is the dimension of the image $L(V)$ as a subspace of W. If L is represented by a matrix A relative to a basis for V and a basis for W, then the rank of L is the same as the rank of A, because the image $L(V)$ is simply the column space of A.

Now consider a smooth map $f\colon N \to M$ of manifolds. Its *rank* at a point p in N, denoted by $\operatorname{rk} f(p)$, is defined as the rank of the differential $f_{*,p}$: $T_p(N) \to T_{f(p)}(M)$. Relative to the coordinate neighborhoods $(U, x^1, \dots, x^n)$ at p and $(V, y^1, \dots, y^m)$ at $f(p)$, the differential is represented by the Jacobian matrix $[\partial f^i/\partial x^j(p)]$ (Proposition 8.12), so

$$\operatorname{rk} f(p) = \operatorname{rk}\left[\frac{\partial f^i}{\partial x^j}(p)\right].$$

Since the differential of a map is independent of the coordinate chart, so is its rank.

Definition 8.19. A point p in N is a *critical point* of f if the differential

$$f_{*,p}\colon T_pN \to T_{f(p)}M$$

fails to be surjective. It is a *regular point* of f if the differential $f_{*,p}$ is surjective. A point in M is a *critical value* if it is the image of a critical point; otherwise it is a *regular value*.

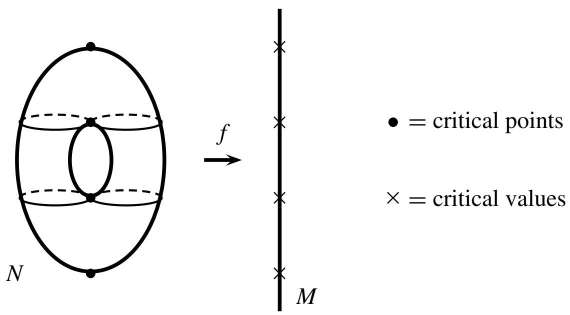


Fig. 8.4. Critical points and critical values.

Two aspects of this definition merit elaboration:

(i) We do *not* define a regular value to be the image of a regular point. In fact, a regular value need not be in the image of f at all. Any point of M not in the image of f is automatically a regular value because it is not the image of a critical point.

(ii) A point c in M is a critical value if and only if *some* point in the preimage $f^{-1}(\{c\})$ is a critical point. A point c in the image of f is a regular value if and only if *every* point in the preimage $f^{-1}(\{c\})$ is a regular point.

Proposition 8.20. *For a real-valued function $f\colon M \to \mathbb{R}$, a point p in M is a critical point if and only if relative to some chart $(U, x^1, \dots, x^n)$ containing p, all the partial derivatives*

$$\frac{\partial f}{\partial x^j}(p) = 0, \quad j = 1, \dots, n.$$

Proof. By Proposition 8.12 the differential $f_{*,p}\colon T_pM \to T_{f(p)}\mathbb{R} \simeq \mathbb{R}$ is represented by the matrix

$$\left[\frac{\partial f}{\partial x^1}(p) \ \dots \ \frac{\partial f}{\partial x^n}(p)\right].$$

Since the image of $f_{*,p}$ is a linear subspace of $\mathbb{R}$, it is either zero-dimensional or one-dimensional. In other words, $f_{*,p}$ is either the zero map or a surjective map. Therefore, $f_{*,p}$ fails to be surjective if and only if all the partial derivatives $\partial f/\partial x^i(p)$ are zero. $\square$

Problems

8.1.* Differential of a map
Let $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the map

$$(u, v, w) = F(x, y) = (x, y, xy).$$

Compute $F_*(\partial/\partial x)$ as a linear combination of $\partial/\partial u$, $\partial/\partial v$, and $\partial/\partial w$.

8.2. Differential of a map
Fix a real number α and define $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{bmatrix} u \\ v \end{bmatrix} = (u, v) = F(x, y) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Let $X = -y\,\partial/\partial x + x\,\partial/\partial y$ be a vector field on $\mathbb{R}^2$. If $F_*(X) = a\,\partial/\partial u + b\,\partial/\partial v$, find a and b in terms of x, y, and α.

8.3. Transition matrix for coordinate vectors
Let x, y be the standard coordinates on $\mathbb{R}^2$, and let U be the open set

$$U = \mathbb{R}^2 - \{(x, 0) \mid x \geq 0\}.$$

On U the polar coordinates r, θ are uniquely defined by

$$x = r\cos\theta,$$
$$y = r\sin\theta,\ r > 0,\ 0 < \theta < 2\pi.$$

Find $\partial/\partial r$ and $\partial/\partial\theta$ in terms of $\partial/\partial x$ and $\partial/\partial y$.

8.4.* Velocity of a curve in local coordinates
Prove Proposition 8.15.

8.5. Velocity vector
Let $p = (x, y)$ be a point in $\mathbb{R}^2$. Then

$$c_p(t) = \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad t \in \mathbb{R},$$

is a curve with initial point p in $\mathbb{R}^2$. Compute the velocity vector $c_p'(0)$.

8.6. Differential of a linear map
Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. For any $p \in \mathbb{R}^n$, there is a canonical identification: $T_p(\mathbb{R}^n) \xrightarrow{\sim} \mathbb{R}^n$ given by

$$\sum a^i \frac{\partial}{\partial x^i} \mapsto \mathbf{a} = \langle a^1, \dots, a^n \rangle.$$

Show that the differential $L_{*,p}\colon T_p(\mathbb{R}^n) \to T_{f(p)}(\mathbb{R}^m)$ is the map $L\colon \mathbb{R}^n \to \mathbb{R}^m$ itself, with the identification of the tangent spaces as above.

8.7.* Tangent space to a product
If M and N are manifolds, let $\pi_1\colon M \times N \to M$ and $\pi_2\colon M \times N \to N$ be the two projections. Prove that for $(p, q) \in M \times N$,

$$\pi_{1*} \times \pi_{2*}\colon T_{(p,q)}(M \times N) \to T_pM \times T_qN$$

is an isomorphism.

8.8. Differentials of multiplication and inverse
Let G be a Lie group with multiplication map $\mu\colon G \times G \to G$, inverse map $\iota\colon G \to G$, and identity element e.

(a) Show that the differential at the identity of the multiplication map μ is addition:

$$\mu_{*,(e,e)}\colon T_eG \times T_eG \to T_eG,$$
$$\mu_{*,(e,e)}(X_e, Y_e) = X_e + Y_e.$$

(*Hint*: First, compute $\mu_{*,(e,e)}(X_e, 0)$ and $\mu_{*,(e,e)}(0, Y_e)$ using Proposition 8.17.)

(b) Show that the differential at the identity of ι is the negative:

$$\iota_{*,e}\colon T_eG \to T_eG,$$
$$\iota_{*,e}(X_e) = -X_e.$$

(*Hint*: Take the differential of $\mu(c(t), (\iota \circ c)(t)) = e$.)

8.9.* Transforming vectors to coordinate vectors
Let $X_1, \ldots, X_n$ be n vector fields on an open subset U of a manifold of dimension n. Suppose that at $p \in U$, the vectors $(X_1)_p, \ldots, (X_n)_p$ are linearly independent. Show that there is a chart $(V, x^1, \ldots, x^n)$ about p such that $(X_i)_p = (\partial/\partial x^i)_p$ for all $i = 1, \ldots, n$.

9

Submanifolds

We now have two ways of showing that a given topological space is a manifold:

(a) by checking directly that the space is Hausdorff, second countable, and has a C^∞ atlas;
(b) by exhibiting it as an appropriate quotient space. Chapter 7 lists some conditions under which a quotient space is a manifold.

In this chapter we introduce the concept of a *regular submanifold* of a manifold, a subset that is locally defined by the vanishing of some of the coordinate functions. Using the inverse function theorem, we derive a criterion, called the *regular level set theorem*, that can often be used to show that a level set of a C^∞ map of manifolds is a regular submanifold and therefore a manifold.

Although the regular level set theorem is a simple consequence of the constant rank theorem to be discussed in Chapter 11, deducing it directly from the inverse function theorem has the advantage of producing explicit coordinate functions on the submanifold.

9.1 Submanifolds

The xy-plane in $\mathbb{R}^3$ is the prototype of a *regular submanifold* of a manifold. It is defined by the vanishing of a coordinate function z.

Definition 9.1. A subset S of a manifold N of dimension n is a *regular submanifold* of dimension k if for every $p \in S$ there is a coordinate neighborhood $(U, \phi) = (U, x^1, \ldots, x^n)$ of p in the atlas of N such that $U \cap S$ is defined by the vanishing of $n - k$ of the coordinate functions. By renumbering the coordinates, we may assume these $n - k$ coordinate functions are $x^{k+1}, \ldots, x^n$.

We call such a chart (U, ϕ) in N an *adapted chart* relative to S. On $U \cap S$, $\phi = (x^1, \ldots, x^k, 0, \ldots, 0)$. Let

$$\phi_S \colon U \cap S \to \mathbb{R}^k$$

be the restriction of the first k components of ϕ to $U \cap S$, that is, $\phi_S = (x^1, \dots, x^k)$.

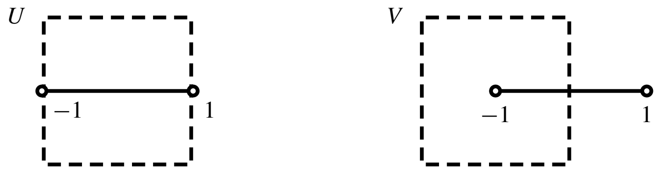
Definition 9.2. If S is a regular submanifold of dimension k in a manifold N of dimension n, then $n - k$ is said to be the *codimension* of S in N.

Remark 9.3. As a topological space, a regular submanifold of N is required to have the subspace topology.

Example 9.4. In the definition of a regular submanifold, the dimension k of the submanifold may be equal to n, the dimension of the manifold. In this case, $U \cap S$ is defined by the vanishing of none of the coordinate functions and so $U \cap S = U$. An open subset of a manifold is a regular submanifold of the same dimension.

Remark 9.5. There are other types of submanifolds, but for now by a "submanifold" we will always mean a "regular submanifold."

Example 9.6. The interval $S := (-1, 1)$ on the x-axis is a regular submanifold of the xy-plane (Figure 9.1). As an adapted chart, we can take the open square $U = (-1, 1) \times (-1, 1)$ with coordinates x, y. Then $U \cap S$ is precisely the zero set of y on U.



U is an adapted chart $\quad$ V is not an adapted chart

Fig. 9.1.

Note that if $V = (-2, 0) \times (-1, 1)$, then (V, x, y) would not be an adapted chart relative to S, since $V \cap S$ is the open interval $(-1, 0)$ on the x-axis, while the zero set of y on V is the open interval $(-2, 0)$ on the x-axis.

Example 9.7. Let Γ be the graph of the function $f(x) = \sin(1/x)$ on the interval $(0, 1)$, and let S be the union of Γ and the open interval

$$I = \{(0, y) \in \mathbb{R}^2 \mid -1 < y < 1\}.$$

The subset S of $\mathbb{R}^2$ is not a regular submanifold for the following reason: if p is in the interval I, then there is no adapted chart containing p, since any sufficiently small neighborhood U of p in $\mathbb{R}^2$ intersects S in infinitely many components. (The closure of Γ in $\mathbb{R}^2$ is called the *topologist's sine curve* (Figure 9.2). It differs from S in including the endpoints $(1, \sin 1)$, $(0, 1)$, and $(0, -1)$.)

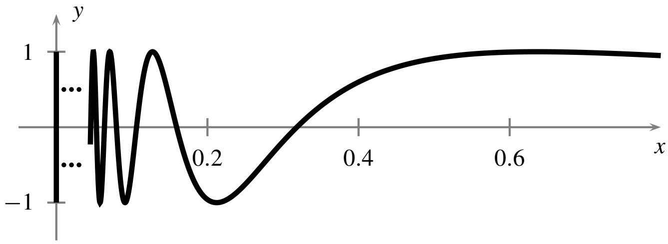


Fig. 9.2. The topologist's sine curve.

Proposition 9.8. *Let S be a regular submanifold of N and $\mathfrak{A} = \{(U, \phi)\}$ a collection of compatible adapted charts of N that covers S. Then $\{(U \cap S, \phi_S)\}$ is an atlas for S. Therefore, a regular submanifold is itself a manifold. If N has dimension n and S is locally defined by the vanishing of $n - k$ coordinates, then* $\dim S = k$.

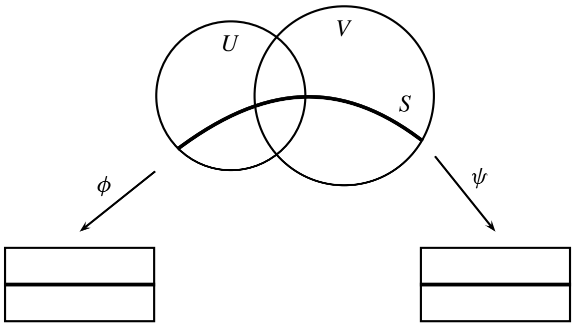


Fig. 9.3. Overlapping adapted charts relative to a regular submanifold S.

Proof. Let $(U, \phi) = (U, x^1, \ldots, x^n)$ and $(V, \psi) = (V, y^1, \ldots, y^n)$ be two adapted charts in the given collection (Figure 9.3). Assume that they intersect. Then for $p \in U \cap V \cap S$,

$$\phi(p) = (x^1, \ldots, x^k, 0, \ldots, 0) \quad \text{and} \quad \psi(p) = (y^1, \ldots, y^k, 0, \ldots, 0),$$

so

$$\phi_S(p) = (x^1, \ldots, x^k) \quad \text{and} \quad \psi_S(p) = (y^1, \ldots, y^k).$$

Therefore,

$$\psi_S \circ \phi_S^{-1}(x^1, \ldots, x^k) = (y^1, \ldots, y^k).$$

Since $y^1, \ldots, y^k$ are C^∞ functions of $x^1, \ldots, x^k$, the transition function $\psi_S \circ \phi_S^{-1}$ is C^∞. Hence, any two charts in $\{(U \cap S, \phi_S)\}$ are C^∞ compatible. Since $\{U \cap S\}_{U \in \mathfrak{A}}$ is an open cover of S, the collection $\{(U \cap S, \phi_S)\}$ is an atlas for S. □

9.2 The Zero Set of a Function

A *level set* of a map $f: N \to M$ is the subset

$$f^{-1}(\{c\}) = \{p \in N \mid f(p) = c\}$$

for some $c \in M$. The usual notation for a level set is $f^{-1}(c)$, rather than the more correct $f^{-1}(\{c\})$. If $f: N \to \mathbb{R}^m$, then $Z(f) := f^{-1}(0)$ is the *zero set* of f. The inverse image $f^{-1}(c)$ of a regular value c is called a *regular level set*.

Example 9.9 (*The* 2*-sphere in* $\mathbb{R}^3$). Define

$$f(x, y, z) = x^2 + y^2 + z^2 - 1$$

on $\mathbb{R}^3$. Then the level set

$$f^{-1}(0) = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = x^2 + y^2 + z^2 - 1 = 0\}$$

is the unit 2-sphere S^2. We will use the inverse function theorem to find adapted charts of $\mathbb{R}^3$ that cover S^2.

Since

$$\frac{\partial f}{\partial x} = 2x, \qquad \frac{\partial f}{\partial y} = 2y, \qquad \frac{\partial f}{\partial z} = 2z,$$

the only critical point of f is $(0, 0, 0)$, which does not lie on the sphere S^2. Thus, all points on the sphere are regular points of f and 0 is a regular value of f.

Let p be a point of S^2 at which $(\partial f/\partial x)(p) = 2x(p) \neq 0$. Then the Jacobian matrix of the map $(f, y, z): \mathbb{R}^3 \to \mathbb{R}^3$ is

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and the Jacobian determinant at p is $\partial f/\partial x(p) \neq 0$. By the inverse function theorem (Theorem 6.13), there is a neighborhood U_p of p in $\mathbb{R}^3$ so that (U_p, f, y, z) is a chart in the atlas of $\mathbb{R}^3$. In this chart, the set $U_p \cap S^2$ is defined by the vanishing of the first coordinate $f = 0$. Thus, (U_p, f, y, z) is an adapted chart relative to S^2, and $(U_p \cap S^2, y, z)$ is a chart for S^2.

Similarly, if $(\partial f/\partial y)(p) \neq 0$, then there is an adapted chart (V_p, x, f, z) containing p in which the set $V_p \cap S^2$ is the zero set of the second coordinate f. If $(\partial f/\partial z)(p) \neq 0$, then there is an adapted chart (W_p, x, y, f) containing p. As p varies over all points of the sphere, we obtain a collection of adapted charts of $\mathbb{R}^3$ covering S^2. Therefore, S^2 is a regular submanifold of $\mathbb{R}^3$. By Proposition 9.8, S is a manifold of dimension 2.

This is an important example because one can generalize its proof almost verbatim to prove that a regular level set of a function $f\colon N \to \mathbb{R}$ is a regular submanifold of N. The idea is that in a coordinate neighborhood $(U, x^1, \dots, x^n)$ if a partial derivative $\partial f/\partial x^i(p) \neq 0$, then we replace the coordinate x^i by f.

Theorem 9.10. *Let $f\colon N \to \mathbb{R}$ be a C^∞ function on the manifold N. Then a nonempty regular level set $S = f^{-1}(c)$ is a regular submanifold of N of codimension* 1.

Proof. Replacing f by $f - c$ if necessary, we may assume $c = 0$. Let $p \in S$. Since p is a regular point of f, there is a chart $(U, x^1, \dots, x^n)$ containing p relative to which $(\partial f/\partial x^i)(p) \neq 0$ for some i. By renumbering $x^1, \dots, x^n$, we may assume that $(\partial f/\partial x^1)(p) \neq 0$.

The Jacobian matrix of the C^∞ map $(f, x^2, \dots, x^n)\colon U \to \mathbb{R}^n$ is

$$\begin{bmatrix} \dfrac{\partial f}{\partial x^1} & \dfrac{\partial f}{\partial x^2} & \cdots & \dfrac{\partial f}{\partial x^n} \\ \dfrac{\partial x^2}{\partial x^1} & \dfrac{\partial x^2}{\partial x^2} & \cdots & \dfrac{\partial x^2}{\partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial x^n}{\partial x^1} & \dfrac{\partial x^n}{\partial x^2} & \cdots & \dfrac{\partial x^n}{\partial x^n} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial x^1} & * & \cdots & * \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

So the Jacobian determinant at p is $\partial f/\partial x^1(p) \neq 0$. By the inverse function theorem, there is a neighborhood U_p of p on which $f, x^2, \dots, x^n$ form a coordinate system. Relative to the chart $(U_p, f, x^2, \dots, x^n)$ the level set $U_p \cap S$ is defined by setting the first coordinate $f = 0$, so $(U_p, f, x^2, \dots, x^n)$ is an adapted chart relative to S. Therefore, S is a regular submanifold of dimension $n - 1$ in N. □

9.3 The Regular Level Set Theorem

The generalization of Theorem 9.10 to a level set of a function to $\mathbb{R}^m$ is straightforward.

Theorem 9.11. *Let $f\colon N \to \mathbb{R}^m$ be a C^∞ map on a manifold N of dimension n. Then a nonempty regular level set $S = f^{-1}(c)$ is a regular submanifold of dimension $n - m$ of N.*

Proof. As in the proof of Theorem 9.10, by replacing f by $f - c$, we may assume that $c = 0 \in \mathbb{R}^m$. Let p be any point of S and let $(U, x^1, \dots, x^n)$ be a chart of N containing p. Since p is a regular point of f, the matrix $[\partial f^i/\partial x^j(p)]$ has rank m, so $n \geq m$. By renumbering the f^i and x^j's, we may assume that the first $m \times m$ block $[\partial f^i/\partial x^j(p)]_{1 \leq i, j \leq m}$ is nonsingular.

Replace the first m coordinates $x^1, \dots, x^m$ in the chart (U, x) by $f^1, \dots, f^m$. We claim that there is a neighborhood U_p of p so that $(U_p, f^1, \dots, f^m, x^{m+1}, \dots, x^n)$ is a chart in the atlas of N. It suffices to compute its Jacobian matrix at p:

$$\begin{bmatrix} \dfrac{\partial f^i}{\partial x^j} & \dfrac{\partial f^i}{\partial x^\beta} \\ \dfrac{\partial x^\alpha}{\partial x^j} & \dfrac{\partial x^\alpha}{\partial x^\beta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f^i}{\partial x^j} & * \\ 0 & I \end{bmatrix},$$

where $1 \le i, j \le m$ and $m + 1 \le \alpha, \beta \le n$. Since this matrix has determinant

$$\det \left[\frac{\partial f^i}{\partial x^j}(p) \right]_{1 \le i, j \le m} \neq 0,$$

the inverse function theorem implies the claim.

In the chart $(U_p, f^1, \dots, f^m, x^{m+1}, \dots, x^n)$, the set S is obtained by setting the first m coordinate functions $f^1, \dots, f^m$ equal to 0. So $(U_p, f^1, \dots, f^m, x^{k+1}, \dots, x^n)$ is an adapted chart for $\mathbb{R}^n$ relative to S. Therefore, S is a regular submanifold of N of dimension $n - m$. □

The proof of Theorem 9.11 gives the following useful lemma.

Lemma 9.12. *Let $f: N \to \mathbb{R}^m$ be a C^∞ map on a manifold N of dimension n and let S be the level set $f^{-1}(0)$. If relative to some coordinate chart $(U, x^1, \dots, x^n)$ about $p \in S$, the Jacobian determinant $\partial(f^1, \dots, f^m)/\partial(x^{j_1}, \dots, x^{j_m}) \neq 0$, then in some neighborhood of p one may replace $x^{j_1}, \dots, x^{j_m}$ by $f^1, \dots, f^m$ to obtain an adapted chart for N relative to S.*

The next step is to extend Theorem 9.11 to a regular level set of a map between smooth manifolds. This very useful theorem does not seem to have an agreed-upon name in the literature. It is known variously as the implicit function theorem, the preimage theorem [9], the regular level set theorem [11], among other nomenclatures. We will follow [11] and call it the regular level set theorem.

Theorem 9.13 (Regular level set theorem). *Let $f: N \to M$ be a C^∞ map of manifolds, with $\dim N = n$ and $\dim M = m$. Then a nonempty regular level set $f^{-1}(c)$ is a regular submanifold of N of dimension equal to $n - m$.*

Proof. Choose a chart (V, ψ) about c in M with $\psi(c) = 0$. Consider the map $\psi \circ f: f^{-1}(V) \to \mathbb{R}^m$. Since $(\psi \circ f)_* = \psi_* \circ f_*$ and ψ_* is an isomorphism at every point of V,

$$\operatorname{rk}(\psi \circ f)_{*,q} = \operatorname{rk} f_{*,q}$$

for all $q \in f^{-1}(V)$. Hence, c is a regular value of f if and only if 0 is a regular value of $\psi \circ f$. Moreover, $(\psi \circ f)^{-1}(0) = f^{-1}(c)$. By Theorem 9.11, the level set $(\psi \circ f)^{-1}(0)$ is a regular submanifold of $f^{-1}(V)$ of dimension $n - m$. Since $f^{-1}(V)$ is open in N, the adapted charts of $f^{-1}(V)$ that cover $f^{-1}(c)$ are also adapted charts of N that cover $f^{-1}(c)$. It follows that $f^{-1}(c)$ is a regular submanifold of N of dimension $n - m$. □

9.4 Examples of Regular Submanifolds

Example 9.14 (*Hypersurface*). Show that the solution set S of $x^3 + y^3 + z^3 = 1$ in $\mathbb{R}^3$ is a manifold of dimension 2.

Solution. Let $f(x, y, z) = x^3 + y^3 + z^3$. Then $S = f^{-1}(1)$. Since $\partial f/\partial x = 3x^2$, $\partial f/\partial y = 3y^2$, and $\partial f/\partial z = 3z^2$, the only critical point of f is $(0, 0, 0)$, which is not in S. Thus, 1 is a regular value of $f\colon \mathbb{R}^3 \to \mathbb{R}$. By the regular level set theorem (Theorem 9.13), S is a regular submanifold of $\mathbb{R}^3$ of dimension 2. So S is a manifold (Proposition 9.8). □

Example 9.15 (*Solution set of two polynomial equations*). Decide if the subset S of $\mathbb{R}^3$ defined by the two equations

$$\begin{aligned} x^3 + y^3 + z^3 &= 1, \\ x + y + z &= 0 \end{aligned}$$

is a regular submanifold of $\mathbb{R}^3$.

Solution. Define $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ by

$$(u, v) = F(x, y, z) = (x^3 + y^3 + z^3, x + y + z).$$

Then S is the level set $F^{-1}(1, 0)$. The Jacobian matrix of F is

$$J(F) = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \end{bmatrix} = \begin{bmatrix} 3x^2 & 3y^2 & 3z^2 \\ 1 & 1 & 1 \end{bmatrix},$$

where $u_x = \partial u/\partial x$ and so forth. The critical points of F are the points (x, y, z) where the matrix $J(F)$ has rank < 2. That is precisely where all 2×2 minors of $J(F)$ are zero:

$$\begin{vmatrix} 3x^2 & 3y^2 \\ 1 & 1 \end{vmatrix} = 0, \qquad \begin{vmatrix} 3x^2 & 3z^2 \\ 1 & 1 \end{vmatrix} = 0. \tag{9.1}$$

(The third condition

$$\begin{vmatrix} 3y^2 & 3z^2 \\ 1 & 1 \end{vmatrix} = 0$$

is a consequence of these two.) Solving (9.1), we get $y = \pm x$, $z = \pm x$. Since $x + y + z = 0$ on S, this implies that $(x, y, z) = (0, 0, 0)$. Since $(0, 0, 0)$ does not satisfy the first equation $x^3 + y^3 + z^3 = 1$, there are no critical points of F on S. Therefore, S is a regular level set. By the regular level set theorem, S is a regular submanifold of $\mathbb{R}^3$ of dimension 1. □

Example 9.16 (*Special linear group*). As a set, the *special linear group* $\mathrm{SL}(n, \mathbb{R})$ is the subset of $\mathrm{GL}(n, \mathbb{R})$ consisting of matrices of determinant 1. Since

$$\det(AB) = (\det A)(\det B) \quad \text{and} \quad \det(A^{-1}) = \frac{1}{\det A},$$

$\mathrm{SL}(n,\mathbb{R})$ is a subgroup of $\mathrm{GL}(n,\mathbb{R})$. To show that it is a regular submanifold, we let $f\colon \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}$ be the determinant map,

$$f(A) = \det A,$$

and apply the regular level set theorem to $f^{-1}(1) = \mathrm{SL}(n,\mathbb{R})$. We need to check that 1 is a regular value of f.

Let S_{ij} denote the submatrix of $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ obtained by deleting its ith row and jth column. Then $m_{ij} := \det S_{ij}$ is the (i, j)-*minor* of A. From linear algebra we have a formula for computing the determinant by expanding along any row or any column; if we expand along the ith row,

$$f(A) = \det A = (-1)^{i+1} a_{i1} m_{i1} + (-1)^{i+2} a_{i2} m_{i2} + \cdots + (-1)^{i+n} a_{in} m_{in}. \tag{9.2}$$

Therefore

$$\frac{\partial f}{\partial a_{ij}} = (-1)^{i+j} m_{ij}.$$

Hence, a matrix $A \in \mathrm{GL}(n,\mathbb{R})$ is a critical point of f if and only if all the $(n-1)\times(n-1)$ minors m_{ij} of A are 0. By (9.2) such a matrix A has determinant 0. Since every matrix in $\mathrm{SL}(n,\mathbb{R})$ has determinant 1, all the matrices in $\mathrm{SL}(n,\mathbb{R})$ are regular points of the determinant function. By the regular level set theorem (Theorem 9.13), $\mathrm{SL}(n,\mathbb{R})$ is a regular submanifold of $\mathrm{GL}(n,\mathbb{R})$ of codimension 1, i.e.,

$$\dim \mathrm{SL}(n,\mathbb{R}) = \dim \mathrm{GL}(n,\mathbb{R}) - 1 = n^2 - 1.$$

Problems

9.1. Regular values
Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = x^3 - 6xy + y^2.$$

Find all values $c \in \mathbb{R}$ for which the level set $f^{-1}(c)$ is a regular submanifold of $\mathbb{R}^2$.

9.2. Solution set of one equation
Let x, y, z, w be the standard coordinates on $\mathbb{R}^4$. Is the solution set of $x^5 + y^5 + z^5 + w^5 = 1$ in $\mathbb{R}^4$ a manifold? Explain why or why not. (Assume that the subset is given the subspace topology.)

9.3. Solution set of two equations
Is the solution set of the system of equations

$$x^3 + y^3 + z^3 = 1, \quad z = xy,$$

in $\mathbb{R}^3$ a C^∞ manifold? Prove your answer.

9.4.* Regular submanifolds
A subset S of $\mathbb{R}^2$ has the property that locally on S one of the coordinates is a C^∞ function of the other coordinate. Show that S is a regular submanifold of $\mathbb{R}^2$. (Note that the unit circle defined by $x^2 + y^2 = 1$ has this property. At every point of the circle, there is a neighborhood in which y is a C^∞ function of x or x is a C^∞ function of y.)

9.5. Graph of a smooth function
Show that the graph $\Gamma(f)$ of a smooth function $f\colon \mathbb{R}^2 \to \mathbb{R}$,

$$\Gamma(f) = \{(x, y, f(x, y)) \in \mathbb{R}^3\},$$

is a regular submanifold of $\mathbb{R}^3$.

9.6. Euler's formula
A polynomial $F(x_0, \dots, x_n) \in \mathbb{R}[x_0, \dots, x_n]$ is *homogeneous of degree k* if it is a linear combination of monomials $x_0^{i_0} \cdots x_n^{i_n}$ of degree $\sum_{j=0}^n i_j = k$. Let $F(x_0, \dots, x_n)$ be a homogeneous polynomial of degree k. Clearly, for any $t \in \mathbb{R}$,

$$F(tx_0, \dots, tx_n) = t^k F(x_0, \dots, x_n). \tag{9.3}$$

Show that

$$\sum_{i=0}^n x_i \frac{\partial F}{\partial x_i} = kF.$$

9.7. Smooth projective hypersurface
On the projective space $\mathbb{R}P^n$ a homogeneous polynomial $F(x_0, \dots, x_n)$ of degree k is not a function since its value at a point $[a_0, \dots, a_n]$ is not unique. However, the zero set in $\mathbb{R}P^n$ of a homogeneous polynomial $F(x_0, \dots, x_n)$ is well defined, since $F(a_0, \dots, a_n) = 0$ iff

$$F(ta_0, \dots, ta_n) = t^k F(a_0, \dots, a_n) = 0 \quad \text{for all } t \in \mathbb{R}^\times.$$

The zero set of finitely many homogeneous polynomials in $\mathbb{R}P^n$ is called a (*real*) *projective variety*. A projective variety defined by a single homogeneous polynomial of degree k is called a *hypersurface* of degree k. Show that the hypersurface $Z(F)$ defined by $F(x_0, x_1, x_2) = 0$ is smooth if $\partial F/\partial x_0$, $\partial F/\partial x_1$, and $\partial F/\partial x_2$ are not simultaneously zero on $Z(F)$. (*Hint*: In (U_0, x, y), $F(x_0, x_1, x_2) = x_0^k F(1, x_1/x_0, x_2/x_0) = x_0^k F(1, x, y)$, where we set $x = x_1/x_0$ and $y = x_2/x_0$. Define $f(x, y) = F(1, x, y)$. Then f and F have the same zero set in U_0.)

9.8. Product of regular submanifolds
If S_i is a regular submanifold of the manifold M_i for $i = 1, 2$, prove that $S_1 \times S_2$ is a regular submanifold of $M_1 \times M_2$.

9.9. The transversality theorem
A C^∞ map $f\colon N \to M$ is said to be *transversal* to a manifold $S \subset M$ if for every $p \in f^{-1}(S)$,

$$f_*(T_pN) + T_{f(p)}S = T_{f(p)}M. \tag{9.4}$$

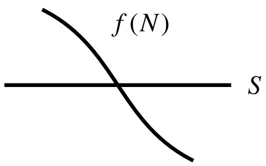
(If A and B are subspaces of a vector space, their sum $A + B$ is the subspace consisting of all $a + b$ with $a \in A$ and $b \in B$. The sum need not be a direct sum.) The goal of this exercise is to prove the *transversality theorem*: if a C^∞ map $f\colon N \to M$ is transversal to a regular submanifold S of codimension k in M, then $f^{-1}(S)$ is a regular submanifold of codimension k in N.

When S consists of a single point c, transversality of f to S simply means that $f^{-1}(c)$ is a regular level set. Thus the transversality theorem is a generalization of the regular level set theorem. It is especially useful in giving conditions under which the intersection of two submanifolds is a submanifold.

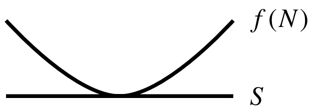
Let $p \in f^{-1}(S)$ and $(U, x^1, \dots, x^m)$ be an adapted chart centered at $f(p)$ for M relative to S such that $U \cap S = Z(x^{m-k+1}, \dots, x^m)$, the zero set of the functions $x^{m-k+1}, \dots, x^m$. Define $g\colon U \to \mathbb{R}^k$ to be the map

$$g = (x^{m-k+1}, \dots, x^m).$$

(a) Show that $f^{-1}(U) \cap f^{-1}(S) = (g \circ f)^{-1}(0)$.
(b) Show that $f^{-1}(U) \cap f^{-1}(S)$ is a regular level set of the function $g \circ f\colon f^{-1}(U) \to \mathbb{R}^k$.
(c) Prove the transversality theorem.



f transversal to S in $\mathbb{R}^2$



f not transversal to S in $\mathbb{R}^2$

10

Categories and Functors

10.1 Categories

Many of the problems in mathematics share common features. For example, in topology one is interested in knowing if two topological spaces are homeomorphic and in group theory one is interested in knowing if two groups are isomorphic. This has given rise to the theory of categories and functors, which tries to clarify the structural similarities among different areas of mathematics.

A *category* consists of a collection of elements, called *objects*, and for any two objects A and B, a set $\mathrm{Hom}(A, B)$ of *morphisms* from A to B, such that given any morphism $f \in \mathrm{Hom}(A, B)$ and any morphism $g \in \mathrm{Hom}(B, C)$, the *composite* $g \circ f \in \mathrm{Hom}(A, C)$ is defined. Furthermore, the composition of morphisms is required to satisfy two properties:

(i) the identity axiom: for each object A, there is an identity morphism $1_A \in \mathrm{Hom}(A, A)$ such that for any $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, A)$,

$$f \circ 1_A = f \quad \text{and} \quad 1_A \circ g = g;$$

(ii) the associative axiom: for $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$, and $h \in \mathrm{Hom}(C, D)$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

If $f \in \mathrm{Hom}(A, B)$, we often write $f\colon A \to B$.

Example 10.1. The category of groups and group homomorphisms is the category in which the objects are groups and for any two groups A and B, $\mathrm{Hom}(A, B)$ is the set of group homomorphisms from A to B.

Example 10.2. The collection of all vector spaces over $\mathbb{R}$ together with linear maps between vector spaces is a category.

Example 10.3. The collection of all topological spaces together with continuous maps between them is called the *continuous category*.

Example 10.4. The collection of smooth manifolds together with smooth maps between them is called the *smooth category*.

Example 10.5. We call a pair (M, q), where M is a manifold and q a point in M, a *pointed manifold*. Given any two such pairs (N, p) and (M, q), let $\mathrm{Hom}((N, p), (M, q))$ be the set of all smooth maps $F: N \to M$ such that $F(p) = q$. This gives rise to the *category of pointed manifolds*.

Definition 10.6. Two objects A and B in a category are said to be *isomorphic* if there are morphisms $f: A \to B$ and $g: B \to A$ such that

$$g \circ f = 1_A \quad \text{and} \quad f \circ g = 1_B.$$

In this case both f and g are called *isomorphisms*.

10.2 Functors

Definition 10.7. A *(covariant) functor* F from one category $\mathcal{C}$ to another category $\mathcal{D}$ is a map that associates to each object A in $\mathcal{C}$ an object $F(A)$ in $\mathcal{D}$ and to each morphism $f: A \to B$ a morphism $F(f): F(A) \to F(B)$ such that

(i) $F(1_A) = 1_{F(A)}$;
(ii) $F(f \circ g) = F(f) \circ F(g)$.

Example 10.8. The tangent space construction is a functor from the category of pointed manifolds to the category of vector spaces. To each pointed manifold (N, p) we associate the tangent space $T_p(N)$ and to each smooth map $f: (N, p) \to (M, f(p))$, we associate the differential $f_*: T_pN \to T_{f(p)}M$.

The functorial property (i) holds because if $1: N \to N$ is the identity map, then its differential $1_*: T_pN \to T_pN$ is also the identity map.

The functorial property (ii) holds because in this context it is the chain rule

$$(g \circ f)_* = g_* \circ f_*.$$

Proposition 10.9. *Let $F: \mathcal{C} \to \mathcal{D}$ be a functor from a category $\mathcal{C}$ to a category $\mathcal{D}$. If $f: A \to B$ is an isomorphism in $\mathcal{C}$, then $F(f): F(A) \to F(B)$ is an isomorphism in $\mathcal{D}$.*

Proof. Problem 10.2. □

Note that we can recast Corollary 8.7 in a more functorial form. Suppose $f: N \to M$ is a diffeomorphism. Then (N, p) and $(M, f(p))$ are isomorphic objects in the category of pointed manifolds. By Proposition 10.9, the tangent spaces T_pN and $T_{f(p)}M$ must be isomorphic as vector spaces.

If in the definition of a covariant functor, we reverse the direction of the arrow for the morphism $F(f)$, then we obtain a *contravariant functor*. More precisely, the definition is as follows.

Definition 10.10. A *contravariant functor* F from one category $\mathcal{C}$ to another category $\mathcal{D}$ is a map that associates to each object A in $\mathcal{C}$ an object $F(A)$ in $\mathcal{D}$ and to each morphism $f\colon A \to B$ a morphism $F(f)\colon F(B) \to F(A)$ such that

(i) $F(1_A) = 1_{F(A)}$;
(ii) $F(f \circ g) = F(g) \circ F(f)$. (Note the reversal of order.)

An example of a contravariant functor is the dual of a vector space, which we review in the next section.

10.3 Dual Maps

Let V be a real vector space. Recall that its dual space V^* is the vector space of all *linear functionals* on V, i.e., linear functions $\alpha\colon V \to \mathbb{R}$. We also write

$$V^* = \mathrm{Hom}(V, \mathbb{R}).$$

If V is a finite-dimensional vector space with basis $\{e_1, \dots, e_n\}$, then by Proposition 3.1 its dual space V^* has as a basis the collection of linear functionals $\{\alpha^1, \dots, \alpha^n\}$ defined by

$$\alpha^i(e_j) = \delta^i_j, \quad 1 \le i, \quad j \le n.$$

Since a linear function on V is determined by what it does on a basis of V, this set of equations defines α^i uniquely.

A linear map $L\colon V \to W$ of vector spaces induces a linear map L^*, called the *dual* of L, on the dual spaces:

$$\begin{aligned} &L^*\colon W^* \to V^*, \\ &(L^*\alpha)(v) = \alpha(L(v)), \quad \text{for } \alpha \in W^*, \quad v \in V. \end{aligned}$$

Thus, $L^*\alpha = \alpha \circ L$. Note that the dual of L reverses the direction of the arrow.

Proposition 10.11 (Functorial properties of the dual). *Suppose V, W, and S are real vector spaces.*

(i) *If $1_V\colon V \to V$ is the identity map on V, then $1_V^*\colon V^* \to V^*$ is the identity map on V^*.*
(ii) *If $f\colon V \to W$ and $g\colon W \to S$ are linear maps, then $(g \circ f)^* = f^* \circ g^*$.*

Proof. Problem 10.3. □

According to this proposition, the dual construction $(\ ) \mapsto (\ )^*$ is a contravariant functor from the category of vector spaces to itself. Consequently, if $f\colon V \to W$ is an isomorphism, then so is its dual $f^*\colon W^* \to V^*$ (cf. Proposition 10.9).

Problems

10.1. Differential of the inverse map
If $F\colon N \to M$ is a diffeomorphism of manifolds and $p \in N$, prove that $(F^{-1})_{*,F(p)} = (F_{*,p})^{-1}$.

10.2. Isomorphism under a functor
Prove Proposition 10.9.

10.3. Functorial properties of the dual
Prove Proposition 10.11.

10.4. Matrix of the dual map
Suppose a linear transformation $L\colon V \to \bar{V}$ is represented by the matrix $A = [a^i_j]$ relative to a basis $e_1, \dots, e_n$ for V and $\bar{e}_1, \dots, \bar{e}_m$ for $\bar{V}$:

$$L(e_j) = \sum_i a^i_j \bar{e}_i.$$

Let $\alpha^1, \dots, \alpha^n$ and $\bar{\alpha}^1, \dots, \bar{\alpha}^m$ be the dual bases for V^* and $\bar{V}^*$, respectively. Prove that if $L^*(\bar{\alpha}^i) = \sum_j b^i_j \alpha^j$, then $b^i_j = a^i_j$.

10.5. Injectivity of the dual map

(a) Suppose V and W are vector spaces of possibly infinite dimension over a field K. Show that if a linear map $L\colon V \to W$ is surjective, then its dual $L^*\colon W^* \to V^*$ is injective.
(b) Suppose V and W are finite-dimensional vector spaces of a field K. Prove the converse of the implication in (a).

11

The Rank of a Smooth Map

In this chapter we analyze the local structure of a smooth map on the basis of its rank. Recall that the rank of a smooth map $f: N \to M$ at a point $p \in N$ is the rank of its differential at p. Two cases are of special interest: when the map f has maximal rank at a point or constant rank in a neighborhood. Let $n = \dim N$ and $m = \dim M$. In case $f: N \to M$ has maximal rank at p, there are three not mutually exclusive possibilities:

(i) If $n = m$, then by the inverse function theorem, f is a local diffeomorphism at p.

(ii) If $n \le m$, then the maximal rank is n and f is an *immersion* at p.

(iii) If $n \ge m$, then the maximal rank is m and f is a *submersion* at p.

Because manifolds are locally Euclidean, theorems on the rank of a smooth map between Euclidean spaces (Appendix B) translate easily to theorems about manifolds. This leads to the constant rank theorem for manifolds, which gives a simple normal form for a smooth map having constant rank on an open set (Theorem 11.1). As an immediate consequence, we obtain a criterion for a level set to be a regular submanifold which, following [11], we call the constant-rank level set theorem. As we explain in Section 11.2, maximal rank at a point implies constant rank in a neighborhood, so immersions and submersions are maps of constant rank. The constant rank theorem specializes to the immersion theorem and the submersion theorem, giving simple normal forms for an immersion and a submersion. The regular level set theorem, which we encountered in Section 9.3, is now seen to be a special case of the constant-rank level set theorem.

By the regular level set theorem the *preimage* of a regular value of a smooth map is a manifold. The *image* of a smooth map, on the other hand, does not generally have a nice structure. Using the immersion theorem we derive conditions under which the image of a smooth map is a manifold.

11.1 Constant Rank Theorem

Suppose $f : N \to M$ is a C^∞ map of manifolds and we want to show that the level set $f^{-1}(c)$ is a manifold for some c in M. In order to apply the regular level set theorem, we need the differential f_* to have maximal rank at every point of $f^{-1}(c)$. Sometimes this is not true; even if true, it may be difficult to show. In such cases, the constant-rank level set theorem may be helpful. It has one cardinal virtue: it is not necessary to know precisely the rank of f; it suffices that the rank be constant.

Because manifolds are locally Euclidean, the constant rank theorem for Euclidean spaces (Theorem B.6) has an immediate analogue for manifolds.

Theorem 11.1 (Constant rank theorem). *Let N and M be manifolds of respective dimensions n and m. Suppose $f\colon N \to M$ has constant rank k in a neighborhood of a point p in N. Then there are charts (U, ϕ) centered at p in N and (V, ψ) centered at $f(p)$ in M such that in a neighborhood of $\phi(p)$,*

$$\psi \circ f \circ \phi^{-1}(r^1, \dots r^n) = (r^1, \dots, r^k, 0, \dots, 0).$$

Proof. Choose a chart $(\bar{U}, \bar{\phi})$ about p in N and $(\bar{V}, \bar{\psi})$ about $f(p)$ in M. Then $\bar{\psi} \circ f \circ \bar{\phi}^{-1}$ is a map between open subsets of Euclidean spaces. Because $\bar{\phi}$ and $\bar{\psi}$ are diffeomorphisms, $\bar{\psi} \circ f \circ \bar{\phi}^{-1}$ has the same constant rank k as f in a neighborhood of $\bar{\phi}(p)$ in $\mathbb{R}^n$. By the constant rank theorem for Euclidean spaces (Theorem B.6) there are a diffeomorphism G of a neighborhood of $\phi(p)$ in $\mathbb{R}^n$ and a diffeomorphism F of a neighborhood of $(\bar{\psi} \circ f)(p)$ in $\mathbb{R}^m$ such that

$$F \circ \bar{\psi} \circ f \circ \bar{\phi}^{-1} \circ G^{-1}(r^1, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0).$$

Let $\phi = G \circ \bar{\phi}$ and $\psi = F \circ \bar{\psi}$. □

From this the constant-rank level set theorem follows easily.

Theorem 11.2 (Constant-rank level set theorem). *Let $f : N \to M$ be a C^∞ map of manifolds and $c \in M$. If f has constant rank k in a neighborhood of the level set $f^{-1}(c)$ in N, then $f^{-1}(c)$ is a regular submanifold of N of codimension k.*

Proof. Let p be an arbitrary point in $f^{-1}(c)$. By the constant rank theorem there are a coordinate chart $(U, \phi) = (U, x^1, \dots, x^n)$ centered at $p \in N$ and a coordinate chart $(V, \psi) = (V, y^1, \dots, y^m)$ centered at $f(p) = c \in M$ such that

$$\psi \circ f \circ \phi^{-1}(r^1, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0) \in \mathbb{R}^m.$$

This shows that the level set $(\psi \circ f \circ \phi^{-1})^{-1}(0)$ is defined by the vanishing of the coordinates $r^1, \dots, r^k$.

The image of the level $f^{-1}(c)$ under ϕ is the level set $(\psi \circ f \circ \phi^{-1})^{-1}(0)$ (Figure 11.1), since

$$\phi(f^{-1}(c)) = \phi(f^{-1}(\psi^{-1}(0)) = (\psi \circ f \circ \phi^{-1})^{-1}(0).$$

Thus, the level set $f^{-1}(c)$ in U is defined by the vanishing of the coordinate functions $x^1, \dots, x^k$. This proves that $f^{-1}(c)$ is a regular submanifold of N of codimension k. □

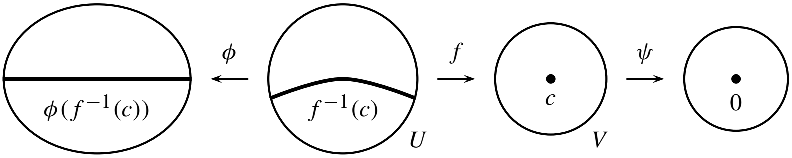


Fig. 11.1. Constant-rank level set.

Example 11.3 (*Orthogonal group*). The *orthogonal group* $O(n)$ is defined to be the subgroup of $GL(n, \mathbb{R})$ consisting of matrices A such that $A^T A = I$, the $n \times n$ identity matrix. Using the constant rank theorem, prove that $O(n)$ is a regular submanifold of $GL(n, \mathbb{R})$.

Solution. Define $f\colon \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ by $f(A) = A^T A$. Then $O(n)$ is the level set $f^{-1}(I)$. If A and B are two elements of $\mathrm{GL}(n, \mathbb{R})$, then $B = AC$ for some matrix C in $\mathrm{GL}(n, \mathbb{R})$. Denote by ℓ_C and $r_C : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ the left and right multiplication by C, respectively. Since

$$f(AC) = (AC)^T AC = C^T A^T AC = C^T f(A)C,$$

we have

$$f \circ r_C(A) = \ell_{C^T} \circ r_C \circ f(A).$$

Hence,

$$f \circ r_C = \ell_{C_T} \circ r_C \circ f.$$

By the chain rule,

$$f_{*,AC} \circ (r_C)_{*,A} = (\ell_{C^T})_{*,A^T AC} \circ (r_C)_{*,A^T A} \circ f_{*,A}. \tag{11.1}$$

Since left and right multiplications are diffeomorphisms, their differentials are isomorphisms. Composition with an isomorphism does not change the rank of a linear map. Hence, in (11.1),

$$\mathrm{rk}\, f_{*,AC} = \mathrm{rk}\, f_{*,A}.$$

As AC and A are two arbitrary points of $\mathrm{GL}(n, \mathbb{R})$, this proves that the differential of f has constant rank on $\mathrm{GL}(n, \mathbb{R})$. By the constant-rank level set theorem, the orthogonal group $O(n) = f^{-1}(I)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$. □

11.2 Immersions and Submersions

Definition 11.4. A smooth map $f\colon N \to M$ of manifolds is an *immersion* if the differential $f_{*,p}\colon T_pN \to T_{f(p)}M$ is injective for every p in N. It is a *submersion* if $f_{*,p}$ is surjective for every p in N.

We can also speak of a smooth map $f\colon N \to M$ being an *immersion* or a *submersion at a single point* p in N; this would mean that its differential $f_{*,p}$ at p is injective or surjective, respectively. Note that $f\colon N \to M$ is a submersion at p if and only if p is a regular point of f.

Example 11.5. The prototype of an immersion is the inclusion of $\mathbb{R}^n$ in a higher-dimensional $\mathbb{R}^m$:

$$i(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0).$$

The prototype of a submersion is the projection of $\mathbb{R}^n$ onto a lower-dimensional $\mathbb{R}^m$:

$$\pi(x^1, \ldots, x^m, x^{m+1}, \ldots, x^n) = (x^1, \ldots, x^m).$$

According to Theorem 11.8 below, every immersion is locally an inclusion and every submersion is locally a projection.

Example 11.6. If U is an open subset of a manifold M, then the inclusion $i : U \to M$ is both an immersion and a submersion. This example shows in particular that a submersion need not be onto.

Consider a C^∞ map $f\colon N \to M$. Let $(U, x^1, \ldots, x^n)$ be a chart about p in N and $(V, y^1, \ldots, y^m)$ a chart about $f(p)$ in M. Write $f^i = y^i \circ f$ for the ith component of f in the chart $(V, y^1, \ldots, y^m)$. Then the linear map $f_{*,p}$ is represented by the matrix $[\partial f^i/\partial x^j(p)]$ (Proposition 8.12). Hence,

$$\begin{aligned} f_{*,p} \text{ is injective} \quad &\Leftrightarrow \quad n \le m \ \text{ and } \ \operatorname{rk}[\partial f^i/\partial x^j(p)] = n, \\ f_{*,p} \text{ is surjective} \quad &\Leftrightarrow \quad n \ge m \ \text{ and } \ \operatorname{rk}[\partial f^i/\partial x^j(p)] = m. \end{aligned} \tag{11.2}$$

The rank of a matrix is the number of linearly independent rows of the matrix; it is also the number of linearly independent columns. Thus, for an m by n matrix the maximum possible rank of a matrix is the minimum of m and n. It follows from (11.2) that being an immersion or a submersion at p is equivalent to the maximality of $\operatorname{rk}[\partial f^i/\partial x^j(p)]$.

Having maximal rank at a point is an *open condition* in the sense that the set

$$D_{\max}(f) = \{p \in U \mid f_{*,p} \text{ has maximal rank at } p\}$$

is an open subset of U. This is because if k is the maximal rank, then

$$\begin{aligned} \operatorname{rk} f_{*,p} = k \quad &\text{iff} \quad \operatorname{rk}[\partial f^i/\partial x^j\,(p)] = k \\ &\text{iff} \quad \operatorname{rk}[\partial f^i/\partial x^j\,(p)] \ge k \quad \text{(since } k \text{ is maximal).} \end{aligned}$$

So the complement $U - D_{\max}(f)$ is defined by

$$\operatorname{rk}[\partial f^i/\partial x^j\,(p)] < k,$$

which is equivalent to the vanishing of all $k \times k$ minors of the matrix $[\partial f^i/\partial x^j\,(p)]$. As the zero set of finitely many continuous functions, $U - D_{\max}(f)$ is closed and so $D_{\max}(f)$ is open. In particular, if f has maximal rank at p, then it has maximal rank at all points in some neighborhood of p.

Proposition 11.7. *Let N and M be manifolds of respective dimensions n and m. If a C^∞ map $f: N \to M$ is an immersion at a point $p \in N$, then it has constant rank n in a neighborhood of p. If a C^∞ map $f: N \to M$ is a submersion at a point $p \in N$, then it has constant rank m in a neighborhood of p.*

The following theorems are therefore simply special cases of the constant rank theorem.

Theorem 11.8. *Let N and M be manifolds of respective dimensions n and m.*

(i) (Immersion theorem) *Suppose $f: N \to M$ is an immersion at $p \in N$. Then there are charts (U, ϕ) centered at p in N and (V, ψ) centered at $f(p)$ in M such that in a neighborhood of $\phi(p)$,*

$$\psi \circ f \circ \phi^{-1}(r^1, \dots, r^n) = (r^1, \dots, r^n, 0, \dots, 0).$$

(ii) (Submersion theorem) *Suppose $f: N \to M$ is a submersion at p in N. Then there are charts (U, ϕ) centered at p in N and (V, ψ) centered at $f(p)$ in M such that in a neighborhood of $\phi(p)$:*

$$\psi \circ f \circ \phi^{-1}(r^1, \dots, r^m, r^{m+1}, \dots, r^n) = (r^1, \dots, r^m).$$

Corollary 11.9. *A submersion $f: N \to M$ of manifolds is an open map.*

Proof. Let W be an open subset of N. We need to show that its image $f(W)$ is open in M. Choose a point $f(p)$ in $f(W)$, with $p \in W$. By the submersion theorem, f is locally a projection. Since a projection is an open map (Problem A.4), there is an open neighborhood U of p in W such that $f(U)$ is open in M. Clearly,

$$f(p) \in f(U) \subset f(W).$$

Hence, $f(W)$ is open in M. □

There is a close connection between submersions and regular level sets. Indeed, for a C^∞ map $f: N \to M$ of manifolds, a level set $f^{-1}(c)$ is regular if and only if f is a submersion at every point of $f^{-1}(c)$. Since the maximality of the rank of f is an open condition, a regular level set $f^{-1}(c)$ has a neighborhood on which f has constant rank m. This shows that the regular level theorem (Theorem 9.13) is in fact a special case of the constant-rank level set theorem (Theorem 11.2).

11.3 Images of Smooth Maps

The following are all examples of C^∞ maps $f: N \to M$, with $N = \mathbb{R}$ and $M = \mathbb{R}^2$.

Example 11.10. $f(t) = (t^2, t^3)$.

This map f is one-to-one, because $t \mapsto t^3$ is one-to-one. Since $f'(0) = (0, 0)$, the differential $f_{*,0}\colon T_0\mathbb{R} \to T_{(0,0)}\mathbb{R}^2$ is the zero map and hence not injective; so f is not an immersion at 0. Its image is the cuspidal cubic $y^2 = x^3$ (Figure 11.2).

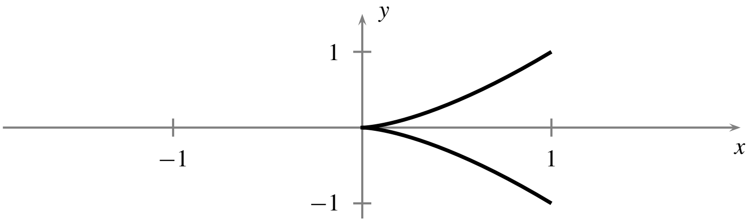


Fig. 11.2. Cuspidal cubic, not an immersion.

Example 11.11. $f(t) = (t^2 - 1, t^3 - t)$.

Since the equation $f'(t) = (2t, 3t^2 - 1) = (0, 0)$ has no solution in t, this map f is an immersion. It is not one-to-one, because it maps both $t = 1$ and $t = -1$ to the origin. To find an equation for the image $f(N)$, let $x = t^2 - 1$ and $y = t^3 - t$. Then $y = t(t^2 - 1) = tx$; so

$$y^2 = t^2x^2 = (x + 1)x^2.$$

Thus the image $f(N)$ is the nodal cubic $y^2 = x^2(x + 1)$ (Figure 11.3).

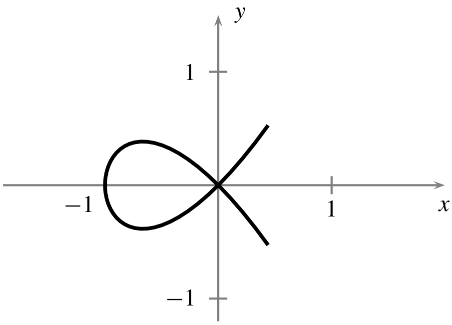


Fig. 11.3. Nodal cubic, an immersion but not one-to-one.

Example 11.12. The map f in Figure 11.4 is a one-to-one immersion but its image, with the subspace topology of $\mathbb{R}^2$, is not homeomorphic to the domain $\mathbb{R}$, because there are points near $f(p)$ in the image that correspond to points in $\mathbb{R}$ far away from p. More precisely, if U is an interval about p as shown, there is no neighborhood V of $f(p)$ in $f(N)$ such that $f^{-1}(V) \subset U$; hence, f^{-1} is not continuous.

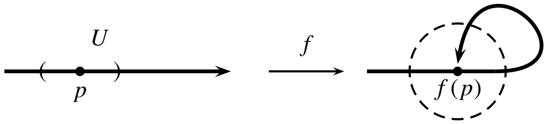


Fig. 11.4. A one-to-one immersion that is not an embedding.

Example 11.13. The manifold M in Figure 11.5 is the union of the graph of $y = \sin(1/x)$ on the interval $(0, 1)$, the open line segment from $y = 0$ to $y = 1$ on the y-axis, and a smooth curve joining $(0, 0)$ and $(1, \sin 1)$. The map f is a one-to-one immersion whose image with the subspace topology is not homeomorphic to $\mathbb{R}$.

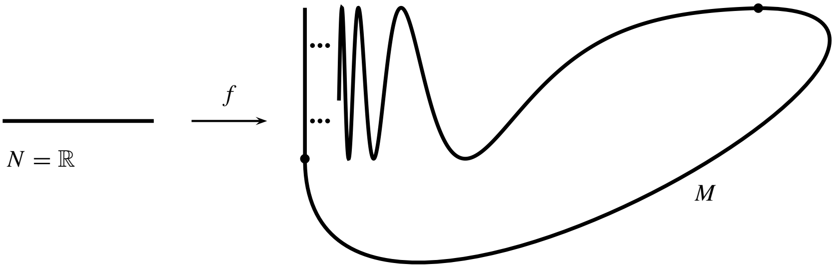


Fig. 11.5. A one-to-one immersion that is not an embedding.

Notice that in these examples the image $f(N)$ is not a regular submanifold of $M = \mathbb{R}^2$. We would like conditions on the map f so that its image $f(N)$ would be a regular submanifold of M.

Definition 11.14. A C^∞ map $f : N \to M$ is called an *embedding* if

(i) it is a one-to-one immersion and
(ii) the image $f(N)$ with the subspace topology is homeomorphic to N under f. (The phrase "one-to-one" in this definition is redundant since a homeomorphism is necessarily one-to-one.)

Remark 11.15. Unfortunately, there is quite a bit of confusion about terminology in the literature concerning the use of the word "submanifold." Many authors give the image $f(N)$ of a one-to-one immersion $f : N \to M$ not the subspace topology, but the topology inherited from f, i.e., a subset $f(U)$ of $f(N)$ is said to be open if and only if U is open in N. With this topology, $f(N)$ is by definition homeomorphic to N. These authors define a submanifold to be the image of any one-to-one immersion with the topology and differentiable structure inherited from f. Such a set is sometimes called an *immersed submanifold* of M. Figures 11.4 and 11.5 show two examples of immersed submanifolds. If the underlying set of an immersed submanifold is given the subspace topology, then the resulting space need not be a manifold at all!

For us, a submanifold without any qualifying adjective is always a *regular submanifold*. To recapitulate, a regular submanifold of a manifold M is a subset S of M with the subspace topology such that every point of S has a neighborhood $U \cap S$ that is defined by the vanishing of coordinate functions on U, where U is a chart in M.

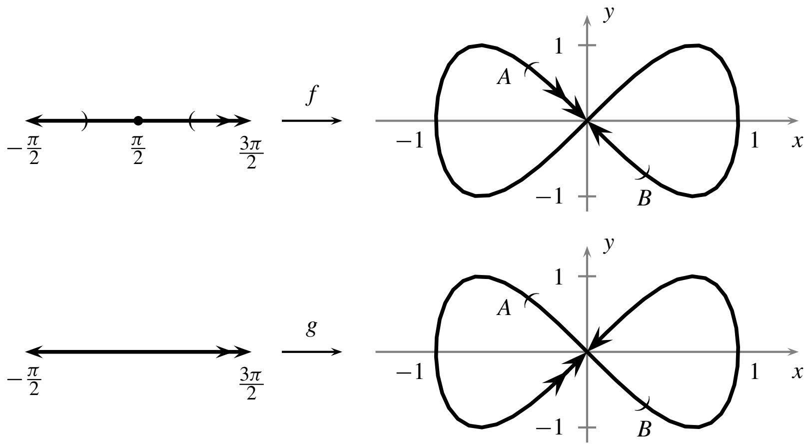


Fig. 11.6. The figure-eight as two distinct immersed submanifolds of $\mathbb{R}^2$.

Example 11.16 (*The figure-eight*). The figure-eight is the image of a one-to-one immersion

$$f(t) = (\cos t, \sin 2t), \quad -\pi/2 < t < 3\pi/2$$

(Figure 11.6). As such, it is an immersed submanifold of $\mathbb{R}^2$, with a topology and manifold structure induced from the open interval $(-\pi/2, 3\pi/2)$ by f. Because of the presence of a cross at the origin, it cannot be a regular submanifold of $\mathbb{R}^2$. In fact, with the subspace topology of $\mathbb{R}^2$, the figure-eight is not even a manifold.

The figure-eight is also the image of the one-to-one immersion

$$g(t) = (\cos t, -\sin 2t), \quad -\pi/2 < t < 3\pi/2$$

(Figure 11.6). The maps f and g induce distinct immersed submanifold structures on the figure-eight. For example, the open interval from A to B in Figure 11.6 is an open set in the topology induced from g, but it is not an open set in the topology induced from f.

Theorem 11.17. *If $f\colon N \to M$ is an embedding, then its image $f(N)$ is a regular submanifold of M.*

Proof. Let $p \in N$. By the immersion theorem (Theorem 11.8), there are local coordinates $(U, x^1, \ldots, x^n)$ near p and $(V, y^1, \ldots, y^m)$ near $f(p)$ so that $f\colon U \to V$ has the form

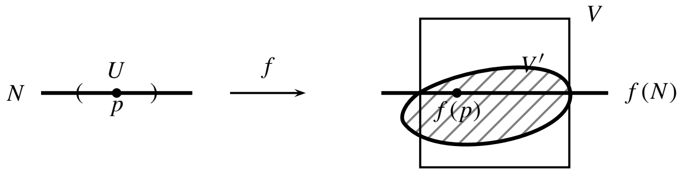


Fig. 11.7. The image of an embedding is a regular submanifold.

$$(x^1, \ldots, x^n) \to (x^1, \ldots, x^n, 0, \ldots, 0).$$

Thus, $f(U)$ is defined in V by the vanishing of the coordinates $y^{n+1}, \ldots, y^m$. This alone does not prove that $f(N)$ is a regular submanifold, since $V \cap f(N)$ may be larger than $f(U)$. (Think about Examples 11.12 and 11.13.) We need to show that in some neighborhood of $f(p)$ in V, the set $f(N)$ is defined by the vanishing of $m - n$ coordinates.

Since $f(N)$ with the subspace topology is homeomorphic to N, the image $f(U)$ is open in $f(N)$. By the definition of the subspace topology, there is an open set V' in M such that $V' \cap f(N) = f(U)$ (Figure 11.7). In $V \cap V'$,

$$V \cap V' \cap f(N) = V \cap f(U) = f(U),$$

and $f(U)$ is defined by the vanishing of $y^{n+1}, \ldots, y^m$. Thus, $(V \cap V', y^1, \ldots, y^m)$ is an adapted chart containing $f(p)$ for $f(N)$. Since $f(p)$ is an arbitrary point of $f(N)$, this proves that $f(N)$ is a regular submanifold of M. □

Theorem 11.18. *If N is a regular submanifold of M, then the inclusion $i\colon N \to M$, $i(p) = p$, is an embedding.*

Proof. Since a regular submanifold has the subspace topology and $i(N)$ also has the subspace topology, $i\colon N \to i(N)$ is a homeomorphism. It remains to show that $i\colon N \to M$ is an immersion.

Let $p \in N$. Choose an adapted chart $(V, y^1, \ldots, y^n, y^{n+1}, \ldots, y^m)$ for M about p such that $V \cap N$ is the zero set of $y^{n+1}, \ldots, y^m$. Relative to the charts $(V \cap N, y^1, \ldots, y^n)$ for N and $(V, y^1, \ldots, y^m)$ for M, the inclusion i is given by

$$(y^1, \ldots, y^n) \mapsto (y^1, \ldots, y^n, 0, \ldots, 0),$$

which shows that i is an immersion. □

In the literature the image of an embedding is often called an *embedded submanifold*. Theorems 11.17 and 11.18 show that an embedded submanifold and a regular submanifold are one and the same thing.

11.4 Smooth Maps into a Submanifold

Suppose $f\colon N \to M$ is a C^∞ map whose image $f(N)$ lies in a subset $S \subset M$. If S is a manifold, is the induced map $\tilde{f}\colon N \to S$ also C^∞? This question is more subtle than it looks, because the answer depends on whether S is a regular submanifold or an immersed submanifold of M.

Example 11.19. Consider the one-to-one immersions f and $g\colon I \to R^2$ in Example 11.16, where I is the open interval $(-\pi/2, 3\pi/2)$ in $\mathbb{R}$. Let S be the figure-eight in $\mathbb{R}^2$ with the immersed submanifold structure induced from g. Because the image of $f\colon I \to \mathbb{R}^2$ lies in S, the C^∞ map f induces a map $\tilde{f}\colon I \to S$.

The open interval from A to B in Figure 11.6 is an open neighborhood of 0 in S. Its inverse image under $\tilde{f}$ contains 0 as an isolated point and is therefore not open. This shows that although $f\colon I \to \mathbb{R}^2$ is C^∞, the induced map $\tilde{f}\colon I \to S$ is not continuous and therefore not C^∞.

Theorem 11.20. *Suppose $F\colon N \to M$ is C^∞ and the image of F lies in a subset S of M. If S is a regular submanifold of M, then the induced map $\tilde{F}\colon N \to S$ is C^∞.*

Proof. Let $p \in N$. Denote the dimensions of N, M, and S by n, m, and s, respectively. By hypothesis, $F(p) \in S \subset M$. Since S is a regular submanifold of M, there is an adapted coordinate chart $(V, \psi) = (V, y^1, \dots, y^m)$ for M about $F(p)$ such that $S \cap V$ is the zero set of $y^{s+1}, \dots, y^m$, with coordinate map $\psi_S = (y^1, \dots, y^s)$. By the continuity of F, it is possible to choose a coordinate chart $(U, \phi) = (U, x^1, \dots, x^n)$ about p such that $F(U) \subset V$. Then $F(U) \subset V \cap S$ so that on $\phi(U)$,

$$\psi \circ F \circ \phi^{-1}(x^1, \dots, x^n) = (y^1, \dots, y^s, 0, \dots, 0)$$

and

$$\psi_S \circ \tilde{F} \circ \phi^{-1}(x^1, \dots, x^n) = (y^1, \dots, y^s),$$

which shows that $\tilde{F}$ is C^∞ on U. □

Example 11.21. The multiplication map

$$\begin{aligned} \mu\colon \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}) &\to \mathrm{GL}(n, \mathbb{R}) \\ (A, B) &\mapsto AB \end{aligned}$$

is clearly C^∞ because

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

is a polynomial and hence a C^∞ function of the coordinates a_{ik} and b_{kj}. However, one cannot conclude in the same way that the multiplication map

$$\bar{\mu}\colon \mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$$

is C^∞. This is because $\{a_{ij}\}_{1 \le i, j \le n}$ is not a coordinate system on $\mathrm{SL}(n, \mathbb{R})$; there is one coordinate too many.

Since $\mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R})$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$, the inclusion map

$$i\colon\ \mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R})\to\mathrm{GL}(n,\mathbb{R})\times\mathrm{GL}(n,\mathbb{R})$$

is C^∞ by Theorem 11.18; therefore, the composition

$$\mu\circ i\colon\ \mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R})\to\mathrm{GL}(n,\mathbb{R})$$

is also C^∞. Because the image of $\mu\circ i$ lies in $\mathrm{SL}(n,\mathbb{R})$, and $\mathrm{SL}(n,\mathbb{R})$ is a regular submanifold of $\mathrm{GL}(n,\mathbb{R})$ (see Example 9.16), by Theorem 11.20 the induced map

$$\bar\mu\colon\ \mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R})\to\mathrm{SL}(n,\mathbb{R})$$

is C^∞.

11.5 The Tangent Plane to a Surface in $\mathbb{R}^3$

Suppose $f(x^1,x^2,x^3)$ is a real-valued function on $\mathbb{R}^3$ with no critical points on its zero set N. By the regular level set theorem, N is a regular submanifold of $\mathbb{R}^3$. By Theorem 11.18 the inclusion $i\colon N\to\mathbb{R}^3$ is an embedding, so at any point p in N, $i_{*,p}\colon T_pN\to T_p\mathbb{R}^3$ is injective. We may therefore think of the tangent plane T_pN as a plane in $T_p\mathbb{R}^3\simeq\mathbb{R}^3$ (Figure 11.8). We would like to find the equation of this plane.

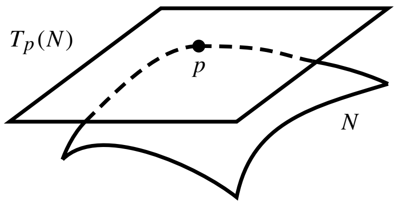


Fig. 11.8. Tangent plane to a surface N at p.

Suppose $v=\sum v^i\,\partial/\partial x^i$ is a vector in T_pN. Under the isomorphism $T_p\mathbb{R}^3\simeq\mathbb{R}^3$, we identify v with the vector $\langle v^1,v^2,v^3\rangle$ in $\mathbb{R}^3$. Let $c(t)$ be a curve lying in N with $c(0)=p$ and $c'(0)=\langle v^1,v^2,v^3\rangle$. Since $c(t)$ lies in N, $f(c(t))=0$ for all t. By the chain rule,

$$0=\frac{d}{dt}f(c(t))=\sum_{i=1}^{3}\frac{\partial f}{\partial x^i}(c(t))(c^i)'(t).$$

At $t=0$,

$$0=\sum_{i=1}^{3}\frac{\partial f}{\partial x^i}(c(0))(c^i)'(0)=\sum_{i=1}^{3}\frac{\partial f}{\partial x^i}(p)v^i.$$

One usually translates the tangent plane from the origin to p by making the substitution $v^i=x^i-p^i$. Then the tangent plane to N at p is defined by the equation

$$\sum_{i=1}^{3} \frac{\partial f}{\partial x^i}(p)(x^i - p^i) = 0. \tag{11.3}$$

Example 11.22 (*Tangent plane to a sphere*). Let $f(x, y, z) = x^2 + y^2 + z^2 - 1$. To get the equation of the tangent plane to the unit sphere S^2 in $\mathbb{R}^3$ at $(a, b, c) \in S^2$, we compute

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = 2z.$$

At $p = (a, b, c)$,

$$\frac{\partial f}{\partial x}(p) = 2a, \quad \frac{\partial f}{\partial y}(p) = 2b, \quad \frac{\partial f}{\partial z}(p) = 2c.$$

By (11.3) the equation of the tangent plane to the sphere at (a, b, c) is

$$2a(x - a) + 2b(y - b) + 2c(z - c) = 0,$$

or

$$ax + by + cz = 1,$$

since $a^2 + b^2 + c^2 = 1$.

Problems

11.1. Tangent vector to a sphere

The unit sphere S^n in $\mathbb{R}^{n+1}$ is defined by the equation $\sum_{i=1}^{n+1}(x^i)^2 = 1$. For $p \in S^n$, show that a necessary and sufficient condition for $X_p = \sum a^i \partial/\partial x^i|_p$ to be tangent to S^n at p is $\sum a^i p^i = 0$.

11.2.* Critical points of a smooth map on a compact manifold

Show that a smooth map f from a compact manifold N to $\mathbb{R}^m$ has a critical point. (*Hint*: Use Corollary 11.9 and the connectedness of $\mathbb{R}^m$.)

11.3. Differential of an inclusion map

On the upper hemisphere of the unit sphere S^2, we have the coordinate map $\phi = (u, v)$, where

$$u(a, b, c) = a \quad \text{and} \quad v(a, b, c) = b.$$

So the derivations $\partial/\partial u|_p, \partial/\partial v|_p$ are tangent vectors of S^2 at any point $p = (a, b, c)$ on the upper hemisphere. Let $i\colon S^2 \to \mathbb{R}^3$ be the inclusion and x, y, z the standard coordinates on $\mathbb{R}^3$. The differential $i_*\colon T_pS^2 \to T_p\mathbb{R}^3$ maps $\partial/\partial u|_p, \partial/\partial v|_p$ into $T_p\mathbb{R}^3$. Thus,

$$i_*\left(\left.\frac{\partial}{\partial u}\right|_p\right) = \alpha^1 \left.\frac{\partial}{\partial x}\right|_p + \beta^1 \left.\frac{\partial}{\partial y}\right|_p + \gamma^1 \left.\frac{\partial}{\partial z}\right|_p,$$

$$i_*\left(\left.\frac{\partial}{\partial v}\right|_p\right) = \alpha^2 \left.\frac{\partial}{\partial x}\right|_p + \beta^2 \left.\frac{\partial}{\partial y}\right|_p + \gamma^2 \left.\frac{\partial}{\partial z}\right|_p,$$

for some constants $\alpha^i, \beta^i, \gamma^i$. Find $(\alpha^i, \beta^i, \gamma^i)$ for $i = 1, 2$.

11.4. One-to-one immersion of a compact manifold
Let $f: N \to M$ be a one-to-one immersion. Prove that if N is compact, then $f(N)$ is a regular submanifold of M.

11.5. Multiplication map in SL(n, $\mathbb{R}$)
Let $f: \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$ be the determinant map $f(A) = \det A = \det[a_{ij}]$. For $A \in \mathrm{SL}(n, \mathbb{R})$, at least one partial derivative $\partial f/\partial a_{k\ell}(A) \neq 0$ for some (k, ℓ) (Example 9.16). Use Lemma 9.12 and the implicit function theorem to prove that

(a) there is a neighborhood of A in $\mathrm{SL}(n, \mathbb{R})$ in which a_{ij}, $(i, j) \neq (k, \ell)$, form a coordinate system, and $a_{k\ell}$ is a C^∞ function of the other entries a_{ij}, $(i, j) \neq (k, \ell)$;
(b) the multiplication map

$$\bar{\mu}: \mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$$

is C^∞.

12

The Tangent Bundle

Let M be a manifold. For each point $p \in M$, the *tangent space* T_pM is the vector space of all point-derivations of $C_p^\infty(M)$, the algebra of germs of C^∞ functions at p. The *tangent bundle* of M is the disjoint union of all the tangent spaces of M:

$$TM = \coprod_{p \in M} T_pM = \bigcup_{p \in M} \{p\} \times T_pM.$$

To form the disjoint union here, we attach a label p to each element of T_pM. So defined, TM is simply a set, with no topology or manifold structure. We will make it into a smooth manifold and show that it is a C^∞ vector bundle over M. The first step is to give it a topology.

12.1 The Topology of the Tangent Bundle

Let $(U, \phi) = (U, x^1, \dots, x^n)$ be a coordinate chart for M. At a point $p \in U$, a basis for T_pM is $(\partial/\partial x^1)|_p, \dots, (\partial/\partial x^n)|_p$, so a tangent vector $X_p \in T_pM$ is uniquely a linear combination

$$X_p = \sum_i^n a^i \left.\frac{\partial}{\partial x^i}\right|_p,$$

where $a^i = a^i(X_p) \in \mathbb{R}$ depends on X_p. Since $\phi_*(X_p) = \sum a^i\, \partial/\partial r^i|_{\phi(p)} \in T_{\phi(p)}(\mathbb{R}^n)$, we may identify $\phi_*(X_p)$ with the column vector $\langle a^1, \dots, a^n \rangle$ in $\mathbb{R}^n$. Let

$$TU = \coprod_{p \in U} T_pU = \coprod_{p \in U} T_pM.$$

(We saw in Remark 8.2 that $T_pU = T_pM$.) If we define

$$\begin{aligned} \tilde{\phi} = (\phi, \phi_*)\colon TU &\to \phi(U) \times \mathbb{R}^n \\ (p, X_p) &\mapsto (x^1(p), \dots, x^n(p), a^1(X_p), \dots, a^n(X_p)), \end{aligned} \tag{12.1}$$

then $\tilde{\phi}$ is a bijection, with inverse

$$(\phi(p), a^1, \ldots, a^n) \mapsto \left(p, \sum a^i \left. \frac{\partial}{\partial x^i} \right|_p \right).$$

We can therefore use $\tilde{\phi}$ to transfer the topology of $\phi(U) \times \mathbb{R}^n$ to TU: a set A in TU is open if and only if $\tilde{\phi}(A)$ is open in $\phi(U) \times \mathbb{R}^n$.

Let $\mathcal{B}$ be the collection of all open subsets of $T(U_\alpha)$ as U_α runs over all coordinate open sets in M.

Lemma 12.1. *Let U and V be coordinate open sets in M. If A is open in TU and B is open in TV, then $A \cap B$ is open in $T(U \cap V)$.*

Proof. Since $T(U \cap V)$ is a subspace of TU, $A \cap T(U \cap V)$ is open in $T(U \cap V)$. Similarly, $B \cap T(U \cap V)$ is open in $T(U \cap V)$. But

$$A \cap B \subset TU \cap TV = T(U \cap V).$$

Hence,

$$A \cap B = (A \cap T(U \cap V)) \cap (B \cap T(U \cap V))$$

is open in $T(U \cap V)$. □

It follows from this lemma that the collection $\mathcal{B}$ satisfies the conditions (i) and (ii) of Proposition A.14 for a collection of subsets to be a basis for some topology on TM. We give the tangent bundle TM the topology generated by the basis $\mathcal{B}$.

Lemma 12.2. *A manifold M has a countable basis consisting of coordinate open sets.*

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be the maximal atlas on M and $\mathcal{B} = \{B_i\}$ a countable basis for M. For each coordinate open set U_α and a point $p \in U_\alpha$, choose a basic open set $B_{p,\alpha} \in \mathcal{B}$ such that

$$p \in B_{p,\alpha} \subset U_\alpha.$$

The collection $\{B_{p,\alpha}\}$, without duplicate elements, is a subcollection of $\mathcal{B}$ and is therefore countable.

Given any open set U in M and a point $p \in U$, there is a coordinate open set U_α such that

$$p \in U_\alpha \subset U.$$

Hence,

$$p \in B_{p,\alpha} \subset U,$$

which shows that $\{B_{p,\alpha}\}$ is a basis for M. □

Proposition 12.3. *The tangent bundle TM of a manifold M is second countable.*

Proof. Let $\{U_i\}_{i=1}^\infty$ be a countable basis of M consisting of coordinate open sets. Since $TU_i \simeq U_i \times \mathbb{R}^n$, it is diffeomorphic to an open subset of $\mathbb{R}^{2n}$ and is therefore second countable. For each i, choose a countable basis $\{B_{i,j}\}_{j=1}^\infty$ for TU_i. Then $\{B_{i,j}\}_{i,j=1}^\infty$ is a countable basis for the tangent bundle. □

Proposition 12.4. *The tangent bundle TM of a manifold M is Hausdorff.*

Proof. Problem 12.1. □

12.2 The Manifold Structure on the Tangent Bundle

Next we show that if $\{(U_\alpha, \phi_\alpha)\}$ is a C^∞ atlas for M, then $\{(TU_\alpha, \tilde{\phi}_\alpha)\}$ is a C^∞ atlas for TM. It is clear that $TM = \cup_\alpha TU_\alpha$. It remains to check that on $(TU_\alpha) \cap (TU_\beta)$, $\tilde{\phi}_\alpha$ and $\tilde{\phi}_\beta$ are C^∞ compatible.

Recall that if $(U, x^1, \dots, x^n)$, $(V, y^1, \dots, y^n)$ are two charts on M, then for any $p \in U \cap V$ there are two bases singled out for the tangent space $T_p(M)$: $\{\partial/\partial x^j|_p\}_{j=1}^n$ and $\{\partial/\partial y^i|_p\}_{i=1}^n$. So any tangent vector $X_p \in T_p(M)$ has two descriptions:

$$X_p = \sum_j a^j \left.\frac{\partial}{\partial x^j}\right|_p = \sum_i b^i \left.\frac{\partial}{\partial y^i}\right|_p. \tag{12.2}$$

It is easy to compare them. By applying both sides to x^k, we find that

$$a^k = \left(\sum_j a^j \frac{\partial}{\partial x^j}\right) x^k = \left(\sum_i b^i \frac{\partial}{\partial y^i}\right) x^k = \sum_i b^i \frac{\partial x^k}{\partial y^i}.$$

Similarly, applying both sides of (12.2) to y^k gives

$$b^k = \sum_j a^j \frac{\partial y^k}{\partial x^j}.$$

Write $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Then

$$\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1} \colon \phi_\alpha(U_{\alpha\beta}) \times \mathbb{R}^n \to \phi_\beta(U_{\alpha\beta}) \times \mathbb{R}^n$$

is given by

$$(x, a^1, \dots, a^n) \mapsto \left(\phi_\alpha^{-1}(x), \sum_j a^j \frac{\partial}{\partial x^j}\right) \mapsto (\phi_\beta \circ \phi_\alpha^{-1}(x), b^1, \dots, b^n),$$

where

$$b^i = \sum_j a^j \frac{\partial y^i}{\partial x^j}.$$

By the definition of an atlas, $\phi_\beta \circ \phi_\alpha^{-1}(x)$ is C^∞; its components are simply the y^i's. So the y^i's are C^∞ functions of the x^j's. This implies that all the partial derivatives $\partial y^i/\partial x^j$ are C^∞ functions. Therefore, $\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1}$ is C^∞. This completes the proof that the tangent bundle TM is a C^∞ manifold, with $\{(TU_\alpha, \tilde{\phi}_\alpha)\}$ as a C^∞ atlas.

12.3 Vector Bundles

On the tangent bundle TM of a smooth manifold M, there is a natural projection map $\pi\colon TM \to M$, $\pi(p, X_p) = p$. This makes the tangent bundle into a C^∞ *vector bundle*, which we now define.

Given any map $\pi: E \to M$, we call the inverse image $\pi^{-1}(p) := \pi^{-1}(\{p\})$ of a point $p \in M$ the *fiber at* p. The fiber at p is often written E_p. A surjective smooth map $\pi: E \to M$ of manifolds is said to be *locally trivial of rank* r if

(i) each fiber $\pi^{-1}(p)$ has the structure of a vector space of dimension r;
(ii) for each $p \in M$, there are an open neighborhood U of p and a fiber-preserving diffeomorphism

$$\phi: \pi^{-1}(U) \to U \times \mathbb{R}^r$$

that maps each fiber $\pi^{-1}(q)$ to the corresponding fiber $\{q\} \times \mathbb{R}^r$ isomorphically as vector spaces, for all $q \in U$. Such an open set is called a *trivializing open set* for E.

The collection $\{(U, \phi)\}$, with $\{U\}$ an open cover of M, is called a *local trivialization* for E, and $\{U\}$ is called a *trivializing open cover* of M for E.

A C^∞ *vector bundle of rank* r is a triple (E, M, π) consisting of manifolds E, M, and a surjective smooth map $\pi: E \to M$ that is locally trivial of rank r. The manifold E is called the *total space* of the vector bundle and M the *base space*. By abuse of language, we say that E is a *vector bundle over* M.

Example 12.5 (*The product bundle*). Given a manifold M, let $\pi: M \times \mathbb{R}^r \to M$ be the projection to the first factor. Then $M \times \mathbb{R}^r$ is a vector bundle of rank r over M, called the *product bundle* of rank r. It has a local trivialization given by the identity map $1_{M\times\mathbb{R}}: M \times \mathbb{R} \to M \times \mathbb{R}$. The infinite cylinder $S^1 \times \mathbb{R}$ is the product bundle of rank 1 over the circle (Figure 12.1).

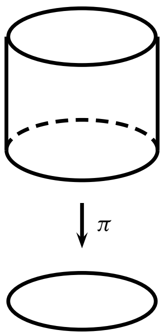


Fig. 12.1. A circular cylinder is a product bundle over a circle.

Let $\pi_E: E \to M$, $\pi_F: F \to N$ be two vector bundles, possibly of different ranks. A *bundle map* from E to F is a pair of maps $(f, \tilde{f})$, $f: M \to N$ and $\tilde{f}: E \to F$ such that

(i) the diagram

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

is commutative, meaning $\pi_F \circ \tilde{f} = f \circ \pi_E$;

(ii) $\tilde{f}$ is linear on each fiber, i.e., for each $p \in M$, $\tilde{f}\colon E_p \to F_{f(p)}$ is a linear map of vector spaces.

The collection of all vector bundles together with bundle maps between them forms a category. Thus, it makes sense to speak of an isomorphism of vector bundles. Any bundle isomorphic to a product bundle is called a *trivial bundle*.

A smooth map $f\colon N \to M$ of manifolds induces a bundle map $(f, \tilde{f})$, where $\tilde{f} : TN \to TM$ is given by

$$\tilde{f}(p, X_p) = (f(p), f_*(X_p)) \in T_{f(p)}M$$

for all $X_p \in T_pN$.

12.4 Smooth Sections

A *section* of a vector bundle $\pi : E \to M$ is a map $s\colon M \to E$ such that $\pi \circ s = 1_M$. This condition means precisely that for each p in M, $s(p) \in E_p$. Pictorially we visualize a section as a cross-section of the bundle (Figure 12.2). We say that a section is *smooth* if it is smooth as a map from M to E.

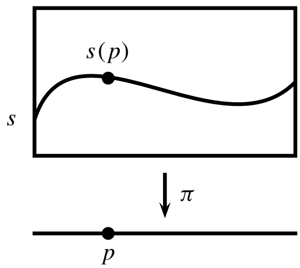


Fig. 12.2. A section of a vector bundle.

Definition 12.6. A *vector field* X on a manifold M is a function that assigns a tangent vector $X_p \in T_pM$ to each point $p \in M$. In terms of the tangent bundle, a vector field on M is simply a section of the tangent bundle $\pi : TM \to M$ and the vector field is *smooth* if it is smooth as a map from M to TM.

Example 12.7. The formula

$$X_{(x,y)} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

defines a smooth vector field on $\mathbb{R}^2$ (Figure 12.3).

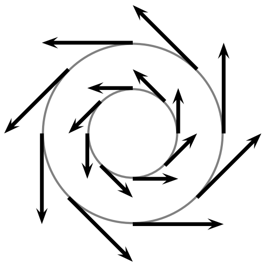


Fig. 12.3. The vector field $(-y, x)$ in $\mathbb{R}^2$.

Proposition 12.8. *Let s and t be C^∞ sections of a C^∞ vector bundle $\pi: E \to M$ and let f be a C^∞ function on M. Then*

(i) *the sum $s + t$ defined by*

$$(s + t)(p) = s(p) + t(p) \in E_p, \ \ p \in M,$$

is a C^∞ section of E.

(ii) *the product fs defined by*

$$(fs)(p) = f(p)s(p), \ \ p \in M,$$

is a C^∞ section of E.

Proof.
(i) It is clear that $s + t$ is a section of E. To show that it is C^∞, fix a point $p \in M$ and let V be a trivializing open set for E containing p, with C^∞ trivialization

$$\phi: \pi^{-1}(V) \to V \times \mathbb{R}^r.$$

Suppose

$$\phi \circ s(q) = (q, a^1(q), \ldots, a^r(q))$$

and

$$\phi \circ t(q) = (q, b^1(q), \ldots, b^r(q))$$

for $q \in V$. Because s and t are C^∞ maps, a^i and b^i are C^∞ functions on V. Since ϕ is linear on each fiber,

$$\phi \circ (s+t)(q) = (q, a^1(q) + b^1(q), \dots, a^r(q) + b^r(q)),\ q \in V.$$

This proves that $s+t$ is a C^∞ map on V and hence at p. Since p is an arbitrary point of M, the section $s+t$ is C^∞ on M.

(ii) We omit the proof as it is similar to that of (i). □

Denote the set of all C^∞ sections of E by $\Gamma(E)$. The proposition shows that $\Gamma(E)$ is not only a vector space over $\mathbb{R}$, but also a module over $C^\infty(M)$. For any open set U, one can also consider the vector space $\Gamma(U, E)$ of C^∞ sections of E over U. Then $\Gamma(U, E)$ is both a vector space over $\mathbb{R}$ and a $C^\infty(U)$-module. Note that $\Gamma(M, E) = \Gamma(E)$.

12.5 Smooth Frames

A *frame* for a vector bundle $\pi : E \to M$ over an open set U is a collection of sections $s_1, \dots, s_r$ of E over U such that at each point $p \in U$, the elements $s_1(p), \dots, s_r(p)$ form a basis for the fiber $E_p := \pi^{-1}(p)$. A frame $s_1, \dots, s_r$ is said to be *smooth* or C^∞ if $s_1, \dots, s_r$ are C^∞ as sections of E over U. A frame for the tangent bundle $TM \to M$ over an open set U is called simply a *frame on* U.

Example 12.9. The collection of vector fields $\partial/\partial x, \partial/\partial y, \partial/\partial z$ is a smooth frame on $\mathbb{R}^3$.

Proposition 12.10 (Characterization of C^∞ sections). *Let $\pi : E \to M$ be a C^∞ vector bundle and U an open subset of M. Suppose $s_1, \dots, s_r$ is a C^∞ frame for E over U. Then a section $s = \sum c^j s_j$ of E over U is C^∞ if and only if the coefficients c^j are C^∞ functions on U.*

Proof. If the c^j's are C^∞ functions on U, then $s = \sum c^j s_j$ is a C^∞ section on U by Proposition 12.8.

Conversely, fix a point $p \in U$ and choose a trivializing open set V for E containing p, with C^∞ trivialization $\phi \colon \pi^{-1}(V) \to V \times \mathbb{R}^r$. Since $\phi \circ s_j \colon V \to V \times \mathbb{R}^r$ is C^∞, if

$$\phi \circ s_j(q) = (q, a_j^1(q), \dots, a_j^r(q)),$$

then $a_j^1, \dots, a_j^r$ are C^∞ functions on V. Similarly, if

$$\phi \circ s(q) = (q, a^1(q), \dots, a^r(q)),$$

then $a^1, \dots, a^r$ are C^∞ functions on V.

Since $s = \sum c^j s_j$ and ϕ is linear on fibers,

$$\phi(s(q)) = \sum_j c^j(q)\phi(s_j(q)) = \left(q, \sum_j c^j(q) a_j^1(q), \dots, \sum_j c^j(q) a_j^r(q)\right).$$

Thus,

$$a^i = \sum_j c^j a^i_j.$$

In matrix notation,

$$a = \begin{bmatrix} a^1 \\ \vdots \\ a^r \end{bmatrix} = A \begin{bmatrix} c^1 \\ \vdots \\ c^r \end{bmatrix} = Ac.$$

By Cramer's rule, A^{-1} is a matrix of C^∞ functions on V. Hence, $c = A^{-1}a$ is a column vector of C^∞ functions on V. This proves that $c^1, \dots, c^r$ are C^∞ functions at $p \in U$. Since p is an arbitrary point of U, the coefficients c^j are C^∞ functions on U. $\square$

Problems

12.1.* Hausdorff condition on the tangent bundle
Prove Proposition 12.4.

12.2. Transition functions for the total space of the tangent bundle
Let $(U, \phi) = (U, x^1, \dots, x^n)$ and $(V, \psi) = (V, y^1, \dots, y^n)$ be overlapping coordinate charts on a manifold M. They induce coordinate charts $(TU, \tilde{\phi})$ and $(TV, \tilde{\psi})$ on the total space TM of the tangent bundle (see equation (12.1)), with transition function $\tilde{\psi} \circ \tilde{\phi}^{-1}$:

$$(x^1, \dots, x^n, a^1, \dots, a^n) \mapsto (y^1, \dots, y^n, b^1, \dots, b^n).$$

(a) Compute the Jacobian matrix of the transition function $\tilde{\psi} \circ \tilde{\phi}^{-1}$ at $\phi(p)$.
(b) Show that the Jacobian determinant of the transition function $\tilde{\psi} \circ \tilde{\phi}^{-1}$ at $\phi(p)$ is $(\det[\partial y^i/\partial x^j])^2$.

13

Bump Functions and Partitions of Unity

The existence of a C^∞ partition of unity is one of the most important technical tools in the theory of C^∞ manifolds. It is the single feature that makes the behavior of C^∞ manifolds so different from real-analytic or complex manifolds. In this chapter we construct C^∞ bump functions on any manifold and prove the existence of a C^∞ partition of unity on a compact manifold. The proof of the existence of a C^∞ partition of unity on a general manifold is more technical and is postponed to Appendix C.

13.1 C^∞ Bump Functions

The *support* of a C^∞ function f on a manifold M is defined to be the closure of the set on which $f \neq 0$:

$$\operatorname{supp} f = \text{closure of } \{p \in M \mid f(p) \neq 0\}.$$

Let q be a point in M, and U a neighborhood of q. By a *bump function at q supported in U* we mean any continuous function f that is 1 in a neighborhood of q with $\operatorname{supp} f \subset U$.

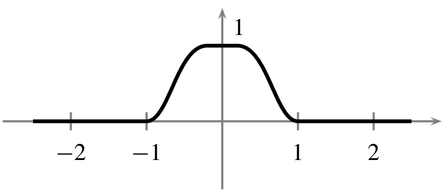


Fig. 13.1. A bump function at 0.

For example, Figure 13.1 is the graph of a bump function at 0 with support in $(-2, 2)$. The function is nonzero on the open interval $(-1, 1)$ and is zero otherwise. Its support is the closed interval $[-1, 1]$.

The only bump functions that interest us are C^∞ bump functions. While the continuity of a function can often be seen by inspection, the smoothness of a function always requires a formula. Our goal in this section is to find a formula for a C^∞ bump function as in Figure 13.1.

Example 13.1. The graph of $y = x^{5/3}$ looks perfectly smooth (Figure 13.2), but it is in fact not smooth at $x = 0$, since its second derivative $y'' = (10/9)x^{-1/3}$ is not defined at $x = 0$.

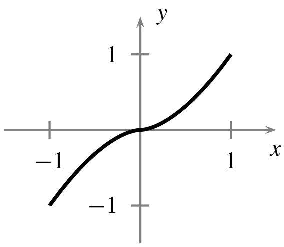


Fig. 13.2. The graph of $y = x^{5/3}$.

In Example 1.3 we introduced the C^∞ function

$$f(t) = \begin{cases} e^{-1/t} & \text{for } t > 0, \\ 0 & \text{for } t \le 0 \end{cases}$$

with graph as in Figure 13.3.

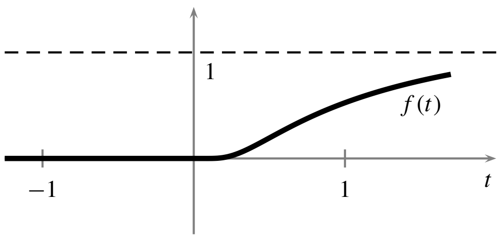


Fig. 13.3. The graph of $f(t)$.

Define

$$g(t) = \frac{f(t)}{f(t) + f(1-t)}. \tag{13.1}$$

We first show that the denominator $f(t)+f(1-t)$ is never zero. For $t > 0$, $f(t) > 0$ and therefore

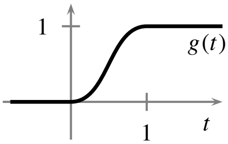
$$f(t) + f(1-t) \geq f(t) > 0.$$

For $t \leq 0$, $1 - t \geq 1$ and therefore

$$f(t) + f(1-t) \geq f(1-t) > 0.$$

In either case, $f(t) + f(1-t) \neq 0$. This proves that $g(t)$ is defined for all t. As the quotient of two C^∞ functions with the denominator never zero, $g(t)$ is C^∞ for all t.

Moreover, for $t \leq 0$, $f(t) = 0$ and $f(1-t) > 0$, so $g(t) \equiv 0$ for $t \leq 0$. For $t \geq 1$, $1 - t \leq 0$ and $f(1-t) = 0$, so $g(t) \equiv 1$ for $t \geq 1$. Thus, g is a C^∞ function with graph as in Figure 13.4.



$$g(t) = \begin{cases} 0 & \text{for } t \leq 0, \\ 1 & \text{for } t \geq 1. \end{cases}$$

Fig. 13.4. The graph of $g(t)$.

Given two positive real numbers $a < b$, we make a linear change of variables to map $[a^2, b^2]$ to $[0, 1]$:

$$x \mapsto \frac{x - a^2}{b^2 - a^2}.$$

Let

$$h(x) = g\left(\frac{x - a^2}{b^2 - a^2}\right).$$

Then $h\colon \mathbb{R} \to [0, 1]$ is a C^∞ function such that

$$h(x) = \begin{cases} 0 & \text{for } x \leq a^2, \\ 1 & \text{for } x \geq b^2. \end{cases}$$

(See Figure 13.5.)

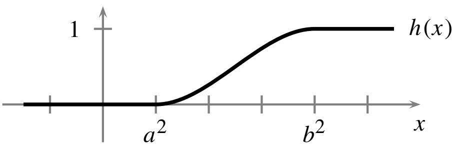


Fig. 13.5. The graph of $h(x)$.

Replace x by x^2 to make the function symmetric in x: $k(x) = h(x^2)$ (Figure 13.6). Finally, set

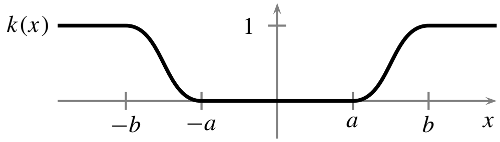


Fig. 13.6. The graph of $k(x)$.

$$\rho(x) = 1 - k(x) = 1 - g\left(\frac{x^2 - a^2}{b^2 - a^2}\right).$$

This $\rho(x)$ is a C^∞ bump function at 0 in $\mathbb{R}$ (Figure 13.7). For any $q \in \mathbb{R}$, $\rho(x - q)$ is a C^∞ bump function at q.

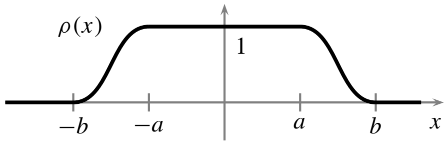


Fig. 13.7. A bump function at 0 on $\mathbb{R}$.

It is easy to extend the construction of a bump function from $\mathbb{R}$ to $\mathbb{R}^n$. To get a bump function at 0 in $\mathbb{R}^n$ which is 1 on the closed ball $\overline{B}(0, a)$ and 0 outside the closed ball $\overline{B}(0, b)$, set

$$\sigma(x) = \rho(|x|) = 1 - g\left(\frac{|x|^2 - a^2}{b^2 - a^2}\right). \tag{13.2}$$

As a composition of C^∞ functions, σ is C^∞. To get a C^∞ bump function at q in $\mathbb{R}^n$, take $\sigma(x - q)$.

Exercise 13.2 (Bump function supported in an open set). Let q be a point and U any neighborhood of q in a manifold. Construct a C^∞ bump function at q supported in U.

Proposition 13.3 (C^∞ extension of a function). *Suppose f is a C^∞ function defined on a neighborhood U of q in a manifold M. Then there is a C^∞ function $\tilde{f}$ on M which agrees with f in some possibly smaller neighborhood of q.*

Proof. Choose a C^∞ bump function ρ which is supported in U and is identically 1 in a neighborhood V of q (Figure 13.8). Define

$$\tilde{f}(x) = \begin{cases} \rho(x) f(x) & \text{for } x \text{ in } U, \\ 0 & \text{for } x \text{ not in } U. \end{cases}$$

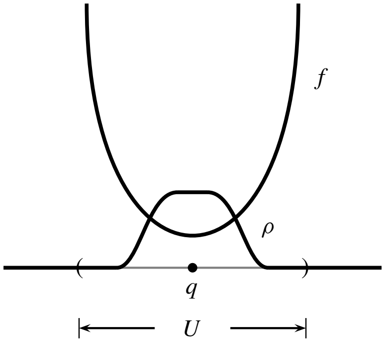


Fig. 13.8. Extending the domain of a function by multiplying by a bump function.

As the product of two C^∞ functions on U, $\tilde{f}$ is C^∞ on U. If $x \notin U$, then $x \notin \operatorname{supp} \rho$, and so there is an open set containing x on which $\tilde{f}$ is 0 since $\operatorname{supp} \rho$ is closed. Therefore, $\tilde{f}$ is also C^∞ at every point $x \notin U$.

Finally, since $\rho \equiv 1$ on V, the function $\tilde{f}$ agrees with f on V. □

13.2 Partitions of Unity

If $\{U_i\}_{i \in I}$ is a finite open cover of M, a C^∞ *partition of unity subordinate to* $\{U_i\}$ is a collection of nonnegative C^∞ functions $\{\rho_i\}_{i \in I}$ satisfying

(a) $\sum \rho_i = 1$;
(b) $\operatorname{supp} \rho_i \subset U_i$.

When I is an infinite set, for Condition (a) to make sense, we will need to impose a *locally finite* condition. A collection $\{A_\alpha\}$ of subsets of a topological space S is *locally finite* if every point q in S has a neighborhood that intersects only finitely many of the A_α's. (A *neighborhood* of a point q is an open set containing q.) In particular, every q in S is contained in finitely many of the A_α's.

Example 13.4 (*An open cover that is not locally finite*). Let $U_{r,n}$ be the open interval $(r - (1/n), r + (1/n))$ in the real line $\mathbb{R}$. The open cover $\{U_{r,n} \mid r \in \mathbb{Q}, n \in \mathbb{Z}^+\}$ of $\mathbb{R}$ is not locally finite.

Definition 13.5. A C^∞ *partition of unity* on a manifold is a collection of C^∞ functions $\{\rho_\alpha\}_{\alpha \in A}$ such that

(i) the collection of supports, $\{\operatorname{supp} \rho_\alpha\}_{\alpha \in A}$, is locally finite;
(ii) $\sum \rho_\alpha = 1$.

Given an open cover $\{U_\alpha\}_{\alpha \in A}$ of M, we say that a partition of unity $\{\rho_\alpha\}_{\alpha \in A}$ is *subordinate to the open cover* $\{U_\alpha\}$ if $\operatorname{supp} \rho_\alpha \subset U_\alpha$ for every $\alpha \in A$.

Since the collection of supports, $\{\operatorname{supp}\rho_\alpha\}$, is locally finite (Condition (i)), every point q lies in finitely many of the sets $\operatorname{supp}\rho_\alpha$. Hence $\rho_\alpha(q) \neq 0$ for only finitely many α. It follows that the sum in (ii) is a finite sum at every point.

Example 13.6. Let U, V be the open intervals $(-\infty, 2)$, $(-1, \infty)$ in $\mathbb{R}$, and let ρ_V be a C^∞ function with graph as in Figure 13.9, e.g., the function $g(t)$ of (13.1). Define $\rho_U = 1 - \rho_V$. Then $\operatorname{supp}\rho_V \subset V$ and $\operatorname{supp}\rho_U \subset U$. Thus, $\{\rho_U, \rho_V\}$ is a partition of unity subordinate to the open cover $\{U, V\}$.

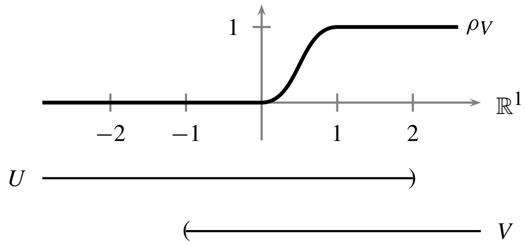


Fig. 13.9. A partition of unity $\{\rho_U, \rho_V\}$ subordinate to an open cover $\{U, V\}$.

Remark 13.7. Suppose $\{f_\alpha\}_{\alpha\in A}$ is a collection of C^∞ functions on a manifold M such that the collection of its supports, $\{\operatorname{supp} f_\alpha\}_{\alpha\in A}$, is locally finite. Then every point q in M has a neighborhood W_q that intersects $\operatorname{supp} f_\alpha$ for only finitely many α. Thus, on W_q the sum $\sum_{\alpha\in A} f_\alpha$ is actually a finite sum. This shows that the function $f = \sum f_\alpha$ is well defined and C^∞ on the manifold M. We call such a sum a *locally finite* sum.

13.3 Existence of a Partition of Unity

Because the case of a compact manifold is somewhat easier and already has some of the features of the general case, for pedagogical reasons we give a separate proof for the compact case.

Lemma 13.8. *If $\rho_1, \dots, \rho_m$ are real-valued functions on a manifold M, then*

$$\operatorname{supp}\left(\sum \rho_i\right) \subset \bigcup \operatorname{supp}\rho_i.$$

Proof. Problem 13.1. □

Proposition 13.9. *Let M be a compact manifold and $\{U_\alpha\}_{\alpha\in A}$ an open cover of M. There exists a C^∞ partition of unity $\{\rho_\alpha\}_{\alpha\in A}$ subordinate to $\{U_\alpha\}_{\alpha\in A}$.*

Proof. For each $q \in M$, find an open set U_α containing q from the given cover and let ψ_q be a C^∞ bump function at q supported in U_α (Exercise 13.2). Because $\psi_q(q) > 0$, there is a neighborhood W_q of q on which $\psi_q > 0$. By the compactness of M, the open cover $\{W_q \mid q \in M\}$ has a finite subcover, say $\{W_{q_1}, \ldots, W_{q_m}\}$. Let $\psi_{q_1}, \ldots, \psi_{q_m}$ be the corresponding bump functions. Then $\psi := \sum \psi_{q_i}$ is positive at every point q in M because $q \in W_{q_i}$ for some i. Define

$$\varphi_i = \frac{\psi_{q_i}}{\psi}, \quad i = 1, \ldots, m.$$

Clearly, $\sum \varphi_i = 1$. Moreover, since $\psi > 0$, $\varphi_i(q) \neq 0$ if and only if $\psi_{q_i}(q) \neq 0$, so

$$\operatorname{supp} \varphi_i = \operatorname{supp} \psi_{q_i} \subset U_\alpha$$

for some $\alpha \in A$. This shows that $\{\varphi_i\}$ is a partition of unity for which for every i, $\operatorname{supp} \varphi_i \subset U_\alpha$ for some $\alpha \in A$.

The next step is to make the index set of the partition of unity the same as that of the open cover. For each $i = 1, \ldots, m$, choose $\tau(i) \in A$ to be an index such that

$$\operatorname{supp} \varphi_i \subset U_{\tau(i)}.$$

We group the collection of functions $\{\varphi_i\}$ into subcollections according to $\tau(i)$ and define for each $\alpha \in A$

$$\rho_\alpha = \sum_{\tau(i)=\alpha} \varphi_i;$$

if there is no i for which $\tau(i) = \alpha$, define $\rho_\alpha = 0$. Then

$$\sum_{\alpha \in A} \rho_\alpha = \sum_{\alpha \in A} \sum_{\tau(i)=\alpha} \varphi_i = \sum_{i=1}^{m} \varphi_i = 1.$$

Moreover, by Lemma 13.8,

$$\operatorname{supp} \rho_\alpha \subset \bigcup_{\tau(i)=\alpha} \operatorname{supp} \varphi_i \subset U_\alpha.$$

So $\{\rho_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$. □

To generalize the proof of Proposition 13.9 to an arbitrary manifold, it will be necessary to find an appropriate substitute for compactness. As the proof is rather technical and is not necessary for the rest of the book, we put it in Appendix C. The statement is as follows.

Theorem 13.10 (Existence of a C^∞ partition of unity). *Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of a manifold M.*

(i) *Then there is a C^∞ partition of unity $\{\varphi_k\}_{k=1}^{\infty}$ with compact support such that for each k, $\operatorname{supp} \varphi_k \subset U_\alpha$ for some $\alpha \in A$.*
(ii) *If we do not require compact support, then there is a C^∞ partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$.*

Problems

13.1.* Support of a finite sum
Prove Lemma 13.8.

13.2.* Locally finite family and compact set
Let $\{A_\alpha\}$ be a locally finite family of subsets of a topological space S. Show that every compact set K in S has a neighborhood W that intersects only finitely many of the A_α.

13.3. Smooth Urysohn lemma
Let A and B be two disjoint closed sets in a manifold. Find a C^∞ function f such that f is identically 1 on A and identically 0 on B. (*Hint*: Consider a C^∞ partition of unity $\{\rho_{M-A}, \rho_{M-B}\}$ subordinate to the open cover $\{M - A, M - B\}$. This lemma is needed in Section 28.3.)

13.4.* Support of the pullback of a function
Let $f\colon M \to \mathbb{R}$ be a C^∞ function on a manifold M. If N is another manifold and $\pi : M \times N \to M$ is the projection onto the first factor, prove that

$$\operatorname{supp}(\pi^* f) = (\operatorname{supp} f) \times N.$$

13.5. Pullback of a partition of unity
Suppose $\{\rho_\alpha\}$ is a partition of unity on a manifold M subordinate to an open cover $\{U_\alpha\}$ of M and $F\colon N \to M$ is a C^∞ map. Prove that

(a) the collection of supports $\{\operatorname{supp} F^*\rho_\alpha\}$ is locally finite;
(b) the collection of functions $\{F^*\rho_\alpha\}$ is a partition of unity on N subordinate to the open cover $\{F^{-1}(U_\alpha)\}$ of N.

13.6.* Closure of a locally finite union
If $\{A_\alpha\}$ is a locally finite collection of subsets in a topological space, then

$$\overline{\bigcup A_\alpha} = \bigcup \overline{A_\alpha}, \tag{13.3}$$

where $\overline{A}$ denotes the closure of the subset A.

Remark. For any collection of subset A_α, one always has

$$\bigcup \overline{A_\alpha} \subset \overline{\bigcup A_\alpha}.$$

However, the reverse inclusion is in general not true. For example, suppose A_n is the closed interval $[0, 1 - (1/n)]$ in $\mathbb{R}$. Then

$$\overline{\bigcup_{n=1}^{\infty} A_n} = \overline{[0, 1)} = [0, 1],$$

but

$$\bigcup_{n=1}^{\infty} \overline{A_n} = \bigcup_{n=1}^{\infty} \left[0, 1 - \frac{1}{n}\right] = [0, 1).$$

If $\{A_\alpha\}$ is a finite collection, the equality (13.3) is easily shown to be true.

14

Vector Fields

In Section 12.4 we defined a vector field X on a manifold M as the assignment of a tangent vector $X_p \in T_pM$ at each point $p \in M$. More formally a vector field on M is a section of the tangent bundle $TM \to M$, and a vector field is smooth if and only if it is smooth as a section of the tangent bundle. In this chapter we give two more characterizations of smooth vector fields (Section 14.1).

If X is a vector field on a manifold M, then through each point of M there is a curve, called an *integral curve of* X, whose velocity vector field is given by X. The collection of integral curves through the points of M may be thought of as a motion of the manifold, called a *local flow* of the vector field.

After discussing local flows, we collect together a few facts about vector fields—the Lie bracket, related vector fields, and the push-forward.

14.1 Smoothness of a Vector Field

Suppose X is a vector field on a manifold M. At a point p in a coordinate chart $(U, \phi) = (U, x^1, \ldots, x^n)$ for M, the value of the vector field X is a linear combination

$$X_p = \sum a^i(p) \frac{\partial}{\partial x^i}\bigg|_p .$$

Using the chart $(TU, \tilde{\phi})$ for TM (see Section 12.1),

$$\tilde{\phi}(X_p) = (\phi(p), a^1(p), \ldots, a^n(p)).$$

Thus, the vector field $X = \sum a^i\, \partial/\partial x^i$ is smooth on U if and only if the coefficients a^i are smooth functions on U. This gives a second characterization of a smooth vector field on M: in any coordinate chart $(U, x^1, \ldots, x^n)$ in the atlas of the manifold, a vector field $X = \sum a^i\, \partial/\partial x^i$ is smooth if and only if the coefficient functions a^i are C^∞ on U.

Just as in Section 2.5, a vector field X on a manifold M gives rise to a linear map on the algebra $C^\infty(M)$ of C^∞ functions on M: for $f \in C^\infty(M)$, define Xf to be the function

$$(Xf)(p) = X_p f, \quad p \in M.$$

Finally there is still a third characterization of a smooth vector field, in terms of its action as an operator on C^∞ functions.

Proposition 14.1. *A vector field X on M is smooth if and only if for every smooth function f on M, the function Xf is smooth on M.*

Proof.
($\Rightarrow$) Suppose X is smooth, $f \in C^\infty(M)$, and $p \in M$. Relative to a chart $(U, x^1, \ldots, x^n)$ about p, $X = \sum a^i\, \partial/\partial x^i$, where by the second characterization of a smooth vector field, the a^i are C^∞ functions on U. Then $Xf = \sum a^i\, \partial f/\partial x^i$ is C^∞ on U. Since p is arbitrary, Xf is C^∞ on M.

($\Leftarrow$) On the chart $(U, x^1, \ldots, x^n)$, $X = \sum a^i\, \partial/\partial x^i$. Let $p \in U$. By Proposition 13.3, each x^i can be extended to a C^∞ function $\tilde{x}^i$ on M that agrees with x^i in a neighborhood V of p. Therefore, on V,

$$X\tilde{x}^k = \left(\sum a^i \frac{\partial}{\partial x^i}\right)\tilde{x}^k = \left(\sum a^i \frac{\partial}{\partial x^i}\right)x^k = a^k.$$

This proves that a^k is C^∞ at p. Since p is an arbitrary point in U, the function a^k is C^∞ on U. By the second characterization of a smooth vector field, X is smooth. □

By Proposition 14.1, we may view a C^∞ vector field X as a linear operator $X\colon C^\infty(M) \to C^\infty(M)$ of the algebra of C^∞ functions on M. As in Proposition 2.6, this linear operator $X\colon C^\infty(M) \to C^\infty(M)$ is a derivation. In the following we think of C^∞ vector fields on M alternately as C^∞ sections of the tangent bundle TM or as derivations on the algebra $C^\infty(M)$ of C^∞ functions. In fact, it can be shown that these two descriptions of C^∞ vector fields are equivalent (Problem 19.11).

14.2 Integral Curves

In Example 12.7, it appears that through each point in the plane one can draw a circle whose velocity at any point is the given vector field at that point. Such a curve is an example of an *integral curve* of the vector field, which we now define.

Definition 14.2. Let X be a C^∞ vector field on a manifold M, and $p \in M$. An *integral curve* of X starting at p is a curve $c\colon (a,b) \to M$ defined on an open interval (a,b) containing 0 such that $c(0) = p$ and $c'(t) = X_{c(t)}$. To show its dependence on the initial point p, we also write $c_t(p)$ instead of $c(t)$.

Definition 14.3. An integral curve is *maximal* if its domain cannot be extended to a larger interval.

Example 14.4. Recall the vector field $X_{(x,y)} = \langle -y, x\rangle$ on $\mathbb{R}^2$ (Figure 12.3). We will find an integral curve $c(t)$ of X starting at the point $(1,0) \in \mathbb{R}^2$. The condition for $c(t) = (x(t), y(t))$ to be an integral curve is $c'(t) = X_{c(t)}$ or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -y(t) \\ x(t) \end{bmatrix},$$

so we need to solve the system of first-order ordinary differential equations

$$x' = -y, \tag{14.1}$$

$$y' = x, \tag{14.2}$$

with initial condition $(x(0), y(0)) = (1, 0)$. From (14.1), $y = -x'$. So $y' = -x''$. Substituting into (14.2) gives

$$x'' = -x.$$

It is well known that the solutions are

$$\begin{aligned} x &= A\cos t + B\sin t, \\ y &= -x' = A\sin t - B\cos t. \end{aligned} \tag{14.3}$$

The initial condition forces $A = 1$, $B = 0$, so the integral curve starting at $(1,0)$ is $c(t) = (\cos t, \sin t)$, which parametrizes the unit circle.

More generally, if the initial point of the integral curve, corresponding to $t = 0$, is $p = (x_0, y_0)$, then (14.3) gives

$$A = x_0, \qquad B = -y_0,$$

and the general solution is

$$\begin{aligned} x &= x_0\cos t - y_0 \sin t, \\ y &= x_0 \sin t + y_0 \cos t, \quad t \in \mathbb{R}. \end{aligned}$$

This can be written in matrix notation as

$$c(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} p,$$

which shows that the integral curve of X starting at p can be obtained by rotating the point p counterclockwise about the origin through an angle t. Notice that

$$c_s(c_t(p)) = c_{s+t}(p),$$

since a rotation through an angle t followed by a rotation through an angle s is the same as a rotation through the angle $s + t$. For each $t \in \mathbb{R}$, $c_t \colon M \to M$ is a diffeomorphism with inverse c_{-t}.

Let $\mathrm{Diff}(M)$ be the group of diffeomorphisms of a manifold M. A homomorphism $c \colon \mathbb{R} \to \mathrm{Diff}(M)$ is called a *one-parameter group of diffeomorphisms* of M. In this example the integral curves of the vector field $X_{(x,y)} = \langle -y, x\rangle$ on $\mathbb{R}^2$ give rise to a one-parameter group of diffeomorphisms of $\mathbb{R}^2$.

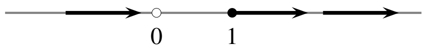


Fig. 14.1. The vector field d/dx on $\mathbb{R} - \{0\}$.

Example 14.5. Let M be $\mathbb{R} - \{0\}$ and let X be the vector field d/dx on M. Find the maximal integral curve starting at $x = 1$.

Solution. In column vector notation, the vector field X is simply 1. If $x(t)$ is an integral curve starting at 1, then

$$x'(t) = X_{x(t)} = 1, \quad x(0) = 1.$$

So $x(t) = t + 1$. Since 0 is not in M, the domain of the maximal integral curve is the open interval $(-1, \infty)$. □

From this example we see that it may not be possible to extend the domain of definition of an integral curve to the entire real line.

14.3 Local Flows

The two examples in the preceding section illustrate the fact that finding an integral curve of a vector field amounts to solving a system of first-order ordinary differential equations with initial conditions. In general, if X is a smooth vector field on a manifold M, to find an integral $c(t)$ of X starting at p, we first choose a coordinate chart $(U, \phi) = (U, x^1, \dots, x^n)$ about p. In terms of the local coordinates,

$$X = \sum a^i(x) \frac{\partial}{\partial x^i},$$

and by Proposition 8.15,

$$c'(t) = \sum (c^i)'(t) \frac{\partial}{\partial x^i},$$

where $c^i(t) = x^i \circ c(t)$ is the ith component of $c(t)$ in the chart (U, ϕ). The condition $c'(t) = X_{c(t)}$ is thus equivalent to

$$(c^i)'(t) = a^i(c(t)) \quad \text{for } i = 1, \dots, n. \tag{14.4}$$

This is a system of ordinary differential equations (ODE); the initial condition $c(0) = p$ translates to $(c^1(0), \dots, c^n(0)) = (p^1, \dots, p^n)$. By an existence and uniqueness theorem from the theory of ODE, such a system always has a unique solution in the following sense.

Theorem 14.6. *Let V be an open subset of $\mathbb{R}^n$, p_0 a point in V, and $f\colon V \to \mathbb{R}^n$ a C^∞ function. Then the differential equation*

$$dy/dt = f(y), \quad y(0) = p_0,$$

has a unique C^∞ solution $y\colon (a(p_0), b(p_0)) \to V$, where $(a(p_0), b(p_0))$ is the maximal open interval containing 0 on which y is defined.

The uniqueness of the solution means that if $z\colon (\delta, \epsilon) \to V$ satisfies the same differential equation

$$dz/dt = f(z), \quad z(0) = p_0,$$

then the domain of definition (δ, ϵ) of z is a subset of $(a(p_0), b(p_0))$ and $z(t) = y(t)$ on the interval (δ, ϵ).

For a vector field X on a chart U of a manifold and a point $p \in U$, this theorem guarantees the existence and uniqueness of a maximal integral curve starting at p.

Next we would like to study the dependence of an integral curve on its initial condition. Again we study the problem locally on $\mathbb{R}^n$. The function y will now be a function of two arguments t and q, and the condition for y to be an integral curve starting at the point q is

$$\frac{\partial y}{\partial t}(t, q) = f(y(t, q)), \quad y(0, q) = q.$$

The following theorem from the theory of ODE guarantees the smooth dependence of the solution on the initial condition.

Theorem 14.7. *Let V be an open subset of $\mathbb{R}^n$ and $f\colon V \to \mathbb{R}^n$ a C^∞ function on V. For each point $p_0 \in V$, there is a neighborhood W of p_0 in V, a number $\epsilon > 0$, and a C^∞ function*

$$y\colon (-\epsilon, \epsilon) \times W \to V$$

such that

$$\frac{\partial y}{\partial t}(t, q) = f(y(t, q)), \quad y(0, q) = q$$

for all $(t, q) \in (-\epsilon, \epsilon) \times W$.

For a proof of these two theorems, see [4, Appendix C, pp. 359–366].

It follows from Theorem 14.7 that if X is any C^∞ vector field on a chart U and $p \in U$, then there are a neighborhood W of p in U, an $\epsilon > 0$, and a C^∞ map

$$F\colon (-\epsilon, \epsilon) \times W \to U \tag{14.5}$$

such that for each $q \in W$, the function $F(t, q)$ is an integral curve of X starting at q. In particular, $F(0, q) = q$. We usually write $F_t(q)$ for $F(t, q)$.

Suppose s, t in the interval $(-\epsilon, \epsilon)$ are such that both $F_t(F_s(q))$ and $F_{t+s}(q)$ are defined. Then both $F_t(F_s(q))$ and $F_{t+s}(q)$ as functions of t are integral curves of X with initial point $F_s(q)$, which is the point corresponding to $t = 0$. By the uniqueness of the integral curve,

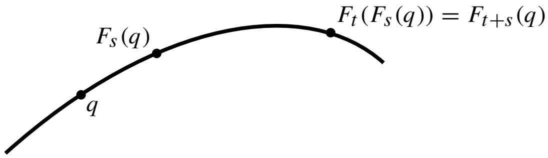


Fig. 14.2. The flow line through q of a local flow.

$$F_t(F_s(q)) = F_{t+s}(q). \tag{14.6}$$

The map F in (14.5) is called a *local flow* generated by the vector field X. For each $q \in U$, the function $F_t(q)$ of t is called a *flow line* of the local flow. Each flow line is an integral curve of X. If a local flow F is defined on $(-\infty, \infty) \times M$, then it is called a *global flow*. Every smooth vector field has a local flow about any point, but not necessarily a global flow. A vector field having a global flow is called a *complete vector field*. If F is a global flow, then $F_t \colon M \to M$ is a diffeomorphism for every $t \in \mathbb{R}$, since it has inverse F_{-t}. Thus, a global flow on M gives rise to a one-parameter group of diffeomorphisms of M.

This discussion suggests the following definition.

Definition 14.8. A *local flow* about a point p in an open set U of a manifold is a C^∞ function

$$F \colon (-\epsilon, \epsilon) \times W \to U,$$

where ϵ is a positive real number and W is a neighborhood of p in U, such that writing $F_t(q) = F(t, q)$, we have

(i) $F_0(q) = q$ for all $q \in W$,
(ii) $F_t(F_s(q)) = F_{t+s}(q)$ whenever both sides are defined.

If $F(t, q)$ is a local flow of the vector field X on U, then

$$F(0, q) = q \quad \text{and} \quad \frac{\partial F}{\partial t}(0, q) = X_{F(0,q)} = X_q.$$

Thus, one can recover the vector field from its flow.

Example 14.9. The function $F \colon (-\infty, \infty) \times \mathbb{R}^2 \to \mathbb{R}^2$,

$$F\left(t, \begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

is the global flow on $\mathbb{R}^2$ generated by the vector field

$$\begin{aligned} X_{(x,y)} &= \frac{\partial F}{\partial t}(t, (x, y))\bigg|_{t=0} = \begin{bmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}\bigg|_{t=0} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \end{aligned}$$

This is Example 12.7 again.

14.4 The Lie Bracket

Suppose X and Y are smooth vector fields on an open subset U of a manifold M, which we view as derivations on $C^\infty(U)$. For a C^∞ function f on U, by Proposition 14.1 the function Yf is C^∞ on U, and the function $(XY)f := X(Yf)$ is also C^∞ on U. Moreover, because X and Y are both $\mathbb{R}$-linear maps from $C^\infty(U)$ to $C^\infty(U)$, the map $XY\colon C^\infty(U) \to C^\infty(U)$ is $\mathbb{R}$-linear. However, XY does not satisfy the derivation property: if $f, g \in C^\infty(U)$, then

$$\begin{aligned} XY(fg) &= X((Yf)g + fYg) \\ &= (XYf)g + (Yf)(Xg) + (Xf)(Yg) + f(XYg). \end{aligned}$$

Looking more closely at this formula we see that the two extra terms $(Yf)(Xg)$ and $(Xf)(Yg)$ that make XY not a derivation are symmetric in X and Y. Thus, if we compute $YX(fg)$ as well and subtract it from $XY(fg)$, the extra terms will disappear, and $XY - YX$ will be a derivation of $C^\infty(U)$.

Given two smooth vector fields X and Y on U and $p \in U$, we define their *Lie bracket* $[X, Y]$ at p to be

$$[X, Y]_p f = (X_p Y - Y_p X) f$$

for any germ f of a C^∞ function at p. By the same calculation as above, but now evaluated at p, it is easy to check that $[X, Y]_p$ is a derivation of $C_p^\infty(U)$ and is a tangent vector at p (Definition 8.1). As p varies over U, $[X, Y]$ becomes a vector field on U.

Proposition 14.10. *If X and Y are smooth vector fields on M, then the vector field $[X, Y]$ is also smooth on M.*

Proof. By Proposition 14.1 it suffices to check that if f is a C^∞ function on M, then so is $[X, Y]f$. But

$$[X, Y]f = (XY - YX)f,$$

which is clearly C^∞ on M since both X and Y are as well. □

Denoting the vector space of all smooth vector fields on M by $\mathfrak{X}(M)$, we see that the Lie bracket provides a product operation on $\mathfrak{X}(M)$.

Clearly

$$[Y, X] = -[X, Y].$$

Exercise 14.11 (Jacobi identity). Check the *Jacobi identity*:

$$\sum_{\text{cyclic}} [X, [Y, Z]] = 0.$$

This notation means that one permutes X, Y, Z cyclically and one takes the sum of the resulting terms. Written out,

$$\sum_{\text{cyclic}} [X, [Y, Z]] = [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]].$$

Definition 14.12. A *Lie algebra* is a real vector space V together with a product, called the *bracket*, $[\ ,\ ]\colon V \times V \to V$, satisfying the following properties: for all $a, b \in \mathbb{R}$ and $X, Y, Z \in V$,

(i) bilinearity:

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$
$$[Z, aX + bY] = a[Z, X] + b[Z, Y],$$

(ii) anticommutativity: $[Y, X] = -[X, Y]$,
(iii) Jacobi identity: $\sum_{\text{cyclic}} [X, [Y, Z]] = 0$.

Example 14.13. If M is a manifold, then the vector space $\mathfrak{X}(M)$ of C^∞ vector fields on M is a Lie algebra with the Lie bracket $[\ ,\ ]$ as the bracket.

Example 14.14. Let $K^{n\times n}$ be the vector space of all $n \times n$ matrices over a field K. Define for $X, Y \in K^{n\times n}$,

$$[X, Y] = XY - YX,$$

where XY is the matrix product of X and Y. With this bracket, $K^{n\times n}$ becomes a Lie algebra. The bilinearity and anticommutativity of $[\ ,\ ]$ are immediate, while the Jacobi identity follows from the same computation as Exercise 14.11.

More generally, if A is any associative algebra, then the product

$$[x, y] = xy - yx, \quad x, y \in A,$$

makes A into a Lie algebra.

Definition 14.15. A *derivation* of a Lie algebra V is a linear map $D\colon V \to V$ satisfying the product rule

$$D[Y, Z] = [DY, Z] + [Y, DZ].$$

Example 14.16. Let V be a Lie algebra. For each X in V, define $\mathrm{ad}_X\colon V \to V$ by

$$\mathrm{ad}_X(Y) = [X, Y].$$

We may rewrite the Jacobi identity in the form

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$$

or

$$\mathrm{ad}_X[Y, Z] = [\mathrm{ad}_X Y, Z] + [Y, \mathrm{ad}_X Z],$$

which shows that $\mathrm{ad}_X\colon V \to V$ is a derivation of V.

14.5 Related Vector Fields

Definition 14.17. Let $F\colon N \to M$ be a smooth map of manifolds. A vector field X on N is *F-related* to a vector field $\tilde{X}$ on M if for all $p \in N$,

$$F_{*,p}(X_p) = \tilde{X}_{F(p)}. \tag{14.7}$$

We may reformulate this condition as follows.

Proposition 14.18. *A vector field X on N and a vector field $\tilde{X}$ on M are F-related if and only if for all $g \in C^\infty(M)$,*

$$X(g \circ F) = (\tilde{X}g) \circ F.$$

Proof.
($\Rightarrow$) Suppose X on N and $\tilde{X}$ on M are F-related. By (14.7), for any $g \in C^\infty(M)$,

$$\begin{aligned} F_{*,p}(X_p)g &= \tilde{X}_{F(p)}g, && \text{(definition of } F\text{-related)} \\ X_p(g \circ F) &= (\tilde{X}g)(F(p)), && \text{(definition of } F_* \text{ and } \tilde{X}g) \\ (X(g \circ F))(p) &= (\tilde{X}g)(F(p)). \end{aligned}$$

Since this is true for all $p \in N$,

$$X(g \circ F) = (\tilde{X}g) \circ F.$$

($\Leftarrow$) Reversing the set of equations above proves the converse. □

Proposition 14.19. *Let $F\colon N \to M$ be a smooth map of manifolds. If the C^∞ vector fields X and Y on N are F-related to the C^∞ vector fields $\tilde{X}$ and $\tilde{Y}$, respectively, on M, then $[X, Y]$ is F-related to $[\tilde{X}, \tilde{Y}]$.*

Proof. For any $g \in C^\infty(M)$,

$$\begin{aligned} [X, Y](g \circ F) &= XY(g \circ F) - YX(g \circ F) && \text{(definition of } [X, Y]) \\ &= X((\tilde{Y}g) \circ F) - Y((\tilde{X}g) \circ F) && \text{(Proposition 14.18)} \\ &= (\tilde{X}\tilde{Y}g) \circ F - (\tilde{Y}\tilde{X}g) \circ F && \text{(Proposition 14.18)} \\ &= ((\tilde{X}\tilde{Y} - \tilde{Y}\tilde{X})g) \circ F \\ &= ([\tilde{X}, \tilde{Y}]g) \circ F. \end{aligned}$$

By Proposition 14.18 again, this proves that $[X, Y]$ on N and $[\tilde{X}, \tilde{Y}]$ on M are F-related. □

14.6 The Push-Forward of a Vector Field

Let $F\colon N \to M$ be a smooth map of manifolds and let $F_*\colon T_pN \to T_{F(p)}M$ be its differential at a point p in N. If $X_p \in T_pN$, we call $F_*(X_p)$ the *push-forward* of the vector X_p at p. This notion does not extend in general to vector fields, since if X is a vector field on N and $z = F(p) = F(q)$ for two distinct points $p, q \in N$, then X_p and X_q are both pushed forward to tangent vectors at $z \in M$, but there is no reason why $F_*(X_p)$ and $F_*(X_q)$ should be equal.

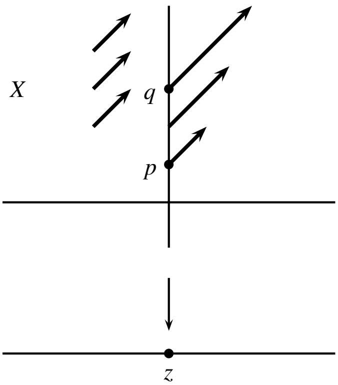


Fig. 14.3. The vector field X cannot be pushed forward..

In one important special case, the push-forward F_*X of any vector field X on N always makes sense, namely, when $F\colon N \to M$ is a diffeomorphism. In this case, since F is injective, there is no ambiguity about the meaning of $(F_*X)_{F(p)} = F_{*,p}(X_p)$, and since F is surjective, F_*X is defined everywhere on M.

Problems

14.1. Equality of vector fields
Show that two C^∞ vector fields X and Y on a manifold M are equal if and only if for every C^∞ function f on M, we have $Xf = Yf$.

14.2. Vector field on an odd sphere
Let $x^1, y^1, \dots, x^n, y^n$ be the standard coordinates on $\mathbb{R}^{2n}$. The unit sphere S^{2n-1} in $\mathbb{R}^{2n}$ is defined by the equation $\sum_{i=1}^n (x^i)^2 + (y^i)^2 = 1$. Show that

$$X = \sum_{i=1}^n -y^i \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial y^i}$$

is a nowhere-vanishing smooth vector field on S^{2n-1}. Since all spheres of the same dimension are diffeomorphic, this proves that on every odd-dimensional sphere there is a nowhere-vanishing smooth vector field. It is a classic theorem of differential and algebraic topology that on an even-dimensional sphere every continuous vector field must vanish somewhere (see [13, Section 5, p. 31] or [8, Theorem 16.5, p. 70]). (*Hint*: Use Problem 11.1.)

14.3. Integral curves in the plane
Find the integral curves of the vector field

$$X_{(x,y)} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} = \begin{bmatrix} x \\ -y \end{bmatrix} \quad \text{on } \mathbb{R}^2.$$

14.4. Maximal integral curve in the plane
Find the maximal integral curve $c(t)$ starting at the point $(a, b) \in \mathbb{R}^2$ of the vector field $X_{(x,y)} = \langle 1, x \rangle$ on $\mathbb{R}^2$.

14.5. Integral curve starting at a zero of a vector field
Suppose the smooth vector field X on M vanishes at a point $p \in M$. Show that the integral curve of X with initial point p is the constant curve $c(t) = p$ for all $t \in \mathbb{R}$.

14.6. Maximal integral curve
Let X be the vector field $x\, d/dx$ on $\mathbb{R}$. Find the maximal integral curve $c(t)$ with $c(0) = 2$.

14.7. Maximal integral curve
Let X be the vector field $x^2\, d/dx$ on the real line $\mathbb{R}$. For each $p > 0$ in $\mathbb{R}$, find the maximal integral curve of X with initial point p.

14.8. Reparametrization of an integral curve
Suppose $c\colon (a, b) \to M$ is an integral curve of the smooth vector field X on M. Show that for any real number s, the map

$$c_s : (a+s, b+s) \to M, \quad c_s(t) = c(t-s)$$

is also an integral curve of X.

14.9. Lie bracket in local coordinates
Consider the two vector fields X, Y on $\mathbb{R}^n$:

$$X = \sum a^i \frac{\partial}{\partial x^i}, \qquad Y = \sum b^j \frac{\partial}{\partial x^j},$$

where $a^i(x), b^j(x)$ are C^∞ functions on $\mathbb{R}^n$. Since $[X, Y]$ is also a C^∞ vector field on $\mathbb{R}^n$,

$$[X, Y] = \sum c^k \frac{\partial}{\partial x^k}$$

for some C^∞ functions c^k. Find the formula for c^k in terms of a^i and b^j.

14.10. Lie bracket of vector fields
If f and g are C^∞ functions and X and Y are C^∞ vector fields on a manifold M, show that

$$[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X.$$

14.11. Lie bracket of vector fields on $\mathbb{R}^2$
Compute the Lie bracket

$$\left[-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right]$$

on $\mathbb{R}^2$.

14.12. Vector field under a diffeomorphism
Let $F\colon N \to M$ be a C^∞ diffeomorphism of manifolds. Prove that if g is a C^∞ function and X a C^∞ vector field on N, then

$$F_*(gX) = (g \circ F^{-1})F_*X.$$

14.13. Lie bracket under a diffeomorphism
Let $F\colon N \to M$ be a C^∞ diffeomorphism of manifolds. Prove that if X and Y are C^∞ vector fields on N, then

$$F_*[X,Y] = [F_*X, F_*Y].$$

15

Lie Groups

Certain manifolds such as the circle have in addition to their C^∞ structure also a group structure; moreover, the group operations are C^∞. Manifolds such as these are called Lie groups. This chapter is a compendium of a few important examples of Lie groups, the *classical groups*.

15.1 Examples of Lie Groups

We recall here the definition of a Lie group, which first appeared in Section 6.1.

Definition 15.1. A *Lie group* is a C^∞ manifold G which is also a group such that the two group operations, multiplication

$$\mu\colon G \times G \to G, \quad \mu(a, b) = ab$$

and inverse

$$\iota\colon G \to G, \quad \iota(a) = a^{-1}$$

are C^∞.

NOTATION. We use capital letters to denote matrices, but generally lower-case letters to denote their entries. Thus, the (i, j)-entry of the matrix AB is $(AB)_{ij} = \sum a_{ik}b_{kj}$.

Example 15.2. In Example 5.14 we defined the general linear group

$$\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det A \neq 0\}.$$

As an open subset of $\mathbb{R}^{n\times n}$, it is a manifold. Matrix multiplication

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

is a polynomial in the coordinates of $\mathrm{GL}(n, \mathbb{R})$ and is clearly C^∞.

Recall that the (i, j)-*minor* of a matrix A is the determinant of the submatrix of A obtained by deleting the ith row and the jth column of A. By Cramer's rule, the (i, j)-entry of A^{-1} is

$$(A^{-1})_{ij} = \frac{1}{\det A} \cdot (-1)^{i+j}((j, i)\text{-minor of } A),$$

which is a C^∞ function of the a_{ij}'s provided $\det A \neq 0$. Therefore, the inverse map $\iota\colon \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ is also C^∞. This proves that $\mathrm{GL}(n, \mathbb{R})$ is a Lie group.

Example 15.3 (*Special linear group*). The special linear group $\mathrm{SL}(n, \mathbb{R})$ is the subgroup of $\mathrm{GL}(n, \mathbb{R})$ consisting of matrices of determinant 1. By Example 9.16, $\mathrm{SL}(n, \mathbb{R})$ is a regular submanifold of dimension $n^2 - 1$ of $\mathrm{GL}(n, \mathbb{R})$. By Example 11.21, the multiplication map

$$\bar{\mu}\colon \mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$$

is C^∞.

To see that the inverse map

$$\bar{\iota}\colon \mathrm{SL}(n, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$$

is C^∞, let $i : \mathrm{SL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ be the inclusion map and $\iota : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ the inverse map of $\mathrm{GL}(n, \mathbb{R})$. As the composite of two C^∞ maps,

$$\iota \circ i\colon \mathrm{SL}(n, \mathbb{R}) \xrightarrow{i} \mathrm{GL}(n, \mathbb{R}) \xrightarrow{\iota} \mathrm{GL}(n, \mathbb{R})$$

is a C^∞ map. Since its image is contained in the regular submanifold $\mathrm{SL}(n, \mathbb{R})$, the induced map $\bar{\iota} : \mathrm{SL}(n, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$ is C^∞ by Theorem 11.20. Thus, $\mathrm{SL}(n, \mathbb{R})$ is a Lie group.

Example 15.4 (*Orthogonal group*). Recall that the orthogonal group $O(n)$ is the subgroup of $\mathrm{GL}(n, \mathbb{R})$ consisting of all matrices A satisfying $A^T A = I$. Thus, $O(n)$ is the inverse image of I under the map $f(A) = A^T A$.

In Example 11.3 we showed that $f\colon \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ has constant rank on $\mathrm{GL}(n, \mathbb{R})$. By the constant-rank level set theorem, $O(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$. One drawback of this approach is that it does not tell us what the rank of f is, and so the dimension of $O(n)$ remains unknown.

In this example we will apply the regular level set theorem to prove that $O(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$. This will at the same time determine the dimension of $O(n)$. To accomplish this, we must first redefine the target space of f. Since $A^T A$ is a symmetric matrix, the image of f lies in S_n, the vector space of all $n \times n$ real symmetric matrices. Note that S_n is a vector space of dimension $(n^2+n)/2$. Consider $f\colon GL(n, \mathbb{R}) \to S_n$.

The tangent space of S_n at any point is canonically isomorphic to S_n itself, because S_n is a vector space. Thus, the image of the differential

$$f_{*,A}\colon T_A(\mathrm{GL}(n, \mathbb{R})) \to T_{A^T A}(S_n) \simeq S_n$$

is a vector space of dimension at most $(n^2+n)/2$.

While it is true that f also maps $\mathrm{GL}(n,\mathbb{R})$ to $\mathrm{GL}(n,\mathbb{R})$ or $\mathbb{R}^{n\times n}$, if we had taken $\mathrm{GL}(n,\mathbb{R})$ or $\mathbb{R}^{n\times n}$ as the target space of f, the differential $f_{*,A}$ would never be surjective for any $A \in \mathrm{GL}(n,\mathbb{R})$ when $n \geq 2$, since $f_{*,A}$ factors through the proper subspace S_n of $\mathbb{R}^{n\times n}$. This illustrates a general principle: for the differential $f_{*,A}$ to be surjective, the target space of f should be as small as possible.

To show that the differential of

$$f\colon\ \mathrm{GL}(n,\mathbb{R}) \to S_n, \quad f(A) = A^T A,$$

is surjective, we compute explicitly the differential $f_{*,A}$. Since $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$, its tangent space at any $A \in \mathrm{GL}(n,\mathbb{R})$ is

$$T_A(\mathrm{GL}(n,\mathbb{R})) = T_A(\mathbb{R}^{n\times n}) = \mathbb{R}^{n\times n}.$$

For any matrix $X \in \mathbb{R}^{n\times n}$, there is a curve $c(t)$ in $\mathrm{GL}(n,\mathbb{R})$ with $c(0) = A$ and $c'(0) = X$ (Proposition 8.16). By Proposition 8.17,

$$\begin{aligned}
f_{*,A}(X) &= \frac{d}{dt} f(c(t))\bigg|_{t=0} \\
&= \frac{d}{dt} c(t)^T c(t)\bigg|_{t=0} \\
&= (c'(t)^T c(t) + c(t)^T c'(t))|_{t=0} \\
&= X^T A + A^T X.
\end{aligned}$$

The surjectivity of $f_{*,A}$ becomes the following question: if $A \in O(n)$ and B is any symmetric matrix in S_n, does there exist an $n \times n$ matrix X such that

$$X^T A + A^T X = B?$$

Note that since $(X^T A)^T = A^T X$, it is enough to solve

$$A^T X = \frac{1}{2} B, \tag{15.1}$$

for then

$$X^T A + A^T X = \frac{1}{2} B^T + \frac{1}{2} B = B.$$

Equation (15.1) clearly has a solution: $X = \frac{1}{2}(A^T)^{-1} B$. So $f_{*,A}\colon T_A\,\mathrm{GL}(n,\mathbb{R}) \to S_n$ is surjective for all $A \in \mathrm{GL}(n,\mathbb{R})$. By the regular level set theorem, $O(n)$ is a regular submanifold of $\mathrm{GL}(n,\mathbb{R})$ of dimension

$$\dim O(n) = n^2 - \dim S_n = n^2 - (n^2+n)/2 = (n^2-n)/2. \tag{15.2}$$

Example 15.5. The *complex general linear group* $\mathrm{GL}(n,\mathbb{C})$ is defined to be the group of nonsingular $n \times n$ complex matrices. Since an $n \times n$ matrix A is nonsingular if and only if $\det A \neq 0$, $\mathrm{GL}(n,\mathbb{C})$ is an open subset of $\mathbb{C}^{n\times n}$, the vector space of $n \times n$ complex matrices. For the same reason as in the real case, $\mathrm{GL}(n,\mathbb{C})$ is a Lie group of dimension $2n^2$.

15.2 Lie Subgroups

Definition 15.6. A *Lie subgroup* of a Lie group G is (i) an abstract subgroup H which is (ii) an *immersed submanifold* via the inclusion map so that (iii) the group operations on H are C^∞.

An *abstract subgroup* simply means a subgroup in the algebraic sense, in contrast to a *Lie subgroup*. For an explanation of why a Lie subgroup is defined to be an immersed submanifold instead of a regular submanifold, see Remark 16.13.

Because a Lie subgroup is an immersed submanifold, it need not have the relative topology. In particular, the inclusion map $i\colon H \to G$ need not be continuous.

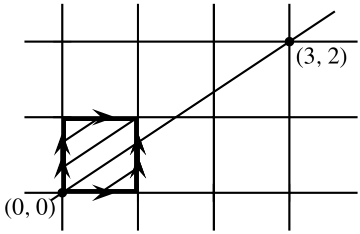


Fig. 15.1. An embedded Lie subgroup of the torus.

Example 15.7 (*Lines with irrational slope in a torus*). Let G be the torus $\mathbb{R}^2/\mathbb{Z}^2$ and L a line through the origin in $\mathbb{R}^2$. The torus can also be represented by the unit square with the opposite edges identified. The image H of L in $\mathbb{R}^2/Z^2$ is a closed curve if and only if the line L goes through another lattice point, say $(m, n) \in Z^2$. This is the case if and only if the slope of L is n/m, a rational number; then H consists of finitely many lines segments on the unit square and is a regular submanifold of $\mathbb{R}^2/\mathbb{Z}^2$ (Figure 15.1).

If the slope of L is irrational, then its image H on the torus will never close up. Indeed, it can be shown that H is a dense subset of the torus [2, Example III.6.15, p. 86]. Thus, H is an immersed submanifold but not a regular submanifold of the torus $\mathbb{R}^2/\mathbb{Z}^2$.

Whatever the slope of L, its image H in $\mathbb{R}^2/\mathbb{Z}^2$ is an abstract subgroup of the torus, an immersed submanifold, and a Lie group. Therefore, H is a Lie subgroup of the torus.

Proposition 15.8. *If H is an abstract subgroup and a regular submanifold of a Lie group G, then it is a Lie subgroup of G.*

Proof. Since a regular submanifold is the image of an embedding (Theorem 11.18), it is also an immersed submanifold.

Let $\mu\colon G \times G \to G$ be the multiplication map on G. Since H is a regular submanifold of G, the inclusion map $H \hookrightarrow G$ is C^∞. Hence, the inclusion map

$H \times H \hookrightarrow G$ is C^∞, and the composition $\mu \circ i \colon H \times H \to G$ is C^∞. By Theorem 11.20, the induced map $\bar{\mu} \colon H \times H \to H$ is C^∞, again because H is a regular submanifold.

The smoothness of the inverse map $\bar{\iota} \colon H \to H$ can be deduced from the smoothness of $\iota \colon G \to G$ in the same way. □

A subgroup H as in Proposition 15.8 is called an *embedded Lie subgroup*, because the inclusion map $i \colon H \to G$ of a regular submanifold is an embedding (Theorem 11.18).

Example 15.9. We showed in Examples 15.3 and 15.4 that the subgroups $\mathrm{SL}(n, \mathbb{R})$ and $O(n)$ of $\mathrm{GL}(n, \mathbb{R})$ are both regular submanifolds. By Proposition 15.8 they are embedded Lie subgroups.

We state without proof an important theorem about Lie subgroups. If G is a Lie group, then an abstract subgroup that is a closed subset in the topology of G is called a *closed subgroup*.

Theorem 15.10 (Closed subgroup theorem). *A closed subgroup of a Lie group is an embedded Lie subgroup.*

For a proof of the closed subgroup theorem, see [19, Theorem 3.42, p. 110].

Example 15.11.

(i) The lines with irrational slope in the torus $\mathbb{R}^2/\mathbb{Z}^2$ are not closed subgroups, since they are not regular submanifolds.

(ii) The special linear group $\mathrm{SL}(n, \mathbb{R})$ and the orthogonal group $O(n)$ are the zero sets of polynomial equations on $\mathrm{GL}(n, \mathbb{R})$. As such, they are closed subsets of $\mathrm{GL}(n, \mathbb{R})$. By the closed subgroup theorem, $\mathrm{SL}(n, \mathbb{R})$ and $O(n)$ are embedded Lie subgroups of $\mathrm{GL}(n, \mathbb{R})$.

15.3 The Matrix Exponential

Given an $n \times n$ matrix X, we define its exponential e^X by the same formula as the exponential of a real number:.

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots, \tag{15.3}$$

where I is the $n \times n$ identity matrix. For this formula to make sense, we need to show that the series on the right converges in the normed vector space $\mathbb{R}^{n\times n} \simeq \mathbb{R}^{n^2}$, with the Euclidean norm

$$\|A\| = \left(\sum a_{ij}^2\right)^{\frac{1}{2}}.$$

By a standard theorem of real analysis (cf. [12, Proposition 2.7.4, p. 121]), the convergence of a series of matrices is equivalent to the convergence of the (i, j)-entry of the series as a series of real numbers for every (i, j); in this case, it is the convergence of the series

$$\delta_{ij} + x_{ij} + \frac{1}{2!}(X^2)_{ij} + \frac{1}{3!}(X^3)_{ij} + \cdots . \tag{15.4}$$

Let $a = \max_{1 \le i,j \le n} |x_{ij}|$. Then

$$|(X^2)_{ij}| = \left| \sum_{k=1}^{n} x_{ik}x_{kj} \right| \le \sum |x_{ik}||x_{kj}| \le \sum_{k=1}^{n} a^2 = na^2,$$
$$|(X^3)_{ij}| = \left| \sum (X^2)_{ik}x_{kj} \right| \le \sum |(X^2)_{ik}||x_{kj}| \le \sum na^2a = n^2a^3.$$

By induction, one shows that

$$|(X^\ell)_{ij}| = \left| \sum_{k=1}^{n} (X^{\ell-1})_{ik}x_{kj} \right| \le \sum_{k=1}^{n} n^{\ell-2}a^{\ell-1}a = n^{\ell-1}a^\ell \le (na)^\ell.$$

So the series (15.4) is bounded by

$$1 + (na) + \frac{1}{2!}(na)^2 + \frac{1}{3!}(na)^3 + \cdots = e^{na}.$$

By the comparison test for series, the series (15.3) converges absolutely for any $n \times n$ matrix X.

NOTATION. Following standard convention we use the letter e for the exponential map and for the identity element of a general Lie group. The context should prevent any confusion. We sometimes write $\exp(X)$ for e^X.

Unlike the exponential of real numbers, when A and B are $n \times n$ matrices with $n > 1$, it is not necessarily true that

$$e^A e^B = e^{A+B}.$$

Exercise 15.12 (Exponentials of commuting matrices). Prove that if A and B are commuting $n \times n$ matrices, then

$$e^A e^B = e^{A+B}.$$

Proposition 15.13. *For* $X \in \mathbb{R}^{n\times n}$,

$$\frac{d}{dt}e^{tX} = Xe^{tX} = e^{tX}X.$$

Proof. Because each (i, j)-entry of the series for the exponential function e^{tX} is a power series in t, it is possible to differentiate term by term [17, Theorem 8.1, p. 173]. Hence

$$\begin{aligned}
\frac{d}{dt}e^{tX} &= \frac{d}{dt}\left(I + tX + \frac{1}{2!}t^2X^2 + \frac{1}{3!}t^3X^3 + \cdots\right) \\
&= X + tX^2 + \frac{1}{2!}t^2X^3 + \cdots \\
&= X\left(I + tX + \frac{1}{2!}t^2X^2 + \cdots\right) = Xe^{tX}.
\end{aligned}$$

In the second equality above, one could have factored out X as the second factor:

$$\begin{aligned}\frac{d}{dt}e^{tX} &= X + tX^2 + \frac{1}{2!}t^2X^3 + \cdots \\ &= \left(I + tX + \frac{1}{2!}t^2X^2 + \cdots\right)X = e^{tX}X. \qquad \square\end{aligned}$$

The definition of the matrix exponential e^X makes sense even if X is a complex matrix. All the arguments so far carry over word for word; one merely has to interpret $|a_{ij}|$ not as absolute value, but as the modulus of a complex number a_{ij}.

15.4 The Trace of a Matrix

Define the *trace* of an $n \times n$ matrix X to be the sum of its diagonal entries:

$$\operatorname{tr}(X) = \sum_{i=1}^{n} x_{ii}.$$

Lemma 15.14.
(i) *For any two* $A, B \in \mathbb{R}^{n\times n}$, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.
(ii) *For* $X \in \mathbb{R}^{n\times n}$ *and* $A \in \mathrm{GL}(n, \mathbb{R})$, $\operatorname{tr}(AXA^{-1}) = \operatorname{tr}(X)$.

Proof.
(i)

$$\begin{aligned}\operatorname{tr}(AB) &= \sum_i (AB)_{ii} = \sum_i \sum_k a_{ik}b_{ki}, \\ \operatorname{tr}(BA) &= \sum_k (BA)_{kk} = \sum_k \sum_i b_{ki}a_{ik}.\end{aligned}$$

(ii) Set $B = XA^{-1}$ in (i). $\square$

The eigenvalues of an $n \times n$ matrix A are the roots of the polynomial equation $\det(\lambda I - A) = 0$. Over the field of complex numbers, which is algebraically closed, such an equation necessarily has n roots, counted with multiplicity. Thus, the advantage of allowing complex numbers is that every $n \times n$ matrix, real or complex, has n complex eigenvalues, counted with multiplicity, whereas a real matrix need not have any real eigenvalue.

Example 15.15. The real matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

has no real eigenvalues. It has two complex eigenvalues $\pm i$.

By a theorem from algebra, any complex matrix X can be triangularized; more precisely, there exists a nonsingular complex matrix A so that AXA^{-1} is upper triangular. Since the eigenvalues $\lambda_1, \dots, \lambda_n$ of X are the same as the eigenvalues of AXA^{-1}, the triangular matrix T must have the eigenvalues of X along its diagonal:

$$\begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

Proposition 15.16. *The trace of a matrix, real or complex, is equal to the sum of its* complex *eigenvalues.*

Proof. Suppose X has complex eigenvalues $\lambda_1, \dots, \lambda_n$. Then there exists a nonsingular matrix $A \in \mathrm{GL}(n, \mathbb{C})$ such that

$$AXA^{-1} = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

By Lemma 15.14,

$$\mathrm{tr}(X) = \mathrm{tr}(AXA^{-1}) = \sum \lambda_i. \qquad \square$$

Proposition 15.17. *For any* $X \in \mathbb{R}^{n \times n}$, $\det(e^X) = e^{\mathrm{tr}\, X}$.

Proof.
Case 1. Assume that X is upper triangular:

$$X = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

Then

$$e^X = \sum \frac{1}{k!} X^k = \sum \frac{1}{k!} \begin{bmatrix} \lambda_1^k & & * \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} & & * \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix}.$$

Hence, $\det e^X = \prod e^{\lambda_i} = e^{\sum \lambda_i} = e^{\mathrm{tr}\, X}$.

Case 2. Given a general matrix X, with eigenvalues $\lambda_1, \dots, \lambda_n$, we can find a nonsingular complex matrix A so that

$$AXA^{-1} = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix},$$

an upper triangular matrix. Then

$$\begin{aligned}
e^{AXA^{-1}} &= I + AXA^{-1} + \frac{1}{2!}(AXA^{-1})^2 + \frac{1}{3!}(AXA^{-1})^3 + \cdots \\
&= I + AXA^{-1} + A\left(\frac{1}{2!}X^2\right)A^{-1} + A\left(\frac{1}{3!}X^3\right)A^{-1} + \cdots \\
&= Ae^X A^{-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
\det e^X &= \det(Ae^X A^{-1}) = \det(e^{AXA^{-1}}) \\
&= e^{\operatorname{tr}(AXA^{-1})} \quad \text{(by Case 1, since } AXA^{-1} \text{ is upper triangular)} \\
&= e^{\operatorname{tr} X} \quad \text{(by Lemma 15.14).}
\end{aligned}$$ □

It follows from this proposition that the matrix exponential e^X is always nonsingular because $\det(e^X) = e^{\operatorname{tr} X}$ is never 0. This is one reason why the matrix exponential is so useful, for it allows us to write down explicitly a curve in $\mathrm{GL}(n, \mathbb{R})$ with a given initial point and a given initial velocity. For example, $c(t) = e^{tX} : \mathbb{R} \to \mathrm{GL}(n, \mathbb{R})$ is a curve in $\mathrm{GL}(n, \mathbb{R})$ with initial point I and initial velocity X, since

$$c(0) = e^{0X} = e^0 = I \quad \text{and} \quad c'(0) = \left.\frac{d}{dt}e^{tX}\right|_{t=0} = \left.Xe^{tX}\right|_{t=0} = X. \tag{15.5}$$

15.5 The Differential of det at the Identity

Let $\det\colon \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$ be the determinant map. The tangent space at I of $\mathrm{GL}(n, \mathbb{R})$ is the vector space $\mathbb{R}^{n\times n}$ and the tangent space to $\mathbb{R}$ at 1 is $\mathbb{R}$. So

$$\det_{*,I} \colon \mathbb{R}^{n\times n} \to \mathbb{R}.$$

Proposition 15.18. *For any* $X \in \mathbb{R}^{n\times n}$, $\det_{*,I}(X) = \operatorname{tr} X$.

Proof. We use curves at I to compute the differential (Proposition 8.17). As a curve $c(t)$ with $c(0) = I$ and $c'(0) = X$, we choose the matrix exponential $c(t) = e^{tX}$. Then

$$\begin{aligned}
\det_{*,I}(X) &= \left.\frac{d}{dt}\det(e^{tX})\right|_{t=0} = \left.\frac{d}{dt}e^{t\operatorname{tr} X}\right|_{t=0} \\
&= \left.(\operatorname{tr} X)e^{t\operatorname{tr} X}\right|_{t=0} = \operatorname{tr} X.
\end{aligned}$$ □

Problems

15.1. Product rule for matrix-valued functions
Let (a, b) be an open interval in $\mathbb{R}$. Suppose that $A\colon (a, b) \to \mathbb{R}^{m\times n}$ and $B\colon (a, b) \to \mathbb{R}^{n\times p}$ are differentiable maps. Prove that for $t \in (a, b)$,

$$\frac{d}{dt}A(t)B(t) = A'(t)B(t) + A(t)B'(t),$$

where $A'(t) = (dA/dt)(t)$.

15.2. Identity component of a Lie group
The *identity component* C_e of a Lie group G is the connected component of the identity element e in G. Let μ and ι be the multiplication map and the inverse map of G.

(a) For any $x \in C_e$, show that $\mu(\{x\} \times C_e) \subset C_e$. (*Hint*: Apply Proposition A.44.)
(b) Show that $\iota(C_e) \subset C_e$.
(c) Show that C_e is an open subset of G. (*Hint*: Apply Problem A.11.)
(d) Prove that C_e is itself a Lie group.

15.3.* Open subgroup of a connected Lie group
Prove that an open subgroup H of a connected Lie group G is equal to G.

15.4. Differential of the multiplication map
Let G be a Lie group with multiplication map $\mu : G \times G \to G$, and let $\ell_a : G \to G$ and $r_b : G \to G$ be left and right multiplication by a and $b \in G$, respectively. Show that the differential of μ at $(a,b) \in G \times G$ is

$$\mu_{*,(a,b)}(X_a, Y_b) = (r_b)_*(X_a) + (\ell_a)_*(Y_b) \quad \text{for } X_a \in T_a(G), \quad Y_b \in T_b(G).$$

15.5. Differential of the inverse map
Let G be a Lie group with multiplication map $\mu : G \times G \to G$, inverse map $\iota : G \to G$, and identity element e. Show that the differential of the inverse map at $a \in G$,

$$\iota_{*,a} : T_a G \to T_{a^{-1}} G,$$

is given by

$$\iota_{*,a}(Y_a) = -(r_{a^{-1}})_*(\ell_{a^{-1}})_* Y_a,$$

where $(r_{a^{-1}})_* = (r_{a^{-1}})_{*,e}$ and $(\ell_{a^{-1}})_* = (\ell_{a^{-1}})_{*,a}$.

15.6.* Differential of the determinant map at A
Show that the differential of the determinant map $\det : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}$ at $A \in \mathrm{GL}(n,\mathbb{R})$ is given by

$$\det{}_{*,A}(AX) = (\det A)(\operatorname{tr} X) \quad \text{for } X \in \mathbb{R}^{n\times n}. \tag{15.6}$$

15.7.* Special linear group
Use Problem 15.6 to show that 1 is a regular value of the determinant map. This gives a quick proof that the special linear group $\mathrm{SL}(n,\mathbb{R})$ is a regular submanifold of $\mathrm{GL}(n,\mathbb{R})$.

15.8. General linear group
For $r \in \mathbb{R}^\times := \mathbb{R} - \{0\}$, let M_r be the $n \times n$ matrix

$$M_r = \begin{bmatrix} r & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = [re_1\ e_2\ \cdots\ e_n],$$

where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$. Prove that the map

$$\begin{aligned} f\colon \mathrm{GL}(n, \mathbb{R}) &\to \mathrm{SL}(n, \mathbb{R}) \times \mathbb{R}^\times, \\ A &\mapsto (AM_{1/\det A}, \det A), \end{aligned}$$

is a diffeomorphism.

15.9. Orthogonal group
Show that the orthogonal group $O(n)$ is compact by proving the following two statements.

(a) $O(n)$ is a closed subset of $\mathbb{R}^{n\times n}$.
(b) $O(n)$ is a bounded subset of $\mathbb{R}^{n\times n}$.

15.10. Special orthogonal group SO(2)
The *special orthogonal group* $\mathrm{SO}(n)$ is defined to be the subgroup of $O(n)$ consisting of matrices of determinant 1. Show that every matrix $A \in \mathrm{SO}(2)$ can be written in the form

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some real number θ. Then prove that $\mathrm{SO}(2)$ is diffeomorphic to the circle S^1.

15.11. Unitary group
The *unitary group* $U(n)$ is defined to be

$$U(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid \bar{A}^T A = I\},$$

where $\bar{A}$ denotes the complex conjugate of A, the matrix obtained from A by conjugating every entry of A: $(\bar{A})_{ij} = \overline{a_{ij}}$. Show that $U(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{C})$ and that $\dim U(n) = n^2$.

15.12. Special unitary group SU(2)
The *special unitary group* $\mathrm{SU}(n)$ is defined to be the subgroup of $U(n)$ consisting of matrices of determinant 1.

(a) Show that $\mathrm{SU}(2)$ can also be described as the set

$$\mathrm{SU}(2) = \left\{ \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} \in \mathbb{C}^{2\times 2} \mid a\bar{a} + b\bar{b} = 1 \right\}.$$

(*Hint*: Write out the condition $A^{-1} = \bar{A}^T$ in terms of the entries of A.)

(b) Show that $\mathrm{SU}(2)$ is diffeomorphic to the three-dimensional sphere

$$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}.$$

15.13. A matrix exponential
Compute

$$\exp\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right).$$

15.14. Complex symplectic group
Let J be the $2n \times 2n$ matrix

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

where I_n denotes the $n \times n$ identity matrix. The *complex symplectic group* $\mathrm{Sp}(2n, \mathbb{C})$ is defined to be

$$\mathrm{Sp}(2n, \mathbb{C}) = \{A \in \mathrm{GL}(2n, \mathbb{C}) \mid A^T J A = J\}.$$

Show that $\mathrm{Sp}(2n, \mathbb{C})$ is a regular submanifold of $\mathrm{GL}(2n, \mathbb{C})$ and compute its dimension. (*Hint*: Mimic Example 15.4. It is crucial to choose a correct target space for the map $f(A) = A^T J A$.)

15.15. The compact symplectic group
The *compact symplectic group* $\mathrm{Sp}(n)$ is defined to be $U(2n) \cap \mathrm{Sp}(2n, \mathbb{C})$. Let $f\colon U(2n) \to U(2n)$ be the map $f(A) = A^T J A$. Show that f has constant rank on $U(n)$ and prove that $\mathrm{Sp}(n)$ is a regular submanifold of $U(2n)$. (*Hint*: Mimic Example 11.3.)

16

Lie Algebras

16.1 Tangent Space at the Identity of a Lie Group

Because of the existence of a multiplication, a Lie group is a very special kind of manifold. Let $\ell_g : G \to G$ denote left multiplication by $g \in G$:

$$\ell_g(x) = gx.$$

Then ℓ_g is a diffeomorphism with inverse $\ell_{g^{-1}}$. The diffeomorphism ℓ_g takes the identity e to the element g and induces an isomorphism of tangent spaces

$$\ell_{g*} = \ell_{g*,e} : T_e(G) \to T_g(G).$$

Thus, if we can describe the tangent space $T_e(G)$ at the identity, then $\ell_{g*}T_e(G)$ will give a description of the tangent space $T_g(G)$ at any point $g \in G$.

The tangent space T_eG at the identity of a Lie group canonically has the structure of a Lie algebra. This Lie algebra encodes in it much information about the Lie group. The goal of this chapter is to define the Lie algebra structure on T_eG and to identify this Lie algebra for a few classical groups.

Example 16.1 (*The tangent space to* $\mathrm{GL}(n, \mathbb{R})$ *at* I). Since $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$, the vector space of all $n \times n$ real matrices, the tangent space to $\mathrm{GL}(n, \mathbb{R})$ at the identity I is $\mathbb{R}^{n\times n}$ itself.

16.2 The Tangent Space to $\mathrm{SL}(n, \mathbb{R})$ at I

We begin by finding a condition that a tangent vector X in $T_I(\mathrm{SL}(n, \mathbb{R}))$ must satisfy. By Proposition 8.16 there is a curve $c : (-\epsilon, \epsilon) \to \mathrm{SL}(n, \mathbb{R})$ with $c(0) = I$ and $c'(0) = X$. Being in $\mathrm{SL}(n, \mathbb{R})$, this curve satisfies

$$\det c(t) = 1$$

for all t in the domain $(-\epsilon, \epsilon)$. We now differentiate both sides with respect to t and evaluate at $t = 0$. On the left-hand side,

$$\begin{aligned}
\frac{d}{dt}\det(c(t))\bigg|_{t=0} &= (\det \circ c)_* \left(\frac{d}{dt}\bigg|_0\right) \\
&= \det_{*,I}\left(c_* \frac{d}{dt}\bigg|_0\right) && \text{(by the chain rule)} \\
&= \det_{*,I}(c'(0)) \\
&= \det_{*,I}(X) \\
&= \operatorname{tr}(X) && \text{(by Proposition 15.18)}
\end{aligned}$$

Thus,

$$\operatorname{tr}(X) = \frac{d}{dt}1\bigg|_{t=0} = 0.$$

So the tangent space $T_I(\mathrm{SL}(n, \mathbb{R}))$ is contained in the subspace V of $\mathbb{R}^{n\times n}$ defined by

$$V = \{X \in \mathbb{R}^{n\times n} \mid \operatorname{tr} X = 0\}.$$

Since $\dim V = n^2 - 1 = \dim T_I(\mathrm{SL}(n, \mathbb{R}))$, the two spaces must be equal.

Proposition 16.2. *The tangent space $T_I(\mathrm{SL}(n, \mathbb{R}))$ is the subspace of $\mathbb{R}^{n\times n}$ consisting of all $n \times n$ matrices of trace* 0.

16.3 The Tangent Space to $O(n)$ at I

Let X be a tangent vector to $O(n)$ at the identity I. Choose a curve $c(t)$ in $O(n)$ with $c(0) = I$ and $c'(0) = X$. Since $c(t)$ is in $O(n)$,

$$c(t)^T c(t) = I.$$

Differentiating both sides with respect to t gives

$$c'(t)^T c(t) + c(t)^T c'(t) = 0.$$

Evaluating at $t = 0$ gives

$$X^T + X = 0.$$

Thus, X is a skew-symmetric matrix.

Let K_n be the space of all $n \times n$ real skew-symmetric matrices. For example, for $n = 3$, these are matrices of the form

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}, \quad \text{where } a, b, c, \in \mathbb{R}.$$

The diagonal entries of such a matrix are all 0 and the entries below the diagonal are determined by those above the diagonal. So

$$\begin{aligned}\dim K_n &= \frac{n^2 - \#\text{ diagonal entries}}{2} \\ &= \frac{1}{2}(n^2 - n).\end{aligned}$$

We have shown that

$$T_I(O(n)) \subset K_n. \tag{16.1}$$

By an earlier computation (15.2),

$$\dim T_I(O(n)) = \dim O(n) = \frac{n^2 - n}{2}.$$

Since the two vector spaces in (16.1) have the same dimension, equality holds in (16.1).

16.4 Left-Invariant Vector Fields on a Lie Group

Let X be a vector field on a Lie group G. We do not assume X to be C^∞. For any $g \in G$, because left multiplication $\ell_g : G \to G$ is a diffeomorphism, the pushforward $\ell_{g*}X$ is a well-defined vector field on G. We say that the vector field X is *left-invariant* if

$$\ell_{g*}X = X$$

for every $g \in G$; this means for any $h \in G$,

$$\ell_{g*}(X_h) = X_{gh}.$$

In other words, a vector field X is left-invariant if and only if it is ℓ_g-related to itself for all $g \in G$.

Clearly, a left-invariant vector field X is completely determined by its value X_e at the identity, since

$$X_g = \ell_{g*}(X_e). \tag{16.2}$$

Conversely, given a tangent vector $X_e \in T_e(G)$ we can define a vector field X on G by (16.2). So defined, the vector field X is left-invariant, since

$$\begin{aligned}\ell_{g*}(X_h) &= \ell_{g*}\ell_{h*}X_e \\ &= (\ell_g \circ \ell_h)_* X_e \quad \text{(by the chain rule)} \\ &= (\ell_{gh})_*(X_e) \\ &= X_{gh}.\end{aligned}$$

Thus, there is a one-to-one correspondence

$$\begin{aligned} T_e(G) &\leftrightarrow L(G) := \{\text{left-invariant vector fields on } G\}, \\ X_e &\leftrightarrow X, \quad \text{with } X_g = \ell_{g*}(X_e). \end{aligned} \tag{16.3}$$

If $X_g = \ell_{g*}(X_e)$ for all $g \in G$, we call X the *left-invariant vector field on G generated by* X_e. The set $L(G)$ of left-invariant vector fields on G is obviously a vector space and the correspondence above is an isomorphism of vector spaces.

Example 16.3 (*Left-invariant vector fields on* $\mathbb{R}$). On the Lie group $\mathbb{R}$, the group operation is addition and the identity element is 0. So "left multiplication" ℓ_g is actually addition:

$$\ell_g(x) = g + x.$$

Let us compute $\ell_{g*}(d/dx|_0)$. Since $\ell_{g*}(d/dx|_0)$ is a tangent vector at g, it is a scalar multiple of $d/dx|_g$:

$$\ell_{g*}\left(\left.\frac{d}{dx}\right|_0\right) = a \left.\frac{d}{dx}\right|_g. \tag{16.4}$$

To evaluate a, apply both sides of (16.4) to x:

$$a = a\left.\frac{d}{dx}\right|_g x = \ell_{g*}\left(\left.\frac{d}{dx}\right|_0\right) x = \left.\frac{d}{dx}\right|_0 x \circ \ell_g = \left.\frac{d}{dx}\right|_0 g + x = 1.$$

Thus,

$$\ell_{g*}\left(\left.\frac{d}{dx}\right|_0\right) = \left.\frac{d}{dx}\right|_g.$$

This shows that d/dx is a left-invariant vector field on $\mathbb{R}$. Therefore, the left-invariant vector fields on $\mathbb{R}$ are constant multiples of d/dx.

Example 16.4 (*Left-invariant vector fields on* $\mathrm{GL}(n,\mathbb{R})$). Since $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$, at any $g \in \mathrm{GL}(n,\mathbb{R})$ there is a canonical identification of the tangent space $T_g(\mathrm{GL}(n,\mathbb{R}))$ with $\mathbb{R}^{n\times n}$:

$$\sum a_{ij} \left.\frac{\partial}{\partial x_{ij}}\right|_g \leftrightarrow [a_{ij}]. \tag{16.5}$$

Let $B = \sum b_{ij}\partial/\partial x_{ij}|_I \in T_I(\mathrm{GL}(n,\mathbb{R}))$ and let $\tilde{B}$ be the left-invariant vector field on $\mathrm{GL}(n,\mathbb{R})$ generated by B. By Example 8.18,

$$\tilde{B}_g = (\ell_g)_* B \leftrightarrow gB$$

under the identification (16.5). In terms of the standard basis $\partial/\partial x_{ij}|_g$,

$$\tilde{B}_g = \sum_{i,j} (gB)_{ij} \left.\frac{\partial}{\partial x_{ij}}\right|_g = \sum_{i,j}\left(\sum_k g_{ik}b_{kj}\right)\left.\frac{\partial}{\partial x_{ij}}\right|_g.$$

Proposition 16.5. *Any left-invariant vector field X on a Lie group G is C^∞.*

Proof. By Proposition 14.1 it suffices to show that for any C^∞ function f on G, the function Xf is also C^∞. Choose a C^∞ curve $c\colon \mathbb{R} \to G$ such that $c(0) = e$ and $c'(0) = X_e$. If $g \in G$, then $gc(t)$ is a curve at g with initial vector X_g, for $gc(0) = ge = g$ and

$$(gc)'(0) = \ell_{g*}c'(0) = \ell_{g*}X_e = X_g.$$

By Proposition 8.17,

$$(Xf)(g) = X_g f = \left.\frac{d}{dt}\right|_{t=0} f(gc(t)).$$

Now the function $f(gc(t))$ is a composition of C^∞ functions

$$\begin{array}{ccccccc} G \times \mathbb{R} & \xrightarrow{1\times c} & G \times G & \xrightarrow{\mu} & G & \xrightarrow{f} & \mathbb{R}, \\ (g,t) & \mapsto & (g, c(t)) & \mapsto & gc(t) & \mapsto & f(gc(t)); \end{array}$$

as such, it is C^∞. Its derivative with respect to t,

$$F(g,t) := \frac{d}{dt} f(gc(t)),$$

is therefore also C^∞. Since $(Xf)(g)$ is the composition of C^∞ functions,

$$\begin{array}{ccccc} G & \to & G \times \mathbb{R} & \xrightarrow{F} & \mathbb{R}, \\ g & \mapsto & (g,0) & \mapsto & F(g,0) = \left.\frac{d}{dt}\right|_{t=0} f(gc(t)), \end{array}$$

it is a C^∞ function on G. This proves that X is a C^∞ vector field on G. □

It follows from this proposition that the vector space $L(G)$ of left-invariant vector fields on G is a subspace of the vector space $\mathfrak{X}(G)$ of all C^∞ vector fields on G.

Proposition 16.6. *If X and Y are left-invariant vector fields on G, then so is* $[X, Y]$.

Proof. For any g in G, X is ℓ_g-related to X and Y is ℓ_g-related to Y. By Proposition 14.19, $[X, Y]$ is ℓ_g-related to $[X, Y]$. □

16.5 The Lie Algebra of a Lie Group

A *Lie subalgebra* of a Lie algebra $\mathfrak{g}$ is a vector subspace $\mathfrak{h} \subset \mathfrak{g}$ that is closed under the bracket $[\ ,\ ]$. By Proposition 16.6, the space $L(G)$ of left-invariant vector fields on a Lie group G is closed under the Lie bracket $[\ ,\ ]$ and thus is a Lie subalgebra of the Lie algebra $\mathfrak{X}(G)$, the Lie algebra of all C^∞ vector fields on G.

Since the tangent space $T_e(G)$ is isomorphic to $L(G)$ as a vector space, it inherits a Lie bracket from $L(G)$. For $A \in T_eG$, denote by $\tilde{A}$ the left-invariant vector field generated by A:

$$\tilde{A}_g = \ell_{g*}(A) \quad \text{for any } g \in G.$$

If $A, B \in T_eG$, then their Lie bracket $[A, B] \in T_eG$ is defined to be

$$[A, B] = [\tilde{A}, \tilde{B}]_e.$$

Proposition 16.7. *If $A, B \in T_eG$ and $\tilde{A}, \tilde{B}$ are the left-invariant vector fields they generate, then*

$$[\tilde{A}, \tilde{B}] = [A, B]\tilde{}.$$

Proof. By Proposition 16.6, $[\tilde{A}, \tilde{B}]$ is a left-invariant vector field. Thus both $[\tilde{A}, \tilde{B}]$ and $[A, B]\tilde{}$ are left-invariant vector fields whose value at e is $[A, B]$. Since a left-invariant vector field is determined by its value at e, the two vector fields are equal. □

With the Lie bracket [,], the tangent space $T_e(G)$ becomes a Lie algebra, called the *Lie algebra* of the Lie group G. As a Lie algebra, $T_e(G)$ is usually denoted by $\mathfrak{g}$.

16.6 The Lie Bracket on $\mathfrak{gl}(n, \mathbb{R})$

For the general linear group $\mathrm{GL}(n, \mathbb{R})$, the tangent space at the identity I can be identified with the vector space $\mathbb{R}^{n\times n}$ of all $n \times n$ real matrices. We identified a tangent vector in $T_I(\mathrm{GL}(n, \mathbb{R}))$ with a matrix $A \in \mathbb{R}^{n\times n}$ via

$$\sum a_{ij} \frac{\partial}{\partial x_{ij}}\bigg|_I \longleftrightarrow [a_{ij}]. \tag{16.6}$$

The tangent space $T_I\,\mathrm{GL}(n, \mathbb{R})$ with its Lie algebra structure is denoted $\mathfrak{gl}(n, \mathbb{R})$. Let $\tilde{A}$ be the left-invariant vector field on $\mathrm{GL}(n, \mathbb{R})$ generated by A. Then on the Lie algebra $\mathfrak{gl}(n, \mathbb{R})$ we have the Lie bracket $[A, B] = [\tilde{A}, \tilde{B}]_I$ coming from the Lie bracket of left-invariant vector fields. In the next proposition, we identify the Lie bracket in terms of matrices.

Proposition 16.8. *Let*

$$A = \sum a_{ij} \frac{\partial}{\partial x_{ij}}\bigg|_I, \qquad B = \sum b_{ij} \frac{\partial}{\partial x_{ij}}\bigg|_I \in T_I(\mathrm{GL}(n, \mathbb{R})).$$

If

$$[A, B] = [\tilde{A}, \tilde{B}]_I = \sum c_{ij} \frac{\partial}{\partial x_{ij}}\bigg|_I, \tag{16.7}$$

then

$$c_{ij} = \sum_k a_{ik}b_{kj} - b_{ik}a_{kj}.$$

Thus, if derivations are identified with matrices via (16.6)*, then*

$$[A, B] = AB - BA.$$

Proof. Applying both sides of (16.7) to x_{ij},

$$\begin{aligned}
c_{ij} &= [\tilde{A}, \tilde{B}]_I x_{ij} \\
&= \tilde{A}_I \tilde{B} x_{ij} - \tilde{B}_I \tilde{A} x_{ij} \\
&= A\tilde{B} x_{ij} - B\tilde{A} x_{ij} \quad (\text{because } \tilde{A}_I = A,\ \tilde{B}_I = B).
\end{aligned}$$

So it is necessary to find a formula for the function $\tilde{B}x_{ij}$.

In Example 16.4 we found that the left-invariant vector field $\tilde{B}$ on $\mathrm{GL}(n, \mathbb{R})$ is given by

$$\tilde{B}_g = \sum_{i,j} (gB)_{ij} \frac{\partial}{\partial x_{ij}}\bigg|_g \qquad \text{at } g \in \mathrm{GL}(n, \mathbb{R}).$$

Hence,

$$\tilde{B}_g x_{ij} = (gB)_{ij} = \sum_k g_{ik} b_{kj} = \sum_k b_{kj} x_{ik}(g).$$

This gives the formula for $\tilde{B}x_{ij}$ as

$$\tilde{B}x_{ij} = \sum_k b_{kj} x_{ik}.$$

It follows that

$$\begin{aligned}
A\tilde{B}x_{ij} &= \sum_{p,q} a_{pq} \frac{\partial}{\partial x_{pq}}\bigg|_I \left(\sum_k b_{kj} x_{ik}\right) = \sum_{p,q,k} a_{pq} b_{kj} \delta_{ip} \delta_{kq} \\
&= \sum_k a_{ik} b_{kj} = (AB)_{ij}.
\end{aligned}$$

Interchanging A and B gives

$$B\tilde{A}x_{ij} = \sum_k b_{ik} a_{kj} = (BA)_{ij}.$$

Therefore,

$$c_{ij} = \sum_k a_{ik} b_{kj} - b_{ik} a_{kj} = (AB - BA)_{ij}. \qquad \square$$

16.7 The Push-Forward of a Left-Invariant Vector Field

As we noted in Section 14.6, if $F\colon N \to M$ is a C^∞ map of manifolds and X is a C^∞ vector field on N, the push-forward $F_* X$ is in general not defined except when F is a diffeomorphism. However, by the correspondence between left-invariant vector fields on a Lie group and tangent vectors at the identity of the Lie group, one can push forward a left-invariant vector field under a C^∞ map of Lie groups. We show this now.

Recall that if H is a Lie group and $h \in H$, then $\ell_h\colon H \to H$ is left multiplication by h. From Section 16.4, every left-invariant vector field on a Lie group H is of the form $\tilde{A}$ for some $A \in T_e H$, with $\tilde{A}_h = (\ell_h)_* A$.

Definition 16.9. Let $F\colon H \to G$ be a C^∞ map of Lie groups. Define $F_*\colon L(H) \to L(G)$ by

$$F_*(\tilde{A}) = (F_*A)\tilde{}$$

for all $A \in T_eH$.

Definition 16.10. A map $F\colon H \to G$ between two Lie groups H and G is a *Lie group homomorphism* if it is a C^∞ map and a group homomorphism.

The group homomorphism condition means that for all $h, x \in H$,

$$F(hx) = F(h)F(x). \tag{16.8}$$

This may be rewritten in functional notation as

$$F \circ \ell_h = \ell_{F(h)} \circ F \quad \text{for all } h \in H. \tag{16.9}$$

Let e_H and e_G be the identity elements of H and G, respectively. Taking h and x in (16.8) to be the identity e_H, it follows that $F(e_H) = e_G$. So a group homomorphism always maps the identity to the identity.

Proposition 16.11. *If $F\colon H \to G$ is a Lie group homomorphism and $A \in T_eH$ is a tangent vector of H at the identity e of H, then the left-invariant vector field $F_*\tilde{A}$ on G is F-related to the left-invariant vector field $\tilde{A}$ on H.*

Proof. For $h \in H$,

$$\begin{aligned}
F_*(\tilde{A}_h) &= F_*(\ell_{h*}A) && \text{(definition of } \tilde{A}) \\
&= (F \circ \ell_h)_*A && \text{(chain rule)} \\
&= (\ell_{F(h)} \circ F)_*A && (F \text{ is a Lie group homomorphism}) \\
&= \ell_{F(h)*}F_*A && \text{(chain rule again)} \\
&= ((F_*A)\tilde{})_{F(h)} && \text{(definition of } (\ )\tilde{}) \\
&= (F_*\tilde{A})_{F(h)}.
\end{aligned}$$

□

16.8 The Differential as a Lie Algebra Homomorphism

Proposition 16.12. *If $F\colon H \to G$ is a Lie group homomorphism, then its differential at the identity,*

$$F_* = F_{*,e}\colon T_eH \to T_eG,$$

is a Lie algebra homomorphism, i.e., a linear map such that for all $A, B \in T_eH$,

$$F_*[A, B] = [F_*A, F_*B].$$

Proof. By Proposition 16.11, the vector field $F_*\tilde{A}$ on G is F-related to the vector field $\tilde{A}$ on H, and the vector field $F_*\tilde{B}$ is F-related to $\tilde{B}$ on H. Hence, the bracket $[F_*\tilde{A}, F_*\tilde{B}]$ on G is F-related to the bracket $[\tilde{A}, \tilde{B}]$ on H (Proposition 14.19). This means that

$$F_*\left([\tilde{A}, \tilde{B}]_e\right) = [F_*\tilde{A}, F_*\tilde{B}]_{F(e)} = [F_*\tilde{A}, F_*\tilde{B}]_e.$$

The left-hand side of this equality is $F_*[A, B]$, while the right-hand side is

$$\begin{aligned} [F_*\tilde{A}, F_*\tilde{B}]_e &= [(F_*A)\tilde{\ }, (F_*B)\tilde{\ }]_e \quad &&\text{(definition of } F_*\tilde{A}) \\ &= [F_*A, F_*B] &&\text{(definition of } [\ ,\ ] \text{ on } T_eG). \end{aligned}$$

Equating the two sides gives

$$F_*[A, B] = [F_*A, F_*B]. \qquad \square$$

Suppose H is a Lie subgroup of a Lie group G, with inclusion map $i\colon H \to G$. Since i is an immersion, its differential

$$i_*\colon T_eH \to T_eG$$

is injective. By Proposition 16.12, for $X, Y \in T_eH$,

$$i_*([X, Y]_{T_eH}) = [i_*X, i_*Y]_{T_eG}. \tag{16.10}$$

This shows that if T_eH is identified with a subspace of T_eG via i_*, then the bracket on T_eH is the restriction of the bracket on T_eG to T_eH. Thus, the Lie algebra of a Lie subgroup H may be identified with a Lie subalgebra of the Lie algebra of G.

In general, the Lie algebras of the classical groups are denoted by gothic letters. For example, the Lie algebras of $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R})$, $O(n)$, and $U(n)$ are denoted $\mathfrak{gl}(n, \mathbb{R})$, $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{o}(n)$, and $\mathfrak{u}(n)$, respectively. By (16.10) and Proposition 16.8, the Lie algebra structures on $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{o}(n)$, and $\mathfrak{u}(n)$ are given by

$$[A, B] = AB - BA, \quad \text{as on } \mathfrak{gl}(n, \mathbb{R}).$$

Remark 16.13. A fundamental theorem in Lie group theory asserts the existence of a one-to-one correspondence between the connected Lie subgroups of a Lie group G and the Lie subalgebras of its Lie algebra $\mathfrak{g}$ [19, Theorem 3.19, Corollary (a), p. 95]. For the torus $\mathbb{R}^2/\mathbb{Z}^2$, the Lie algebra $\mathfrak{g}$ is $\mathbb{R}^2$ and the one-dimensional Lie subalgebras are all the lines through the origin. According to the theorem, the one-dimensional connected Lie subgroups of the torus are the images of all the lines through the origin. It is because of this theorem that a Lie subgroup is defined to be an *immersed* submanifold. In the example of the torus, the one-dimensional embedded Lie subgroups correspond to only the lines with rational slope through the origin in $\mathbb{R}^2$, not to all one-dimensional subalgebras of the Lie algebra.

Problems

16.1. Skew-Hermitian matrices

A complex matrix $X \in \mathbb{C}^{n\times n}$ is said to be *skew-Hermitian* if its transpose conjugate $\bar{X}^T = -X$. Let V be the vector space of $n \times n$ skew-Hermitian matrices. Show that $\dim V = n^2$.

16.2. Tangent space at I of a unitary group

Show that the tangent space at the identity I of the unitary group $U(n)$ is the vector space of $n \times n$ skew-Hermitian matrices.

16.3. Lie algebra of a complex symplectic group

(a) Show that the tangent space at the identity I of $\mathrm{Sp}(2n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{C})$ is the vector space of all $2n \times 2n$ complex matrices X such that JX is symmetric.
(b) Calculate the dimension of $\mathrm{Sp}(2n, \mathbb{C})$.

16.4. Lie algebra of a compact symplectic group

Refer to Problem 15.15 for the definition and notations concerning the compact symplectic group $\mathrm{Sp}(n)$.

(a) Show that if $X \in T_I(\mathrm{Sp}(n))$, then X is skew-Hermitian and JX is symmetric.
(b) Let V be the vector space of $n \times n$ complex matrices X such that X is skew-Hermitian and JX is symmetric. For $X \in V$, prove that the curve $c(t) = e^{tX}$ lies in $\mathrm{Sp}(n)$.
(c) Prove that $T_I(\mathrm{Sp}(n)) = V$.
(d) Suppose $a, b, c, d \in \mathbb{C}^{n\times n}$ and
$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{C}^{2n\times 2n}.$$
Show that $X \in V$ iff
$$X = \begin{bmatrix} a & b \\ -\bar{b} & -a \end{bmatrix},$$
with a skew-Hermitian and b symmetric.
(e) Compute the dimension of $\mathrm{Sp}(n)$ by computing $\dim V$.

16.5. Left-invariant vector fields on $\mathbb{R}^n$

Find the left-invariant vector fields on $\mathbb{R}^n$.

16.6. Tangent spaces to $\mathrm{GL}(n, \mathbb{R})$

Show that the tangent space to $\mathrm{GL}(n, \mathbb{R})$ at a point A is the left translate by A of the Lie algebra $\mathfrak{gl}(n, \mathbb{R})$.

16.7. Integral curves of a left-invariant vector field

Let $A \in \mathfrak{gl}(n, \mathbb{R})$ and let $\tilde{A}$ be the left-invariant vector field on $\mathrm{GL}(n, \mathbb{R})$ generated by A. Show that $c(t) = e^{tA}$ is the integral curve of $\tilde{A}$ starting at A. Find the integral curve of $\tilde{A}$ starting at $g \in \mathrm{GL}(n, \mathbb{R})$.

16.8. Parallelizable manifolds
A manifold whose tangent bundle is trivial is said to be *parallelizable*. If M is a manifold of dimension n, show that parallelizability is equivalent to the existence of a smooth frame $X_1, \ldots, X_n$ on M.

16.9. Parallelizability of a Lie group
Show that every Lie group is parallelizable.

16.10. The adjoint representation
Let G be a Lie group of dimension n with Lie algebra $\mathfrak{g}$.

(a) For each $a \in G$, the differential at the identity of the conjugation map $c(a) := \ell_a \circ r_{a^{-1}} \colon G \to G$ is a linear isomorphism $c(a)_* \colon \mathfrak{g} \to \mathfrak{g}$. Hence, $c(a)_* \in \mathrm{GL}(\mathfrak{g})$. Show that the map $\mathrm{Ad} \colon G \to \mathrm{GL}(\mathfrak{g})$ defined by $\mathrm{Ad}(a) = c(a)_*$, is a group homomorphism. It is called the *adjoint representation* of the Lie group G.
(b) Show that $\mathrm{Ad} \colon G \to \mathrm{GL}(\mathfrak{g})$ is C^∞.

17

Differential 1-Forms

Let M be a smooth manifold and p a point in M. The *cotangent space* of M at p, denoted by $T_p^*(M)$ or T_p^*M, is the dual space of the tangent space T_pM. An element of the cotangent space T_p^*M is called a *covector* at p. Thus, a covector ω_p at p is a linear function

$$\omega_p \colon T_pM \to \mathbb{R}.$$

A *covector field*, a *differential* 1*-form*, or more simply a 1*-form* on M, is a function that assigns to each point p in M a covector at p. In this sense it is dual to a vector field on M, which assigns to each point in M a tangent vector at p. The great utility of differential forms in manifold theory arises from the fact that they can be pulled back under a map. This is in contrast to vector fields, which in general cannot be pushed forward under a map.

17.1 The Differential of a Function

Definition 17.1. If f is a C^∞ function on a manifold M, its *differential* is defined to be the 1-form df on M such that for any $p \in M$ and $X_p \in T_pM$,

$$(df)_p(X_p) = X_p f.$$

In Section 8.2 we encountered another notion of the differential, for a map between manifolds. Let us compare the two notions of the differential.

Proposition 17.2. *If $f \colon M \to \mathbb{R}$ is a C^∞ function, then for $p \in M$ and $X_p \in T_pM$,*

$$f_*(X_p) = (df)_p(X_p) \left.\frac{\partial}{\partial x}\right|_{f(p)}.$$

Proof. Since $f_*(X_p) \in T_{f(p)}\mathbb{R}$, there is a real number a such that

$$f_*(X_p) = a \left.\frac{\partial}{\partial x}\right|_{f(p)}. \tag{17.1}$$

To evaluate a, apply both sides of (17.1) to x:

$$a = f_*(X_p)(x) = X_p(x \circ f) = X_p f = (df)_p(X_p). \qquad \square$$

This proposition shows that under the canonical identification of the tangent space $T_{f(p)}\mathbb{R}$ with $\mathbb{R}$ via

$$a \left.\frac{\partial}{\partial x}\right|_{f(p)} \leftrightarrow a,$$

f_* is the same as df. For this reason, we are justified in calling both of them the *differential* of f.

17.2 Local Expression for a Differential 1-Form

Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a coordinate chart on a manifold M. Then the differentials $dx^1, \ldots, dx^n$ are 1-forms on U.

Proposition 17.3. *At each point $p \in U$, the covectors $(dx^1)_p, \ldots, (dx^n)_p$ form a basis for the cotangent space T_p^*M dual to the basis $(\partial/\partial x^1)_p, \ldots, (\partial/\partial x^n)_p$ for the tangent space T_pM.*

Proof. The proof is just like the Euclidean case (Proposition 4.1):

$$(dx^i)_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \left.\frac{\partial}{\partial x^j}\right|_p x^i = \delta^i_j. \qquad \square$$

Thus, every 1-form ω on U can be written as a linear combination

$$\omega = \sum a_i\, dx^i,$$

where the coefficients a_i are functions on U.

In particular, if f is a C^∞ function on M, then the 1-form df, when restricted to U, must be a linear combination

$$df = \sum a_i\, dx^i.$$

To find a_j, we apply the usual trick of evaluating both sides on $\partial/\partial x^j$:

$$a_j = df\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j}.$$

This gives the local expression for df:

$$df = \sum \frac{\partial f}{\partial x^i}\, dx^i. \tag{17.2}$$

17.3 The Cotangent Bundle

The underlying set of the *cotangent bundle* T^*M of a manifold M is the disjoint union of the cotangent spaces at all the points of M:

$$T^*M := \coprod_{p\in M} T_p^*M = \bigcup_{p\in M} \{p\} \times T_p^*M.$$

Mimicking the construction of the tangent bundle, we can give T^*M a topology as follows. If $(U, \phi = (x^1, \dots, x^n))$ is a chart on M and $p \in U$, then each $\omega_p \in T_p^*M$ can be written uniquely as a linear combination

$$\omega_p = \sum c_i(\omega_p)\, dx^i|_p.$$

This gives rise to a bijection

$$\begin{aligned} \tilde{\phi}\colon T^*U &\to \phi(U) \times \mathbb{R}^n \\ (p, \omega_p \in T_p^*M) &\mapsto (\phi(p), c_1(\omega_p), \dots, c_n(\omega_p)). \end{aligned} \tag{17.3}$$

Using this bijection, we can transfer the topology of $\phi(U) \times \mathbb{R}^n$ to T^*U.

Now let $\mathcal{B}$ be the collection of all open subsets of T^*U, as U varies over all charts in the maximal atlas of M. As in Section 12.1, $\mathcal{B}$ satisfies the conditions for a collection of subsets of T^*M to be a basis. We give T^*M the topology generated by the basis $\mathcal{B}$. Just as for the tangent bundle, with the maps $\tilde{\phi}$ of (17.3) as coordinate maps, T^*M becomes a C^∞ manifold and in fact, a vector bundle of rank n over M, justifying its name as the *cotangent bundle*. It has a natural projection $\pi : T^*M \to M$ mapping (p, ω_p) to p.

In terms of the cotangent bundle, a 1-form on M is simply a section of the cotangent bundle T^*M, i.e., it is a map $\omega\colon M \to T^*M$ such that $\pi \circ \omega = 1_M$, the identity map on M. We say that a 1-form ω is C^∞ if it is C^∞ as a map: $M \to T^*M$.

17.4 Characterization of C^∞ 1-Forms

By definition a 1-form ω on an open set U in a manifold M is C^∞ if it is C^∞ as a section of the cotangent bundle T^*M over U. The following two propositions give alternate characterizations of a C^∞ 1-form.

Proposition 17.4. *A 1-form ω on a manifold M is C^∞ if and only if either of the following conditions holds:*

(i) *for every point $p \in M$, there is a chart $(U, x^1, \dots, x^n)$ about p such that if $\omega = \sum a_i\, dx^i$ on U, then the functions a_i are C^∞ on U;*
(ii) *for any chart $(U, x^1, \dots, x^n)$ on M, if $\omega = \sum a_i\, dx^i$ on U, then the functions a_i are C^∞ on U.*

Proof. A chart (U, ϕ) on M gives rise to a chart $(T^*U, \tilde{\phi})$ on the cotangent bundle T^*M where

$$\tilde{\phi}\colon T^*U \to \phi(U) \times \mathbb{R}^n,$$
$$(q, \sum c_i\, (dx^i)_q) \mapsto (\phi(q), c_1, \dots, c_n).$$

In this chart,

$$(\tilde{\phi} \circ \omega)_q = (\phi(q), c_1(\omega_q), \dots, c_n(\omega_q)).$$

If $\omega = \sum a_i\, dx^i$, then $a_i(q) = c_i(\omega_q)$. Note that the c_i's are function on T^*U, while the a_i's are functions on U. By the definition of a C^∞ map, the section $\omega\colon U \to T^*U$ is C^∞ if and only if the functions $a_1, \dots, a_n$ are C^∞ on U. Now (i) follows from the definition of a C^∞ map (Definition 6.3), and (ii) follows from Problem 6.6. □

As a corollary, if f is a C^∞ function on M, then df is a C^∞ 1-form, since the coefficients $\partial f/\partial x^i$ are all C^∞.

Proposition 17.5. *A* 1*-form ω on a manifold M is C^∞ if and only if for every C^∞ vector field X on M, the function $\omega(X)$ is C^∞ on M.*

Proof.
($\Rightarrow$) Suppose ω is a C^∞ 1-form and X is a C^∞ vector field on M. For any $p \in M$, choose a coordinate neighborhood $(U, x^1, \dots, x^n)$ about p. Then $\omega = \sum a_i\, dx^i$ and $X = \sum b^j \partial/\partial x^j$ for C^∞ functions a_i, b^j on U. On U

$$\omega(X) = \left(\sum a_i\, dx^i\right)\left(\sum b^j \frac{\partial}{\partial x^j}\right) = \sum a_i b^i,$$

which is C^∞ at p. Since p is an arbitrary point of M, the function $\omega(X)$ is C^∞ on M.

($\Leftarrow$) Suppose ω is a 1-form on M such that the function $\omega(X)$ is C^∞ for every C^∞ vector field X on M. For $p \in M$, choose a coordinate neighborhood $(U, x^1, \dots, x^n)$ about p. Then $\omega = \sum a_i\, dx^i$ on U.

Fix an integer j, $1 \le j \le n$. We can extend the C^∞ vector field $\partial/\partial x^j$ on U to a C^∞ vector field X on M that agrees with $\partial/\partial x^j$ in a neighborhood of p, as follows. Let $\sigma : M \to \mathbb{R}$ be a C^∞ bump function which is identically 1 on a neighborhood V of p and which has support contained in U. Define

$$X_q = \begin{cases} \sigma(q) \left.\dfrac{\partial}{\partial x^j}\right|_q & \text{for } q \in U, \\ 0 & \text{for } q \notin U. \end{cases}$$

Then X is C^∞ on M and so by hypothesis, $\omega(X)$ is C^∞ on M. Restricted to the open set V,

$$\omega(X) = \left(\sum a_i\, dx^i\right)\left(\frac{\partial}{\partial x^j}\right) = a_j.$$

This proves that a_j is C^∞ on the coordinate chart $(V, x^1, \dots, x^n)$. By Proposition 17.4, the 1-form ω is C^∞ on M. □

17.5 Pullback of 1-forms

Just as the differential of a smooth map $F\colon N \to M$ pushes forward a tangent vector at a point $p \in N$, so the codifferential (the dual of the differential)

$$F^*\colon T^*_{F(p)}M \to T^*_p N$$

pulls back a covector at the point $F(p)$.

However, while *vector fields* on N in general cannot be pushed forward to M, every covector field on M can be pulled back to N. If ω is a 1-form on M, we define its *pullback* $F^*\omega$ to be the 1-form on N given by

$$(F^*\omega)_p(X_p) = \omega_{F(p)}(F_*X_p)$$

for any $p \in N$ and $X_p \in T_pN$.

Note that $(F^*\omega)_p$ is simply the image of the covector $\omega_{F(p)}$ under the codifferential $F^*\colon T^*_{F(p)}M \to T^*_p N$.

Problems

17.1. A 1-form on $\mathbb{R}^2 - \{(0,0)\}$
Denote the standard coordinates on $\mathbb{R}^2$ by x, y, and let

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \quad \text{and} \quad Y = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

be vector fields on $\mathbb{R}^2$. Find a 1-form ω on $\mathbb{R}^2 - \{(0,0)\}$ such that $\omega(X) = 1$ and $\omega(Y) = 0$.

17.2. $\mathfrak{F}$-linearity of a 1-form
Let ω be a C^∞ 1-form on a manifold M. Show that if f is a C^∞ function and X a C^∞ vector field on M, then

$$\omega(fX) = f\omega(X).$$

Thus, a 1-form is linear over the C^∞ functions.

17.3. Transition formula for 1-forms
Suppose $(U, x^1, \dots, x^n)$ and $(V, y^1, \dots, y^n)$ are two charts on M with nonempty overlap $U \cap V$. Then a C^∞ 1-form ω on $U \cap V$ has two different local expressions:

$$\omega = \sum a_j\, dx^j = \sum b_i\, dy^i.$$

Find a formula for a_j in terms of b_i.

18

Differential k-Forms

We now generalize the construction of 1-forms on a manifold to k-forms. Recall that a *k-tensor* on a vector space V is a k-linear function

$$f\colon V \times \cdots \times V \to \mathbb{R}.$$

The k-tensor f is *alternating* if for any permutation $\sigma \in S_k$,

$$f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = (\operatorname{sgn} \sigma) f(v_1, \ldots, v_k). \tag{18.1}$$

When $k = 1$, the only element of the permutation group S_1 is the identity permutation. So for 1-tensors the condition (18.1) is vacuous and all 1-tensors are alternating (and symmetric too). An alternating k-tensor on V is also called a *k-covector* on V.

Denote by $A_k(V)$ the vector space of alternating k-tensors on V. From Section 3.10, if $\alpha^1, \ldots, \alpha^n$ is a basis for the 1-tensors on V, then a basis for $A_k(V)$ is

$$\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}, \quad 1 \le i_1 < \cdots < i_k \le n.$$

We apply this construction to the tangent space T_pM of a manifold M at a point p. The vector space $A_k(T_pM)$, usually denoted $\bigwedge^k(T_p^*M)$, is the space of all alternating k-tensors on the tangent space T_pM. A *k-covector field*, a *differential k-form*, or simply a *k-form* on M is a function ω that assigns to each point $p \in M$ a k-covector $\omega_p \in \bigwedge^k(T_p^*M)$. An n-form on a manifold of dimension n is also called a *top form.*

Example 18.1. On $\mathbb{R}^n$, at each point p there is a standard basis for the tangent space $T_p(\mathbb{R}^n)$:

$$\left.\frac{\partial}{\partial r^1}\right|_p, \ldots, \left.\frac{\partial}{\partial r^n}\right|_p.$$

Let $(dr^1)_p, \ldots, (dr^n)_p$ be the dual basis; this means that

$$(dr^i)_p \left(\left.\frac{\partial}{\partial r^j}\right|_p \right) = \delta^i_j.$$

As p varies over points in $\mathbb{R}^n$, we get differential forms $dr^1, \dots, dr^n$ on $\mathbb{R}^n$.

By Proposition 3.29 a basis for the alternating k-tensors in $\bigwedge^k(T_p^*\mathbb{R}^n)$ is

$$(dr^{i_1})_p \wedge \cdots \wedge (dr^{i_k})_p, \quad 1 \le i_1 < \cdots < i_k \le n.$$

If ω is a k-form on $\mathbb{R}^n$, then at each point $p \in \mathbb{R}^n$, ω_p is a linear combination:

$$\omega_p = \sum a_{i_1 \cdots i_k}(p)\, (dr^{i_1})_p \wedge \cdots \wedge (dr^{i_k})_p.$$

Omitting the point p, we write

$$\omega = \sum a_{i_1 \cdots i_k}\, dr^{i_1} \wedge \cdots \wedge dr^{i_k}.$$

In this expression the coefficients $a_{i_1 \cdots i_k}$ are functions on $\mathbb{R}^n$ because they vary with the point p. To simplify the notation, we introduce the *multi-index* $I = (i_1, \dots, i_k)$ and write

$$\omega = \sum a_I\, dr^I, \quad 1 \le i_1 < \cdots < i_k \le n,$$

where dr^I stands for $dr^{i_1} \wedge \cdots \wedge dr^{i_k}$.

18.1 Local Expression for a k-Form

Suppose $(U, x^1, \dots, x^n)$ is a coordinate chart on a manifold M. We have already defined the 1-forms $dx^1, \dots, dx^n$ on U. Since at each point $p \in U$, $(dx^1)_p, \dots, (dx^n)_p$ is a basis for T_p^*M, by Proposition 3.29 a basis for $\bigwedge^k(T_p^*M)$ is the set

$$(dx^{i_1})_p \wedge \cdots \wedge (dx^{i_k})_p, \quad 1 \le i_1 < \cdots < i_k \le n.$$

Thus, locally a k-form on U will be a linear combination $\omega = \sum a_I dx^I$, where the I are multi-indices and the a_I are functions on U.

Exercise 18.2 (Transition formula for a 2-form). We suppose $(U, x^1, \dots, x^n)$ and $(V, y^1, \dots, y^n)$ are two coordinate charts on M with $U \cap V \neq \varnothing$. Then a C^∞ 2-form ω on $U \cap V$ has two local expressions:

$$\omega = \sum_{i<j} a_{ij}\, dx^i \wedge dx^j = \sum_{k<\ell} b_{k\ell}\, dy^k \wedge dy^\ell.$$

Find the formula for a_{ij} in terms of $b_{k\ell}$ and the coordinate functions $x^1, \dots, x^n, y^1, \dots, y^n$. (*Hint*: If ω and τ are 1-forms and X and Y are vector fields, then by Example 3.20, $\omega \wedge \tau(X, Y) = \omega(X)\tau(Y) - \omega(Y)\tau(X)$.)

18.2 The Bundle Point of View

If V is a vector space, another common notation for the space $A_k(V)$ of alternating k-linear functions on V is $\bigwedge^k(V^*)$. Thus,

$$\begin{aligned}\textstyle\bigwedge^0(V^*) &= A_0(V) = \mathbb{R},\\ \textstyle\bigwedge^1(V^*) &= A_1(V) = V^*,\\ \textstyle\bigwedge^2(V^*) &= A_2(V), \quad \text{and so on.}\end{aligned}$$

To better understand differential forms, we mimic the construction of the tangent and cotangent bundles and form the set

$$\textstyle\bigwedge^k(T^*M) = \coprod_{p\in M} \bigwedge^k(T_p^*M) = \coprod_{p\in M} A_k(T_pM) = \bigcup_{p\in M}\{p\} \times A_k(T_pM)$$

of all alternating k-tensors at all points of M. This set is called the kth *exterior power* of the cotangent bundle. If (U, ϕ) is a coordinate chart on M, then there is a bijection

$$\begin{aligned}\textstyle\bigwedge^k(T^*U) = \bigcup_{p\in U}\{p\} \times \bigwedge^k(T_p^*U) &\simeq \phi(U) \times \mathbb{R}^{\binom{n}{k}},\\ (p, \omega_p) &\mapsto (\phi(p), \{a_I(\omega_p)\}_I),\end{aligned}$$

where $\omega_p = \sum a_I(\omega_p)\, dx^I$ on U and $I = (1 \le i_1 < \cdots < i_k \le n)$. In this way we can give $\bigwedge^k(T^*U)$ and hence $\bigwedge^k(T^*M)$ a topology and even a differentiable structure. The details are just like the construction of the tangent bundle, so we omit them. The upshot is that

$$\pi\colon \textstyle\bigwedge^k(T^*M) \to M$$

is a C^∞ vector bundle of rank $\binom{n}{k}$, where $n = \dim M$, and that a differential k-form is simply a section of this bundle. Evidently, we define a k-form to be C^∞ if it is C^∞ as a section of the bundle $\bigwedge^k(T^*M)$.

NOTATION. If $E \to M$ is a C^∞ vector bundle, then the vector space of C^∞ sections of E is denoted $\Gamma(E)$ or $\Gamma(M, E)$. The vector space of all C^∞ k-forms on M is usually denoted $\Omega^k(M)$. Thus,

$$\Omega^k(M) = \Gamma(\textstyle\bigwedge^k(T^*M)) = \Gamma(M, \bigwedge^k(T^*M)).$$

18.3 C^∞ k-Forms

There are several equivalent characterizations of C^∞ k-forms.

Proposition 18.3. *Let ω be a k-form on a manifold M. The following are equivalent:*

(i) *The k-form ω is C^∞ on M.*
(ii) *For any coordinate chart $(U, x^1, \ldots, x^n)$ on M, if $\omega = \sum a_I\, dx^I$, then the coefficients a_I are all C^∞ functions on U.*

(iii) *For any k smooth vector fields $X_1, \dots, X_k$ on M, the function $\omega(X_1, \dots, X_k)$ is C^∞ on M.*

Since the proofs are similar to those for 1-forms, we omit them.

Example 18.4. We defined the 0-tensors and the 0-covectors to be the constants, that is, $L_0(V) = A_0(V) = \mathbb{R}$. Therefore, the bundle $\bigwedge^0(T^*M)$ is simply $M \times \mathbb{R}$ and a 0-form on M is a function on M. A C^∞ 0-form on M is thus the same as a C^∞ function on M. In our new notations,

$$\Omega^0(M) = \Gamma\big(\textstyle\bigwedge^0(T^*M)\big) = \Gamma(M \times \mathbb{R}) = C^\infty(M).$$

18.4 Pullback of k-Forms

Just as one can pull back 1-forms under a smooth map $F\colon N \to M$, so one can pull back k-forms as well. For 0-forms, i.e., functions, the pullback F^* is defined to be the composition:

$$N \xrightarrow{F} M \xrightarrow{f} \mathbb{R}, \qquad F^*(f) = f \circ F \in \Omega^0(N).$$

For a k-form ω on M, we define its *pullback* $F^*\omega$, a k-form on N, as follows: if $p \in N$ and $v_1, \dots, v_k \in T_pN$, then

$$(F^*\omega)_p(v_1, \dots, v_k) = \omega_{F(p)}(F_*v_1, \dots, F_*v_k).$$

In a sense this is also a composition:

$$T_pN \times \cdots \times T_pN \xrightarrow{F_*} T_{F(p)}M \times \cdots \times T_{F(p)}M \xrightarrow{\omega} \mathbb{R}.$$

Proposition 18.5 (Linearity of the pullback). *Let $F\colon N \to M$ be a C^∞ map. If ω, τ are k-forms on M and a is a real number, then*

(i) $F^*(\omega + \tau) = F^*\omega + F^*\tau$;
(ii) $F^*(a\omega) = aF^*\omega$.

Proof. Problem 18.1. $\square$

18.5 The Wedge Product

We learned in Chapter 3 that if ω and τ are alternating tensors of degree k and ℓ, respectively on a vector space V, then their wedge product $\omega \wedge \tau$ is the alternating $(k + \ell)$-tensor on V defined by

$$\omega \wedge \tau(v_1, \dots, v_{k+\ell}) = \sum (\operatorname{sgn} \sigma)\omega(v_{\sigma(1)}, \dots, v_{\sigma(k)})\tau(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}),$$

where σ runs over all (k, ℓ)-shuffles of $1, \dots, k+\ell$ and all $v_i \in V$. For example, if ω and τ are 1-tensors, then

$$\omega \wedge \tau(v_1, v_2) = \omega(v_1)\tau(v_2) - \omega(v_2)\tau(v_1).$$

The wedge product extends pointwise to differential forms on a manifold: if ω is a k-form and τ an ℓ-form on M, define $\omega \wedge \tau$ to be the $(k+\ell)$-form such that

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p$$

at all $p \in M$.

Proposition 18.6. *If ω and τ are C^∞ forms on M, then $\omega \wedge \tau$ is also C^∞.*

Proof. Let $(U, x^1, \dots, x^n)$ be a chart on M. On U,

$$\omega = \sum a_I \, dx^I, \quad \tau = \sum b_J \, dx^J$$

for C^∞ function a_I, b_J on U. Their wedge product is

$$\begin{aligned} \omega \wedge \tau &= \left(\sum a_I \, dx^I\right) \wedge \left(\sum b_J \, dx^J\right) \\ &= \sum a_I b_J \, dx^I \wedge dx^J. \end{aligned}$$

In this sum, $dx^I \wedge dx^J = 0$ if I and J have an index in common. If I and J are disjoint, then $dx^I \wedge dx^J = dx^K$, where $K = I \cup J$ but reordered as an increasing sequence. Thus,

$$\omega \wedge \tau = \sum_K \left(\sum_{I \cup J = K} a_I b_J \right) dx^K.$$

Since the coefficient of dx^K is C^∞ on U, by Proposition 18.3, $\omega \wedge \tau$ is C^∞. □

Proposition 18.7 (Pullback of a wedge product). *If $F\colon N \to M$ is a C^∞ map of manifolds and ω and τ are differential forms on M, then*

$$F^*(\omega \wedge \tau) = (F^*\omega) \wedge (F^*\tau).$$

Proof. Problem 18.2. □

We define the vector space $\Omega^*(M)$ of C^∞ differential forms on a manifold M of dimension n to be the direct sum

$$\Omega^*(M) = \oplus_{k=0}^n \Omega^k(M).$$

What this means is that each element of $\Omega^*(M)$ is uniquely a sum $\sum_{i=1}^r \omega_{k_i}$, where $\omega_{k_i} \in \Omega^{k_i}(M)$. With the wedge product, the vector space $\Omega^*(M)$ becomes a graded algebra, the grading being the degree of a differential form. By Propositions 18.5 and 18.7, if $F\colon N \to M$ is a C^∞ map of manifolds, then the pullback map $F^*\colon \Omega^*(M) \to \Omega^*(N)$ is a homomorphism of graded algebras.

18.6 Invariant Forms on a Lie Group

Just as there are left-invariant vector fields on a Lie group G, so there are also left-invariant differential forms. For $g \in G$, let $\ell_g \colon G \to G$ be left multiplication by g. A k-form ω on G is said to be *left-invariant* if $\ell_g^*\omega = \omega$ for all $g \in G$. This means for all $g, x \in G$,

$$\ell_g^*(\omega_{gx}) = \omega_x.$$

Thus, a left-invariant k-form is uniquely determined by its value at the identity, since for any $g \in G$,

$$\omega_g = (\ell_{g^{-1}})^*\omega_e. \tag{18.2}$$

We have the following analogue of Proposition 16.5.

Proposition 18.8. *Every left-invariant k-form ω on a Lie group G is C^∞.*

Proof. By Proposition 18.3, it suffices to prove that for any k smooth vector fields $X_1, \dots, X_k$ on G, the function $\omega(X_1, \dots, X_k)$ is C^∞ on G. Let $(Y_1)_e, \dots, (Y_n)_e$ be a basis for the tangent space T_eG and $Y_1, \dots, Y_n$ the left-invariant vector fields they generate. Then $Y_1, \dots, Y_n$ is a C^∞ frame over G (Proposition 16.5). Each X_j can be written as a linear combination $X_j = \sum a_j^i Y_i$. By Proposition 12.10, the functions a_j^i are C^∞. Hence, to prove that ω is C^∞, it suffices to show that $\omega(Y_{i_1}, \dots, Y_{i_k})$ is C^∞ for *left-invariant* vector fields $Y_{i_1}, \dots, Y_{i_k}$. But

$$\begin{aligned}(\omega(Y_{i_1}, \dots, Y_{i_k}))(g) &= \omega_g((Y_{i_1})_g, \dots, (Y_{i_k})_g) \\ &= ((\ell_{g^{-1}})^*\omega_e)((\ell_g)_*(Y_{i_1})_e, \dots, (\ell_g)_*(Y_{i_k})_e) \\ &= \omega_e((Y_{i_1})_e, \dots, (Y_{i_k})_e),\end{aligned}$$

which is a constant, independent of g. Being the constant function, $\omega(Y_{i_1}, \dots, Y_{i_k})$ is C^∞ on G. □

Similarly, a k-form ω on G is said to be *right-invariant* if $r_g^*\omega = \omega$ for all $g \in G$. The analogue of Proposition 18.8, that every right-invariant form on a Lie group is C^∞, is proved in the same way.

Let $\Omega^k(G)^G$ denote the vector space of left-invariant k-forms on G. The linear map

$$\Omega^k(G)^G \xrightarrow{\sim} \textstyle\bigwedge^k(\mathfrak{g}^*), \quad \omega \mapsto \omega_e,$$

has an inverse defined by (18.2) and is therefore an isomorphism. It follows that $\dim \Omega^k(G)^G = \binom{n}{k}$.

Problems

18.1. Linearity of the pullback
Prove Proposition 18.5.

18.2. Pullback of a wedge product
Prove Proposition 18.7.

18.3.* Vertical plane
Let x, y, z be the standard coordinates on $\mathbb{R}^3$. A plane in $\mathbb{R}^3$ is *vertical* if it is defined by $ax + by = 0$ for some $a, b \in \mathbb{R}$. Prove that on a vertical plane, $dx \wedge dy = 0$.

18.4.* Support of a sum or product
Generalizing the support of a function, we define the *support of a k-form* $\omega \in \Omega^k(M)$ to be

$$\operatorname{supp}\omega = \text{closure of } \{p \in M \mid \omega_p \neq 0\}.$$

Let ω and τ be differential forms on a manifold M. Prove that

(a) $\operatorname{supp}(\omega + \tau) \subset \operatorname{supp}\omega \cup \operatorname{supp}\tau$.
(b) $\operatorname{supp}(\omega \wedge \tau) \subset \operatorname{supp}\omega \cap \operatorname{supp}\tau$.

18.5.* Locally finite collection of supports
Let $\{\rho_\alpha\}_{\alpha\in A}$ be a collection of functions on M and ω a C^∞ k-form with compact support on M. If the collection of supports, $\{\operatorname{supp}\rho_\alpha\}_{\alpha\in A}$, is locally finite, prove that $\rho_\alpha\omega \equiv 0$ for all but finitely many α.

18.6. Locally finite sums
We say that a sum $\sum \omega_\alpha$ of differential k-forms on a manifold M is *locally finite* if $\{\omega_\alpha\}$ is a collection of k-forms such that $\{\operatorname{supp}\omega_\alpha\}$ is a locally finite family. Suppose $\sum \omega_\alpha$ and $\sum \tau_\alpha$ are locally finite sums and f is a C^∞ function on M.

(a) Show that every point $p \in M$ has a neighborhood U on which $\sum \omega_\alpha$ is a finite sum.
(b) Show that $\sum \omega_\alpha + \tau_\alpha$ is a locally finite sum and

$$\sum \omega_\alpha + \tau_\alpha = \sum \omega_\alpha + \sum \tau_\alpha.$$

(c) Show that $\sum f\omega_\alpha$ is a locally finite sum and

$$\sum f\omega_\alpha = f\left(\sum \omega_\alpha\right).$$

18.7.* Pullback by a surjective submersion
If $\pi\colon \tilde{M} \to M$ is a surjective submersion, then the pullback map $\pi^*\colon \Omega^*(M) \to \Omega^*(\tilde{M})$ is an injective algebra homomorphism.

18.8. Bi-invariant top forms on a compact connected Lie group
Suppose G is a compact connected Lie group of dimension n. This exercise proves that every left-invariant n-form on G is right-invariant.

(a) Let ω be a left-invariant n-form on G. For any $a \in G$, show that $r_a^*\omega$ is also left-invariant, where $r_a\colon G \to G$ is right multiplication by a.
(b) Since $\dim \Omega^n(G)^G = \dim \bigwedge^n(\mathfrak{g}^*) = 1$, $r_a^*\omega = f(a)\omega$ for some nonzero real constant $f(a)$ depending on $a \in G$. Show that $f\colon G \to \mathbb{R}^\times$ is a group homomorphism.

(c) Show that $f: G \to \mathbb{R}^\times$ is C^∞. (*Hint*: Note that $f(a)\omega_e = (r_a^*\omega)_e = r_a^*(\omega_a) = r_a^*\ell_{a^{-1}}^*(\omega_e)$. Thus, $f(a)$ is induced by the adjoint representation $\mathrm{Ad}(a): \mathfrak{g} \to \mathfrak{g}$. See Problem 16.10.)

(d) As the continuous image of a compact connected set G, the set $f(G) \subset \mathbb{R}^\times$ is compact connected. Prove that $f(G) = 1$. Hence, $r_a^*\omega = \omega$ for all $a \in G$.

19

The Exterior Derivative

In contrast to undergraduate calculus, where the basic objects of study are functions, the basic objects in calculus on manifolds are differential forms. Our program now is to learn how to integrate and differentiate differential forms.

Recall that an *antiderivation* on a graded algebra $A = \oplus_{k=0}^{\infty} A^k$ is an $\mathbb{R}$-linear map $D\colon A \to A$ such that

$$D(\omega \cdot \tau) = (D\omega) \cdot \tau + (-1)^k \omega \cdot D\tau.$$

for $\omega \in A^k$ and $\tau \in A^\ell$. In the graded algebra A, an element of A^k is called a *homogeneous element* of degree k. The antiderivation is *of degree m* if

$$\deg D\omega = \deg \omega + m$$

for all homogeneous elements $\omega \in A$.

Let M be a manifold and $\Omega^*(M)$ the graded algebra of C^∞ differential forms on M. The extraordinary usefulness of differential forms comes from the fact that on the graded algebra $\Omega^*(M)$ there is a uniquely and intrinsically defined antiderivation called the *exterior derivative*.

Definition 19.1. An *exterior differentiation* or *exterior derivative* on a manifold M is an $\mathbb{R}$-linear map

$$D\colon \Omega^*(M) \to \Omega^*(M)$$

such that

(i) D is an antiderivation of degree 1;
(ii) $D \circ D = 0$;
(iii) if f is a C^∞ function and X a C^∞ vector field on M, then $(Df)(X) = Xf$.

Condition (iii) says that on 0-forms an exterior derivative agrees with the differential df of a function f. Hence, by (17.2), in a coordinate chart $(U, x^1, \dots, x^n)$,

$$Df = \sum \frac{\partial f}{\partial x^i}\, dx^i.$$

In this chapter we prove the existence and uniqueness of exterior differentiation on a manifold.

19.1 Exterior Derivative on a Coordinate Chart

We showed in Section 4.4 the existence and uniqueness of exterior differentiation on an open subset of $\mathbb{R}^n$. The same proof carries over to any coordinate chart on a manifold.

More precisely, suppose $(U, x^1, \ldots, x^n)$ is a coordinate chart on a manifold M. Then any k-form ω on U is uniquely a linear combination

$$\omega = \sum a_I \, dx^I, \quad a_I \in C^\infty(U).$$

If d is an exterior differentiation on U, then

$$\begin{aligned} d\omega &= \sum (da_I) \wedge dx^I + \sum a_I \, d\, dx^I && \text{(by (i))} \\ &= \sum (da_I) \wedge dx^I && \text{(by (ii), } d^2 = 0) \\ &= \sum_I \sum_j \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I && \text{(by (iii)).} \end{aligned} \tag{19.1}$$

Hence, if an exterior differentiation d exists on U, then it is uniquely defined by (19.1).

To show existence, we define d by the formula (19.1). The proof that d satisfies (i), (ii), and (iii) is the same as in Proposition 4.13.

Like the derivative of a function on $\mathbb{R}^n$, an antiderivation D on $\Omega^*(M)$ has the property that for a k-form ω, the value of $D\omega$ at a point p depends only on the values of ω in a neighborhood of p. To explain this, we make a digression on local operators.

19.2 Local Operators

An endomorphism of a vector space W is often called an *operator* on W. For example, if $W = C^\infty(\mathbb{R})$ is the vector space of C^∞ functions on $\mathbb{R}$, then the derivative d/dx is an operator on W:

$$\frac{d}{dx} f(x) = f'(x).$$

The derivative has the property that the value of $f'(x)$ at a point p depends only on the values of f in a small neighborhood of p. More precisely, if $f = g$ on an open set U in $\mathbb{R}$, then $f' = g'$ on U. We say that the derivative is a *local operator on* $C^\infty(\mathbb{R})$.

Definition 19.2. An operator $D\colon \Omega^*(M) \to \Omega^*(M)$ is said to be *local* if for all $k \geq 0$, whenever a k-form $\omega \in \Omega^k(M)$ restricts to 0 on an open set U, then $D\omega \equiv 0$ on U.

Here by restricting to 0 on U, we mean that $\omega_p = 0$ at every point p in U, and the symbol "$\equiv 0$" means "identically zero": $(D\omega)_p = 0$ at every point p in U. An equivalent definition of a local operator is that for all $k \geq 0$, whenever two k-forms $\omega, \tau \in \Omega^k(M)$ agree on an open set U, then $D\omega \equiv D\tau$ on U.

Example 19.3. Define the integral operator

$$I\colon C^\infty([a,b]) \to C^\infty([a,b])$$

by

$$I(f) = \int_a^b f(t)dt.$$

Here $I(f)$ is a number, which we view as a constant function on $[a,b]$. The integral is not a local operator since the value of $I(f)$ at any point p depends on the values of f over the entire interval $[a,b]$.

Proposition 19.4. *Any antiderivation D on $\Omega^*(M)$ is a local operator.*

Proof. Suppose $\omega \in \Omega^k(M)$ and $\omega \equiv 0$ on an open subset U. Let p be an arbitrary point in U. It suffices to prove that $(D\omega)_p = 0$.

Choose a C^∞ bump function f at p supported in U. In particular, $f \equiv 1$ in a neighborhood of p in U. Then $f\omega \equiv 0$ on M, since if a point q is in U, then $\omega_q = 0$, and if q is not in U, then $f(q) = 0$. Applying the antiderivation property of D to $f\omega$, we get

$$0 = D(0) = D(f\omega) = (Df) \wedge \omega + (-1)^0 f \wedge (D\omega).$$

We now evaluate the right-hand side at p, noting that $\omega_p = 0$ and $f(p) = 1$. This gives $0 = (D\omega)_p$. Since p is an arbitrary point of U, $D\omega \equiv 0$ on U. □

Remark 19.5. The same proof shows that a derivation on $\Omega^*(M)$ is also a local operator.

19.3 Extension of a Local Form to a Global Form

Sometimes we are given a differential form τ that is defined only on an open subset U of a manifold M. We can use a bump function to extend τ to a global form $\tilde{\tau}$ on M that agrees with τ near some point. (By a global form, we mean a differential form defined at every point of M.)

Proposition 19.6. *Suppose τ is a C^∞ differential form on an open subset U of M. For any $p \in U$, there is a C^∞ global form $\tilde{\tau}$ on M that agrees with τ on a neighborhood of p in U.*

The proof is almost identical to that of Proposition 13.3. We leave it as an exercise.

Of course, the extension $\tilde{\tau}$ is not unique. In the proof it depends on p and on the choice of a bump function at p.

19.4 Existence of an Exterior Differentiation

To define an exterior derivative $d\colon \Omega^*(M) \to \Omega^*(M)$, let ω be a k-form on M and $p \in M$. Choose a chart $(U, x^1, \dots, x^n)$ about p. Suppose $\omega = \sum a_I \, dx^I$ on U. Define

$$(d\omega)_p = \left(\sum da_I \wedge dx^I\right)_p. \tag{19.2}$$

We now show that this definition is independent of the chart. If $(V, y^1, \dots, y^n)$ is another chart about p and $\omega = \sum b_J dy^J$ on V, then on $U \cap V$,

$$\sum a_I \, dx^I = \sum b_J \, dy^J.$$

As shown in Section 19.1, on $U \cap V$ there is a unique exterior differentiation

$$d_{U\cap V}\colon \Omega^*(U \cap V) \to \Omega^*(U \cap V).$$

By the properties of the exterior derivative,

$$\begin{aligned} &d_{U\cap V}\left(\sum a_I \, dx^I\right) = d_{U\cap V}\left(\sum b_J \, dy^J\right) \\ &\Longrightarrow \sum da_I \wedge dx^I = \sum db_J \wedge dy^J \end{aligned}$$

at all points of $U \cap V$. In particular,

$$\left(\sum da_I \wedge dx^I\right)_p = \left(\sum db_J \wedge dy^J\right)_p.$$

Thus, $(d\omega)_p$ is well defined, independent of the chart.

As p varies over all points of M, this defines an operator

$$d\colon \Omega^*(M) \to \Omega^*(M).$$

To check properties (i), (ii), and (iii), it suffices to check them at each point $p \in M$. Using the definition (19.2), the verification is the same as for the exterior derivative on $\mathbb{R}^n$ (Proposition 4.14).

19.5 Uniqueness of Exterior Differentiation

Suppose $D\colon \Omega^*(M) \to \Omega^*(M)$ is an exterior differentiation. We will show that D coincides with the exterior differentiation d defined in Section 19.4.

To this end, let $\omega \in \Omega^k(M)$ and $p \in M$. Choose a chart $(U, x^1, \dots, x^n)$ about p and suppose $\omega = \sum a_I \, dx^I$ on U. Extend the functions $a_I, x^1, \dots, x^n$ on U to C^∞ functions $\tilde{a}_I, \tilde{x}^1, \dots, \tilde{x}^n$ on M that agree with $a_I, x^1, \dots, x^n$ on a neighborhood of V of p (by Proposition 19.6). Define

$$\tilde{\omega} = \sum \tilde{a}_I \, d\tilde{x}^I \in \Omega^k(M).$$

Then

$$\omega \equiv \tilde{\omega} \quad \text{on } V.$$

Since D is a local operator,

$$D\omega = D\tilde{\omega} \quad \text{on } V.$$

Thus,

$$\begin{aligned}
(D\omega)_p = (D\tilde{\omega})_p = (D \sum \tilde{a}_I \, d\tilde{x}^I)_p &= \left(\sum d\tilde{a}_I \wedge d\tilde{x}^I\right)_p \\
&= \left(\sum da_I \wedge dx^I\right)_p \quad (\text{since } \tilde{a}_I = a_I \text{ and } \tilde{x}^i = x^i \text{ on } V) \\
&= (d\omega)_p.
\end{aligned}$$

19.6 The Restriction of a k-Form to a Submanifold

If S is a regular submanifold of a manifold M and ω is a k-form on M, then the *restriction* of ω to S is the k-form $\omega|_S$ on S defined by

$$(\omega|_S)_p(X_1, \ldots, X_k) = \omega_p(X_1, \ldots, X_k)$$

for $X_1, \ldots, X_k \in T_pS \subset T_pM$. Thus, $(\omega|_S)_p$ is obtained from ω_p by restricting the domain of ω_p to $T_pS \times \cdots \times T_pS$ (k times).

A nonzero form on M may restrict to the zero form on a submanifold S. For example, if S is a smooth curve in $\mathbb{R}^2$ defined by the nonconstant function $f(x, y)$, then $df = (\partial f/\partial x)\, dx + (\partial f/\partial y)\, dy$ is a nonzero 1-form on $\mathbb{R}^2$, but since f is identically zero ($f \equiv 0$) on S, the differential df is also identically zero on S. Thus, $(df)|_S \equiv 0$.

To avoid too cumbersome a notation, we sometimes write df to mean $(df)|_S$, relying on the context to make clear that it is the restriction of df to S.

One should distinguish between a *nonzero* form and a *nowhere-zero* or *nowhere-vanishing* form. For example, $x\, dy$ is a nonzero form on $\mathbb{R}^2$, meaning that it is not the identically zero form. However, it is not nowhere-zero, because it vanishes on the y-axis. On the other hand, dx and dy are nowhere-zero 1-forms on $\mathbb{R}^2$.

19.7 A Nowhere-Vanishing 1-Form on the Circle

As an application of the exterior derivative, we will construct a nowhere-vanishing 1-form on the circle.

Example 19.7. Let S^1 be the unit circle defined by $x^2 + y^2 = 1$ in $\mathbb{R}^2$. The 1-form dx restricts from $\mathbb{R}^2$ to a 1-form on S^1. When restricted to S^1, at each point $p \in S^1$ the domain of $(dx)|_{S^1,p}$ is $T_p(S^1)$ instead of $T_p(\mathbb{R}^2)$:

$$(dx)|_{S^1,p}\colon T_p(S^1) \to \mathbb{R}.$$

At $p = (1,0)$, a basis for the tangent space $T_p(S^1)$ is $\partial/\partial y$ (Figure 19.1). Since

$$(dx)_p\left(\frac{\partial}{\partial y}\right) = 0,$$

we see that although dx is a nowhere-vanishing 1-form on $\mathbb{R}^2$, it vanishes at $(1,0)$ when restricted to S^1.

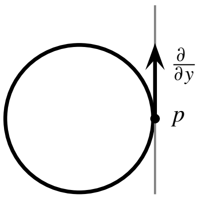


Fig. 19.1. The tangent space to S^1 at $p = (1,0)$.

To find a nowhere-vanishing 1-form on S^1, we take the exterior derivative of both sides of the equation

$$x^2 + y^2 = 1.$$

Using the antiderivation property of d, we get

$$2x\,dx + 2y\,dy = 0. \tag{19.3}$$

Let

$$U_x = \{(x,y) \in S^1 \mid x \neq 0\} \quad \text{and} \quad U_y = \{(x,y) \in S^1 \mid y \neq 0\}.$$

By (19.3), on $U_x \cap U_y$,

$$\frac{dy}{x} = -\frac{dx}{y}.$$

Define a 1-form ω on S^1 by

$$\omega = \begin{cases} \dfrac{dy}{x} & \text{on } U_x, \\ -\dfrac{dx}{y} & \text{on } U_y. \end{cases}$$

Since these two 1-forms agree on $U_x \cap U_y$, ω is a well-defined 1-form on $S^1 = U_x \cup U_y$.

To show that ω is C^∞ and nowhere-vanishing, we need charts. Let

$$U_x^+ = \{(x,y) \in S^1 \mid x > 0\}.$$

We define similarly U_x^-, U_y^+, U_y^-. On U_x^+, y is a local coordinate and so dy is a basis for the cotangent space $T_p^*(S^1)$ at each point $p \in U_x^+$. Since $\omega = dy/x$ on U_x^+, ω is C^∞ and nowhere-zero on U_x^+. A similar argument applies to dy/x on U_x^- and $-dx/y$ on U_y^+ and U_y^-. Hence, ω is C^∞ and nowhere-vanishing on S^1.

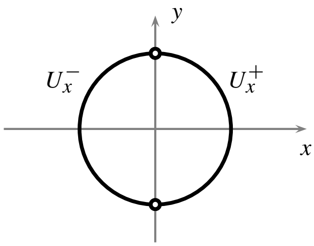


Fig. 19.2. Two charts on the unit circle.

19.8 Exterior Differentiation Under a Pullback

The pullback of differential forms commutes with the exterior derivative.

Theorem 19.8. *Let $F: N \to M$ be a smooth map of manifolds. If $\omega \in \Omega^k(M)$, then $dF^*\omega = F^*d\omega$.*

Proof. We first check the case $k = 0$ when ω is a C^∞ function h on M. For $p \in N$ and $X_p \in T_pN$,

$$\begin{aligned}
(dF^*h)_p(X_p) &= X_p(F^*h) && \text{(property (iii) of } d\text{)}\\
&= X_p(h \circ F) && \text{(definition of the pullback of a function)}
\end{aligned}$$

and

$$\begin{aligned}
(F^*dh)_p(X_p) &= (dh)_{F(p)}(F_*X_p) && \text{(definition of the pullback of a 1-form)}\\
&= (F_*X_p)h && \text{(definition of the differential } dh\text{)}\\
&= X_p(h \circ F) && \text{(definition of } F_*\text{).}
\end{aligned}$$

Now consider the general case of a C^∞ k-form ω on M. It suffices to verify $dF^*\omega = F^*d\omega$ at an arbitrary point $p \in N$. This reduces the proof to a local computation. If $(V, y^1, \dots, y^m)$ is a chart of M at $F(p)$, then on V,

$$\omega = \sum a_I \, dy^{i_1} \wedge \cdots \wedge dy^{i_k}, \quad I = (i_1, \dots, i_k),$$

for some C^∞ functions a_I on V and

$$\begin{aligned}
F^*\omega &= \sum (F^*a_I) F^*dy^{i_1} \wedge \cdots \wedge F^*dy^{i_k} && \text{(Proposition 18.7)}\\
&= \sum (a_I \circ F)\, dF^{i_1} \wedge \cdots \wedge dF^{i_k} && (F^*dy^i = dF^*y^i = d(y^i \circ F)\\
& && \qquad = dF^i).
\end{aligned}$$

So

$$dF^*\omega = \sum d(a_I \circ F) \wedge dF^{i_1} \wedge \cdots \wedge dF^{i_k}.$$

On the other hand,

$$\begin{aligned}
F^*d\omega &= F^*\left(\sum da_I \wedge dy^{i_1} \wedge \cdots \wedge dy^{i_k}\right)\\
&= \sum F^*da_I \wedge F^*dy^{i_1} \wedge \cdots \wedge F^*dy^{i_k}\\
&= \sum d(F^*a_I) \wedge dF^{i_1} \wedge \cdots \wedge dF^{i_k} \quad \text{(by the case } k=0\text{)}\\
&= \sum d(a_I \circ F) \wedge dF^{i_1} \wedge \cdots \wedge dF^{i_k}.
\end{aligned}$$

Therefore,

$$dF^*\omega = F^*d\omega. \qquad \square$$

Example 19.9. Let U be the open set $(0, \infty) \times (0, 2\pi)$ in the (r, θ)-plane $\mathbb{R}^2$. Define $F\colon U \subset \mathbb{R}^2 \to \mathbb{R}^2$ by

$$(x, y) = F(r, \theta) = (r\cos\theta, r\sin\theta).$$

Compute the pullback $F^*(dx \wedge dy)$.

Solution. We first compute F^*dx:

$$\begin{aligned}
F^*dx &= dF^*x && \text{(Theorem 19.8)}\\
&= d(x \circ F) && \text{(definition of the pullback of a function)}\\
&= d(r\cos\theta)\\
&= (\cos\theta)\,dr - r\sin\theta\,d\theta.
\end{aligned}$$

Similarly,

$$F^*dy = dF^*y = d(r\sin\theta) = (\sin\theta)\,dr + r\cos\theta\,d\theta.$$

Since the pullback commutes with the wedge product (Proposition 18.7),

$$\begin{aligned}
F^*(dx \wedge dy) &= (F^*dx) \wedge (F^*dy)\\
&= ((\cos\theta)\,dr - r\sin\theta\,d\theta) \wedge ((\sin\theta)\,dr + r\cos\theta\,d\theta)\\
&= (r\cos^2\theta + r\sin^2\theta)\,dr \wedge d\theta\\
&= r\,dr \wedge d\theta. \qquad \square
\end{aligned}$$

Problems

19.1.* Extension of a C^∞ form
Prove Proposition 19.6.

19.2. Transition formula for an n-form
Let $(U, x^1, \ldots, x^n)$ be a chart on a manifold and $f^1, \ldots, f^n$ smooth functions on U. Prove that

$$df^1 \wedge \cdots \wedge df^n = \det\left[\frac{\partial f^i}{\partial x^j}\right] dx^1 \wedge \cdots \wedge dx^n.$$

19.3. Pullback of a differential form
Let U be the open set $(0, \infty) \times (0, \pi) \times (0, 2\pi)$ in the (ρ, ϕ, θ)-space $\mathbb{R}^3$. Define $F: U \to \mathbb{R}^2$ by

$$(x, y, z) = F(\rho, \phi, \theta) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi).$$

Show that $F^*(dx \wedge dy \wedge dz) = \rho^2 \sin\phi \, d\rho \wedge d\phi \wedge d\theta$.

19.4. Pullback of a differential form
Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$F(x, y) = (u, v) = (x^2 + y^2, xy).$$

Compute $F^*(u\, du + v\, dv)$.

19.5. Pullback of a differential form by a curve
Let ω be the 1-form $\omega = (-y\, dx + x\, dy)/(x^2 + y^2)$ on $\mathbb{R}^2 - \{\mathbf{0}\}$. Define $c: \mathbb{R} \to \mathbb{R}^2 - \{\mathbf{0}\}$ by $c(t) = (\cos t, \sin t)$. Compute $c^*\omega$.

19.6. Coordinate functions and differential forms
Let $f^1, \ldots, f^n$ be C^∞ functions on a neighborhood U of a point p in a manifold of dimension n. Show that there is a neighborhood W of p on which $f^1, \ldots, f^n$ form a coordinate system if and only if $(df^1 \wedge \cdots \wedge df^n)_p \neq 0$.

19.7. Local operators
An operator $L: \Omega^*(M) \to \Omega^*(M)$ is *support-decreasing* if $\operatorname{supp} L(\omega) \subset \operatorname{supp} \omega$ for every k-form $\omega \in \Omega^*(M)$ for all $k \geq 0$. Show that an operator on $\Omega^*(M)$ is local if and only if it is support-decreasing.

19.8. Derivations of C^∞ functions are local operators
Let M be a smooth manifold. The definition of a *local operator* D on $C^\infty(M)$ is similar to that of a local operator on $\Omega^*(M)$: D is *local* if whenever a function $f \in C^\infty(M)$ vanishes identically on an open subset U, then $Df \equiv 0$ on U. Prove that a derivation of $C^\infty(M)$ is a local operator on $C^\infty(M)$.

19.9. Global formula for the exterior derivative of a 1-form
Prove that if ω is a C^∞ 1-form and X and Y are C^∞ vector fields on a manifold M, then

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

19.10. A nowhere-vanishing form on a smooth hypersurface

(a) Let $f(x, y)$ be a C^∞ function on $\mathbb{R}^2$ and assume that 0 is a regular value of f. By the regular level set theorem, the zero set M of $f(x, y)$ is a one-dimensional submanifold of $\mathbb{R}^2$. Construct a nowhere-vanishing 1-form on M.
(b) Let $f(x, y, z)$ be a C^∞ function on $\mathbb{R}^3$ and assume that 0 is a regular value of f. By the regular level set theorem, the zero set M of $f(x, y, z)$ is a two-dimensional submanifold of $\mathbb{R}^3$. Let f_x, f_y, f_z be the partial derivatives of f with respect to x, y, z, respectively. Show that the equalities

$$\frac{dx \wedge dy}{f_z} = \frac{dy \wedge dz}{f_x} = \frac{dz \wedge dx}{f_y}$$

hold on M whenever they make sense, and therefore piece together to give a nowhere-vanishing 2-form on M.

(c) Generalize this problem to a regular level set of $f(x^1, \dots, x^{n+1})$ in $\mathbb{R}^{n+1}$.

19.11. Vector fields as derivations of C^∞ functions

In Section 14.4 we showed that a C^∞ vector field X on a manifold M gives rise to a derivation of $C^\infty(M)$. To distinguish the vector field from the derivation, we will temporarily denote the derivation arising from X by $\varphi(X)$. Thus, for any $f \in C^\infty(M)$,

$$(\varphi(X)f)(p) = X_p f \quad \text{for all } p \in M.$$

(a) Let $\mathcal{F} = C^\infty(M)$. Prove that $\varphi\colon \mathfrak{X}(M) \to \operatorname{Der}(C^\infty(M))$ is an $\mathcal{F}$-linear map.

(b) Show that φ is injective.

(c) If D is a derivation of $C^\infty(M)$ and $p \in M$, define $D_p\colon C_p^\infty(M) \to C_p^\infty(M)$ by

$$D_p[f] = [D\tilde{f}] \in C_p^\infty(M),$$

where $[f]$ is the germ of f at p and $\tilde{f}$ is a global extension of f given by Proposition 19.6. Show that $D_p[f]$ is well defined. (*Hint*: Apply Problem 19.8.)

(d) Show that D_p is a derivation of $C_p^\infty(M)$.

(e) Prove that $\varphi\colon \mathfrak{X}(M) \to \operatorname{Der}(C^\infty(M))$ is an isomorphism of $\mathcal{F}$-modules.

19.12. Twentieth-century formulation of Maxwell's equations

In Maxwell's theory of electricity and magnetism, developed in the late nineteenth century, the electric field $\mathbf{E} = \langle E_1, E_2, E_3 \rangle$ and the magnetic field $\mathbf{B} = \langle B_1, B_2, B_3 \rangle$ in a vacuum $\mathbb{R}^3$ with no charge or current, satisfy the following equations:

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, & \nabla \times \mathbf{B} &= \frac{\partial \mathbf{E}}{\partial t}, \\ \operatorname{div} \mathbf{E} &= 0, & \operatorname{div} \mathbf{B} &= 0. \end{aligned}$$

By the correspondence in Section 4.6, the 1-form E on $\mathbb{R}^3$ corresponding to the vector field $\mathbf{E}$ is

$$E = E_1\, dx + E_2\, dy + E_3\, dz$$

and the 2-form B on $\mathbb{R}^3$ corresponding to the vector field $\mathbf{B}$ is

$$B = B_1\, dy \wedge dz + B_2\, dz \wedge dx + B_3\, dx \wedge dy.$$

Let $\mathbb{R}^4$ be space-time with coordinates (x, y, z, t). Then both E and B can be viewed as differential forms on $\mathbb{R}^4$. Define F to be the 2-form on space-time

$$F = E \wedge dt + B.$$

Decide which two of Maxwell's equations are equivalent to the equation

$$dF = 0.$$

Prove your answer. (The other two are equivalent to $d * F = 0$ for a star-operator $*$ defined in differential geometry. See [1, Section 19.1, p. 689].)

20

Orientations

20.1 Orientations on a Vector Space

On $\mathbb{R}^1$ an orientation is one of two directions (Figure 20.1).

Fig. 20.1. Orientations on a line.

On $\mathbb{R}^2$ an orientation is either counterclockwise or clockwise (Figure 20.2).

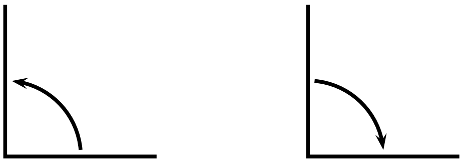


Fig. 20.2. Orientations on a plane.

On $\mathbb{R}^3$ an orientation is either right-handed (Figure 20.3) or left-handed (Figure 20.4). The right-handed orientation on $\mathbb{R}^3$ is the choice of a Cartesian coordinate system so that if you hold out your right hand with the index finger curling from the x-axis to the y-axis, then your thumb points in the direction of the z-axis.

How should one define an orientation on $\mathbb{R}^4$? If we analyze the three examples above, we see that an orientation can be specified by an ordered basis for $\mathbb{R}^n$. Let $e_1, \dots, e_n$ be the standard basis for $\mathbb{R}^n$. For $\mathbb{R}^1$ an orientation could be given by either e_1 or $-e_1$. For $\mathbb{R}^2$ the counterclockwise orientation is (e_1, e_2), while the clockwise

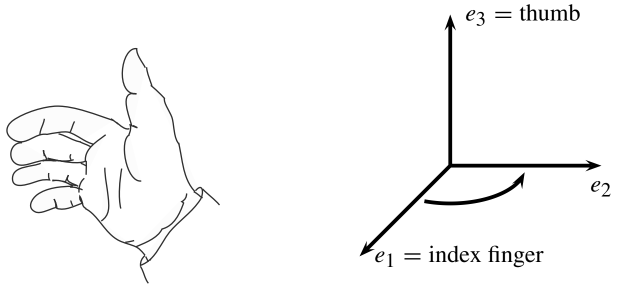


Fig. 20.3. Right-handed orientation (e_1, e_2, e_3) on $\mathbb{R}^3$.

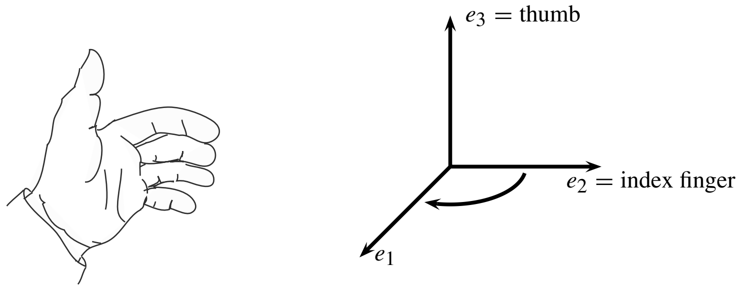


Fig. 20.4. Left-handed orientation (e_1, e_2, e_3) on $\mathbb{R}^3$.

orientation is (e_2, e_1). For $\mathbb{R}^3$ the right-handed orientation is (e_1, e_2, e_3), and the left-handed orientation is (e_2, e_1, e_3).

For any two ordered bases $(u_1, u_2)$and (v_1, v_2) for $\mathbb{R}^2$, there is a unique nonsingular 2 by 2 matrix $A = [a_{ij}]$ such that

$$u_j = \sum_{i=1}^{2} v_i a_{ij}, \quad j = 1, 2,$$

called the *change of basis matrix* from (v_1, v_2) to (u_1, u_2). In matrix notation, if we write ordered basis as row vectors, for example, $[u_1\, u_2]$ for the basis (u_1, u_2), then

$$[u_1\, u_2] = [v_1\, v_2]A.$$

We say that two ordered bases are *equivalent* if the change of basis matrix A has positive determinant. It is easy to check that this is indeed an equivalence relation on the set of all ordered bases for $\mathbb{R}^2$. It therefore partitions the ordered bases into two equivalence classes. Each equivalence class is called an *orientation* on $\mathbb{R}^2$.

The equivalence class containing the ordered basis (e_1, e_2) is the counterclockwise orientation and the equivalence class of (e_2, e_1) is the clockwise orientation.

The general case is similar. Two ordered bases $u = [u_1 \cdots u_n]$ and $v = [v_1 \cdots v_n]$ of a vector space V are said to be *equivalent* if $u = vA$ for an n by n matrix A with positive determinant. An *orientation* on V is an equivalence class of ordered bases.

The zero-dimensional vector space $\{0\}$ is a special case because it does not have a basis. We define an orientation on $\{0\}$ to be one of the two numbers ± 1.

20.2 Orientations and n-Covectors

Instead of using an ordered basis, we can also use an n-covector to specify an orientation on an n-dimensional vector space V. This is based on the fact that the space $\bigwedge^n(V^*)$ of n-covectors on V is one dimensional.

Lemma 20.1. *Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be vectors in a vector space V. Suppose*

$$u_j = \sum_{i=1}^{n} a_{ij} v_i, \quad j = 1, \ldots, n,$$

for a matrix $A = [a_{ij}]$ of real numbers. If ω is an n-covector on V, then

$$\omega(u_1, \ldots, u_n) = (\det A)\, \omega(v_1, \ldots, v_n).$$

Proof. By hypothesis,

$$u_j = \sum a_{ij} v_i.$$

Since ω is n-linear,

$$\begin{aligned} \omega(u_1, \ldots, u_n) &= \omega\left(\sum a_{i_1 1} v_{i_1}, \ldots, \sum a_{i_n n} v_{i_n}\right) \\ &= \sum a_{i_1 1} \cdots a_{i_n n}\, \omega(v_{i_1}, \ldots, v_{i_n}). \end{aligned}$$

For $\omega(v_{i_1}, \ldots, v_{i_n})$ to be nonzero, $i_1, \ldots, i_n$ must all be distinct. This means $i_1, \ldots, i_n$ is a permutation of $1, \ldots, n$. Since ω is an alternating n-tensor,

$$\omega(v_{i_1}, \ldots, v_{i_n}) = (\operatorname{sgn} i)\omega(v_1, \ldots, v_n).$$

Thus,

$$\begin{aligned} \omega(u_1, \ldots, u_n) &= \sum_{i \in S_n} (\operatorname{sgn} i) a_{i_1 1} \cdots a_{i_n n}\, \omega(v_1, \ldots, v_n) \\ &= (\det A)\, \omega(v_1, \ldots, v_n). \end{aligned}$$ □

As a corollary,

$$\begin{aligned} &\operatorname{sgn} \omega(u_1, \dots, u_n) = \operatorname{sgn} \omega(v_1, \dots, v_n) \\ &\text{iff } \det A > 0 \\ &\text{iff } u_1, \dots, u_n \text{ and } v_1, \dots, v_n \text{ are equivalent ordered bases.} \end{aligned}$$

We say that the n-covector ω *represents* the orientation $(v_1, \dots, v_n)$ if $\omega(v_1, \dots, v_n) > 0$. By the preceding corollary, this is a well-defined notion, independent of the choice of ordered basis for the orientation. Moreover, two n-covectors ω and ω' on V represent the same orientation if and only if $\omega = a\omega'$ for some positive real number a.

An isomorphism $\bigwedge^n(V^*) \simeq \mathbb{R}$ identifies the set of nonzero n-covectors on V with $\mathbb{R} - \{0\}$, which has two connected components. Two nonzero n-covectors ω and ω' on V are in the same component if and only if $\omega = a\omega'$ for some real number $a > 0$. Thus, each connected component of $\bigwedge^n(V^*) - \{0\}$ represents an orientation on V.

Example 20.2. Let e_1, e_2 be the standard basis for $\mathbb{R}^2$ and α^1, α^2 its dual basis. Then the 2-covector $\alpha^1 \wedge \alpha^2$ represents the counterclockwise orientation on $\mathbb{R}^2$ since

$$\left(\alpha^1 \wedge \alpha^2\right)(e_1, e_2) = 1 > 0.$$

Example 20.3. Let $\partial/\partial x|_p, \partial/\partial y|_p$ be the standard basis for the tangent space $T_p(\mathbb{R}^2)$, and $(dx)_p, (dy)_p$ its dual basis. Then $(dx)_p \wedge (dy)_p$ represents the counterclockwise orientation on $T_p(\mathbb{R}^2)$.

We define an equivalence relation on the nonzero n-covectors on the n-dimensional vector space V as follows:

$$\omega \sim \omega' \quad \text{iff} \quad \omega = a\omega' \quad \text{for some } a > 0.$$

Then an orientation on V is also given by an equivalence class of nonzero n-covectors on V.

20.3 Orientations on a Manifold

Every vector space of dimension n has two orientations, corresponding to the two equivalence classes of ordered bases or the two equivalence classes of nonzero n-covectors. To orient a manifold M, we orient the tangent space at each point $p \in M$. This can be done by simply assigning a nonzero n-covector to each point of M, in other words, by giving a nowhere-vanishing n-form on M. The assignment of an orientation at each point must be done in a "coherent" way, so that the orientation does not change abruptly in a neighborhood of a point. The simplest way to guarantee this is to require that the n-form on M specifying the orientation at each point be C^∞. (It is enough to require that the n-form be continuous, but we prefer working with C^∞ forms in order to apply the methods of differential calculus.)

Definition 20.4. A manifold M of dimension n is *orientable* if it has a C^∞ nowhere-vanishing n-form.

If ω is a C^∞ nowhere-vanishing n-form on M, then at each point $p \in M$ the n-covector ω_p picks out an equivalence class of ordered bases for the tangent space T_pM.

Example 20.5. The Euclidean space $\mathbb{R}^n$ is orientable as a manifold, because it has the nowhere-vanishing n-form $dx^1 \wedge \cdots \wedge dx^n$.

If ω and ω' are two C^∞ nowhere-vanishing n-forms on a manifold M of dimension n, then $\omega = f\omega'$ for a C^∞ nowhere-vanishing function f on M. On a *connected* manifold M, such a function f is either everywhere positive or everywhere negative. Thus, the C^∞ nowhere-vanishing n-forms on a connected manifold M can be partitioned into two equivalence classes:

$$\omega \sim \omega' \quad \text{iff} \quad \omega = f\omega' \quad \text{with } f > 0.$$

We call either equivalence class an *orientation* on the connected manifold M. By definition a connected manifold has exactly two orientations.

If a manifold is not connected, each connected component can have one of two possible orientations. We call a C^∞ nowhere-vanishing n-form on M that specifies the orientation of M an *orientation form.* An *oriented manifold* is a pair $(M, [\omega])$, where M is a manifold of dimension n and $[\omega]$ is an orientation on M, i.e., the equivalence class of a nowhere-vanishing n-form ω on M. We sometimes write M, instead of $(M, [\omega])$, for an oriented manifold, if it is clear from the context what the orientation is. For example, unless otherwise specified, $\mathbb{R}^n$ is oriented by $dx^1 \wedge \cdots \wedge dx^n$.

Remark 20.6 (*Orientations on a zero-dimensional manifold*). A zero-dimensional manifold is a point. According to the definition above, a zero-dimensional manifold is always orientable. Its two orientations are represented by the two numbers ± 1.

A diffeomorphism $F\colon (N, [\omega_N]) \to (M, [\omega_M])$ of oriented manifolds is said to be *orientation-preserving* if $[F^*\omega_M] = [\omega_N]$; it is *orientation-reversing* if $[F^*\omega_M] = [-\omega_N]$.

Proposition 20.7. *Let U and V be open subsets of $\mathbb{R}^n$. A C^∞ map $F\colon U \to V$ is orientation-preserving if and only if the Jacobian determinant* $\det[\partial F^i/\partial x^j]$ *is everywhere positive on U.*

Proof. Let $x^1, \dots, x^n$ and $y^1, \dots, y^n$ be the standard coordinates on $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$. Then

$$\begin{aligned} F^*(dy^1 \wedge \cdots \wedge dy^n) &= d(F^*y^1) \wedge \cdots \wedge d(F^*y^n) \\ &= d(y^1 \circ F) \wedge \cdots \wedge d(y^n \circ F) \\ &= dF^1 \wedge \cdots \wedge dF^n \\ &= \det\left[\frac{\partial F^i}{\partial x^j}\right] dx^1 \wedge \cdots dx^n \quad \text{(by Problem 19.2).} \end{aligned}$$

Thus, F is orientation-preserving if and only if $\det[\partial F^i/\partial x^j]$ is everywhere positive on U. □

20.4 Orientations and Atlases

Definition 20.8. An atlas on M is said to be *oriented* if for any two overlapping charts $(U, x^1, \dots, x^n)$ and $(V, y^1, \dots, y^n)$ of the atlas, the Jacobian determinant $\det[\partial y^i/\partial x^j]$ is everywhere positive on $U \cap V$.

Proposition 20.9. *A manifold M of dimension n has a C^∞ nowhere-vanishing n-form ω if and only if it has an oriented atlas.*

Proof.
($\Leftarrow$) Given an oriented atlas $\{(U_\alpha, x_\alpha^1, \dots, x_\alpha^n)\}_{\alpha \in A}$, let $\{\rho_\alpha\}$ be a C^∞ partition of unity subordinate to $\{U_\alpha\}$. Define

$$\omega = \sum \rho_\alpha \, dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n. \tag{20.1}$$

For any $p \in M$, there is an open neighborhood U_p of p that intersects only finitely many of the sets $\operatorname{supp} \rho_\alpha$. Thus (20.1) is a finite sum on U_p. This shows that ω is defined and C^∞ at every point of M.

Let $(U, x^1, \dots, x^n)$ be one of the charts about p in the oriented atlas. On $U_\alpha \cap U$, by Problem 19.2,

$$dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n = \det\left[\frac{\partial x_\alpha^i}{\partial x^j}\right] dx^1 \wedge \cdots \wedge dx^n,$$

where the determinant is positive because the atlas is oriented. Then

$$\omega = \sum \rho_\alpha \, dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n = \left(\sum \rho_\alpha \det\left[\frac{\partial x_\alpha^i}{\partial x^j}\right]\right) dx^1 \wedge \cdots \wedge dx^n.$$

In the last sum $\rho_\alpha \geq 0$ and $\det[\partial x_\alpha^i/\partial x^j] > 0$ at p for all α. Moreover, $\rho_\alpha(p) > 0$ for at least one α. Hence,

$$\omega_p = (\text{positive number}) \times (dx^1 \wedge \cdots \wedge dx^n)_p \neq 0.$$

As p is an arbitrary point of M, the n-form ω is nowhere-vanishing on M.

($\Rightarrow$) Suppose ω is a C^∞ nowhere-vanishing n-form on M. Given an atlas for M, we will use ω to modify the atlas so that it becomes oriented. Without loss of generality, we may assume that all the open sets of the atlas are connected.

On a chart $(U, x^1, \dots, x^n)$,

$$\omega = f \, dx^1 \wedge \cdots \wedge dx^n$$

for a C^∞ function f. Since ω is nowhere-vanishing and f is continuous, f is either everywhere positive or everywhere negative on U. If $f > 0$, we leave the chart alone; if $f < 0$, we replace the chart by $(U, -x^1, x^2, \ldots, x^n)$. After all the charts have been checked and replaced if necessary, we may assume that on every chart $(V, y^1, \ldots, y^n)$,

$$\omega = h\, dy^1 \wedge \cdots \wedge dy^n$$

with $h > 0$. This is an oriented atlas, since if $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^n)$ are two charts, then on $U \cap V$

$$\omega = f\, dx^1 \wedge \cdots \wedge dx^n = h\, dy^1 \wedge \cdots \wedge dy^n$$

with $f, h > 0$. By Problem 19.2, $f/h = \det[\partial y^i/\partial x^j]$. It follows that $\det[\partial y^i/\partial x^j] > 0$ on $U \cap V$. □

Definition 20.10. Two oriented atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ on a manifold M are said to be *equivalent* if the transition functions

$$\phi_\alpha \circ \psi_\beta^{-1} \colon \psi_\beta(U_\alpha \cap V_\beta) \to \phi_\alpha(U_\alpha \cap V_\beta)$$

have positive Jacobian determinant for all α, β.

It is not difficult to show that this is an equivalence relation on the set of oriented atlases on M (Problem 20.1).

Suppose M is a connected orientable manifold. To each oriented atlas $\{(U_\alpha, \phi_\alpha)\}$ and partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$, we associate the nowhere-vanishing n-form

$$\omega = \sum \rho_\alpha\, dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$$

on M as in (20.1). In this way, equivalent oriented atlases give rise to equivalent nowhere-vanishing n-forms (Problem 20.2). Since there are two equivalence classes of oriented atlases and two equivalence classes of nowhere-vanishing n-forms on M, this construction is a map from $\{\pm 1\}$ to $\{\pm 1\}$. If the oriented atlas $\{(U_\alpha, x_\alpha^1, x_\alpha^2, \ldots, x_\alpha^n)\}$ gives rise to the n-form ω, then by switching the sign of just one coordinate, we get an oriented atlas, for example, $\{(U_\alpha, -x_\alpha^1, x_\alpha^2, \ldots, x_\alpha^n)\}$, that gives rise to the n-form $-\omega$. Hence, the map: $\{\pm 1\} \to \{\pm 1\}$ is surjective and therefore a bijection. This shows that an orientation on a connected M may also be specified by an equivalence class of oriented atlases. By considering each connected component in turn, we can extend to an arbitrary orientable n-manifold the correspondence between eqivalence classes of oriented atlases and equivalence classes of nowhere-vanishing n-forms. More formally, we say that an *oriented atlas* $\{(U_\alpha, \phi_\alpha)\} = \{(U_\alpha, x_\alpha^1, \ldots, x_\alpha^n)\}$ *gives* or *specifies* the orientation of an oriented n-manifold $(M, [\sigma])$ if for every α, there is an everywhere positive function f_α on U_α such that

$$\sigma = f_\alpha\, \phi_\alpha^*(dr^1 \wedge \cdots \wedge dr^n) = f_\alpha\, dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n.$$

Here $r^1, \ldots, r^n$ are the standard coordinates on the Euclidean space $\mathbb{R}^n$.

If ω is a nowhere-vanishing n-form that orients a manifold M, then on any *connected* chart $(U, x^1, \ldots, x^n)$, there is by continuity an everywhere positive or everywhere negative function f such that

$$\omega = f\, dx^1 \wedge \cdots \wedge dx^n.$$

Thus, on an oriented manifold the orientation at one point of a connected chart determines the orientation at every point of the chart.

Example 20.11 (*The open Möbius band*). Let R be the rectangle

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ -1 < y < 1\}$$

(see Figure 20.5). The open Möbius band M (Figure 20.5 and 20.6) is the quotient of the rectangle R by the equivalence relation

$$(0, y) \sim (1, -y). \tag{20.2}$$

The interior of R is the open rectangle

$$U = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1,\ -1 < y < 1\}.$$

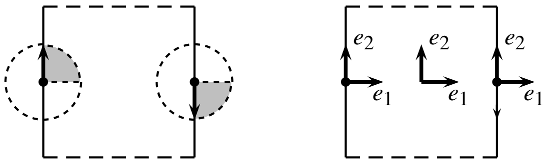


Fig. 20.5. Nonorientability of the Möbius band.

An orientation on M restricts to an orientation on U. To avoid confusion with an ordered pair of numbers, in this example we write an ordered basis without the parentheses. Without loss of generality we may assume the orientation on U to be e_1, e_2. By continuity the orientation at the points $(0, 0)$ and $(1, 0)$ are also e_1, e_2. But under the identification (20.2), the ordered basis e_1, e_2 at $(1, 0)$ maps to $e_1, -e_2$ at $(0, 0)$. Thus, at $(0, 0)$ the orientation has to be both e_1, e_2 and $e_1, -e_2$. This contradiction proves that the Möbius band is not orientable.

Example 20.12. By the regular level set theorem, if 0 is a regular value of a C^∞ function $f(x, y, z)$ on $\mathbb{R}^3$, then the set $M = f^{-1}(0) = \mathrm{Zero}(f)$ is a C^∞ manifold. In Problem 19.10 we constructed a nowhere-vanishing 2-form on M. Thus, M is orientable. Combined with Example 20.11 it follows that an open Möbius band cannot be realized as a regular level set of a C^∞ function on $\mathbb{R}^3$.

Problems

20.1. Equivalence of oriented atlases

Show that the relation in Definition 20.10 is an equivalence relation.

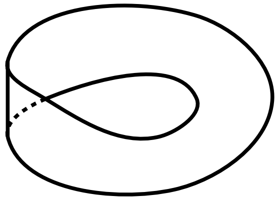


Fig. 20.6. Möbius band.

20.2.* Equivalent nowhere-vanishing n-forms
Show that equivalent oriented atlases give rise to equivalent nowhere-vanishing n-forms.

20.3. Orientation-preserving diffeomorphisms
Let $F\colon (N, \omega_N) \to (M, \omega_M)$ be an orientation-preserving diffeomorphism. If $\{(V, \psi)\} = \{(V, y^1, \dots, y^n)\}$ is an oriented atlas on M that specifies the orientation of M, show that $\{(F^{-1}V, F^*\psi)\} = \{(F^{-1}V, F^1, \dots, F^n)\}$ is an oriented atlas on N that specifies the orientation of N, where $F^i = y^i \circ F$.

20.4. Orientability of a regular level set in $\mathbb{R}^{n+1}$
Suppose $f(x^1, \dots, x^{n+1})$ is a C^∞ function on $\mathbb{R}^{n+1}$ with 0 as a regular value. Show that the zero set of f is an orientable surface in $\mathbb{R}^{n+1}$.

20.5. Orientability of a Lie group
Show that every Lie group G is orientable by constructing a nowhere-vanishing top form on G.

20.6. Orientability of a parallelizable manifold Show that a parallelizable manifold is orientable. (In particular, this shows again that every Lie group is orientable.)

20.7. Orientability of the total space of the tangent bundle
Let M be a smooth manifold and $\pi\colon TM \to M$ its tangent bundle. Show that if $\{(U, \phi)\}$ is any atlas on M, then the atlas $\{(TU, \tilde{\phi})\}$ on TM, with $\tilde{\phi}$ defined in equation (12.1), is oriented. This proves that the total space TM of the tangent bundle is always orientable, regardless of whether or not M is orientable.

21

Manifolds with Boundary

The prototype of a manifold with boundary is the *closed upper half-space*

$$\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\},$$

with the subspace topology of $\mathbb{R}^n$. The points $(x^1, \dots, x^n) \in \mathbb{H}^n$ with $x^n > 0$ are called the *interior points* of $\mathbb{H}^n$, and the points with $x^n = 0$ are called the *boundary points* of $\mathbb{H}^n$. These two sets are denoted $\operatorname{int}(\mathbb{H}^n)$ and $\partial(\mathbb{H}^n)$, respectively (Figure 21.1).

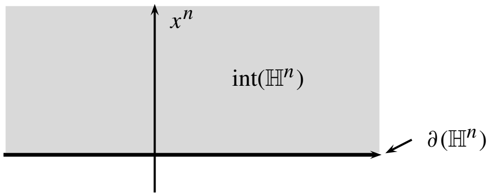


Fig. 21.1. Upper half-space.

In the literature the upper half-space often means the open set

$$\{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n > 0\}.$$

We require that $\mathbb{H}^n$ include the boundary in order for it to serve as a model for manifolds with boundary.

21.1 Invariance of Domain

To discuss C^∞ functions on a manifold with boundary, we need to extend the domain of definition of a C^∞ function to nonopen subsets.

Definition 21.1. Let $S \subset \mathbb{R}^n$ be an arbitrary subset. A function $f: S \to \mathbb{R}^m$ is *smooth at a point* p in S if there exist a neighborhood U of p in $\mathbb{R}^n$ and a C^∞ function $\tilde{f}: U \to \mathbb{R}^m$ such that $\tilde{f} = f$ on $U \cap S$. The function is *smooth on* S if it is smooth at each point of S.

With this definition it now makes sense to speak of an arbitrary subset $S \subset \mathbb{R}^n$ being diffeomorphic to an arbitrary subset $T \subset \mathbb{R}^m$; this will be the case if and only if there are smooth maps $f: S \to T$ and $g: T \to S$ that are inverse to each other.

Exercise 21.2 (Smooth functions on a nonopen set). Using a partition of unity, show that a function $f: S \to \mathbb{R}^m$ is C^∞ on S if and only if there exists an open set U in $\mathbb{R}^n$ containing S and a C^∞ function $\tilde{f}: U \to \mathbb{R}^m$ such that $f = \tilde{f}|_S$.

Theorem 21.3 (C^∞ invariance of domain). *Let $U \subset \mathbb{R}^n$ be an open subset, $S \subset \mathbb{R}^n$ an arbitrary subset, and $f: U \to S$ a diffeomorphism. Then S is open in $\mathbb{R}^n$.*

A diffeomorphism $f: U \to S$ takes an open set in U to an open set in S. Thus, a priori we know only that $f(U)$ is open in S, not that $f(U)$ is open in $\mathbb{R}^n$. Because f is onto $f(U) = S$.

Proof. Let $p \in U$. Since $f: U \to S$ is a diffeomorphism, there is an open set V containing S and a C^∞ map $g: V \to \mathbb{R}^n$ such that $g|_S = f^{-1}$. Thus,

$$U \xrightarrow{f} V \xrightarrow{g} \mathbb{R}^n$$

satisfies

$$g \circ f = 1_U : U \to U \subset \mathbb{R}^n,$$

the identity map on U. By the chain rule,

$$g_{*,f(p)} \circ f_{*,p} = 1_{T_pU} : T_pU \to T_pU \simeq T_p(\mathbb{R}^n),$$

the identity map on the tangent space T_pU. Hence, $f_{*,p}$ is invertible. By the inverse function theorem, f is locally invertible at p. This means there are open neighborhoods U_p of p in U and $V_{f(p)}$ of $f(p)$ in V such that $f: U_p \to V_{f(p)}$ is a diffeomorphism. It follows that

$$V_{f(p)} \subset f(U) = S.$$

Hence, S is open in $\mathbb{R}^n$. □

Proposition 21.4. *Let U and V be open subsets of $\mathbb{H}^n$ and $f: U \to V$ a diffeomorphism. Then f maps interior points to interior points and boundary points to boundary points.*

Proof. Let $p \in U$ be an interior point. Then p is contained in an open ball B, which is actually open in $\mathbb{R}^n$ (not just in $\mathbb{H}^n$). By the invariance of domain, $f(B)$ is open in $\mathbb{R}^n$ (again not just in $\mathbb{H}^n$). Since

$$f(p) \in f(B) \subset V \subset \mathbb{H}^n,$$

$f(p)$ is an interior point of $\mathbb{H}^n$.

If p is a boundary point in $U \cap \partial\mathbb{H}^n$, then $f^{-1}(f(p)) = p$ is a boundary point. Since $f^{-1}\colon V \to U$ is a diffeomorphism, by what has just been proved, $f(p)$ cannot be an interior point. Thus, $f(p)$ is a boundary point. □

21.2 Manifolds with Boundary

In the upper half-space $\mathbb{H}^n$ one may distinguish two kinds of open subsets, depending on whether the set is disjoint from the boundary or intersects the boundary (Figure 21.2).

Fig. 21.2. Two types of open subsets of $\mathbb{H}^n$.

A manifold is locally homeomorphic to only the first kind of open sets. A manifold with boundary generalizes the definition of a manifold by allowing both kinds of open sets.

We say that a topological space M is *locally* $\mathbb{H}^n$ if every point $p \in M$ has a neighborhood U homeomorphic to an open subset of $\mathbb{H}^n$.

Definition 21.5. A *topological n-manifold with boundary* is a second countable Hausdorff topological space which is locally $\mathbb{H}^n$.

For $n \geq 2$, a *chart* on a topological n-manifold with boundary is defined to be a pair (U, ϕ) consisting of an open set U in M and a homeomorphism

$$\phi\colon U \to \phi(U) \subset \mathbb{H}^n$$

of U with an open subset $\phi(U)$ of $\mathbb{H}^n$. As Example 21.8 will show, a slight modification is necessary when $n = 1$: we need to allow ϕ to be a homeomorphism of U with an open subset $\phi(U)$ of $\mathbb{H}^1$ or of the *left half-line*

$$\mathbb{L}^1 := \{x \in \mathbb{R} \mid x \leq 0\}.$$

With this convention, if $(U, x^1, x^2, \ldots, x^n)$ is a chart of an n-dimensional manifold with boundary, then so is $(U, -x^1, x^2, \ldots, x^n)$ for any $n \geq 1$.

A collection $\{(U, \phi)\}$ of charts is a C^∞ *atlas* if for any two charts (U, ϕ) and (V, ψ),

$$\psi \circ \phi^{-1} \colon \phi(U \cap V) \to \psi(U \cap V) \subset \mathbb{H}^n$$

is C^∞. A C^∞ *manifold with boundary* is a topological manifold M with boundary together with a maximal C^∞ atlas.

A point p of M is an *interior point* if in some chart (U, ϕ), $\phi(p)$ is an interior point of $\mathbb{H}^n$. Similarly, p is a *boundary point* of M if $\phi(p)$ is a boundary point of $\mathbb{H}^n$. These concepts are well defined, independent of the charts, because if (V, ψ) is another chart, then the diffeomorphism $\psi \circ \phi^{-1}$ maps $\phi(p)$ to $\psi(p)$. By Proposition 21.4, $\phi(p)$ and $\psi(p)$ are either both interior points or both boundary points (Figure 21.3). The set of boundary points of M is denoted ∂M.

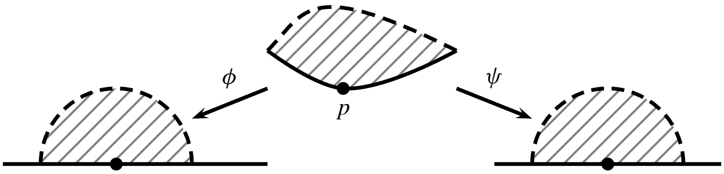


Fig. 21.3. Boundary charts.

Most of the concepts that we introduced for a manifold extend to a manifold with boundary in an obvious way. For example, a function $f \colon M \to \mathbb{R}$ is C^∞ at a boundary point $p \in \partial M$ if there is a chart (U, ϕ) about p such that $f \circ \phi^{-1}$ is C^∞ at $\phi(p) \in \mathbb{H}^n$. This in turn means that $f \circ \phi^{-1}$ has a C^∞ extension to a neighborhood of $\phi(p)$ in $\mathbb{R}^n$.

Remark 21.6. In point-set topology there is another notion of boundary, defined for a subset of a topological space S. If $A \subset S$, a point p in S is said to be a *boundary point of* A if every neighborhood of p contains a point in A and a point not in A. The set of all boundary points of A in S is denoted $\mathrm{bd}(A)$. We call this set the *topological boundary* of A, to distinguish it from the *manifold boundary* ∂A in case A is a manifold with boundary.

Example 21.7. Let A be the open unit disk in $\mathbb{R}^2$:

$$A = \{x \in \mathbb{R}^2 \mid \|x\| < 1\}.$$

Then its topological boundary $\mathrm{bd}(A)$ in $\mathbb{R}^2$ is the unit circle, while its manifold boundary ∂A is the empty set.

If B is the closed unit disk in $\mathbb{R}^2$, then its topological boundary $\mathrm{bd}(B)$, the unit circle, coincides with its manifold boundary ∂B.

21.3 The Boundary of a Manifold with Boundary

Let M be a manifold of dimension n with boundary ∂M. If (U, ϕ) is a chart for M, we denote by $\phi' = \phi|_{U \cap \partial M}$ the restriction of the coordinate map ϕ to the boundary. Since ϕ maps boundary points to boundary points,

$$\phi' \colon U \cap \partial M \to \partial \mathbb{H}^n = \mathbb{R}^{n-1}.$$

Moreover, if (U, ϕ) and (V, ψ) are two charts for M, then

$$\psi' \circ (\phi')^{-1} \colon \phi'(U \cap V \cap \partial M) \to \psi'(U \cap V \cap \partial M)$$

is C^∞. Thus, an atlas $\{(U_\alpha, \phi_\alpha)\}$ for M induces an atlas $\{(U_\alpha \cap \partial M, \phi_\alpha|_{U_\alpha \cap \partial M})\}$ for ∂M, making ∂M into a manifold of dimension $n-1$ without boundary.

21.4 Tangent Vectors, Differential Forms, and Orientations

If M is a manifold with boundary and $p \in \partial M$, we define the *algebra* $C_p^\infty = C_p^\infty(M)$ *of germs of* C^∞ *functions at* p as in Section 2.2. Two C^∞ functions $f \colon U \to \mathbb{R}$ and $g \colon V \to \mathbb{R}$ defined on neighborhoods U and V of p in M are said to be *equivalent* if they agree on some neighborhood W of p contained in $U \cap V$. A *germ* of C^∞ functions at p is an equivalence class of such functions. The *tangent space* T_pM at p is then defined to be the vector space of all derivations on C_p^∞.

For example, for p in the boundary of the upper half-plane $\mathbb{H}^2$, $(\partial/\partial x)_p$ and $(\partial/\partial y)_p$ are both derivations on $C_p^\infty(\mathbb{H}^2)$. The tangent space $T_p(\mathbb{H}^2)$ is represented by a two-dimensional vector space with the origin at p. Since $(\partial/\partial y)_p$ is a tangent vector to $\mathbb{H}^2$ at p, its negative $-(\partial/\partial y)_p$ is also a tangent vector at p (Figure 21.4), although there is no curve through p in $\mathbb{H}^2$ with initial velocity $-(\partial/\partial y)_p$.

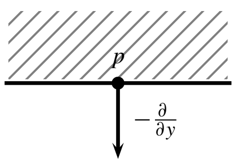


Fig. 21.4. A tangent vector at the boundary.

The *cotangent space* T_p^*M is defined to be the dual of the tangent space:

$$T_p^*M = \operatorname{Hom}(T_pM, \mathbb{R}).$$

Differential k-forms on M are defined as before, as sections of the bundle $\bigwedge^k(T^*M)$. For example, $dx \wedge dy$ is a 2-form on $\mathbb{H}^2$. An *orientation* on an n-manifold M with boundary is given by a C^∞ nowhere-vanishing n-form.

According to Proposition 20.9, for a manifold without boundary the existence of a nowhere-vanishing top form is equivalent to the existence of an oriented atlas. The same proof goes through word for word for a manifold with boundary. At some point in the proof it is necessary to replace the chart $(U, x^1, x^2, \dots, x^n)$ by $(U, -x^1, x^2 \dots, x^n)$. This would not have been possible for $n = 1$ if we had not allowed $\mathbb{L}^1$ as a local model in the definition of a chart for a one-dimensional manifold with boundary.

Example 21.8. The closed interval $[0, 1]$ is a C^∞ manifold with boundary. It has an atlas with two charts (U_1, ϕ_1) and (U_2, ϕ_2), where $U_1 = [0, 1)$, $\phi_1(x) = x$, and $U_2 = (0, 1]$, $\phi_2(x) = 1 - x$. With dx as the orientation form, $[0, 1]$ is an oriented manifold with boundary. However, $\{(U_1, \phi_1), (U_2, \phi_2)\}$ is not an oriented atlas, because the transition function $\phi_2 \circ \phi_1^{-1}(x) = 1 - x$ has negative Jacobian determinant. If we change the sign of ϕ_2, then $\{(U_1, \phi_1), (U_2, -\phi_2)\}$ is an oriented atlas. Note that $-\phi_2(x) = x - 1$ maps $(0, 1]$ into the left half line $\mathbb{L}^1 \subset \mathbb{R}$. If we had allowed only $\mathbb{H}^1$ as a local model for a one-dimensional manifold with boundary, the closed interval $[0, 1]$ would not have an oriented atlas.

21.5 Boundary Orientation for Manifolds of Dimension Greater than One

In this section we show that an orientation on a manifold M with boundary induces in a natural way an orientation on the boundary ∂M. We first consider the case where $\dim M \geq 2$.

Lemma 21.9. *Assume $n \geq 2$. Let (U, ϕ) and (V, ψ) be two charts in an oriented atlas of an orientable manifold M with boundary. Assume that U, V, and ∂M have nonempty intersection. Then the restriction of the transition function to the boundary $B := \phi(U \cap V) \cap \partial\mathbb{H}^n$,*

$$\psi \circ \phi^{-1}|_B \colon \phi(U \cap V) \cap \partial\mathbb{H}^n \to \psi(U \cap V) \cap \partial\mathbb{H}^n,$$

has positive Jacobian determinant.

Proof. Let $\phi = (x^1, \dots, x^n)$ on U and $\psi = (y^1, \dots, y^n)$ on V. Since the transition function

$$\psi \circ \phi^{-1} \colon \phi(U \cap V) \to \psi(U \cap V) \subset \mathbb{H}^n$$

takes boundary points to boundary points and interior points to interior points,

(i) $y^n(x^1, \dots, x^{n-1}, 0) = 0$, and
(ii) $y^n(x^1, \dots, x^{n-1}, x^n) > 0$ for $x^n > 0$,

where $(x^1, x^2, \dots, x^n) \in \phi(U \cap V)$.

Differentiating (i) gives

$$\frac{\partial y^n}{\partial x^i}(x^1, \dots, x^{n-1}, 0) = 0 \quad \text{for } i = 1, \dots, n-1.$$

From (i) and (ii),

$$\begin{aligned}\frac{\partial y^n}{\partial x^n}(x^1,\dots,x^{n-1},0) &= \lim_{t\to 0^+}\frac{y^n(x^1,\dots,x^{n-1},t)-y^n(x^1,\dots,x^{n-1},0)}{t}\\ &= \lim_{t\to 0^+}\frac{y^n(x^1,\dots,x^{n-1},t)}{t}\geq 0,\end{aligned}$$

since both t and $y^n(x^1,\dots,x^{n-1},t)$ are positive.

The Jacobian matrix of $\psi\circ\phi^{-1}$ at a boundary point $(x^1,\dots,x^{n-1},0)$ therefore has the form

$$J(\psi\circ\phi^{-1}|_B)=\begin{pmatrix}\frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^{n-1}} & \frac{\partial y^1}{\partial x^n}\\ \vdots & & \vdots & \vdots\\ \frac{\partial y^{n-1}}{\partial x^1} & \cdots & \frac{\partial y^{n-1}}{\partial x^{n-1}} & \frac{\partial y^{n-1}}{\partial x^n}\\ 0 & \cdots & 0 & \frac{\partial y^n}{\partial x^n}\end{pmatrix}=\begin{pmatrix} & J(\psi\circ\phi^{-1}|_B) & & * \\ & & & \\ 0 & \cdots & 0 & \frac{\partial y^n}{\partial x^n}\end{pmatrix}.$$

It follows that

$$\det J(\psi\circ\phi^{-1})=\det J(\psi\circ\phi^{-1}|_B)\cdot\frac{\partial y^n}{\partial x^n}.$$

Since $\det J(\psi\circ\phi^{-1})$ is positive everywhere by hypothesis, at $(x^1,\dots,x^{n-1},0)$ we have $\partial y^n/\partial x^n>0$ and therefore $\det J(\psi\circ\phi^{-1}|_B)>0$. □

The following proposition is a direct consequence of the lemma.

Proposition 21.10. *If $\{(U_\alpha,\phi_\alpha)\}$ is an oriented atlas for a manifold M with boundary, then the induced atlas $\{(U_\alpha\cap\partial M,\phi_\alpha|_{U_\alpha\cap\partial M})\}$ for ∂M is oriented.*

If $\{(U_\alpha,\phi_\alpha)\}$ is an oriented atlas for a manifold M with boundary, by Proposition 21.10, it would seem most natural to define the boundary orientation on ∂M by the oriented atlas $\{(U_\alpha\cap\partial M,\phi_\alpha|_{U_\alpha\cap\partial M})\}$. This convention, unfortunately, would lead to a sign in Stokes' theorem. In order to have a sign-free Stokes' theorem, we adopt the following convention.

Definition 21.11. Suppose the oriented atlas $\{(U_\alpha,\phi_\alpha)\}$ gives the orientation on a manifold M. If M is even dimensional, then the *boundary orientation* on ∂M is given by the oriented atlas $\{(U_\alpha\cap\partial M,\phi_\alpha|_{U_\alpha\cap\partial M})\}$. If M is odd dimensional, then the *boundary orientation* on ∂M is given by the opposite of the oriented atlas $\{(U_\alpha\cap\partial M,\phi_\alpha|_{U_\alpha\cap\partial M})\}$.

It is clear from Lemma 21.9 that the definition of the boundary orientation is independent of the oriented atlas for M. In Problems 21.4 and 21.5, we describe two other ways of specifying the boundary orientation, in terms of a basis of tangent vectors and in terms of an orientation form.

Example 21.12 (*The boundary orientation on* $\partial\mathbb{H}^n$). The standard orientation on the upper half-space $\mathbb{H}^n$ is given by the oriented atlas with a single chart $(\mathbb{H}^n, x^1, \ldots, x^n)$ corresponding to the n-form $dx^1 \wedge \cdots \wedge dx^n$. The boundary $\partial\mathbb{H}^n \simeq \mathbb{R}^{n-1}$ has an oriented atlas with a single chart $(\partial\mathbb{H}^n, x^1, \ldots, x^{n-1})$, corresponding to the $(n-1)$-form $dx^1 \wedge \cdots \wedge dx^{n-1}$. By Definition 21.11, the boundary orientation on $\partial\mathbb{H}^n$ is given by the $(n-1)$-form $(-1)^n\, dx^1 \wedge \cdots \wedge dx^{n-1}$ (Figures 21.5 and 21.6).

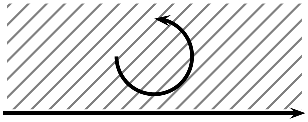


Fig. 21.5. Induced orientation on $\partial\mathbb{H}^2 = \mathbb{R}$.

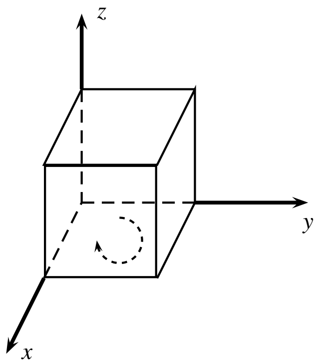


Fig. 21.6. Induced orientation on $\partial\mathbb{H}^3 = \mathbb{R}^2$.

21.6 Boundary Orientation for One-Dimensional Manifolds

An orientation on a point is one of two numbers ± 1. In accordance with Example 21.12, we define the boundary orientation at the boundary point 0 of $\mathbb{H}^1 = [0, \infty)$ to be -1.

Suppose C is a one-dimensional oriented manifold with boundary, and p is a boundary point of C. If $\phi\colon U \to \mathbb{H}^1$ is an orientation-preserving chart about p (Figure 21.7), then the boundary orientation at p is defined to be -1; if $\phi\colon U \to \mathbb{H}^1$ is an orientation-reversing chart about p, then the boundary orientation at p is defined to be 1.

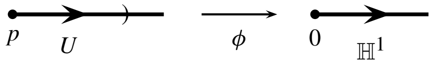


Fig. 21.7. Orientation-preserving chart.

Example 21.13. The closed interval $[a, b]$ in the real line with coordinate x has a standard orientation given by the 1-form dx. A chart centered at a is $([a, b), \phi)$, where $\phi\colon [a, b) \to \mathbb{H}$ is given by $\phi(x) = x - a$. Since ϕ is orientation-preserving, the boundary orientation at a is -1. Similarly, a chart centered at b is $((a, b], \psi)$, where $\psi\colon (a, b] \to \mathbb{H}$ is given by $\psi(x) = b - x$. Since ψ is orientation-reversing, the boundary orientation at b is $-(-1) = +1$.

Example 21.14. Suppose $c : [a, b] \to M$ is a C^∞ map whose image is a one-dimensional manifold C. An orientation on $[a, b]$ induces an orientation on C via the differential $c_*\colon T_p([a, b]) \to T_pC$ at each point $p \in [a, b]$. In a situation like this, we give C the orientation induced from the standard orientation on $[a, b]$. The boundary orientation on the boundary of C is $+1$ at the endpoint $c(b)$ and -1 at the initial point $c(a)$.

Problems

21.1. Topological boundary versus manifold boundary
Let M be the subset $[0, 1) \cup \{2\}$ of the real line. Find its topological boundary $\mathrm{bd}(M)$ and its manifold boundary ∂M.

21.2.* Boundary orientation of the left half-space
Let M be the left half-space

$$\{(y^1, \dots, y^n) \in \mathbb{R}^n \mid y^1 \le 0\},$$

with orientation form $dy^1 \wedge \cdots \wedge dy^n$. Show that an orientation form for the boundary orientation on $\partial M = \{(0, y^2, \cdots, y^n) \in \mathbb{R}^n\}$ is $dy^2 \wedge \cdots \wedge dy^n$.

This exercise shows that if we had used the left half-space as the model of a manifold with boundary, then there would not be a sign in the induced boundary orientation. In fact, certain authors adopt this convention, e.g., [4].

21.3.* Inward-pointing vectors at the boundary
Let M be a manifold with boundary and $p \in \partial M$. We say that a tangent vector $X_p \in T_p(M)$ is *inward-pointing* if $X_p \notin T_p(\partial M)$ and there are a positive real number ϵ and a curve $c\colon [0, \epsilon) \to M$ such that $c(0) = p$, $c((0, \epsilon)) \subset \mathrm{int}(M)$, and $c'(0) = X_p$. A vector $X_p \in T_p(M)$ is *outward-pointing* if $-X_p$ is inward-pointing. For example, on the upper half-plane $\mathbb{H}^2$, $(\partial/\partial y)_p$ is inward-pointing and $-(\partial/\partial y)_p$ is outward-pointing at a point p in the x-axis. Show that $X_p \in T_p(M)$ is inward-pointing iff in any coordinate chart $(U, x^1, \dots, x^n)$ centered at p, the coefficient of $(\partial/\partial x^n)_p$ in X_p is positive.

21.4.* Boundary orientation in terms of tangent vectors
Let M be a manifold with boundary and $p \in \partial M$. Show that an ordered basis $(v_1, \ldots, v_{n-1})$ for the tangent space $T_p(\partial M)$ gives the boundary orientation on ∂M at p iff for any outward-pointing vector $X_p \in T_p(M)$, the ordered basis $(X_p, v_1, \ldots, v_{n-1})$ for $T_p(M)$ gives the orientation on M at p.

21.5.* Orientation form of the boundary orientation
Suppose M is an oriented manifold with boundary with orientation form ω. A vector field *along* ∂M assigns to each point $p \in \partial M$ a vector in the tangent space T_pM (as opposed to $T_p(\partial M)$). Let X be an outward-pointing vector field along ∂M. Show that the contraction $\iota_X\omega$ is a boundary orientation for ∂M. (The contraction is defined in Problem 4.7.)

21.6. Boundary orientation for a cylinder
Let M be the cylinder $S^1 \times [0, 1]$ with the counterclockwise orientation when viewed from the exterior (Figure 21.8). Describe the boundary orientation on $C_0 = S^1 \times \{0\}$ and $C_1 = S^1 \times \{1\}$.

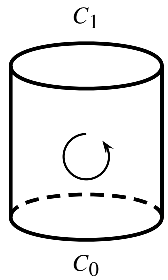


Fig. 21.8. Oriented cylinder.

22

Integration on a Manifold

On a manifold, one integrates not functions as in calculus on $\mathbb{R}^n$ but differential forms. There are actually two theories of integration on a manifold: one where the integration is over a submanifold and the other where the integration is over what is called a *singular chain*. Singular chains allow one to integrate over an object such as a closed rectangle in $\mathbb{R}^2$:

$$[a, b] \times [c, d] := \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \quad c \leq y \leq d\},$$

which is not a submanifold of $\mathbb{R}^2$ because of its corners.

For simplicity we will discuss only integration over a submanifold. For the more general theory of integration over singular chains, the reader may consult the many excellent references in the bibliography, for example, [4, Section 8.2] or [11, Chapter 14].

22.1 The Riemann Integral of a Function on $\mathbb{R}^n$

We assume that the reader is familiar with the theory of Riemann integration in $\mathbb{R}^n$, as in [12] or [17]. What follows is a brief synopsis of the Riemann integral of a bounded function over a bounded set in $\mathbb{R}^n$.

A *closed rectangle* in $\mathbb{R}^n$ is a Cartesian product $R = [a^1, b^1] \times \cdots \times [a^n, b^n]$ of closed intervals in $\mathbb{R}$, where $a^i, b^i \in \mathbb{R}$. Let $f\colon R \to \mathbb{R}$ be a bounded function defined on a closed rectangle R. A *partition* of the closed interval $[a, b]$ is a set of real numbers $\{p_0, \ldots, p_n\}$ such that

$$a = p_0 < p_1 < \cdots < p_n = b.$$

A *partition* of the rectangle R is a collection $P = \{P_1, \ldots, P_n\}$ where P_i is a partition of $[a^i, b^i]$. The partition P divides the rectangle R into closed subrectangles, which we denote by R_j (Figure 22.1).

We define the *lower sum* and the *upper sum* of f with respect to the partition P to be

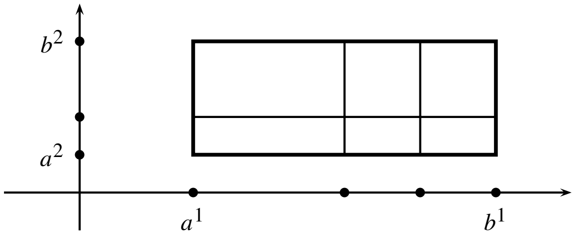


Fig. 22.1. A partition of a closed rectangle.

$$L(f, P) = \sum (\inf_{R_j} f) \operatorname{vol}(R_j), \qquad U(f, P) = \sum (\sup_{R_j} f) \operatorname{vol}(R_j),$$

where each sum runs over all subrectangles of the partition P. For any partition P, clearly $L(f, P) \le U(f, P)$. In fact, more is true: for any two partitions P and P' of the rectangle R,

$$L(f, P) \le U(f, P'),$$

which we show next.

A partition $P' = \{P_1', \dots, P_n'\}$ is a *refinement* of the partition $P = \{P_1, \dots, P_n\}$ if $P_i \subset P_i'$ for all $i = 1, \dots, n$. If P' is a refinement of P, then each subrectangle R_j of P is subdivided into subrectangles R_{jk}' of P', and it is easily seen that

$$L(f, P) \le L(f, P'), \tag{22.1}$$

because if $R_{jk}' \subset R_j$, then $\inf_{R_j} f \le \inf_{R_{jk}'} f$. Similarly, if P' is a refinement of P, then

$$U(f, P') \le U(f, P). \tag{22.2}$$

Any two partitions P and P' of the rectangle R have a common refinement $Q = \{Q_1, \dots, Q_n\}$ with $Q_i = P_i \cup P_i'$. By (22.1) and (22.2),

$$L(f, P) \le L(f, Q) \le U(f, Q) \le U(f, P').$$

It follows that the supremum of the lower sum $L(f, P)$ over all partitions P of R is less than or equal to the infimum of the upper sum $U(f, P)$ over all partitions P of R. We define these two numbers to be the *lower integral* $\underline{\int}_R f$ and the *upper integral* $\overline{\int}_R f$, respectively:

$$\underline{\int}_R f = \sup_P L(f, P), \qquad \overline{\int}_R f = \inf_P L(f, P).$$

Definition 22.1. Let R be a closed rectangle in $\mathbb{R}^n$. A bounded function $f\colon R \to \mathbb{R}$ is said to be *Riemann integrable* if $\underline{\int}_R f = \overline{\int}_R f$; in this case, the Riemann integral of f is this common value, denoted $\int_R f(x)\, |dx^1 \cdots dx^n|$, where $x^1, \dots, x^n$ are the coordinates on $\mathbb{R}^n$.

Remark. When we speak of a rectangle $[a^1, b^1] \times \cdots \times [a^n, b^n]$ in $\mathbb{R}^n$, we have already tacitly chosen n coordinates axes, with coordinates $x^1, \ldots, x^n$. Thus, the definition of the Riemann integrable depends on the coordinates $x^1, \ldots, x^n$. The two vertical bars in the integral serve to emphasize that it is merely a notation for the Riemann integral, not a differential form.

If $f\colon A \subset \mathbb{R}^n \to \mathbb{R}$, then the *extension of f by zero* is the function $\tilde{f}\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in A, \\ 0 & \text{for } x \notin A. \end{cases}$$

Now suppose $f\colon A \to \mathbb{R}$ is a bounded function on a bounded set A in $\mathbb{R}^n$. Enclose A in a closed rectangle and define the Riemann integral of f over A to be

$$\int_A f(x)\,|dx^1 \cdots dx^n| = \int_R \tilde{f}(x)\,|dx^1 \cdots dx^n|$$

if the right-hand side exists. In this way we can deal with the integral of a bounded function whose domain is an arbitrary bounded set in $\mathbb{R}^n$.

The *volume* $v(A)$ of a subset $A \subset \mathbb{R}^n$ is defined to be the integral $\int_A 1$ if the integral exists. For a closed rectangle $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$, the volume is

$$v(R) = \prod_{i=1}^{n} (b_i - a_i).$$

22.2 Integrability Conditions

In this section we describe some conditions under which a function defined on an open subset of $\mathbb{R}^n$ is Riemann integrable.

Definition 22.2. A set $A \subset \mathbb{R}^n$ is said to have *measure zero* if for every $\epsilon > 0$, there is a countable cover $\{R_i\}_{i=1}^{\infty}$ of A by closed rectangles R_i such that $\sum_{i=1}^{\infty} v(R_i) < \epsilon$.

The most useful integrability criterion is the following theorem of Lebesgue [12, Theorem 8.3.1, p. 455].

Theorem 22.3 (Lebesgue's theorem). *A bounded function $f\colon A \to \mathbb{R}$ on a bounded subset $A \subset \mathbb{R}^n$ is Riemann integrable iff the set* $\operatorname{Dist}(\tilde{f})$ *of discontinuities of the extended function $\tilde{f}$ has measure zero.*

Proposition 22.4. *If a continuous function $f\colon U \to \mathbb{R}$ defined on an open subset U of $\mathbb{R}^n$ has compact support, then f is Riemann integrable on U.*

Proof. Being continuous on a compact set, the function f is bounded. Being compact, the set $\operatorname{supp} f$ is closed and bounded in $\mathbb{R}^n$. We claim that the extension $\tilde{f}$ is continuous.

Since $\tilde{f}$ agrees with f on U, the extended function $\tilde{f}$ is continuous on U. If $p \notin U$, then $p \notin \operatorname{supp} f$. As $\operatorname{supp} f$ is a closed subset of $\mathbb{R}^n$, there is an open ball B containing p and disjoint from $\operatorname{supp} f$. Hence, $\tilde{f} \equiv 0$ on B, which proves that $\tilde{f}$ is continuous at $p \notin U$. Thus, $\tilde{f}$ is continuous on $\mathbb{R}^n$. By Lebesgue's theorem, f is Riemann integable on U. □

Proposition 22.5. *A bounded continuous function $f: U \to \mathbb{R}$ defined on an open set U of finite volume is Riemann integrable.*

Proof. Since U has finite volume, the integral $\int_U 1$ exists, where $1: U \to \mathbb{R}$ is the constant function 1 on U. The set of discontinuities of the extended function $\tilde{1}$ is the boundary $\operatorname{bd}(U)$ of U. By Lebesgue's theorem, $\operatorname{bd}(U)$ is a set of measure zero.

Clearly, the set $\operatorname{Dist}(\tilde{f})$ of discontinuities of $\tilde{f}$ is a subset of $\operatorname{bd}(U)$; hence, $\operatorname{Dist}(\tilde{f})$ is also a set of measure zero. By Lebesgue's theorem again, f is Riemann integrable over U. □

22.3 The Integral of an n-Form on $\mathbb{R}^n$

Once a set of coordinates $x^1, \ldots, x^n$ has been fixed on $\mathbb{R}^n$, n-forms on $\mathbb{R}^n$ can be identified with functions on $\mathbb{R}^n$, since every n-form on $\mathbb{R}^n$ can be written as $\omega = f(x)\, dx^1 \wedge \cdots \wedge dx^n$ for a unique function $f(x)$ on $\mathbb{R}^n$. In this way the theory of Riemann integration of functions on $\mathbb{R}^n$ carries over to n-forms on $\mathbb{R}^n$.

Definition 22.6. Let $\omega = f(x)\, dx^1 \wedge \cdots \wedge dx^n$ be a C^∞ n-form on an open subset $U \subset \mathbb{R}^n$, with standard coordinates $x^1, \ldots, x^n$. Its *integral* over U is defined to be the Riemann integral of $f(x)$:

$$\int_U \omega = \int_U f(x)\, dx^1 \wedge \cdots \wedge dx^n = \int_U f(x)\, |dx^1 \cdots dx^n|,$$

if the Riemann integral exists.

In this definition the n-form must be written in the order $dx^1 \wedge \cdots \wedge dx^n$. To integrate for example $\tau = f(x)\, dx^2 \wedge dx^1$ over $U \subset \mathbb{R}^2$, one would write

$$\int_U \tau = \int_U -f(x)\, dx^1 \wedge dx^2 = -\int_U f(x)\, |dx^1\, dx^2|.$$

Let us see how the integral of an n-form on $\mathbb{R}^n$ transforms under a change of variables. A change of variables on U is a diffeomorphism $T: V \subset \mathbb{R}^n \to U \subset \mathbb{R}^n$,

$$x = (x^1, \ldots, x^n) = T(y^1, \ldots, y^n) = T(y) = (T^1(y), \ldots, T^n(y)).$$

Denote by $J(T)$ the Jacobian matrix $[\partial T^i / \partial y^j]$. By Problem 19.2,

$$dT^1 \wedge \cdots \wedge dT^n = \det(J(T))\, dy^1 \wedge \cdots \wedge dy^n.$$

Hence,

$$\begin{aligned}\int_V T^*\omega &= \int_V T^*(f(x))\, T^*dx^1 \wedge \cdots \wedge T^*dx^n \\ &= \int_V f(T(y))\, dT^1 \wedge \cdots \wedge dT^n \\ &= \int_V f(T(y)) \det(J(T))\, dy^1 \wedge \cdots \wedge dy^n \\ &= \int_V f(T(y)) \det(J(T))\, |dy^1 \cdots dy^n|. \end{aligned} \tag{22.3}$$

On the other hand, the change of variables formula from advanced calculus gives

$$\int_U \omega = \int_U f(x)\, |dx^1 \cdots dx^n| = \int_V f(T(y)) |\det(J(T))|\, |dy^1 \cdots dy^n|, \tag{22.4}$$

with an absolute-value sign around the Jacobian determinant. Equations (22.3) and (22.4) differ by the sign of $\det(J(T))$. Hence,

$$\int_V T^*\omega = \pm \int_U \omega, \tag{22.5}$$

depending on whether the Jacobian determinant $\det(J(T))$ is positive or negative.

By Proposition 20.7 a diffeomorphism $T\colon V \subset \mathbb{R}^n \to U \subset \mathbb{R}^n$ is orientation-preserving if and only if its Jacobian determinant $\det(J(T))$ is everywhere positive on V. Equation (22.5) shows that the integral of a differential form is not invariant under all diffeomorphisms of V with U, but only under the orientation-preserving diffeomorphisms.

22.4 The Integral of a Differential Form on a Manifold

The integral of an n-form on $\mathbb{R}^n$ is not so different from the integral of a function. Our approach to integration over a general manifold has several distinguishing features:

(i) The manifold must be oriented (in fact, $\mathbb{R}^n$ has a standard orientation).
(ii) On a manifold of dimension n, one can integrate only n-forms, not functions.
(iii) The n-forms must have compact support.

Let M be an oriented manifold of dimension n, with an oriented atlas $\{(U_\alpha, \phi_\alpha)\}$ giving the orientation of M. Suppose $\{(U, \phi)\}$ is a chart of this atlas. If $\omega \in \Omega^n_c(U)$ is an n-form with compact support on U, then because $\phi : U \to \phi(U)$ is a diffeomorphism, $(\phi^{-1})^*\omega$ is also an n-form with compact support on the open subset $\phi(U) \subset \mathbb{R}^n$. We define the integral of ω on U as

$$\int_U \omega := \int_{\phi(U)} (\phi^{-1})^*\omega.$$

If (U, ψ) is another chart with the same U, then $\phi \circ \psi^{-1}\colon \psi(U) \to \phi(U)$ is an orientation-preserving diffeomorphism, and so

$$\int_{\phi(U)} (\phi^{-1})^*\omega = \int_{\psi(U)} (\phi \circ \psi^{-1})^*(\phi^{-1})^*\omega = \int_{\psi(U)} (\psi^{-1})^*\omega.$$

Thus, the integral $\int_U \omega$ on a chart U of the atlas is well defined, independent of the choice of coordinates on U. By the linearity of the integral on $\mathbb{R}^n$, if $\omega, \tau \in \Omega_c^n(U)$, then

$$\int_U \omega + \tau = \int_U \omega + \int_U \tau.$$

Now let $\omega \in \Omega_c^n(M)$. Choose a partition of unity $\{\rho_\alpha\}$ subordinate to the open cover $\{U_\alpha\}$. Because ω has compact support and a partition of unity has locally finite supports, all except finitely many $\rho_\alpha\omega$ are identically zero by Problem 18.5. In particular,

$$\omega = \sum_\alpha \rho_\alpha \omega$$

is a *finite* sum. Since by Problem 18.4(b),

$$\operatorname{supp}(\rho_\alpha\omega) \subset \operatorname{supp}\rho_\alpha \cap \operatorname{supp}\omega,$$

$\operatorname{supp}(\rho_\alpha\omega)$ is a closed subset of the compact set $\operatorname{supp}\omega$. Hence, $\operatorname{supp}(\rho_\alpha\omega)$ is compact. As $\rho_\alpha\omega$ is an n-form with compact support in the chart U_α, its integral $\int_{U_\alpha} \rho_\alpha\omega$ is defined. Therefore, we can define the integral of ω over M to be the finite sum

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha\omega. \tag{22.6}$$

For this integral to be well defined, we must show that it is independent of the choice of the oriented atlas and of the partition of unity. Let $\{V_\beta\}$ be another oriented atlas of M specifying the orientation of M and $\{\chi_\beta\}$ a partition of unity subordinate to $\{V_\beta\}$. Then $\{(U_\alpha \cap V_\beta, \phi_\alpha)\}$ and $\{(U_\alpha \cap V_\beta, \psi_\beta)\}$ are two new atlases of M specifying the orientation of M, and

$$\begin{aligned}
\sum_\alpha \int_{U_\alpha} \rho_\alpha\omega &= \sum_\alpha \int_{U_\alpha} \rho_\alpha \sum_\beta \chi_\beta\omega && \Big(\text{because } \sum_\beta \chi_\beta = 1\Big)\\
&= \sum_\alpha \sum_\beta \int_{U_\alpha} \rho_\alpha\chi_\beta\omega && (\text{these are } \textit{finite} \text{ sums})\\
&= \sum_\alpha \sum_\beta \int_{U_\alpha \cap V_\beta} \rho_\alpha\chi_\beta\omega,
\end{aligned}$$

where the last line follows from the fact that the support of $\rho_\alpha\chi_\beta$ is contained in $U_\alpha \cap V_\beta$. By symmetry, $\sum_\beta \int_{V_\beta} \chi_\beta\omega$ is equal to the same sum. Hence,

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\beta \int_{V_\beta} \chi_\beta \omega,$$

proving that the integral (22.6) is well defined.

For an oriented manifold M, we indicate by $-M$ the same manifold but with the opposite orientation. If $\{U, \phi)\} = \{(U, x^1, x^2, \ldots, x^n)\}$ is an oriented atlas specifying the orientation of M, then an oriented atlas specifying the orientation of $-M$ is $\{U, \tilde{\phi})\} = \{(U, -x^1, x^2, \ldots, x^n)\}$. Clearly, on n-forms the pullback $\tilde{\phi}^* = -\phi^*$ and therefore $(\tilde{\phi}^{-1})^* = -(\phi^{-1})^*$. This shows that for $\omega \in \Omega_c^n(M)$ and any chart (U, ϕ), $\int_{-U} \omega = -\int_U \omega$ and therefore $\int_{-M} \omega = -\int_M \omega$. Thus, reversing the orientation of M reverses the sign of an integral over M.

The discussion of integration presented above can be extended almost word for word to manifolds with boundary. It has the virtue of simplicity and is of great utility in proving theorems. However, it is not practical for actual computation of integrals; an n-form multiplied by a partition of unity can rarely be integrated as a closed expression. To calculate explicitly integrals over a manifold M, it is best to consider integrals over a *parametrized set*, i.e., a C^∞ map from an open subset of $\mathbb{R}^n$ to the manifold M of dimension n. If $\alpha\colon U \subset \mathbb{R}^n \to M$ is a parametrized set and ω is an n-form on M, not necessarily with compact support, then $\int_{\alpha(U)} \omega$ is defined to be $\int_U \alpha^*\omega$, provided this last integral exists. The integral of a C^∞ n-form over U exists, for example, if U has compact closure and the topological boundary of U is a set of measure zero. We will not delve into this theory of integration (see [16, Theorem 25.4, p. 213] or [11, Proposition 14.7, p. 356]), but will content ourselves with an example.

Example 22.7. In spherical coordinates, ρ is the distance $\sqrt{x^2 + y^2 + z^2}$ of the point $(x, y, z) \in \mathbb{R}^3$ to the origin, ϕ is the angle that the vector $\langle x, y, z\rangle$ makes with the positive z-axis, and θ is the angle that the vector $\langle x, y\rangle$ in the (x, y)-plane makes with the positive x-axis. Let

$$U = \{(x, y, z) \in S^2 \mid 0 < \phi < \pi,\ 0 < \theta < 2\pi\}.$$

Then (U, ϕ, θ) is a chart on the unit sphere S^2. Calculate $\int_U \sin\phi \, d\phi \wedge d\theta$.

Solution. Let $\alpha = (\phi, \theta)$ be the coordinate map on U. Note that ϕ and θ are functions on $U \subset S^2$. Let

$$u = (\alpha^{-1})^*\phi = \phi \circ \alpha^{-1}, \quad v = (\alpha^{-1})^*\theta = \theta \circ \alpha^{-1}$$

be the corresponding functions on the open set

$$\alpha(U) = \{(u, v) \in \mathbb{R}^2 \mid 0 < u < \pi,\ 0 < v < 2\pi\}.$$

By the definition of an integral over U,

$$\begin{aligned}
\int_U \sin\phi\, d\phi \wedge d\theta &= \int_{\alpha(U)} (\alpha^{-1})^*(\sin\phi\, d\phi \wedge d\theta) \\
&= \int_{\alpha(U)} \sin(\phi \circ \alpha^{-1})\, d(\alpha^{-1})^*\phi \wedge d(\alpha^{-1})^*\theta \\
&\qquad \text{(because } d \text{ commutes with } (\alpha^{-1})^*\text{)} \\
&= \int_{\alpha(U)} \sin u\, du \wedge dv \\
&= \int_0^{2\pi}\int_0^{\pi} \sin u\, du\, dv \\
&= 2\pi\Big[\cos u\Big]_0^{\pi} = 4\pi. \qquad \square
\end{aligned}$$

Integration over a zero-dimensional manifold

The discussion of integration so far assumes implicitly that the manifold M has dimension $n \geq 1$. We now treat integration over a zero-dimensional manifold. A compact oriented manifold M of dimension 0 is a finite collection of points, each point oriented by $+1$ or -1. We write this as $M = \sum p_i - \sum q_j$. An integral of a 0-form $f\colon M \to \mathbb{R}$ is defined to be the sum

$$\int_M f = \sum f(p_i) - \sum f(q_j).$$

22.5 Stokes' Theorem

Let M be an oriented manifold of dimension n with boundary ∂M. We give ∂M the boundary orientation.

Theorem 22.8 (Stokes' theorem). *For any* $(n-1)$*-form* ω *with compact support on the oriented* n*-dimensional manifold* M*,*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. Choose an atlas $\{(U_\alpha, \phi_\alpha)\}$ for M in which each U_α is diffeomorphic to either $\mathbb{R}^n$ or $\mathbb{H}^n$ via an orientation-preserving diffeomorphism. This is possible since any open disk is diffeomorphic to $\mathbb{R}^n$ (see Problem 1.4). Let $\{\rho_\alpha\}$ be a C^∞ partition of unity subordinate to $\{U_\alpha\}$. As we showed in the preceding section, the $(n-1)$-form $\rho_\alpha\omega$ has compact support in U_α.

Suppose Stokes' theorem holds for $\mathbb{R}^n$ and for $\mathbb{H}^n$. Then it holds for all the charts U_α in our atlas, which are diffeomorphic to $\mathbb{R}^n$ or $\mathbb{H}^n$. Also, note that

$$(\partial M) \cap U_\alpha = \partial U_\alpha.$$

Therefore,

$$
\begin{aligned}
\int_{\partial M} \omega &= \int_{\partial M} \sum_\alpha \rho_\alpha \omega && \Big(\sum_\alpha \rho_\alpha = 1\Big) \\
&= \sum_\alpha \int_{\partial M} \rho_\alpha \omega && \Big(\sum_\alpha \rho_\alpha \omega \text{ is a finite sum by Problem 18.5}\Big) \\
&= \sum_\alpha \int_{\partial U_\alpha} \rho_\alpha \omega && (\operatorname{supp} \rho_\alpha \omega \text{ is contained in } U_\alpha) \\
&= \sum_\alpha \int_{U_\alpha} d(\rho_\alpha \omega) && (\text{Stokes' theorem for } U_\alpha) \\
&= \sum_\alpha \int_M d(\rho_\alpha \omega) && (\operatorname{supp} d(\rho_\alpha \omega) \subset U_\alpha) \\
&= \int_M d\big(\sum \rho_\alpha \omega\big) && (\rho_\alpha \omega \equiv 0 \text{ for all but finitely many } \alpha) \\
&= \int_M d\omega.
\end{aligned}
$$

Thus, it suffices to prove Stokes' theorem for $\mathbb{R}^n$ and for $\mathbb{H}^n$. We will give a proof only for $\mathbb{H}^2$, as the general case is similar.

Proof of Stokes' theorem for the upper half-plane $\mathbb{H}^2$. Let x, y be the coordinates on $\mathbb{H}^2$. Then the standard orientation on $\mathbb{H}^2$ is given by $dx \wedge dy$, and the boundary orientation on $\partial\mathbb{H}^2$ is given by dx.

The form ω is a linear combination

$$
\omega = f(x, y)\, dx + g(x, y)\, dy \tag{22.7}
$$

for C^∞ functions f, g with compact support in $\mathbb{H}^2$. Since the supports of f and g are compact, we may choose a real number $a > 0$ large enough so that the supports of f and g are contained in the interior of the square $[-a, a] \times [-a, a]$. We will use the notation f_x, f_y to denote the partial derivatives of f with respect to x and y, respectively. Then

$$
d\omega = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy = (g_x - f_y)\, dx \wedge dy,
$$

and

$$
\begin{aligned}
\int_{\mathbb{H}^2} d\omega &= \int_{\mathbb{H}^2} g_x\, |dx\, dy| - \int_{\mathbb{H}^2} f_y\, |dx\, dy| \\
&= \int_0^\infty \int_{-\infty}^\infty g_x\, |dx\, dy| - \int_{-\infty}^\infty \int_0^\infty f_y\, |dy\, dx| \\
&= \int_0^a \int_{-a}^a g_x\, |dx\, dy| - \int_{-a}^a \int_0^a f_y\, |dy\, dx|.
\end{aligned} \tag{22.8}
$$

In this expression,

$$\int_{-a}^{a} g_x(x, y)\, dx = g(x, y)\Big]_{x=-a}^{a} = 0$$

because $\operatorname{supp} g$ lies in the interior of $[-a, a] \times [-a, a]$. Similarly,

$$\int_{0}^{a} f_y(x, y)\, dy = f(x, y)\Big]_{y=0}^{a} = -f(x, 0)$$

because $f(x, a) = 0$. Thus, (22.8) becomes

$$\int_{\mathbb{H}^2} d\omega = \int_{-a}^{a} f(x, 0)\, dx.$$

On the other hand, $\partial\mathbb{H}^2$ is the x-axis and $dy = 0$ on $\partial\mathbb{H}^2$. It follows from (22.7) that $\omega = f(x, 0)\, dx$ when restricted to $\partial\mathbb{H}^2$ and

$$\int_{\partial\mathbb{H}^2} \omega = \int_{-a}^{a} f(x, 0)\, dx.$$

This proves Stokes' theorem for the upper half-plane. □

22.6 Line Integrals and Green's Theorem

We will now show how Stokes' theorem for a manifold unifies some of the theorems of vector calculus on $\mathbb{R}^2$ and $\mathbb{R}^3$.

Theorem 22.9 (Fundamental theorem for line integrals). *Let C be a curve in $\mathbb{R}^3$, parametrized by $\mathbf{r}(t) = (x(t), y(t), z(t))$, $a \le t \le b$, and let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$. If $\mathbf{F} = \operatorname{grad} f$ for some scalar function f, then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Theorem 22.10 (Green's theorem in the plane). *If D is a plane region with boundary ∂D, and P and Q are C^∞ functions on D, then*

$$\int_{\partial D} P\, dx + Q\, dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

In this statement, dA is the usual calculus notation for $|dx\, dy|$.

Suppose in Stokes' theorem we take M to be a curve C with parametrization $\mathbf{r}(t)$, $a \le t \le b$, and ω to be the function f on C. Then

$$\begin{aligned}\int_C d\omega = \int_C df &= \int_C \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz \\ &= \int_C \operatorname{grad} f \cdot d\mathbf{r}\end{aligned}$$

and

$$\int_{\partial C} \omega = f\big]_{\partial C} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

In this case Stokes' theorem specializes to the fundamental theorem for line integrals.

To obtain Green's theorem, let M be a plane region D with boundary ∂D and let ω be the 1-form $P\,dx + Q\,dy$ on D. Then

$$\int_{\partial D} \omega = \int_{\partial D} P\,dx + Q\,dy$$

and

$$\begin{aligned}\int_D d\omega &= \int_D P_y\,dy \wedge dx + Q_x\,dx \wedge dy = \int_D (Q_x - P_y)\,dx \wedge dy \\ &= \int_D (Q_x - P_y)\,|dx\,dy| = \int_D (Q_x - P_y)\,dA.\end{aligned}$$

In this case Stokes' theorem is Green's theorem in the plane.

Problems

22.1. Orientation-preserving or orientation-reversing diffeomorphisms
Let U be the open set $(0, \infty) \times (0, 2\pi)$ in the (r, θ)-plane $\mathbb{R}^2$. We define $F\colon U \subset \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(r, \theta) = (r\cos\theta, r\sin\theta).$$

Decide if F is orientation-preserving or orientation-reversing as a diffeomorphism onto its image.

22.2.* Integral under a diffeomorphism
Suppose N and M are connected oriented n-manifolds and $F\colon N \to M$ is a diffeomorphism. Prove that for any $\omega \in \Omega_c^k(M)$,

$$\int_N F^*\omega = \pm \int_M \omega,$$

where the sign depends on whether F is orientation-preserving or orientation-reversing.

22.3.* Stokes' theorem
Prove Stokes' theorem for $\mathbb{R}^n$ and for $\mathbb{H}^n$.

23

De Rham Cohomology

In vector calculus one often needs to know if a vector field on an open set D in $\mathbb{R}^3$ is the gradient of a function or is the curl of another vector field. By the correspondence of Section 4.6 between vector fields and differential forms, this translates into the question of whether a differential form ω on D is exact. Of course, a necessary condition is that the form ω should be closed. It turns out that whether every closed form on a manifold is exact depends on the topology of the manifold. For example, on $\mathbb{R}^2$ every closed k-form is exact for $k > 0$, but on $\mathbb{R}^2 - \{(0, 0)\}$ there are closed 1-forms that are not exact. The extent to which a closed form is not exact is measured by de Rham cohomology, possibly the most important diffeomorphism invariant of a manifold.

In this chapter we define de Rham cohomology, prove some of its basic properties, and compute two elementary examples, the de Rham cohomology of the real line and the unit circle.

23.1 De Rham Cohomology

Suppose $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is a smooth vector field representing a force on an open subset U of $\mathbb{R}^2$, and C is a parametrized curve $c(t) = (x(t), y(t))$ in U from a point A to a point B, with $a \le t \le b$. Then the work done by the force in moving a particle from A to B along C is given by the line integral $\int_C P\,dx + Q\,dy$.

Such a line integral is easy to compute if the vector field $\mathbf{F}$ is the gradient of a scalar function $f(x, y)$:

$$\mathbf{F} = \operatorname{grad} f = \langle f_x, f_y\rangle,$$

where $f_x = \partial f/\partial x$ and $f_y = \partial f/\partial y$. By Stokes' theorem, the line integral is simply

$$\int_C f_x\,dx + f_y\,dy = \int_C df = f(B) - f(A).$$

A necessary condition for the vector field $\mathbf{F} = \langle P, Q\rangle$ to be a gradient is that

$$P_y = f_{xy} = f_{yx} = Q_x.$$

The question is now the following: if $P_y - Q_x = 0$, is the vector field $\mathbf{F} = \langle P, Q \rangle$ on U the gradient of some scalar function $f(x, y)$ on U?

In Section 4.6 we established a one-to-one correspondence between vector fields and differential 1-forms on an open subset of $\mathbb{R}^3$. There is a similar correspondence on an open subset of any $\mathbb{R}^n$. For $\mathbb{R}^2$, it assumes the following form:

$$\begin{aligned}
\mathfrak{X}(U) &\leftrightarrow \Omega^1(U), \\
\mathbf{F} = \langle P, Q \rangle &\leftrightarrow \omega = P\,dx + Q\,dy, \\
\operatorname{grad} f = \langle f_x, f_y \rangle &\leftrightarrow df = f_x\,dx + f_y\,dy, \\
Q_x - P_y = 0 &\leftrightarrow d\omega = (Q_x - P_y)\,dx \wedge dy = 0.
\end{aligned}$$

In terms of differential forms the question above becomes: if the 1-form $\omega = P\,dx + Q\,dy$ is closed on U, is it exact? The answer to this question is sometimes yes and sometimes no, depending on the topology of U.

Just as for an open subset of $\mathbb{R}^n$, a differential form ω on a manifold M is said to be *closed* if $d\omega = 0$, and *exact* if $\omega = d\tau$ for some form τ of degree one less. Since $d^2 = 0$, every exact form is closed. In general, not every closed form is exact.

Let $Z^k(M)$ be the vector space of all closed k-forms and $B^k(M)$ the vector space of all exact k-forms on the manifold M. Because every exact form is closed, $B^k(M)$ is a subspace of $Z^k(M)$. The quotient vector space $H^k(M) := Z^k(M)/B^k(M)$ measures the extent to which closed k-forms fail to be exact, and is called the *de Rham cohomology* of M in degree k. As explained in Appendix D, the quotient vector space construction introduces an equivalence relation on $Z^k(M)$:

$$\omega' \sim \omega \quad \text{in } Z^k(M) \quad \text{iff} \quad \omega' - \omega \in B^k(M).$$

The equivalence class of a closed form ω is called its *cohomology class* and denoted by $[\omega]$. Two closed forms ω and ω' determine the same cohomology class if and only if they differ by an exact form:

$$\omega' = \omega + d\tau.$$

In this case we say that the two closed forms ω and ω' are *cohomologous.*

Proposition 23.1. *If the manifold M has r connected components, then its de Rham cohomology in degree 0 is $H^0(M) = \mathbb{R}^r$.*

Proof. Since there are no exact 0-forms other than 0,

$$H^0(M) = Z^0(M) = \{\text{closed 0-forms}\}.$$

Supposed f is a closed 0-form on M, i.e., f is a C^∞ function on M such that $df = 0$. On any chart $(U, x^1, \ldots, x^n)$,

$$df = \sum \frac{\partial f}{\partial x^i}\,dx^i.$$

Thus, $df = 0$ on U if and only if all the partial derivatives $\partial f/\partial x^i$ vanish identically on U. This in turn is equivalent to f being locally constant on U. Hence, the closed 0-forms on M are precisely the locally constant functions on M. Such a function must be constant on each connected component of M. If M has r connected components, then a locally constant function on M is simply an ordered set of r real numbers. Thus, $Z^0(M) = \mathbb{R}^r$. □

Proposition 23.2. *On a manifold M of dimension n, the de Rham cohomology $H^k(M) = 0$ for $k > n$.*

Proof. At any point $p \in M$, the tangent space T_pM is a vector space of dimension n. If ω is a k-form on M, then $\omega_p \in A_k(T_pM)$, the space of alternating k-linear functions on T_pM. By Corollary 3.31, if $k > n$, then $A_k(T_pM) = 0$. Hence, for $k > n$, the only k-form on M is the zero form. □

23.2 Examples of de Rham Cohomology

Example 23.3 (*The de Rham cohomology of the real line*). Since $\mathbb{R}$ is connected, by Proposition 23.1,

$$H^0(\mathbb{R}) = \mathbb{R}.$$

For dimension reasons, on $\mathbb{R}$ there are no nonzero 2-forms. This implies that every 1-form on $\mathbb{R}$ is closed. A 1-form $f(x)\,dx$ on $\mathbb{R}$ is exact if and only if there is a C^∞ function $g(x)$ on $\mathbb{R}$ such that

$$f(x)\,dx = dg = g'(x)\,dx.$$

Such a function $g(x)$ is simply an antiderivative of $f(x)$, for example,

$$g(x) = \int_0^x f(t)\,dt.$$

This proves that every 1-form on $\mathbb{R}$ is exact. In summary,

$$H^k(\mathbb{R}) = \begin{cases} \mathbb{R} & \text{for } k = 0; \\ 0 & \text{for } k \geq 1. \end{cases}$$

Example 23.4 (*The de Rham cohomology of a circle*). Let S^1 be the unit circle in the xy-plane. As in the example of $\mathbb{R}$, because S^1 is connected,

$$H^0(S^1) = \mathbb{R},$$

and because S^1 is one dimensional,

$$H^k(S^1) = 0$$

for all $k \geq 2$. It remains to compute $H^1(S^1)$.

In Section 19.7 we found a nowhere-vanishing 1-form on S^1:

$$\omega_{(x,y)} = \begin{cases} \dfrac{dy}{x} & \text{if } (x,y) \in S^1 \text{ and } x \neq 0, \\ -\dfrac{dx}{y} & \text{if } (x,y) \in S^1 \text{ and } y \neq 0. \end{cases} \tag{23.1}$$

Since ω is nowhere-vanishing, it cannot be exact, for an exact 1-form df must vanish at the maximum and minimum of f, and we know that a continuous function on a compact set such as S^1 necessarily has a maximum and a minimum. Thus, the existence of the nowhere-vanishing 1-form ω implies immediately that $H^1(S^1) \neq 0$.

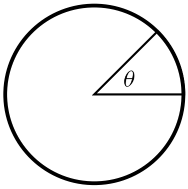


Fig. 23.1. The angle θ on the circle.

On the circle the angle θ relative to the x-axis is defined only up to an integral multiple of 2π; in other words, θ is not a real-valued function on S^1, but a function from S^1 to $\mathbb{R}/(2\pi\mathbb{Z})$ (Figure 23.1).

However, one can make sense of a well-defined 1-form $d\theta$ on S^1 in the following way. The projection $\rho\colon \mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$ is a covering space. A *branch* of $\theta\colon S^1 \to \mathbb{R}/2\pi\mathbb{Z}$ over an open interval I of the circle is a *lift* of $\theta|_I$ to $\mathbb{R}$, i.e., a function $f\colon I \to \mathbb{R}$ such that $\rho \circ f = \theta|_I$. Any two C^∞ lifts $f_1\colon I_1 \to \mathbb{R}$ and $f_2\colon I_2 \to \mathbb{R}$ differ by a locally constant function on $I_1 \cap I_2$, since $f_1 - f_2$ is a continuous function from $I_1 \cap I_2$ into the discrete set $2\pi\mathbb{Z}$. Hence, $df_1 = df_2$ on $I_1 \cap I_2$. By covering the circle with overlapping intervals, we obtain a well-defined 1-form on the circle, which we denote by $d\theta$. In short, on any open interval of the circle, $d\theta$ is the exterior derivative of any C^∞ branch of θ.

Exercise 23.5 (A nowhere-vanishing 1-form on the circle). Show that $d\theta$ is precisely the nowhere-vanishing 1-form ω in (23.1).

Remark 23.6. It spite of the notation, it should be noted that $d\theta$ is not an exact form on the circle, because θ is not a C^∞ function on S^1.

A function on the circle may be identified with a periodic function on the real line of period 2π. Since $d\theta$ is nowhere-vanishing, every 1-form α on the circle is a multiple of $d\theta$:

$$\alpha = f(\theta)\, d\theta,$$

where $f(\theta)$ is a periodic function on $\mathbb{R}$ of period 2π.

Since the circle has dimension 1, all 1-forms on S^1 are closed, so $\Omega^1(S^1) = Z^1(S^1)$. The integration of 1-forms on S^1 defines a linear map

$$\varphi\colon Z^1(S^1) = \Omega^1(S^1) \to \mathbb{R}, \quad \varphi(\alpha) = \int_{S^1} \alpha.$$

To compute the integral $\int_{S^1} d\theta$, note that removing a finite set of points from the domain of integration does not change the value of the integral. If we remove the point 1 from the unit circle, then $\theta\colon S^1 - \{1\} \to (0, 2\pi)$ is a well-defined coordinate function, and $\theta = \theta^* t$, where t is the coordinate on the real line. Hence,

$$\int_{S^1} d\theta = \int_{S^1-\{1\}} d\theta = \int_{S^1-\{1\}} \theta^* dt = \int_0^{2\pi} dt = 2\pi.$$

This shows that the linear map $\varphi\colon \Omega^1(S^1) \to \mathbb{R}$ is onto.

By Stokes' theorem, the exact 1-forms are in $\ker \varphi$. Conversely, suppose $f(\theta)\, d\theta$ is in $\ker \varphi$. Then

$$\int_0^{2\pi} f(\theta)\, d\theta = 0.$$

Define

$$g(\theta) = \int_0^{\theta} f(t)\, dt.$$

Since $f(t)$ is periodic of period 2π:

$$\begin{aligned} g(\theta + 2\pi) &= \int_0^{2\pi} f(t)\, dt + \int_{2\pi}^{2\pi+\theta} f(t)\, dt \\ &= \int_{2\pi}^{2\pi+\theta} f(t)\, dt = \int_0^{\theta} f(t)\, dt = g(\theta). \end{aligned}$$

Hence, $g(\theta)$ is also periodic of period 2π on $\mathbb{R}$ and is therefore a function on S^1. Moreover,

$$dg = g'(\theta)\, d\theta = f(\theta)\, d\theta,$$

which proves that the kernel of φ consists of exact forms. Therefore, integration induces an isomorphism

$$H^1(S^1) \simeq \mathbb{R}.$$

In the next chapter we will develop a tool, the *Mayer–Vietoris sequence*, using which the computation of the cohomology of the circle becomes more or less routine.

23.3 Diffeomorphism Invariance

For any smooth map $F\colon N \to M$ of manifolds, there is a *pullback map* $F^*\colon \Omega^*(M) \to \Omega^*(N)$ of differential forms. Moreover, the pullback F^* commutes with the exterior derivative d (Theorem 19.8).

Lemma 23.7. *The pullback map F^* sends closed forms to closed forms, and sends exact forms to exact forms.*

Proof. Suppose ω is closed. By the commutativity of F^* with d,

$$dF^*\omega = F^*d\omega = 0.$$

Hence, $F^*\omega$ is also closed.

Next suppose $\omega = d\tau$ is exact. Then

$$F^*\omega = F^*d\tau = dF^*\tau.$$

Hence, $F^*\omega$ is exact. □

It follows that F^* induces a linear map of quotient spaces, denoted $F^\#$:

$$F^\# \colon \frac{Z^k(M)}{B^k(M)} \to \frac{Z^k(N)}{B^k(N)}, \quad F^\#([\omega]) = [F^*(\omega)].$$

This is a map in cohomology

$$F^\# \colon H^k(M) \to H^k(N),$$

called the *pullback map in cohomology*.

Remark 23.8. The functorial properties of the pullback map F^* on differential forms easily yield the same functorial properties for the induced map in cohomology:

(i) If $1_M \colon M \to M$ is the identity map, then $1_M^\# \colon H^k(M) \to H^k(M)$ is also the identity map.
(ii) If $F \colon N \to M$ and $G \colon M \to P$ are smooth maps, then

$$(G \circ F)^\# = F^\# \circ G^\#.$$

It follows from (i) and (ii) that $(H^k(\ ), F^\#)$ is a contravariant functor from the category of C^∞ manifolds and C^∞ maps to the category of vector spaces and linear maps. By Proposition 10.9, if $F \colon N \to M$ is a diffeomorphism of manifolds, then $F^\# \colon H^k(M) \to H^k(N)$ is an isomorphism of vector spaces.

In fact, the usual notation for the induced map in cohomology is F^*, the same as the pullback map on differential forms. Henceforth, we will follow this convention. It is usually clear from the context whether F^* is a map in cohomology or on forms.

23.4 The Ring Structure on de Rham Cohomology

The wedge product of differential forms on a manifold M gives the vector space $\Omega^*(M)$ of differential forms a product structure. This product structure induces a product structure in cohomology: if $[\omega] \in H^k(M)$ and $[\tau] \in H^\ell(M)$, define

$$[\omega] \wedge [\tau] = [\omega \wedge \tau] \in H^{k+\ell}(M). \tag{23.2}$$

For the product to be well defined, we need to check three things about closed forms ω and τ:

(i) The wedge product $\omega \wedge \tau$ is a closed form.
(ii) The class $[\omega \wedge \tau]$ is independent of the choice of representative for $[\tau]$. In other words, if τ is replaced by a cohomologous form $\tau' = \tau + d\sigma$, then in the equation

$$\omega \wedge \tau' = \omega \wedge \tau + \omega \wedge d\sigma,$$

we need to show that $\omega \wedge d\sigma$ is exact.
(iii) The class $[\omega \wedge \tau]$ is independent of the choice of representative for $[\omega]$.

These all follow from the antiderivation property of d. For example, in (i), since ω and τ are closed,

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^k \omega \wedge d\tau = 0.$$

In (ii),

$$\begin{aligned} d(\omega \wedge \sigma) &= (d\omega) \wedge \sigma + (-1)^k \omega \wedge d\sigma \\ &= (-1)^k \omega \wedge d\sigma \quad (\text{since } d\omega = 0), \end{aligned}$$

which shows that $\omega \wedge d\sigma$ is exact. Item (iii) is analogous to Item (ii), with the roles of ω and τ reversed.

If M is a manifold of dimension n, we set

$$H^*(M) = \oplus_{k=0}^n H^k(M).$$

What this means is that an element α of $H^*(M)$ is a finite sum of cohomology classes in $H^k(M)$ for various k's:

$$\alpha = \alpha_0 + \cdots + \alpha_n, \quad \alpha_k \in H^k(M).$$

Elements of $H^*(M)$ can be added and multiplied in the same way that one would add or multiply polynomials, except here multiplication is the wedge product. It is easy to check that under addition and multiplication, $H^*(M)$ satisfies all the properties of a ring, called the *cohomology ring* of M. This ring is not commutative, because the wedge product of differential forms is not commutative. However, the ring $H^*(M)$ has a natural grading by the degree of a closed form. In general, a ring A is *graded* if it can be written as a direct sum $A = \oplus_{k=0}^\infty A^k$ so that ring multiplication sends $A^k \times A^\ell$ to $A^{k \times \ell}$. A graded ring $A = \oplus_{k=0}^\infty A^k$ is said to be *anticommutative* if for all $a \in A^k$ and $b \in A^\ell$,

$$a \cdot b = (-1)^{k\ell} b \cdot a.$$

In this terminology, $H^*(M)$ is an anticommutative graded ring.

Suppose $F: N \to M$ is a C^∞ map of manifolds. Because $F^*(\omega \wedge \tau) = F^*\omega \wedge F^*\tau$ for differential forms ω and τ on M (Proposition 18.7), the pullback map $F^*: H^*(M) \to H^*(N)$ is a ring homomorphism. By Remark 23.8, if $F: N \to M$ is a diffeomorphism, then $F^*: H^*(M) \to H^*(N)$ is a ring isomorphism.

To sum up, de Rham cohomology gives a contravariant functor from the category of C^∞ manifolds to the category of anticommutative graded rings. If M and N are diffeomorphic manifolds, then $H^*(M)$ and $H^*(N)$ are isomorphic as anticommutative graded rings. In this way the de Rham cohomology becomes a powerful diffeomorphism invariant of C^∞ manifolds.

Problems

23.1.* Locally constant map on a connected space

A map $f : S \to Y$ between two topological spaces is *locally constant* if for every $p \in S$ there is a neighborhood U of p such that f is constant on U. Show that a locally constant map $f : S \to Y$ on a nonempty connected space S is constant. (*Hint*: Show that for every $y \in Y$, the inverse image $f^{-1}(y)$ is open. Then $S = \bigcup_{y \in Y} f^{-1}(y)$ exhibits S as a disjoint union of open subsets.)

24

The Long Exact Sequence in Cohomology

A *cochain complex* $\mathcal{C}$ is a collection of vector spaces $\{C^k\}_{k\in\mathbb{Z}}$ together with a sequence of linear maps $d_k\colon C^k \to C^{k+1}$

$$\cdots \to C^{-1} \xrightarrow{d_{-1}} C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \cdots$$

such that

$$d_k \circ d_{k-1} = 0 \tag{24.1}$$

for all k. We will call the collection of linear maps $\{d_k\}$ the *differential* of the cochain complex $\mathcal{C}$.

The vector space $\Omega^*(M)$ of differential forms on a manifold M together with the exterior d is a cochain complex, the *de Rham complex* of M:

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots, \quad d \circ d = 0.$$

It turns out that many of the results on the de Rham cohomology of a manifold depend not on the topological properties of the manifold, but on the algebraic properties of the de Rham complex. To better understand de Rham cohomology, it is useful to isolate these algebraic properties. In this chapter we investigate the properties of a cochain complex that constitute the beginning of a subject known as *homological algebra*.

24.1 Exact Sequences

Definition 24.1. A sequence of homomorphisms of vector spaces

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is said to be *exact at B* if $\operatorname{im} f = \ker g$. A sequence of homomorphisms

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$$

that is exact at every term except the first and the last is said simply to be an *exact sequence*. A five-term exact sequence of the form

$$0 \to A \to B \to C \to 0$$

is said to be *short exact*.

The same definition applies to homomorphisms of groups or modules, but we are mainly concerned with vector spaces.

Remark 24.2.
(i) When $A = 0$, the sequence

$$0 \xrightarrow{f} B \xrightarrow{g} C$$

is exact if and only if

$$\ker g = \operatorname{im} f = 0,$$

i.e., g is injective.
(ii) Similarly, when $C = 0$, the sequence

$$A \xrightarrow{f} B \xrightarrow{g} 0$$

is exact if and only if

$$\operatorname{im} f = \ker g = B,$$

i.e., f is surjective.

The following two propositions are very useful when dealing with exact sequences.

Proposition 24.3 (A three-term exact sequence). *Suppose*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is an exact sequence. Then

(i) *the map f is surjective if and only if g is the zero map;*
(ii) *the map g is injective if and only if f is the zero map.*

Proof. Problem 24.1. □

Proposition 24.4 (A four-term exact sequence).
(i) *The four-term sequence of vector spaces $0 \to A \xrightarrow{f} B \to 0$ is exact if and only if $f : A \to B$ is an isomorphism.*
(ii) *If*

$$A \xrightarrow{f} B \to C \to 0$$

is an exact sequence of vector spaces, then

$$C \simeq \operatorname{coker} f = \frac{B}{\operatorname{im} f}.$$

Proof. Problem 24.2. □

24.2 Cohomology of Cochain Complexes

If $\mathcal{C}$ is a cochain complex, then by (24.1),

$$\operatorname{im} d_{k-1} \subset \ker d_k.$$

We can therefore form the quotient space

$$H^k(\mathcal{C}) := \frac{\ker d_k}{\operatorname{im} d_{k-1}},$$

which is called the *kth cohomology vector space* of the cochain complex $\mathcal{C}$. It is a measure of the extent to which the cochain complex $\mathcal{C}$ fails to be exact at C^k. An element of $\ker d_k$ is called a *k-cocycle* and an element of $\operatorname{im} d_{k-1}$ is called a *k-coboundary*. The equivalence class $[c] \in H^k(\mathcal{C})$ of a k-cocycle $c \in \ker d_k$ is called its *cohomology class*.

Example 24.5. In the de Rham complex, a cocycle is a closed form and a coboundary is an exact form.

To simplify the notation we will usually omit the subscript from d_k, and write $d \circ d = 0$ instead of $d_k \circ d_{k-1} = 0$.

If $\mathcal{A}$ and $\mathcal{B}$ are two cochain complexes with differentials d and d', respectively, a *cochain map* $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a collection of linear maps $\varphi_k\colon A^k \to B^k$, one for each k, that commute with d and d':

$$d' \circ \varphi_k = \varphi_{k+1} \circ d.$$

In other words, the following diagram is commutative:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{k-1} \arrow[r, "d"] \arrow[d, "\varphi_{k-1}"] & A^k \arrow[r, "d"] \arrow[d, "\varphi_k"] & A^{k+1} \arrow[r] \arrow[d, "\varphi_{k+1}"] & \cdots \\
\cdots \arrow[r] & B^{k-1} \arrow[r, "d'"'] & B^k \arrow[r, "d'"'] & B^{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

We will usually omit the subscript k in φ_k.

A cochain map $\varphi\colon \mathcal{A} \to \mathcal{B}$ naturally induces a linear map in cohomology

$$\varphi^*\colon H^k(\mathcal{A}) \to H^k(\mathcal{B})$$

by

$$\varphi^*[a] = [\varphi(a)].$$

To show that this is well defined, we need to check that a cochain takes cocycles to cocycles, and coboundaries to coboundaries:

(i) for $a \in Z^k(\mathcal{A})$, $d'(\varphi(a)) = \varphi(da) = 0$.
(ii) for $a' \in A^{k-1}$, $\varphi(d(a')) = d'(\varphi(a'))$.

24.3 The Connecting Homomorphism

A sequence of cochain complexes

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0$$

is *short exact* if i and j are cochain maps and for each k

$$0 \to A^k \xrightarrow{i_k} B^k \xrightarrow{j_k} C^k \to 0$$

is a short exact sequence of vector spaces. To simplify the notation, we usually omit the subscript k from i_k and j_k.

Given a short exact sequence as above, we can construct a linear map $d^*\colon H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A})$, called the *connecting homomorphism*, as follows. Consider the short exact sequences in dimensions k and $k+1$:

$$\begin{tikzcd} 0 \arrow[r] & A^{k+1} \arrow[r, "i"] & B^{k+1} \arrow[r, "j"] & C^{k+1} \arrow[r] & 0 \\ 0 \arrow[r] & A^k \arrow[r, "i"'] \arrow[u, "d"] & B^k \arrow[r, "j"'] \arrow[u, "d"] & C^k \arrow[r] \arrow[u, "d"] & 0. \end{tikzcd}$$

To keep the notation simple, we use the same symbol d to denote the a priori distinct differentials d_A, d_B, d_C of the three cochain complexes. Start with $[c] \in H^k(\mathcal{C})$. Since $j\colon B^k \to C^k$ is onto, there is an element $b \in B^k$ such that $j(b) = c$. Then $db \in B^{k+1}$ is in $\ker j$ because

$$\begin{aligned} jdb &= djb \quad &&\text{(by the commutativity of the diagram)} \\ &= dc \\ &= 0 &&\text{(because } c \text{ is a cocycle).} \end{aligned}$$

By the exactness of the sequence in dimension $k+1$, $\ker j = \operatorname{im} i$. This implies that $db = i(a)$ for some a in A^{k+1}. Once b is chosen, this a is unique because i is injective. The injectivity of i also implies that $da = 0$, since

$$i(da) = d(ia) = ddb = 0.$$

Therefore, a is a cocycle and defines a cohomology class $[a]$. We set

$$d^*[c] = [a] \in H^{k+1}(\mathcal{A}).$$

In defining $d^*[c]$ we made two choices: a cocycle c to represent the cohomology class $[c] \in H^k(\mathcal{C})$ and then an element $b \in B^k$ that maps to c under j. For d^* to be well defined, one must show that the cohomology class $[a] \in H^{k+1}(\mathcal{A})$ does not depend on these choices.

Exercise 24.6 (Connecting homomorphism). Show that the connecting homomorphism

$$d^*\colon H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A})$$

is a well-defined linear map.

24.4 The Long Exact Sequence in Cohomology

Theorem 24.7. *A short exact sequence of cochain complexes*

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0$$

gives rise to a long exact sequence in cohomology:

$$\cdots \xrightarrow{j^*} H^{k-1}(\mathcal{C}) \xrightarrow{d^*} H^k(\mathcal{A}) \xrightarrow{i^*} H^k(\mathcal{B}) \xrightarrow{j^*} H^k(\mathcal{C}) \xrightarrow{d^*} H^{k+1}(\mathcal{A}) \xrightarrow{i^*} \cdots, \tag{24.2}$$

where i^* and j^* are the maps in cohomology induced from the cochain maps i and j, and d^* is the connecting homomorphism.

To prove the theorem one needs to check exactness at $H^k(\mathcal{A})$, $H^k(\mathcal{B})$, and $H^k(\mathcal{C})$ for each k. The proof is a sequence of trivialities involving what is commonly called *diagram-chasing*. As an example, we prove exactness at $H^k(\mathcal{C})$.

Claim. $\operatorname{im} j^* \subset \ker d^*$.

Proof. Let $[b] \in H^k(\mathcal{B})$. Then

$$d^* j^*[b] = d^*[j(b)].$$

In the recipe above for d^*, we can choose the element in B^k that maps to $j(b)$ to be b and take $db \in B^{k+1}$.

$$\begin{tikzcd}
0 \arrow[r] & db = 0 & \\
& b \arrow[u] \arrow[r] & j(b).
\end{tikzcd}$$

Because b is a cocycle, $db = 0$. Following the zig-zag diagram, we see that $d^*[j(b)] = [0]$. So $j^*[b] \in \ker d^*$. □

Claim. $\ker d^* \subset \operatorname{im} j^*$.

Proof. Suppose $d^*[c] = [a] = 0$, where $[c] \in H^k(\mathcal{C})$. This means $a = d(a')$ for some $a' \in A^k$. The calculation of $d^*[c]$ can be represented by the following zig-zag diagram:

$$\begin{tikzcd}
a \arrow[r, "i"] & db & \\
a' \arrow[u, "d"] & b \arrow[u, "d"'] \arrow[r, "j"] & c,
\end{tikzcd}$$

where b is any element in B^k with $j(b) = c$. Then $b - i(a')$ is a cocycle in B^k that maps to c under j:

$$\begin{aligned}
d(b - i(a')) &= db - di(a') = db - id(a') = db - ia = 0, \\
j(b - i(a')) &= j(b) - ji(a') = j(b) = c.
\end{aligned}$$

Therefore,

$$j^*[b - i(a')] = [c].$$

So $[c] \in \operatorname{im} j^*$. $\square$

These two claims together imply the exactness of (24.2) at $H^k(\mathcal{C})$. As for the exactness of the cohomology sequence (24.2) at $H^k(\mathcal{A})$ and at $H^k(\mathcal{B})$, we will leave it to an exercise (Problem 24.3).

Problems

24.1. A three-term exact sequence
Prove Proposition 24.1.

24.2. A four-term exact sequence
Prove Proposition 24.2.

24.3. Long exact cohomology sequence
Prove the exactness of the cohomology sequence (24.2) at $H^k(\mathcal{A})$ and $H^k(\mathcal{B})$.

25

The Mayer–Vietoris Sequence

As the example of $H^1(\mathbb{R})$ illustrates, calculating the de Rham cohomology of a manifold amounts to solving a canonically given system of differential equations on the manifold and in case it is not solvable, to finding the obstructions to its solvability. This is usually quite difficult to do directly. We introduce in this chapter one of the most useful tools in the calculation of de Rham cohomology, the Mayer–Vietoris sequence. Another tool, the homotopy axiom, will come in the next chapter.

25.1 The Mayer–Vietoris Sequence

Let $\{U, V\}$ be an open cover of a manifold M, and let $i_U : U \to M$, $i_U(p) = p$, be the inclusion map. Then the pullback

$$i_U^* : \Omega^k(M) \to \Omega^k(U)$$

is the restriction map that restricts the domain of a k-form on M to U. In fact, there are four inclusion maps that form a commutative diagram:

$$\begin{tikzcd} & U \arrow[dr, hook, "i_U"] & \\ U \cap V \arrow[ur, hook, "j_U"] \arrow[dr, hook, "j_V"'] & & M \\ & V \arrow[ur, hook, "i_V"'] & \end{tikzcd}$$

By restricting to U and to V, we get a homomorphism of vector spaces

$$i : \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V),$$
$$\sigma \mapsto (i_U^*\sigma, i_V^*\sigma).$$

To keep the notation simple, we will often write σ to mean its restriction to an open subset.

Define the difference map

$$j \colon \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V)$$

by

$$j(\omega, \tau) = \tau - \omega,$$

where the right-hand side really means $j_V^*\tau - j_U^*\omega$. If $U \cap V$ is empty, we define $\Omega^k(U \cap V) = 0$. In this case, j is simply the zero map.

Proposition 25.1. *For each integer $k \geq 0$, the sequence*

$$0 \to \Omega^k(M) \xrightarrow{i} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j} \Omega^k(U \cap V) \to 0 \tag{25.1}$$

is exact.

Proof. Exactness at the first two terms $\Omega^k(M)$ and $\Omega^k(U) \oplus \Omega^k(V)$ is straightforward. We leave it as an exercise. We will prove exactness at $\Omega^k(U \cap V)$.

To prove the surjectivity of the difference map

$$j \colon \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V),$$

it is best to consider first the case of functions on $M = \mathbb{R}^1$. Let f be a C^∞ function on $U \cap V$ as in Figure 25.1. We have to write f as the difference of a C^∞ function on V and a C^∞ function on U.

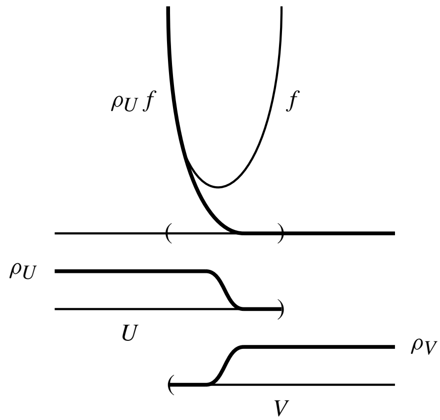


Fig. 25.1. Writing f as the difference of a C^∞ function on V and a C^∞ function on U.

Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to the open cover $\{U, V\}$. Define $f_V \colon V \to \mathbb{R}$ by

$$f_V(x) = \begin{cases} \rho_U(x) f(x) & \text{for } x \in U \cap V, \\ 0 & \text{for } x \in V - (U \cap V). \end{cases}$$

Exercise 25.2 (Smooth extension of a function). Prove that f_V is a C^∞ function on V.

The function f_V is the extension by zero of $\rho_U f$ from $U \cap V$ to V. Similarly, we define f_U to be the extension by zero of $\rho_V f$ from $U \cap V$ to U. Since

$$j(-f_U, f_V) = j_V^* f_V + j_U^* f_U = \rho_U f + \rho_V f = f \quad \text{on } U \cap V,$$

j is surjective.

For differential k-forms on a general manifold M, the formula is similar. For $\omega \in \Omega^k(U \cap V)$, define ω_U to be the extension by zero of $\rho_V \omega$ from $U \cap V$ to U, and ω_V to be the extension by zero of $\rho_U \omega$ from $U \cap V$ to V. On $U \cap V$, $(-\omega_U, \omega_V)$ restricts to $(-\rho_V \omega, \rho_U \omega)$. Hence, j maps $(-\omega_U, \omega_V) \in \Omega^k(U) \oplus \Omega^k(V)$ to

$$\rho_V \omega - (-\rho_U \omega) = \omega \in \Omega^k(U \cap V).$$

This shows that j is surjective and the sequence (25.1) is exact at $\Omega^k(U \cap V)$. □

It follows from Proposition 25.1 that the sequence of cochain complexes

$$0 \to \Omega^*(M) \xrightarrow{i} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{j} \Omega^*(U \cap V) \to 0$$

is short exact. By Theorem 24.7, this short exact sequence of cochain complexes gives rise to a long exact sequence in cohomology, called the *Mayer–Vietoris sequence*:

$$\begin{aligned} \cdots \to H^{k-1}(U \cap V) \xrightarrow{d^*} H^k(M) \xrightarrow{i^*} H^k(U) \oplus H^k(V) \xrightarrow{j^*} H^k(U \cap V) \\ \xrightarrow{d^*} H^{k+1}(M) \to \cdots . \end{aligned}$$

In this sequence i^* and j^* are induced from i and j:

$$\begin{aligned} i^*[\sigma] = [i(\sigma)] &= ([i_U^*\sigma], [i_V^*\sigma]) \in H^k(U) \oplus H^k(V), \\ j^*([\omega], [\tau]) = [j(\omega, \tau)] &= [j_V^*\tau - j_U^*\omega] \in H^k(U \cap V). \end{aligned}$$

The connecting homomorphism $d^* \colon H^k(U \cap V) \to H^{k+1}(M)$ is obtained as follows.

(1) Starting with a closed k-form $\zeta \in \Omega^k(U \cap V)$ and using a partition of unity $\{\rho_U, \rho_V\}$ subordinate to $\{U, V\}$, one can extend $\rho_U \zeta$ from $U \cap V$ to a k-form ζ_V on V and extend $\rho_V \zeta$ from $U \cap V$ to a k-form ζ_U on U (see the proof of Proposition 25.1). Then

$$j(-\zeta_U, \zeta_V) = \zeta_V + \zeta_U = (\rho_U + \rho_V)\zeta = \zeta.$$

(2) By the commutativity $dj = jd$, the pair $(-d\zeta_U, d\zeta_V)$ maps to 0 under j. This means the $(k+1)$-forms $-d\zeta_U$ on U and $d\zeta_V$ on V agree on $U \cap V$.

(3) Therefore, $-d\zeta_U$ on U and $d\zeta_V$ patch together to give a global $(k+1)$-form α on M. Diagram-chasing shows that α is closed. By Section 24.3, $d^*[\zeta] = [\alpha] \in H^{k+1}(M)$. See the two diagrams below:

$$
\begin{tikzcd}
\Omega^{k+1}(M) \arrow[r, "i"] & \Omega^{k+1}(U) \oplus \Omega^{k+1}(V) & \\
 & \Omega^{k}(U) \oplus \Omega^{k}(V) \arrow[u, "d"] \arrow[r, "j"] & \Omega^{k}(U \cap V) \\
\alpha \arrow[r, "i", "(3)"'] & (-d\zeta_U, d\zeta_V) \arrow[r] & 0 \\
 & (-\zeta_U, \zeta_V) \arrow[u, "d", "(2)"'] \arrow[r, "j", "(1)"'] & \zeta \arrow[u]
\end{tikzcd}
$$

Because $\Omega^k(M) = 0$ for $k \leq -1$, the Mayer–Vietoris sequence starts with

$$0 \to H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to \cdots .$$

Proposition 25.3. *In the Mayer–Vietoris sequence if U, V, and $U \cap V$ are connected, then*

(i) *M is connected and*

$$0 \to H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to 0$$

is exact;

(ii) *we may start the Mayer–Vietoris sequence with*

$$0 \to H^1(M) \xrightarrow{i^*} H^1(U) \oplus H^1(V) \xrightarrow{j^*} H^1(U \cap V) \to \cdots .$$

Proof.
(i) The connectedness of M follows from a lemma in point-set topology (Problem A.45). It is also a consequence of the Mayer–Vietoris sequence. On a connected open set the de Rham cohomology in dimension 0 is simply the vector space of constant functions (Proposition 23.1). So the map

$$j^* \colon H^0(U) \oplus H^0(V) \to H^0(U \cap V)$$

is given by

$$(b, c) \mapsto c - b, \quad b, c \in \mathbb{R}.$$

This map is clearly surjective. The surjectivity of j^* implies that

$$\operatorname{im} j^* = H^0(U \cap V) = \ker d^*,$$

from which we conclude that $d^* \colon H^0(U \cap V) \to H^1(M)$ is the zero map. Thus the Mayer–Vietoris sequence starts with

$$0 \to H^0(M) \xrightarrow{i^*} \mathbb{R} \oplus \mathbb{R} \xrightarrow{j^*} \mathbb{R} \xrightarrow{d^*} 0. \tag{25.2}$$

This short exact sequence shows that

$$H^0(M) \simeq \operatorname{im} i^* = \ker j^*.$$

Since

$$(\mathbb{R} \oplus \mathbb{R})/\ker j^* \simeq \operatorname{im} j^* = \mathbb{R},$$

$\ker j^*$ must be one dimensional. So $H^0(M) = \mathbb{R}$, which proves that M is connected.

(ii) From (i) we know that $d^*\colon H^0(U \cap V) \to H^1(M)$ is the zero map. Thus, in the Mayer–Vietoris sequence, the sequence of two maps

$$H^0(U \cap V) \xrightarrow{d^*} H^1(M) \xrightarrow{i^*} H^1(U) \oplus H^1(V)$$

may be replaced by

$$0 \to H^1(M) \xrightarrow{i^*} H^1(U) \oplus H^1(V)$$

without affecting the exactness of the sequence. □

25.2 The Cohomology of the Circle

In Example 23.4 we showed that the integration of 1-forms induces an isomorphism from $H^1(S^1)$ to $\mathbb{R}$. In this section we apply the Mayer–Vietoris sequence to give an alternative computation of the cohomology of the circle.

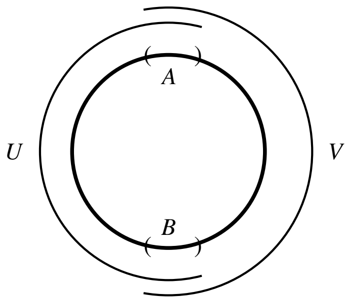


Fig. 25.2. An open cover of the circle.

Cover the circle with two open arcs U and V as in Figure 25.2. The intersection $U \cap V$ is the disjoint union of two open arcs, which we call A and B. Since an open arc is diffeomorphic to an open interval and hence to the real line $\mathbb{R}$, the cohomology rings of U and V are isomorphic to that of $\mathbb{R}$, and the cohomology ring of $U \cap V$ to that of the disjoint union $\mathbb{R} \amalg \mathbb{R}$. They fit into the Mayer–Vietoris sequence, which we arrange in tabular form:

$$
\begin{array}{c|ccccccc}
 & & S^1 & & U \amalg V & & U \cap V \\
\hline
H^2 & \to & 0 & \to & 0 & \to & 0 \\
H^1 & \xrightarrow{d^*} & & \to & 0 & \to & 0 \\
H^0 & 0 \to & \mathbb{R} & \xrightarrow{i^*} & \mathbb{R} \oplus \mathbb{R} & \xrightarrow{j^*} & \mathbb{R} \oplus \mathbb{R}
\end{array}
$$

From the exact sequence

$$0 \to \mathbb{R} \xrightarrow{i^*} \mathbb{R} \oplus \mathbb{R} \xrightarrow{j^*} \mathbb{R} \oplus \mathbb{R} \xrightarrow{d^*} H^1(S^1) \to 0$$

and Problem 25.2, we conclude that $\dim H^1(S^1) = 1$. Hence, the cohomology of the circle is given by

$$H^k(S^1) = \begin{cases} \mathbb{R} & \text{for } k = 0, 1; \\ 0 & \text{otherwise.} \end{cases}$$

By analyzing the maps in the Mayer–Vietoris sequence, it is possible to write down an explicit generator for $H^1(S^1)$. In the Mayer–Vietoris sequence, the map $j^*\colon H^0(U) \oplus H^0(V) \to H^0(U \cap V)$ is given by

$$j^*(b, c) = (c - b, c - b),$$

where b and c are real numbers. Thus, the image of j^* is the diagonal Δ in $\mathbb{R}^2$:

$$\Delta = \{(x, y) \in \mathbb{R}^2 \mid y = x\}.$$

Since $H^1(S^1) \simeq \mathbb{R}$, a generator of $H^1(S^1)$ is simply a nonzero element. As $d^*\colon H^0(U \cap V) \to H^1(S^1)$ is surjective, such a nonzero element is the image of an element $(x, y) \in H^0(U \cap V) \simeq \mathbb{R}^2$ for which $y \neq x$.

So we may start with $(1, 0) \in H^0(U \cap V)$. This corresponds to a function f with value 1 on A and 0 on B. Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to the open cover $\{U, V\}$, and let f_U, f_V be the extensions by zero of $\rho_V f$, $\rho_U f$ from $U \cap V$ to U and to V, respectively. By the proof of Proposition 25.1, $j(-f_U, f_V) = f$ on $U \cap V$. From Section 24.3, $d^*(1, 0)$ is represented by a 1-form on S^1 whose restriction to U is $-df_U$ and whose restriction to V is df_V. Now f_V is the function on V which is ρ_U on A and 0 on $V - A$, so df_V is a 1-form on V whose support is contained entirely in A. A similar analysis shows that $-df_U$ restricts to the same 1-form on A, because $\rho_U + \rho_V = 1$. The extension of either df_V or $-df_U$ by zero to a 1-form on S^1 represents a generator of $H^1(S^1)$. It is a bump 1-form on S^1 supported in A.

25.3 The Euler Characteristic

If the cohomology vector space $H^k(M)$ of an n-manifold is finite-dimensional for every k, we define its *Euler characterisitc* to be the alternating sum

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim H^k(M).$$

As a corollary of the Mayer–Vietoris sequence, the Euler characteristic of M with an open cover $\{U, V\}$ is always computable from those of U, V, and $U \cap V$, as follows.

Exercise 25.4 (Euler characteristics in terms of an open cover). Suppose all the spaces M, U, V, and $U \cap V$ in the Mayer–Vietoris sequence have finite-dimensional cohomology. By applying Problem 25.2 to the Mayer–Vietoris sequence, prove that if $M = U \cup V$, then

$$\chi(M) - (\chi(U) + \chi(V)) + \chi(U \cap V) = 0.$$

Problems

25.1. Short exact Mayer–Vietoris sequence
Prove the exactness of (25.1) at $\Omega^k(M)$ and at $\Omega^k(U) \oplus \Omega^k(V)$.

25.2. Alternating sum of dimensions
Let

$$0 \to A^0 \xrightarrow{d_0} A^1 \xrightarrow{d_1} A^2 \xrightarrow{d_2} \cdots \to A^m \to 0$$

be an exact sequence of finite-dimensional vector spaces. Show that

$$\sum_{k=0}^{m} (-1)^k \dim A^k = 0.$$

26

Homotopy Invariance

The homotopy axiom is a powerful tool for computing de Rham cohomology. Homotopy is normally defined in the continuous category. However, since we are primarily interested in manifolds and smooth maps, our notion of homotopy will be *smooth homotopy*, which differs from the usual homotopy in topology only in that all our maps are assumed to be smooth. In this chapter we define smooth homotopy, state the homotopy axiom for de Rham cohomology, and compute a few examples. We postpone the proof of the homotopy axiom to Chapter 28.

26.1 Smooth Homotopy

Let M and N be manifolds. Two C^∞ maps $f, g\colon M \to N$ are *(smoothly) homotopic* if there is a C^∞ map

$$F\colon M \times \mathbb{R} \to N$$

such that

$$F(x, 0) = f(x), \qquad F(x, 1) = g(x)$$

for all $x \in M$; the map F is called a *homotopy from f to g*. A homotopy F from f to g can also be viewed as a smoothly varying family of maps $\{f_t\colon M \to N \mid t \in \mathbb{R}\}$, where

$$f_t(x) = F(x, t), \quad x \in M,$$

such that $f_0 = f$ and $f_1 = g$. We can think of the parameter t as time and a homotopy as an evolution through time of the map $f_0\colon M \to N$. If f and g are homotopic, we write

$$f \sim g.$$

Since any open interval is diffeomorphic to $\mathbb{R}$ (Problem 1.3), in the definition of homotopy we could have used any open interval containing 0 and 1, instead of $\mathbb{R}$. The advantage of an open interval over the closed interval [0, 1] is that an open interval is a manifold with no boundary.

Example 26.1 (*The straight-line homotopy*). Let f and g be C^∞ maps from a manifold M to $\mathbb{R}^n$. Define $F\colon M \times \mathbb{R} \to \mathbb{R}^n$ by

$$\begin{aligned} F(x,t) &= f(x) + t(g(x) - f(x)) \\ &= (1-t)f(x) + tg(x). \end{aligned}$$

Then F is a homotopy from f to g, called the *straight-line homotopy from f to g* (Figure 26.1).

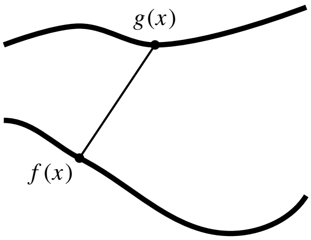


Fig. 26.1. The straight-line homotopy.

In fact, the straight-line homotopy can be defined for any two maps

$$f, g\colon M \to S \subset \mathbb{R}^n,$$

into a subspace S of $\mathbb{R}^n$ as long as for every $x \in M$, the line segment joining $f(x)$ and $g(x)$ lies entirely in S. This is true if, for example, S is a convex subset of $\mathbb{R}^n$.

Exercise 26.2 (Homotopy). Let M and N be manifolds. Prove that being homotopic is an equivalence relation on the set of all C^∞ maps from M to N.

26.2 Homotopy Type

In the following, we write 1_M to denote the identity map on a manifold M.

Definition 26.3. A map $f\colon M \to N$ is a *homotopy equivalence* if it has a *homotopy inverse*, i.e., a map $g\colon N \to M$ such that $g \circ f$ is homotopic to the identity 1_M on M and $f \circ g$ is homotopic to the identity 1_N on N:

$$g \circ f \sim 1_M, \qquad f \circ g \sim 1_N.$$

In this case we say that M is *homotopy equivalent* to N, or that M and N have the same *homotopy type*.

Example 26.4. A diffeomorphism is a homotopy equivalence.

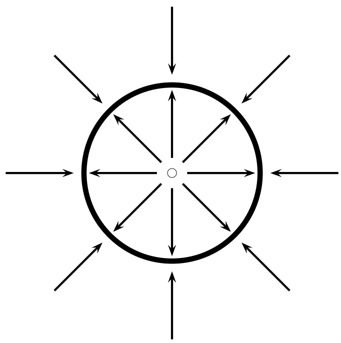


Fig. 26.2. The punctured plane retracts to the unit circle..

Example 26.5 (*The homotopy type of the punctured plane* $\mathbb{R}^2 - \{0\}$). Let $i\colon S^1 \to \mathbb{R}^2 - \{0\}$ be the inclusion map and let $r\colon \mathbb{R}^2 - \{0\} \to S^1$ be given by

$$r(x) = \frac{x}{|x|}.$$

Then $r \circ i$ is the identity map on S^1.

We claim that

$$i \circ r\colon \mathbb{R}^2 - \{0\} \to \mathbb{R}^2 - \{0\}$$

is homotopic to the identity map. Indeed, the straight-line homotopy

$$F\colon (\mathbb{R}^2 - \{0\}) \times \mathbb{R} \to \mathbb{R}^2 - \{0\},$$
$$F(x, t) = (1 - t)x + t\frac{x}{|x|}$$

provides a homotopy between the identity map on $\mathbb{R}^2 - \{0\}$ and $i \circ r$ (Figure 26.2). Therefore, r and i are homotopy inverse to each other, and $\mathbb{R}^2 - \{0\}$ and S^1 have the same homotopy type.

Definition 26.6. A manifold is *contractible* if it has the homotopy type of a point.

In this definition, by "the homotopy type of a point" we mean the homotopy type of a set $\{p\}$ whose single element is a point. Such a set is called a *singleton set*.

Example 26.7 (*The Euclidean space* $\mathbb{R}^n$ *is contractible*). Let p be a point in $\mathbb{R}^n$, $i\colon \{p\} \to \mathbb{R}^n$ the inclusion map, and $r\colon \mathbb{R}^n \to \{p\}$ the constant map. Then $r \circ i = 1_{\{p\}}$, the identity map on $\{p\}$. The straight-line homotopy provides a homotopy between the constant map $i \circ r\colon \mathbb{R}^n \to \mathbb{R}^n$ and the identity map on $\mathbb{R}^n$:

$$F(x, t) = (1 - t)x + t\,r(x) = (1 - t)x + tp.$$

Hence, the Euclidean space $\mathbb{R}^n$ and the set $\{p\}$ have the same homotopy type.

26.3 Deformation Retractions

Let S be a submanifold of a manifold M, with $i\colon S \to M$ the inclusion map.

Definition 26.8. A *retraction* from M to S is a map $r\colon M \to S$ that restricts to the identity map on S; in other words, $r \circ i = 1_S$. If there is a retraction from M to S, we say that S is a *retract* of M.

Definition 26.9. A *deformation retraction* from M to S is a map $F\colon M \times \mathbb{R} \to M$ such that for all $x \in M$,

(i) $F(x, 0) = x$;
(ii) there is a retraction $r\colon M \to S$ such that $F(x, 1) = r(x)$;
(iii) for all $s \in S$ and $t \in \mathbb{R}$, $F(s, t) = s$.

If there is a deformation retraction from M to S, we say that S is a *deformation retract* of M.

Setting $f_t(x) = F(x, t)$, we can think of a deformation retraction $F\colon M \times \mathbb{R} \to M$ as a family of maps $f_t\colon M \to M$ such that

(i) f_0 is the identity map on M;
(ii) $f_1(x) = r(x)$ for some retraction $r\colon M \to S$;
(iii) for every t the map $f_t\colon M \to M$ leaves S pointwise fixed.

We may rephrase Condition (ii) in the definition as follows: there is a retraction $r\colon M \to S$ such that $f_1 = i \circ r$. Thus, a deformation retraction is a homotopy between the identity map 1_M and $i \circ r$ for a retraction $r\colon M \to S$ and this homotopy leaves S fixed for all time t.

Example 26.10. Any point p in a manifold M is a retract of M; simply take a retraction to be the constant map $r\colon M \to \{p\}$.

Example 26.11. The map F in Example 26.5 is a deformation retraction from the punctured plane $\mathbb{R}^2 - \{0\}$ to the unit circle S^1.

Generalizing Example 26.5, we prove the following theorem.

Proposition 26.12. *If $S \subset M$ is a deformation retract of M, then S and M have the same homotopy type.*

Proof. Let $F\colon M \times \mathbb{R} \to M$ be a deformation retraction and let $r(x) = f_1(x) = F(x, 1)$ be the retraction. Because r is a retraction, the composite

$$S \xrightarrow{i} M \xrightarrow{r} S, \quad r \circ i = 1_S,$$

is the identity map on S. By the definition of a deformation retraction, the composite

$$M \xrightarrow{r} S \xrightarrow{i} M$$

is f_1 and the deformation retraction provides a homotopy

$$f_1 = i \circ r \sim f_0 = 1_M.$$

Therefore, $r\colon M \to S$ is a homotopy equivalence, with homotopy inverse $i\colon S \to M$. □

26.4 The Homotopy Axiom for de Rham Cohomology

We state here the homotopy axiom and derive a few consequences. The proof will be given in Chapter 28.

Theorem 26.13 (Homotopy axiom for de Rham cohomology). *Homotopic maps $f_0, f_1\colon M \to N$ induce the same map $f_0^* = f_1^*\colon H^*(N) \to H^*(M)$ in cohomology.*

Corollary 26.14. *If $f\colon M \to N$ is a homotopy equivalence, then the induced map in cohomology*

$$f^*\colon H^*(N) \to H^*(M)$$

is an isomorphism.

Proof (*of Corollary*). Let $g\colon N \longrightarrow M$ be a homotopy inverse to f. Then

$$g \circ f \sim 1_M, \qquad f \circ g \sim 1_N.$$

By the homotopy axiom,

$$(g \circ f)^* = 1_M^*, \qquad (f \circ g)^* = 1_N^*.$$

By functoriality,

$$f^* \circ g^* = 1_{H^*(M)}, \qquad g^* \circ f^* = 1_{H^*(N)}.$$

Therefore, f^* is an isomorphism in cohomology. □

Corollary 26.15. *Suppose S is a submanifold of a manifold M and F is a deformation retraction from M to S. Let $r\colon M \to S$ be the retraction $r(x) = F(x, 1)$. Then r induces an isomorphism in cohomology*

$$r^*\colon H^*(S) \xrightarrow{\sim} H^*(M).$$

Corollary 26.16 (Poincaré lemma). *Since $\mathbb{R}^n$ has the homotopy type of a point, the cohomology of $\mathbb{R}^n$ is*

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{for } k = 0, \\ 0 & \text{for } k > 0. \end{cases}$$

More generally, any contractible manifold will have the same cohomology as a point.

Example 26.17 (*Cohomology of a punctured plane*). For any $p \in \mathbb{R}^2$, the map $x \mapsto x - p$ is a diffeomorphism of $\mathbb{R}^2 - \{p\}$ with $\mathbb{R}^2 - \{0\}$. Because the punctured plane $\mathbb{R}^2 - \{0\}$ and the circle S^1 have the same homotopy type (Example 26.5), they have isomorphic cohomology. Hence, $H^k(\mathbb{R}^2 - \{p\}) \simeq H^k(S^1)$ for all $k \geq 0$.

Example 26.18. The central circle of an open Möbius band M is a deformation retract of M (Figure 26.3). Thus, the open Möbius band has the homotopy type of a circle. By the homotopy axiom,

$$H^k(M) = H^k(S^1) = \begin{cases} \mathbb{R} & \text{for } k = 0, 1, \\ 0 & \text{for } k > 1. \end{cases}$$

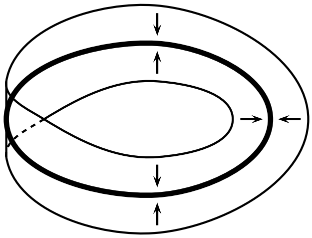


Fig. 26.3. The Möbius band deformation retracts to its central circle..

Problems

26.1. Contractibility and path-connectedness
Show that a contractible manifold is path-connected.

26.2. Contractibility and deformation retraction
Prove that M is contractible if and only if for any $p \in M$, there is a deformation retraction from M to p.

26.3. Deformation retraction from $\mathbb{R}^n$ to a point
Write down a deformation retraction from $\mathbb{R}^n$ to $\{0\}$.

26.4. Deformation retraction from a cylinder to a circle
Show that the circle $S^1 \times \{0\}$ is a deformation retract of the cylinder $S^1 \times \mathbb{R}$.

27

Computation of de Rham Cohomology

With the tools developed so far, we can compute the cohomology of many manifolds. This chapter is a compendium of some examples.

27.1 Cohomology Vector Space of a Torus

Cover a torus M with two open subsets U and V as shown in Figure 27.1.

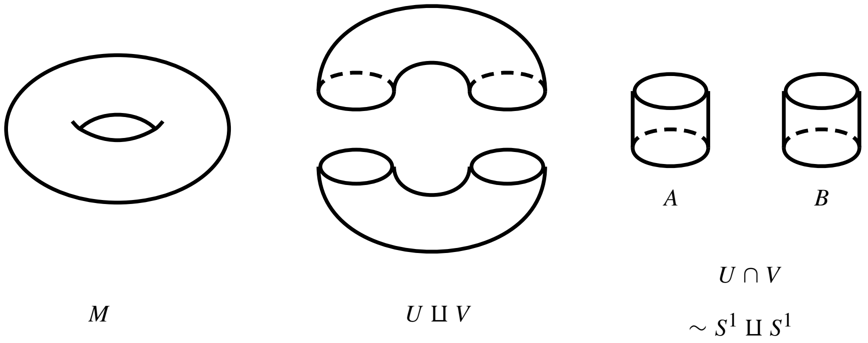


Fig. 27.1. An open cover $\{U, V\}$ of a torus.

Both U and V are diffeomorphic to a cylinder and therefore have the homotopy type of a circle (Problem 26.4). Similarly, the intersection $U \cap V$ is the disjoint union of two cylinders A and B and has the homotopy type of a disjoint union of two circles. Our knowledge of the cohomology of a circle allows us to fill in many terms in the Mayer–Vietoris sequence:

$$
\begin{array}{c|ccccccc}
 & & M & & U \amalg V & & U \cap V \\
\hline
H^2 & \xrightarrow{d_1^*} & H^2(M) & \to & 0 & & \\
H^1 & \xrightarrow{d_0^*} & H^1(M) & \xrightarrow{\gamma} & \mathbb{R} \oplus \mathbb{R} & \xrightarrow{\beta} & \mathbb{R} \oplus \mathbb{R} \\
H^0 & 0 \to & \mathbb{R} & \to & \mathbb{R} \oplus \mathbb{R} & \xrightarrow{\alpha} & \mathbb{R} \oplus \mathbb{R}
\end{array}
\tag{27.1}
$$

Let $j_U : U \cap V \to U$ and $j_V : U \cap V \to V$ be the inclusion maps. If a is the constant function with value a on U, then $j_U^* a$ is the constant function with the value a on each component of $U \cap V$, that is,

$$j_U^* a = (a, a).$$

Therefore, for $(a, b) \in H^0(U) \oplus H^0(V)$,

$$
\begin{aligned}
\alpha(a, b) &= j_V^* b - j_U^* a \\
&= (b, b) - (a, a) \\
&= (b - a, b - a).
\end{aligned}
$$

Similarly, let us now describe the map

$$\beta \colon H^1(U) \oplus H^1(V) \to H^1(U \cap V) = H^1(A) \oplus H^1(B).$$

Since A is a deformation retract of U, the restriction $H^*(U) \to H^*(A)$ is an isomorphism. If ω_U generates $H^1(U)$, then $j_U^* \omega_U$ is a generator of H^1 on A and on B. Identifying $H^1(U \cap V)$ with $\mathbb{R} \oplus \mathbb{R}$, we write $j_U^* \omega_U = (1, 1)$. Let ω_V be a generator of $H^1(V)$. The pair of real numbers

$$(a, b) \in H^1(U) \oplus H^1(V) \simeq \mathbb{R} \oplus \mathbb{R}$$

stands for $(a\omega_U, b\omega_V)$. Then,

$$
\begin{aligned}
\beta(a, b) &= j_V^*(b\omega_V) - j_U^*(a\omega_U) \\
&= (b, b) - (a, a) \\
&= (b - a, b - a).
\end{aligned}
$$

By the exactness of the Mayer–Vietoris sequence,

$$
\begin{aligned}
H^2(M) &= \operatorname{im} d_1^* && \text{(because } H^2(U) \oplus H^2(V) = 0\text{)} \\
&\simeq H^1(U \cap V) / \ker d_1^* && \text{(by the first isomorphism theorem)} \\
&\simeq (\mathbb{R} \oplus \mathbb{R}) / \operatorname{im} \beta \\
&\simeq (\mathbb{R} \oplus \mathbb{R}) / \mathbb{R} \simeq \mathbb{R}.
\end{aligned}
$$

Applying Problem 25.2 to the Mayer–Vietoris sequence (27.1), we get

$$1 - 2 + 2 - \dim H^1(M) + 2 - 2 + \dim H^2(M) = 0.$$

Since $\dim H^2(M) = 1$, this gives $\dim H^1(M) = 2$.

As a check, we can also compute $H^1(M)$ from the Mayer–Vietoris sequence using our knowledge of the maps α and β:

$$\begin{aligned}
H^1(M) &\simeq \ker\gamma \oplus \operatorname{im}\gamma && \text{(by the first isomorphism theorem)}\\
&\simeq \operatorname{im} d_0^* \oplus \ker\beta && \text{(exactness of the M–V sequence)}\\
&\simeq (H^0(U\cap V)/\ker d_0^*) \oplus \ker\beta && \text{(first isomorphism theorem for } d_0^*\text{)}\\
&\simeq ((\mathbb{R}\oplus\mathbb{R})/\operatorname{im}\alpha)\oplus\mathbb{R}\\
&\simeq \mathbb{R}\oplus\mathbb{R}.
\end{aligned}$$

27.2 The Cohomology Ring of a Torus

A torus is diffeomorphic to the quotient of $\mathbb{R}^2$ by the integer lattice $\Lambda = \mathbb{Z}^2$. The quotient map

$$\pi\colon \mathbb{R}^2 \to \mathbb{R}^2/\Lambda$$

induces a pullback map on differential forms,

$$\pi^*\colon \Omega^*(\mathbb{R}^2/\Lambda) \to \Omega^*(\mathbb{R}^2).$$

Since $\pi\colon \mathbb{R}^2 \to \mathbb{R}^2/\Lambda$ is a local diffeomorphism, it is a submersion at each point. By Problem 18.7, $\pi^*\colon \Omega^*(\mathbb{R}^2/\Lambda) \to \Omega^*(\mathbb{R}^2)$ is an inclusion.

For $\lambda \in \Lambda$, define $\ell_\lambda\colon \mathbb{R}^2 \to \mathbb{R}^2$ to be translation by λ,

$$\ell_\lambda(p) = p + \lambda,\ \ p \in \mathbb{R}^2.$$

A differential form $\bar\omega$ on $\mathbb{R}^2$ is said to be *invariant under translation by* $\lambda \in \Lambda$ if $\ell_\lambda^*\bar\omega = \bar\omega$.

Proposition 27.1. *The image of the inclusion map* $\pi^*\colon \Omega^*(\mathbb{R}^2/\Lambda) \to \Omega^*(\mathbb{R}^2)$ *is the subspace of differential forms on* $\mathbb{R}^2$ *invariant under translations by elements of* Λ.

Proof. For all $p \in \mathbb{R}^2$,

$$(\pi\circ\ell_\lambda)(p) = \pi(p+\lambda) = \pi(p).$$

Hence, $\pi\circ\ell_\lambda = \pi$. By the functoriality of the pullback,

$$\pi^* = \ell_\lambda^*\circ\pi^*.$$

Thus, for any $\omega \in \Omega^k(\mathbb{R}^2/\Lambda)$, $\pi^*\omega = \ell_\lambda^*\pi^*\omega$. This proves that $\pi^*\omega$ is invariant under translations ℓ_λ for all $\lambda\in\Lambda$.

Conversely, suppose $\bar\omega \in \Omega^k(\mathbb{R}^2)$ is invariant under translations ℓ_λ for all $\lambda \in \Lambda$. For $p \in \mathbb{R}^2/\Lambda$ and $v_1, \dots, v_k \in T_p(\mathbb{R}^2/\Lambda)$, define

$$\omega_p(v_1,\dots,v_k) = \bar\omega_{\bar p}(\bar v_1,\dots,\bar v_k) \tag{27.2}$$

for any $\bar{p} \in \pi^{-1}(\{p\})$ and $\bar{v}_1, \ldots, \bar{v}_k \in T_{\bar{p}}\mathbb{R}^2$ such that $\pi_*\bar{v}_i = v_i$. Any other point in $\pi^{-1}(\{p\})$ may be written as $\bar{p} + \lambda$ for some $\lambda \in \Lambda$. By invariance,

$$\bar{\omega}_{\bar{p}} = (\ell_\lambda^*\bar{\omega})_{\bar{p}} = \ell_\lambda^*(\bar{\omega}_{\bar{p}+\lambda}).$$

So

$$\begin{aligned}\bar{\omega}_{\bar{p}}(\bar{v}_1, \ldots, \bar{v}_k) &= \ell_\lambda^*(\bar{\omega}_{\bar{p}+\lambda})(\bar{v}_1, \ldots, \bar{v}_k)\\ &= \bar{\omega}_{\bar{p}+\lambda}(\ell_{\lambda*}\bar{v}_1, \ldots, \ell_{\lambda*}\bar{v}_k),\end{aligned}$$

which shows that ω_p is well defined, independent of the choice of $\bar{p}$. Thus, $\omega \in \Omega^k(\mathbb{R}^2/\Lambda)$. Moreover, by (27.2), for any $\bar{p} \in \mathbb{R}^2$ and $\bar{v}_1, \ldots, \bar{v}_k \in T_{\bar{p}}(\mathbb{R}^2)$,

$$\begin{aligned}\bar{\omega}_{\bar{p}}(\bar{v}_1, \ldots, \bar{v}_k) &= \omega_{\pi(\bar{p})}(\pi_*\bar{v}_1, \ldots, \pi_*\bar{v}_k)\\ &= (\pi^*\omega)_{\bar{p}}(\bar{v}_1, \ldots, \bar{v}_k).\end{aligned}$$

Hence, $\bar{\omega} = \pi^*\omega$. □

Let (x, y) be the coordinates on $\mathbb{R}^2$. Since for any $\lambda \in \Lambda$,

$$\ell_\lambda^*(dx) = d(\ell_\lambda^* x) = d(x + \lambda) = dx,$$

by Proposition 27.1 the 1-form dx on $\mathbb{R}^2$ is π^* of a 1-form on the torus $\mathbb{R}^2/\Lambda$. Similarly, dy is also π^* of a 1-form on the torus. We denote these 1-forms on the torus by the same symbols dx and dy.

Proposition 27.2. *Let M be the torus $\mathbb{R}^2/\mathbb{Z}^2$. A basis for the cohomology vector space $H^*(M)$ is $1, dx, dy, dx \wedge dy$.*

Proof. Since $\int_M dx \wedge dy = 1$, the closed 2-form $dx \wedge dy$ defines a nonzero cohomology class. By the computation of Section 27.1, $H^2(M) = \mathbb{R}$. So $dx \wedge dy$ is a basis for $H^2(M)$.

It remains to show that the set of closed 1-forms dx, dy on M is a basis for $H^1(M)$. Define two closed curves C_1, C_2 in $M = \mathbb{R}^2/\mathbb{Z}^2$ as the images of the maps

$$\begin{gathered} c_i \colon [0, 1] \to M,\\ c_1(t) = [(t, 0)], \qquad c_2(t) = [(0, t)],\end{gathered}$$

(see Figure 27.2). Denote by p the point $[(0, 0)]$ in M. Since removing a point does not change the value of an integral and c_1 is a diffeomorphism of the open interval $(0, 1)$ onto $C_1 - \{p\}$,

$$\int_{C_1} dx = \int_{C_1 - \{p\}} dx = \int_{(0,1)} c_1^*\, dx = \int_0^1 dt = 1.$$

In the same way, because $c_1^* dy = 0$,

$$\int_{C_1} dy = \int_{C_1 - \{p\}} dy = \int_0^1 c_1^*\, dy = 0.$$

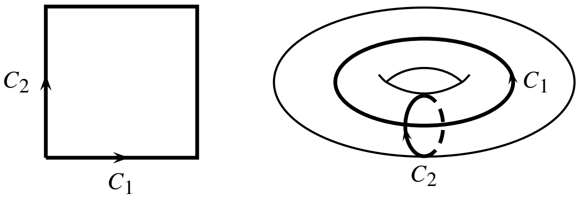


Fig. 27.2. Two closed curves on a torus.

Similarly,

$$\int_{C_2} dx = 0, \qquad \int_{C_2} dy = 1.$$

As x is not a function on the torus M, dx is not necessarily exact on M. In fact, if $dx = df$ for some C^∞ function f on M, then

$$\int_{C_1} dx = \int_{C_1} df = \int_{\partial C_1} f = 0$$

by Stokes' theorem and the fact that $\partial C_1 = \varnothing$. This contradicts the fact that $\int_{C_1} dx = 1$. Thus, dx is not exact on M. By the same reasoning, dy is also not exact on M. Furthermore, the cohomology classes $[dx]$ and $[dy]$ are linearly independent, since if $[dx]$ were a multiple of $[dy]$, then $\int_{C_1} dx$ would have to be a multiple of $\int_{C_1} dy = 0$. By Section 27.1, $H^1(M)$ is two dimensional. Hence, dx, dy is a basis for $H^1(M)$. □

The ring structure of $H^*(M)$ is transparent from this proposition. Abstractly it is the algebra

$$\bigwedge(a,b) := \mathbb{R}[a,b]/(a^2, b^2, ab + ba), \qquad \deg a = 1, \deg b = 1,$$

called the *exterior algebra* on two generators a and b of degree 1.

27.3 The Cohomology of a Surface of Genus g

Using the Mayer–Vietoris sequence to compute the cohomology of a manifold often leads to ambiguities, because there may be several unknown terms in the sequence. We can resolve these ambiguities if we can describe explicitly the maps occurring in the Mayer–Vietoris sequence. Here is an example of how this might be done.

Lemma 27.3. *Suppose p is a point in a compact oriented surface M without boundary, and $i\colon C \to M - \{p\}$ is the inclusion of a small circle around the puncture* (*Figure* 27.3)*. Then the restriction map*

$$i^*\colon H^1(M - \{p\}) \to H^1(C)$$

is the zero map.

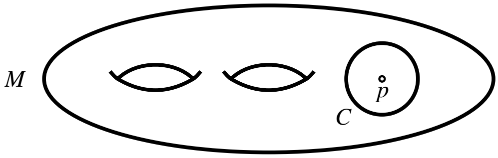


Fig. 27.3. Punctured surface.

Proof. An element $[\omega] \in H^1(M - \{p\})$ is represented by a closed 1-form ω on $M - \{p\}$. Because the linear isomorphism $H^1(C) \simeq H^1(S^1) \simeq \mathbb{R}$ is given by integration over C, to identify $i^*[\omega]$ in $H^1(C)$, it suffices to compute the integral $\int_C i^*\omega$.

If D is the open disk in M bounded by the curve C, then $M - D$ is a compact oriented surface with boundary C. By Stokes' theorem,

$$\int_C i^*\omega = \int_{\partial(M-D)} i^*\omega = \int_{M-D} d\omega = 0$$

because $d\omega = 0$. Hence, $i^* \colon H^1(M - \{p\}) \to H^1(C)$ is the zero map. $\square$

Proposition 27.4. *Let M be a torus, p a point in M, and A the punctured torus $M - \{p\}$. The cohomology of A is*

$$H^k(A) = \begin{cases} \mathbb{R} & \text{for } k = 0, \\ \mathbb{R}^2 & \text{for } k = 1, \\ 0 & \text{for } k > 1. \end{cases}$$

Proof. Cover M with two open sets, A and a disk U containing p. Since A, U, and $A \cap U$ are all connected, we may start the Mayer–Vietoris sequence with the $H^1(M)$ term (Proposition 25.3(ii)). With $H^*(M)$ known from Section 27.1, the Mayer–Vietoris sequence becomes

$$\begin{array}{c|cccc} & M & U \amalg A & U \cap A \sim S^1 \\ \hline H^2 & \overset{d_1^*}{\to} \quad \mathbb{R} & \to H^2(A) \to & 0 \\ H^1 & 0 \to \mathbb{R} \oplus \mathbb{R} & \overset{\beta}{\to} H^1(A) \overset{\alpha}{\to} & H^1(S^1) \end{array}$$

Because $H^1(U) = 0$, the map $\alpha \colon H^1(A) \to H^1(S^1)$ is simply the restriction map i^*. By Lemma 27.3, $\alpha = i^* = 0$. Hence,

$$H^1(A) = \ker \alpha = \operatorname{im} \beta \simeq H^1(M) \simeq \mathbb{R} \oplus \mathbb{R}$$

and there is an exact sequence of linear maps

$$0 \to H^1(S^1) \overset{d_1^*}{\to} \mathbb{R} \to H^2(A) \to 0.$$

Since $H^1(S^1) \simeq \mathbb{R}$, it follows that $H^2(A) = 0$. $\square$

Proposition 27.5. *The cohomology of a compact orientable surface Σ_2 of genus 2 is*

$$H^k(\Sigma_2) = \begin{cases} \mathbb{R} & \textit{for } k = 0, 2, \\ \mathbb{R}^4 & \textit{for } k = 1, \\ 0 & \textit{for } k > 2. \end{cases}$$

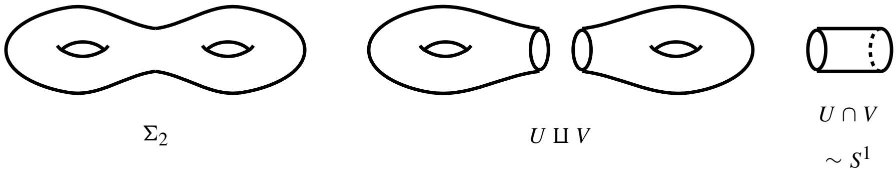


Fig. 27.4. An open cover $\{U, V\}$ of a surface of genus 2.

Proof. Cover Σ_2 with two open sets U and V as in Figure 27.4. The Mayer–Vietoris sequence gives

	M	$U \amalg V$	$U \cap V \sim S^1$
H^2	$\to H^2(\Sigma_2) \to$	0	
H^1	$0 \to H^1(\Sigma_2) \to$	$\mathbb{R}^2 \oplus \mathbb{R}^2 \xrightarrow{\alpha}$	$\mathbb{R}$

The map $\alpha\colon H^1(U) \oplus H^1(V) \to H^1(S^1)$ is the difference map

$$\alpha(\omega_U, \omega_V) = j_V^* \omega_V - j_U^* \omega_U,$$

where j_U and j_V are inclusions of an S^1 in $U \cap V$ into U and V, respectively. By Lemma 27.3, $j_U^* = j_V^* = 0$, so $\alpha = 0$. It then follows from the exactness of the Mayer–Vietoris sequence that

$$H^1(\Sigma_2) \simeq H^1(U) \oplus H^1(V) \simeq \mathbb{R}^4$$

and

$$H^2(\Sigma_2) \simeq H^1(S^1) \simeq \mathbb{R}. \qquad \square$$

A genus 2 surface Σ_2 can be obtained as the quotient space of an octagon with its edges identified following the scheme of Figure 27.5.

To see this, first cut Σ_2 along the circle e as in Figure 27.6. Then the two halves A and B are each a torus minus an open disk (Figure 27.7), so that each half can be represented as a pentagon (Figure 27.8).

When A and B are glued together along e, we obtain the octagon in Figure 27.5.

By Lemma 27.3, if $p \in \Sigma_2$ and $i\colon C \to \Sigma_2 - \{p\}$ is a small circle around p in Σ_2, then the restriction map

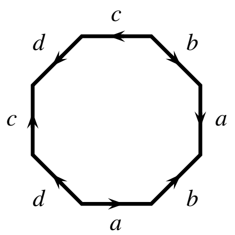


Fig. 27.5. A surface of genus 2 as a quotient space of an octagon.

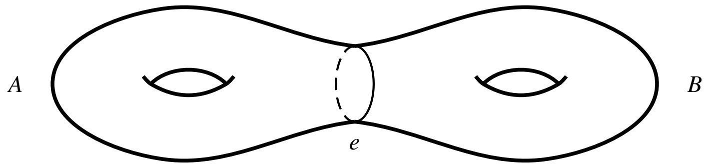


Fig. 27.6. A surface of genus 2 cut along a curve e.

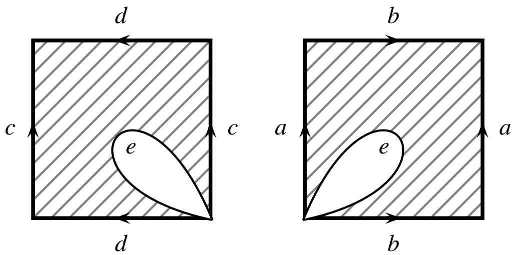


Fig. 27.7. Two halves of a surface of genus 2.

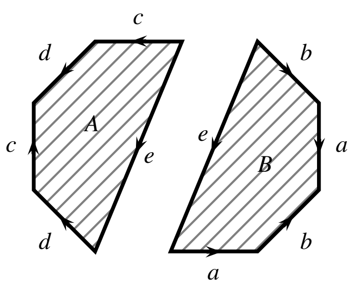


Fig. 27.8. Two halves of a surface of genus 2.

$$i^*\colon H^1(\Sigma_2 - \{p\}) \to H^1(C)$$

is the zero map. This allows us to compute inductively the cohomology of a compact orientable surface Σ_g of genus g.

Exercise 27.6 (Cohomology of a surface). Compute the cohomology vector space of a compact orientable surface Σ_g of genus g.

Problems

27.1. Real projective plane
Compute the cohomology of the real projective plane (Figure 27.9).

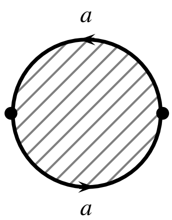


Fig. 27.9. The real projective plane.

27.2. The *n*-sphere
Compute the cohomology of the sphere S^n.

27.3. Cohomology of a multiply punctured plane

(a) Let p, q be distinct points in $\mathbb{R}^2$. Compute the de Rham cohomology of $\mathbb{R}^2 - \{p, q\}$.
(b) Let $p_1, \dots, p_n$ be distinct points in $\mathbb{R}^2$. Compute the de Rham cohomology of $\mathbb{R}^2 - \{p_1, \dots, p_n\}$.

28

Proof of Homotopy Invariance

In this chapter we prove the homotopy invariance of de Rham cohomology.

If $f: M \to N$ is a C^∞ map, the pullback maps on differential forms and on cohomology classes are normally both denoted f^*. Since this might cause confusion in the proof of homotopy invariance, in this chapter we denote the pullback of forms by

$$f^*: \Omega^k(N) \to \Omega^k(M).$$

and the induced map in cohomology by

$$f^\#: H^k(N) \to H^k(M).$$

The relation between these two maps is

$$f^\#[\omega] = [f^*\omega]$$

for $[\omega] \in H^k(N)$.

Theorem 28.1 (Homotopy axiom for de Rham cohomology). *Two smoothly homotopic maps $f, g: M \to N$ of manifolds induce the same map in cohomology:*

$$f^\# = g^\#: H^k(N) \to H^k(M).$$

We first reduce the problem to two special maps i_0 and $i_1: M \to M \times \mathbb{R}$, which are the 0-section and the 1-section, respectively, of the product line bundle $M \times \mathbb{R} \to M$:

$$i_0(x) = (x, 0), \qquad i_1(x) = (x, 1).$$

Then we introduce the all important technique of cochain homotopy. By finding a cochain homotopy between i_0^* and i_1^*, we prove that they induce the same map in cohomology.

28.1 Reduction to Two Sections

Suppose f and $g\colon M \to N$ are smoothly homotopic maps. Let $F\colon M \times \mathbb{R} \to N$ be a homotopy from f to g. This means

$$F(x,0) = f(x), \qquad F(x,1) = g(x), \tag{28.1}$$

for all $x \in M$. For each $t \in \mathbb{R}$, define $i_t\colon M \to M \times \mathbb{R}$ to be the section $i_t(x) = (x,t)$. We can restate (28.1) as

$$F \circ i_0 = f, \qquad F \circ i_1 = g.$$

By the functoriality of the pullback (Remark 23.8),

$$f^{\#} = i_0^{\#} \circ F^{\#}, \qquad g^{\#} = i_1^{\#} \circ F^{\#}.$$

This reduces the proof of homotopy invariance to the special case

$$i_0^{\#} = i_1^{\#}.$$

The two maps $i_0, i_1\colon M \to M \times \mathbb{R}$ are obviously homotopic via the identity map

$$1_{M\times\mathbb{R}}\colon M \times \mathbb{R} \to M \times \mathbb{R}.$$

28.2 Cochain Homotopies

The usual method for showing that two cochain maps

$$f^*,\ g^*\colon \Omega^*(N) \to \Omega^*(M)$$

induce the same map in cohomology is to find a map

$$K\colon \Omega^*(N) \to \Omega^{*-1}(M)$$

of degree -1 such that

$$g^* - f^* = dK \pm Kd.$$

Such a map K is called a *cochain homotopy* from f to g. If ω is any closed form on N, then

$$g^*\omega - f^*\omega = dK\omega \pm Kd\omega = dK\omega,$$

so

$$f^{\#}[\omega] = g^{\#}[\omega].$$

Thus, the existence of a cochain homotopy between f^* and g^* implies $f^{\#} = g^{\#}$.

Remark 28.2. If one could find a map $K\colon \Omega^*(N) \to \Omega^{*-1}(M)$ such that $g^* - f^* = dK$ on $\Omega^*(N)$, then $f^{\#} = g^{\#}$ on $H^*(N)$. However, such a map almost never exists; it is necessary to have the term Kd as well. The cylinder construction in homology theory [15, p. 65] shows why it is natural to consider $dK \pm Kd$.

28.3 Differential Forms on $M \times \mathbb{R}$

Recall that a sum $\sum_\alpha \omega_\alpha$ of C^∞ differential forms on a manifold M is called a *locally finite sum* if the collection of supports, $\{\operatorname{supp}\omega_\alpha\}$, is locally finite. This means every point p in M has a neighborhood V_p such that V_p intersects only finitely many of the sets $\operatorname{supp}\omega_\alpha$. If $\operatorname{supp}\omega_\alpha$ is disjoint from V_p, then $\omega_\alpha \equiv 0$ on V_p. Thus, on V_p the locally finite sum $\sum_\alpha \omega_\alpha$ is actually a finite sum. By definition, a partition of unity $\sum \rho_\alpha$ is a locally finite sum.

Let $\pi : M \times \mathbb{R} \to M$ be the projection to the first factor. In this section we will show that every C^∞ differential form on $M \times \mathbb{R}$ is a locally finite sum of the following two types of forms:

(I) $f(x,t)\,\pi^*\phi$,
(II) $f(x,t)\,dt \wedge \pi^*\phi$,

where $f(x,t)$ is a C^∞ function on $M \times \mathbb{R}$ and ϕ is a C^∞ form on M.

In general, a decomposition of a differential form on $M \times \mathbb{R}$ into a locally finite sum of Type I and Type II forms is far from unique. However, we will show that there is an unambiguous procedure to produce uniquely such a locally finite sum, once we fix an atlas $\{(U_\alpha, x_\alpha^1, \dots, x_\alpha^n)\}$ on M, a partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$, and a collection of C^∞ functions g_α on M such that

$$g_\alpha \equiv 1 \quad \text{on } \operatorname{supp}\rho_\alpha \quad \text{and} \quad \operatorname{supp} g_\alpha \subset U_\alpha.$$

The existence of such functions g_α follows from the smooth Urysohn lemma (Problem 13.3).

Fix U_α, ρ_α, and g_α as above. Then $\{\pi^{-1}U_\alpha\}$ is an open cover of $M \times \mathbb{R}$, and $\{\pi^*\rho_\alpha\}$ is a partition of unity subordinate to $\{\pi^{-1}U_\alpha\}$ (Problem 13.5). On $\pi^{-1}U_\alpha$ we have coordinates $\pi^*x_\alpha^1, \dots \pi^*x_\alpha^n, t$. For the sake of simplicity, we sometimes write x_α^i instead of $\pi^*x_\alpha^i$.

Let ω be any C^∞ k-form on $M \times \mathbb{R}$. Since $\sum \pi^*\rho_\alpha = 1$,

$$\omega = \sum (\pi^*\rho_\alpha)\omega. \tag{28.2}$$

Write ω_α for $(\pi^*\rho_\alpha)\omega$. Then

$$\operatorname{supp}\omega_\alpha \subset \operatorname{supp}\pi^*\rho_\alpha \subset \pi^{-1}U_\alpha.$$

On $\pi^{-1}U_\alpha$ the k-form ω_α may be written uniquely as a linear combination

$$\omega_\alpha = \sum a_I^\alpha\, dx_\alpha^I + \sum b_J^\alpha\, dt \wedge dx_\alpha^J, \tag{28.3}$$

where a_I^α and b_J^α are C^∞ functions on $\pi^{-1}U_\alpha$ with support in $\operatorname{supp}\pi^*\rho_\alpha$. The sum in (28.3) shows that ω_α is a sum of Type I and Type II forms on $\pi^{-1}U_\alpha$. In this sum a_I^α and b_J^α can be extended by zero to C^∞ functions on $M \times \mathbb{R}$, since they have support in $\pi^{-1}U_\alpha$. Unfortunately, dx_α^I and dx_α^J make sense only on U_α and cannot be extended to M, at least not directly.

To extend the decomposition (28.3) to $M \times \mathbb{R}$, note that since $\operatorname{supp} \omega_\alpha \subset \operatorname{supp} \pi^* \rho_\alpha$ and $\pi^* g_\alpha \equiv 1$ on $\operatorname{supp} \pi^* \rho_\alpha$,

$$\begin{aligned} \omega_\alpha = (\pi^* g_\alpha)\omega_\alpha &= \sum a_I^\alpha (\pi^* g_\alpha)\, dx_\alpha^I + \sum b_J^\alpha \, dt \wedge (\pi^* g_\alpha)\, dx_\alpha^J, \\ &= \sum a_I^\alpha (\pi^* g_\alpha \, dx_\alpha^I) + \sum b_J^\alpha \, dt \wedge (\pi^* g_\alpha \, dx_\alpha^J). \end{aligned} \tag{28.4}$$

Since $\operatorname{supp} g_\alpha \subset U_\alpha$, $g_\alpha \, dx_\alpha^I$ can be extended by zero to M. Equations (28.2) and (28.4) prove that ω is a locally finite sum of Type I and Type II forms on $M \times \mathbb{R}$. Moreover, we see that given U_α, ρ_α, and g_α, the decomposition in (28.4) is unique.

28.4 A Cochain Homotopy Between i_0^* and i_1^*

Using the decomposition (28.4), define

$$K \colon \Omega^*(M \times \mathbb{R}) \to \Omega^{*-1}(M)$$

by the following rules:

(i) on Type I forms,

$$K(f\pi^*\omega) = 0,$$

(ii) on Type II forms,

$$K(f\, dt \wedge \pi^*\omega) = \left(\int_0^1 f(x,t)\, dt \right) \omega,$$

and extend by linearity.

28.5 Verification of Cochain Homotopy

We now check that

$$dK + Kd = i_1^* - i_0^*.$$

It suffices to check this equality on any coordinate open set. So fix a coordinate open set $(U \times \mathbb{R}, \pi^* x^1, \dots, \pi^* x^n, t)$ on $M \times \mathbb{R}$. On Type I forms,

$$Kd(f\pi^*\omega) = K\left(\frac{\partial f}{\partial t}\, dt \wedge \pi^*\omega + \sum_i \frac{\partial f}{\partial x^i}\, \pi^* dx^i \wedge \pi^*\omega + f\pi^* d\omega \right).$$

In the sum on the right-hand side, the second and third terms are Type I forms; they map to 0 under K. Thus,

$$
\begin{aligned}
Kd(f\pi^*\omega) &= K\left(\frac{\partial f}{\partial t}\,dt \wedge \pi^*\omega\right) = \left(\int_0^1 \frac{\partial f}{\partial t}\,dt\right)\omega \\
&= (f(x,1) - f(x,0))\omega \\
&= (i_1^* - i_0^*)(f(x,t)\pi^*\omega).
\end{aligned}
$$

Since $dK(f\pi^*\omega) = d(0) = 0$, on Type I forms

$$dK + Kd = i_1^* - i_0^*.$$

On Type II forms, because d is an antiderivation,

$$
\begin{aligned}
dK(f\,dt \wedge \pi^*\omega) &= d\left(\left(\int_0^1 f(x,t)\,dt\right)\omega\right) \\
&= \sum\left(\frac{\partial}{\partial x^i}\int_0^1 f(x,t)\,dt\right)dx^i \wedge \omega + \left(\int_0^1 f(x,t)\,dt\right)d\omega,
\end{aligned}
$$

and

$$
\begin{aligned}
Kd(f\,dt \wedge \pi^*\omega) &= K(d(f\,dt) \wedge \pi^*\omega - (f\,dt) \wedge d\pi^*\omega) \\
&= K\left(\sum_i \frac{\partial f}{\partial x^i}\,\pi^*dx^i \wedge dt \wedge \pi^*\omega\right) - K(f\,dt \wedge \pi^*d\omega) \\
&= -\sum_i\left(\int_0^1 \frac{\partial f}{\partial x^i}\,dt\right)dx^i \wedge \omega - \left(\int_0^1 f(x,t)\,dt\right)d\omega.
\end{aligned}
$$

Since $f(x,t)$ is C^∞, we can differentiate under the integral sign $\int_0^1$. Thus, on Type II forms,

$$dK + Kd = 0.$$

On the other hand,

$$i_1^*(f(x,t)\,dt \wedge \pi^*\omega) = 0$$

because $i_1^*\,dt = di_1^*t = d(1) = 0$. Similarly, i_0^* also vanishes on Type II forms. Therefore,

$$dK + Kd = 0 = i_1^* - i_0^*$$

on Type II forms.

This completes the proof that K is a cochain homotopy between i_0^* and i_1^*. The existence of the cochain homotopy K proves that the induced maps in cohomology $i_0^\#$ and $i_1^\#$ are equal. Therefore,

$$f^\# = i_0^\# \circ F^\# = i_1^\# \circ F^\# = g^\#.$$

Part VIII

Appendices

A

Point-Set Topology

A.1 Topological Spaces

The prototype of a topological space is the Euclidean space $\mathbb{R}^n$. However, the Euclidean space comes with many additional structures, such as a metric, coordinates, an inner product, and an orientation, that are extraneous to its topology. The idea behind the definition of a topological space is to discard all those properties of $\mathbb{R}^n$ that have nothing to do with continuous maps, thereby distilling the notion of continuity to its very essence.

In advanced calculus one learns several characterizations of a continuous map, among which is the following: a map f from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^m$ is continuous if and only if the inverse image $f^{-1}(V)$ of any open set V in $\mathbb{R}^m$ is open in $\mathbb{R}^n$. This shows that continuity can be defined solely in terms of open sets.

To define open sets axiomatically, we look at the properties of open sets in $\mathbb{R}^n$. Recall that in $\mathbb{R}^n$ the *distance* between two points p and q is given by

$$d(p, q) = \left[\sum_{i=1}^{n} (p^i - q^i)^2 \right]^{1/2},$$

and the *open ball* $B(p, r)$ with center $p \in \mathbb{R}^n$ and radius $r > 0$ is the set

$$B(p, r) = \{x \in \mathbb{R}^n \mid d(x, p) < r\}.$$

A set U in $\mathbb{R}^n$ is said to be *open* if for every p in U, there is an open ball $B(p, r)$ with center p and radius r such that $B(p, r) \subset U$. It is clear that the union of an arbitrary collection $\{U_\alpha\}$ of open sets is open, but the same is not true of the intersection of infinitely many open sets.

Example A.1. The intervals $(-1/n, 1/n)$, $n = 1, 2, 3, \ldots$, are all open in $\mathbb{R}^1$, but their intersection $\bigcap_{n=1}^{\infty} (-1/n, 1/n)$ is the singleton set $\{0\}$, which is not open.

What is true is that the intersection of a *finite* collection of open sets in $\mathbb{R}^n$ is open. This leads to the definition of a topology on a set.

Definition A.2. A *topology* on a set S is a collection $\mathcal{T}$ of subsets containing both the empty set $\varnothing$ and the set S such that $\mathcal{T}$ is closed under arbitrary union and finite intersection, i.e., if $U_\alpha \in \mathcal{T}$ for all α in an index set A, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$ and if $\{U_1, \ldots, U_n\} \in \mathcal{T}$, then $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called *open sets* and the pair $(S, \mathcal{T})$is called a *topological space*. To simplify the notation, by a "topological space S" we mean a set S together with a topology $\mathcal{T}$ on S. A*neighborhood* of a point p in S is an open set U containing p.

Example A.3. The open subsets of $\mathbb{R}^n$ as we understand them in advanced calculus form a topology on $\mathbb{R}^n$, the *standard topology* of $\mathbb{R}^n$. In this topology a set U is open in $\mathbb{R}^n$ if and only if for every $p \in U$, there is an open ball $B(p, \epsilon)$ with center p and radius ϵ which is contained in U. Unless stated otherwise, $\mathbb{R}^n$ will always have its standard topology.

Example A.4. For any set S, the collection $\mathcal{T} = \{\varnothing, S\}$ consisting of the empty set $\varnothing$ and the entire set S is a topology on S.

Example A.5. For any set S, let $\mathcal{T}$ be the collection of all subsets of S. Then $\mathcal{T}$ is a topology on S, called the *discrete topology*. The discrete topology can also be characterized as the topology in which every point is open.

The complement of an open set is called a *closed set*. By de Morgan's laws from set theory, arbitrary intersections and finite unions of closed sets are closed (Problem A.3). One may also specify a topology by describing all the closed sets.

Remark A.6. When we say that a topology is *closed* under arbitrary union and finite intersection, the word "closed" has a different meaning from that of a "closed subset."

Example A.7 (*Finite-complement topology on* $\mathbb{R}^1$). Let $\mathcal{T}$ be the collection of subsets of $\mathbb{R}^1$ consisting of the empty set $\varnothing$, the line $\mathbb{R}^1$ itself, and the complements of finite sets. Then $\mathcal{T}$ is closed under arbitrary union and finite intersection and so defines a topology on $\mathbb{R}^1$ called the *finite-complement topology*.

Example A.8. A famous topology in mathematics is the *Zariski topology* in algebraic geometry. Let K be a field and let S be the vector space K^n. Define a subset of K^n to be *closed* if it is the zero set $Z(f_1, \ldots, f_r)$ of finitely many polynomials $f_1, \ldots, f_r$ on K^n. To show that these are indeed the closed subsets of a topology, we need to check that they are closed under arbitrary intersection and finite union.

Let I be the ideal generated by $f_1, \ldots, f_r$ in the polynomial ring $K[x_1, \ldots, x_n]$. Then $Z(f_1, \ldots, f_r) = Z(I)$, the zero set of *all* the polynomials in the ideal I. Conversely, by the Hilbert basis theorem, any ideal in $K[x_1, \ldots, x_n]$ has a finite set of generators. Hence, the zero set of finitely many polynomials is the same as the zero set of an ideal in $K[x_1, \ldots, x_n]$.

Exercise A.9 (Intersection and union of zero sets). Show that

(i)
$$\bigcap_\alpha Z(I_\alpha) = Z\left(\sum_\alpha I_\alpha\right)$$

and

$$\text{(ii)} \qquad Z(\{f_i\}_i) \cup Z(\{g_j\}_j) = Z(\{f_i g_j\}_{i,j}),$$

where $\sum_\alpha I_\alpha$ is the smallest ideal in $K[x_1, \dots, x_n]$ containing all the ideals I_α and i and j run over some finite index sets.

It follows that the complements of the $Z(I)$'s form a topology on K^n, called the *Zariski topology* on K^n. Since the zero set of a polynomial on $\mathbb{R}$ is a finite set, the Zariski topology on $\mathbb{R}$ is precisely the finite-complement topology of Example A.7.

A.2 Subspace Topology

Let $(S, \mathcal{T})$ be a topological space and A a subset of S. Define $\mathcal{T}_A$ to be the collection of subsets

$$\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}.$$

By the distributive property of union and intersection,

$$\bigcup_\alpha (U_\alpha \cap A) = \left(\bigcup_\alpha U_\alpha\right) \cap A$$

and

$$\bigcap_i (U_i \cap A) = \left(\bigcap_i U_i\right) \cap A,$$

which shows that $\mathcal{T}_A$ is closed under arbitrary union and finite intersection. Moreover, $\varnothing, A \in \mathcal{T}_A$. So $\mathcal{T}_A$ is a topology on A, called the *subspace topology* or the *relative topology* of A in S.

Example A.10. Consider the subset $A = [0, 1]$ of $\mathbb{R}$. In the subspace topology, the half-open interval $[0, 1/2)$ is an open subset of A, because

$$[0, \tfrac{1}{2}) = (-\tfrac{1}{2}, \tfrac{1}{2}) \cap A.$$

(See Figure A.1.)

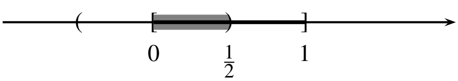


Fig. A.1. An open subset of [0, 1).

A.3 Bases

It is generally difficult to describe directly all the open sets in a topology $\mathcal{T}$. What one can usually do is to describe a subcollection $\mathcal{B}$ of $\mathcal{T}$ so that any open set is expressible as a union of open sets in $\mathcal{B}$. For example, we define open sets in $\mathbb{R}^n$ in terms of open balls.

Definition A.11. A subcollection $\mathcal{B}$ of a topology $\mathcal{T}$ is a *basis* for $\mathcal{T}$ if given any open set U and a point p in U, there is an open set $B \in \mathcal{B}$ such that $p \in B \subset U$. An element of the basis $\mathcal{B}$ is called a *basic open set*.

Example A.12. The collection of all open balls $B(p,r)$ in $\mathbb{R}^n$, with $p \in \mathbb{R}^n$ and $r \in \mathbb{R}$, is a basis for the standard topology of $\mathbb{R}^n$.

Proposition A.13. *A collection $\mathcal{B}$ of open sets of S is a basis if and only if every open set in S is a union of sets in $\mathcal{B}$.*

Proof.
($\Rightarrow$) Suppose $\mathcal{B}$ is a basis and U is an open set in S. For every $p \in U$, there is a basic open set $B_p \in \mathcal{B}$ such that $p \in B_p \subset U$. Therefore, $U = \bigcup_{p \in U} B_p$.

($\Leftarrow$) Suppose every open set in S is a union of open sets in $\mathcal{B}$. Given an open set U and a point p in U, since $U = \bigcup_{B_\alpha \in \mathcal{B}} B_\alpha$, there is a $B_\alpha \in \mathcal{B}$ such that $p \in B_\alpha \subset U$. Hence, $\mathcal{B}$ is a basis. □

The following proposition gives a useful criterion for deciding if a collection $\mathcal{B}$ of subsets is a basis for some topology.

Proposition A.14. *A collection $\mathcal{B}$ of subsets of a set S is a basis for some topology $\mathcal{T}$ on S if and only if*

(i) *S is the union of all the sets in $\mathcal{B}$, and*
(ii) *given any two sets B_1 and $B_2 \in \mathcal{B}$ and $p \in B_1 \cap B_2$, there is a set $B \in \mathcal{B}$ such that $p \in B \subset B_1 \cap B_2$.*

Proof.
($\Rightarrow$) (i) follows from Proposition A.13.

(ii) If $\mathcal{B}$ is a basis, then B_1 and B_2 are open sets and hence so is $B_1 \cap B_2$. By the definition of a basis, there is a $B \in \mathcal{B}$ such that $p \in B \subset B_1 \cap B_2$.

($\Leftarrow$) Define $\mathcal{T}$ to be the collection consisting of all sets that are unions of sets in $\mathcal{B}$. Then the empty set $\varnothing$ and the set S are in $\mathcal{T}$ and $\mathcal{T}$ is clearly closed under arbitrary union. To show that $\mathcal{T}$ is closed under finite intersection, let $U = \bigcup_\mu B_\mu$ and $V = \bigcup_\nu B_\nu$ be in $\mathcal{T}$, where $B_\mu, B_\nu \in \mathcal{B}$. Then

$$\begin{aligned} U \cap V &= \left(\bigcup_\mu B_\mu\right) \cap \left(\bigcup_\nu B_\nu\right) \\ &= \bigcup_{\mu,\nu} (B_\mu \cap B_\nu). \end{aligned}$$

Thus, any p in $U \cap V$ is in $B_\mu \cap B_\nu$ for some μ, ν. By (ii) there is a set B_p in $\mathcal{B}$ such that $p \in B_p \subset B_\mu \cap B_\nu$. Then

$$U \cap V = \bigcup_{p \in U \cap V} B_p \in \mathcal{T}. \qquad \square$$

The topology $\mathcal{T}$ defined in the proof of Proposition A.14 is called the *topology generated by the collection* $\mathcal{B}$.

Proposition A.15. *Let $\mathcal{B} = \{B_\alpha\}$ be a basis for a topological space S, and A a subspace of S. Then $\{B_\alpha \cap A\}$ is a basis for A.*

Proof. Let U' be any open set in A and $p \in U'$. By the definition of subspace topology, $U' = U \cap A$ for some open set U in S. Since $p \in U \cap A \subset U$, there is a basic open set B_α such that $p \in B_\alpha \subset U$. Then

$$p \in B_\alpha \cap A \subset U \cap A = U',$$

which proves that the collection $\{B_\alpha \cap A \mid B_\alpha \in \mathcal{B}\}$ is a basis for A. $\square$

A.4 Second Countability

We say that a point in $\mathbb{R}^n$ is *rational* if all of its coordinates are rational numbers. Let $\mathbb{Q}$ be the set of rational numbers and $\mathbb{Q}^+$ the set of positive rational numbers.

Proposition A.16. *The collection $\mathcal{B}_{\text{rat}}$ of all open balls in $\mathbb{R}^n$ with rational centers and rational radii is a basis for $\mathbb{R}^n$.*

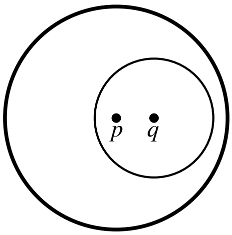


Fig. A.2. A ball with rational center q and rational radius $r/2$.

Proof. Given any open set U and p in U, there is an open ball $B(p,r)$ with $r \in \mathbb{Q}$ such that $p \in B(p,r) \subset U$. Now choose a rational point q in the smaller ball $B(p, r/2)$. We claim that

$$p \in B\left(q, \frac{r}{2}\right) \subset B(p,r). \tag{A.1}$$

(See Figure A.2.) Since $d(p,q) < r/2$, we have $p \in B(q,r/2)$. Next if $x \in B(q,r/2)$, then

$$d(x,p) \le d(x,q) + d(q,p) < \frac{r}{2} + \frac{r}{2} = r.$$

So $x \in B(p,r)$. This proves the claim (A.1) and shows that the collection $\mathcal{B}_{\text{rat}}$ of open balls with rational centers and rational radii is a basis for $\mathbb{R}^n$. □

Both of the sets $\mathbb{Q}$ and $\mathbb{Q}^+$ are countable. Since the centers of the balls in $\mathcal{B}_{\text{rat}}$ are indexed by $\mathbb{Q}^n$, a countable set, and the radii are indexed by $\mathbb{Q}^+$, also a countable set, the collection $\mathcal{B}_{\text{rat}}$ is countable.

Definition A.17. A topological space is said to be *second countable* if it has a countable basis. (See Definition A.58 for first countability.)

Example A.18. Proposition A.16 shows that $\mathbb{R}^n$ is second countable.

Proposition A.19. *A subspace A of a second countable space S is second countable.*

Proof. By Proposition A.15, if $\mathcal{B} = \{B_i\}$ is a countable basis for S, then $\mathcal{B}_A := \{B_i \cap A\}$ is a countable basis for A. □

A.5 Separation Axioms

There are various separation axioms for a topological space. The only ones we will need are the Hausdorff condition and normality.

Definition A.20. A topological space S is *Hausdorff* if given any two distinct points x, y in S, there exist disjoint open sets U, V such that $x \in U$ and $y \in V$. A Hausdorff space is *normal* if given any two disjoint closed sets F, G in S, there exist disjoint open sets U, V such that $F \subset U$ and $G \subset V$ (Figure A.3).

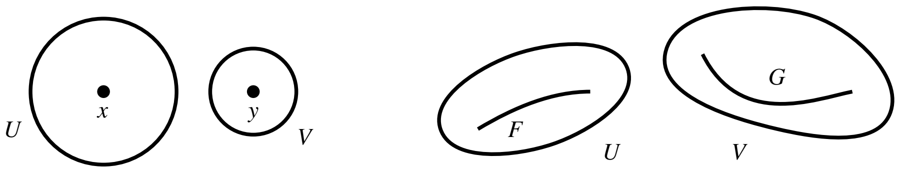


Fig. A.3. The Hausdorff condition and normality.

Proposition A.21. *Every singleton set (a one-point set) in a Hausdorff space S is closed.*

Proof. Let $x \in S$. By the Hausdorff condition, for any $y \neq x$, there exists an open set $U \ni x$ and an open set $V \ni y$ such that U and V are disjoint. In particular,

$$y \in V \subset S - U \subset S - \{x\}.$$

This proves that $S - \{x\}$ is open. Therefore, $\{x\}$ is closed. □

Example A.22. The Euclidean space $\mathbb{R}^n$ is Hausdorff, for given distinct points x, y in $\mathbb{R}^n$, if $\epsilon = \frac{1}{2}d(x, y)$, then the open balls $B(x, \epsilon)$ and $B(y, \epsilon)$ will be disjoint (Figure A.4).

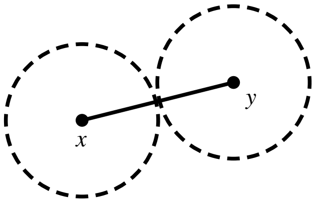


Fig. A.4. Two disjoint neighborhoods in $\mathbb{R}^n$.

Proposition A.23. *Any subspace A of a Hausdorff space S is Hausdorff.*

Proof. Let x and y be distinct points in A. Since S is Hausdorff, there exist disjoint neighborhoods U and V of x and y, respectively, in S. Then $U \cap A$ and $V \cap A$ are disjoint neighborhoods of x and y in A. □

A.6 The Product Topology

The *Cartesian product* of two sets A and B is the set $A \times B$ of all ordered pairs (a, b) with $a \in A$ and $b \in B$. Given two topological spaces X and Y, consider the collection $\mathcal{B}$ of subsets of $X \times Y$ of the form $U \times V$, with U open in X and V open in Y. If $U_1 \times V_1$ and $U_2 \times V_2$ are in $\mathcal{B}$, then

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2),$$

which is also in $\mathcal{B}$ (Figure A.5). From this, it follows easily that $\mathcal{B}$ satisfies the conditions of Proposition A.14 for a basis and generates a topology on $X \times Y$, called the *product topology*. Unless noted otherwise, this will always be the topology on the product of two topological spaces.

Proposition A.24. *Let $\{U_i\}$ and $\{V_j\}$ be bases for the topological spaces X and Y, respectively. Then $\{U_i \times V_j\}$ is a basis for $X \times Y$.*

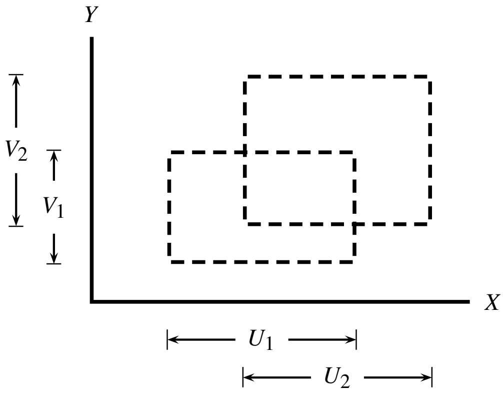


Fig. A.5. Intersection of two basic open subsets in $X \times Y$.

Proof. Given any open set W in $X \times Y$ and a point $(x, y) \in W$, we can find a basic open set $U \times V$ in $X \times Y$ such that $(x, y) \in U \times V \subset W$. Since U is open in X and $\{U_i\}$ is a basis for X,

$$x \in U_i \subset U$$

for some U_i. similarly,

$$y \in V_j \subset V$$

for some V_j. Therefore,

$$(x, y) \in U_i \times V_j \subset U \times V.$$

By the definition of a basis, $\{U_i \times V_j\}$ is a basis for $X \times Y$. □

Corollary A.25. *The product of two second countable spaces is second countable.*

Proposition A.26. *The product of two Hausdorff spaces X and Y is Hausdorff.*

Proof. Given two distinct points (x_1, y_1), (x_2, y_2) in $X \times Y$, without loss of generality we may assume that $x_1 \neq x_2$. Since X is Hausdorff, there exist *disjoint* open sets U_1, U_2 in X such that $x_1 \in U_1$ and $x_2 \in U_2$. Then $U_1 \times Y$ and $U_2 \times Y$ are disjoint neighborhoods of (x_1, y_1) and (x_2, y_2) (Figure A.6). So $X \times Y$ is Hausdorff. □

The product topology can be generalized to the product of an arbitrary collection $\{X_\alpha\}_{\alpha \in A}$ of topological spaces. By definition, the *product topology* on the Cartesian product $\prod_{\alpha \in A} X_\alpha$ is the topology with basis consisting of sets of the form $\prod_{\alpha \in A} U_\alpha$, where U_α is open in X_α and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in A$.

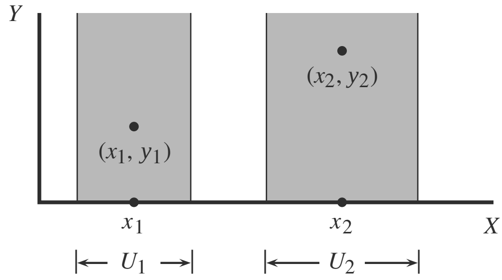


Fig. A.6. Two disjoint neighborhoods in $X \times Y$.

A.7 Continuity

Let $f \colon X \to Y$ be a function of topological spaces. Mimicking the definition from advanced calculus, we say that f is *continuous at a point* p in X if for every neighborhood V of $f(p)$ in Y, there is a neighborhood U of p in X such that $f(U) \subset V$. We say that f is *continuous on* X if it is continuous at every point of X.

Proposition A.27. *A function $f \colon X \to Y$ is continuous if and only if the inverse image of any open set is open.*

Proof.
($\Rightarrow$) Suppose V is open in Y. To show that $f^{-1}(V)$ is open in X, let $p \in f^{-1}(V)$. Then $f(p) \in V$. so there is a neighborhood U of p such that $f(U) \subset V$. Therefore, $p \in U \subset f^{-1}(V)$, which proves that $f^{-1}(V)$ is open in X.

($\Leftarrow$) Let $p \in X$, and V a neighborhood of $f(p)$ in Y. By hypothesis, $f^{-1}(V)$ is open in X. Since $f(p) \in V$, $p \in f^{-1}(V)$. So there is an open set U in X such that $p \in U \subset f^{-1}(V)$. This means $f(U) \subset V$. (In fact, one may take $U = f^{-1}(V)$.)

□

Example A.28. If A is a subspace of X, then the inclusion map

$$i \colon A \to X,$$
$$i(a) = a,$$

is continuous.

Proof. If U is open in X, then $i^{-1}(U) = U \cap A$, which is open in the subspace topology of A. □

Example A.29. The projection $\pi \colon X \times Y \to X$, $\pi(x, y) = x$, is continuous.

Proof. Let U be open in X. Then $\pi^{-1}(U) = U \times Y$, which is open in the product topology of $X \times Y$. □

If A is a subspace of X and $f : X \to Y$ is a function, the *restriction* of f to A,

$$f|_A : A \to Y,$$

is defined by

$$(f|_A)(a) = f(a).$$

Proposition A.30. *The restriction $f|_A$ of a continuous function $f : X \to Y$ to a subspace A is continuous.*

Proof. Let V be open in Y. Then

$$(f|_A)^{-1}(V) = \{a \in A \mid f(a) \in V\} = f^{-1}(V) \cap A.$$

Since f is continuous, $f^{-1}(V)$ is open in X. Hence, $f^{-1}(V) \cap A$ is open in A. By Proposition A.27, $f|_A : A \to Y$ is continuous. □

Continuity may also be phrased in terms of closed sets.

Proposition A.31 (Continuity in terms of closed sets). *A function $f : X \to Y$ is continuous if and only if the inverse image of any closed set is closed.*

Proof. Problem A.6. □

A.8 Compactness

Definition A.32. Let A be a subset of a topological space S. An *open cover of A in S* is a collection $\{U_\alpha\}$ of open sets in S such that $A \subset \bigcup_\alpha U_\alpha$. A *subcover* of an open cover is a subcollection whose union still contains A. The subset A is *compact* in S if every open cover of A in S has a finite subcover.

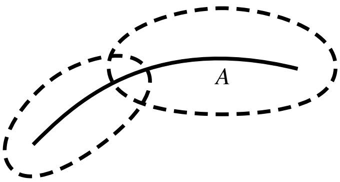


Fig. A.7. An open cover of A.

The subset $A \subset S$ with its subspace topology is a topological space. An *open cover of A in A* is a collection of sets of the form $U_\alpha \cap A$, with U_α open in S, such that

$$A \subset \bigcup_\alpha (U_\alpha \cap A).$$

Thus, we can speak of A being compact in S or in A. The next proposition shows that the two notions of compactness are equivalent.

Proposition A.33. *A subspace A of a topological space S is compact in S if and only if it is compact in A, i.e., in the relative topology on A.*

Proof.
($\Rightarrow$) Let $\{V_\alpha\}$ be an open cover of A by open subsets of A. For each α, since V_α is open in A, there exists an open subset U_α of S such that $V_\alpha = U_\alpha \cap A$. Because

$$A \subset \bigcup_\alpha V_\alpha \subset \bigcup_\alpha U_\alpha,$$

$\{U_\alpha\}$ is an open cover of A in S. By hypothesis, there is a finite subcollection $\{U_{\alpha_i}\}$ such that $A \subset \bigcup_i U_{\alpha_i}$. Then

$$A \subset \left(\bigcup_i U_{\alpha_i}\right) \cap A = \bigcup_i (U_{\alpha_i} \cap A) = \bigcup_i V_{\alpha_i}.$$

So $\{V_{\alpha_i}\}$ is a finite subcover of $\{V_\alpha\}$ that covers A. This proves that A is compact in A.

($\Leftarrow$) Let $\{U_\alpha\}$ be an open cover of A in S. Then $\{U_\alpha \cap A\}$ is an open cover of A in A. By the compactness of A in A, there is a finite subcover $\{U_{\alpha_i} \cap A\}$. Then

$$A \subset \bigcup_i (U_{\alpha_i} \cap A) = \bigcup_i U_{\alpha_i}.$$

Hence, $\{U_{\alpha_i}\}$ is a finite subcover of A in S. This proves that A is compact in S. □

Proposition A.34. *A closed subset F of a compact topological space S is compact.*

Proof. Let $\{U_\alpha\}$ be an open cover of F in S. The collection $\{U_\alpha, S - F\}$ is then an open cover of S. By the compactness of S, there is a finite subcover $\{U_{\alpha_i}, S - F\}$ that covers S. So $F \subset \bigcup_i U_{\alpha_i}$. This proves that F is compact. □

Proposition A.35. *In a Hausdorff space S, it is possible to separate a compact subset K and a point p not in K by disjoint open sets, i.e., there exist an open set $U \supset K$ and an open set $V \ni p$ such that $U \cap V = \varnothing$.*

Proof. By the Hausdorff property, for every $x \in K$, there are disjoint open sets $U_x \ni x$ and $V_x \ni p$. The collection $\{U_x\}_{x \in K}$ is a cover of K by open subsets of S. Since K is compact, it has a finite subcover $\{U_{x_i}\}$.

Let $U = \bigcup_i U_{x_i}$ and $V = \bigcap_i V_{x_i}$. Then U is an open set of S containing K. Being the intersection of finitely many open sets containing p, V is an open set containing p. Moreover, the set

$$U \cap V = \bigcup_i (U_{x_i} \cap V)$$

is empty since each $U_{x_i} \cap V \subset U_{x_i} \cap V_{x_i}$, which is empty. □

Proposition A.36. *Every compact subset K of a Hausdorff space S is closed.*

Proof. By the preceding proposition, for every point p in $S - K$, there is an open set V such that $p \in V \subset S - K$. This proves that $S - K$ is open. Hence, K is closed. □

Exercise A.37 (Compact Hausdorff space). Prove that a compact Hausdorff space is normal. (Normality was defined in Definition A.20.)

Proposition A.38. *The image of a compact set under a continuous map is compact.*

Proof. Let $f\colon X \to Y$ be a continuous map and K a compact subset of X. Suppose $\{U_\alpha\}$ is a cover of $f(K)$ by open subsets of Y. Since f is continuous, the inverse images $f^{-1}(U_\alpha)$ are all open. Moreover,

$$K \subset f^{-1}(f(K)) \subset f^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha f^{-1}(U_\alpha).$$

So $\{f^{-1}(U_\alpha)\}$ is an open cover of K in X. By the compactness of K, there is a finite subcollection $\{f^{-1}(U_{\alpha_i})\}$ such that

$$K \subset \bigcup_i f^{-1}(U_{\alpha_i}) = f^{-1}\left(\bigcup_i U_{\alpha_i}\right).$$

Then $f(K) \subset \bigcup_i U_{\alpha_i}$. Thus, $f(K)$ is compact. □

Recall that a map $f\colon X \to Y$ is said to be *open* if the image of every open set in X is open in Y; similarly, $f\colon X \to Y$ is said to be *closed* if the image of every closed set in X is closed in Y.

Proposition A.39. *A continuous bijection $f\colon X \to Y$ from a compact space X to a Hausdorff space Y is a homeomorphism.*

Proof. It suffices to show that $f^{-1}\colon Y \to X$ is continuous. By Proposition A.31, this is equivalent to $f = (f^{-1})^{-1}\colon X \to Y$ being a closed map. Let F be a closed subset of X. Since X is compact, F is compact (Proposition A.34). By Proposition A.38, $f(F)$ is compact in Y. Since Y is Hausdorff, $f(F)$ is closed (Proposition A.36). This proves that f^{-1} is continuous. □

Exercise A.40 (Finite union of compact sets). Prove that a finite union of compact subsets of a topological space is compact.

We mention without proof an important result. For a proof, see [14, Theorem 26.7, p. 167, and Theorem 37.3, p. 234].

Theorem A.41 (The Tychonoff theorem). *The product of any collection of compact spaces is compact in the product topology.*

A.9 Connectedness

Definition A.42. A topological space S is *disconnected* if it is the union of two disjoint nonempty open subsets $S = U \cup V$ (Figure A.8). It is *connected* if it is not disconnected. A subset A of S is disconnected, if it is disconnected in its subspace topology.

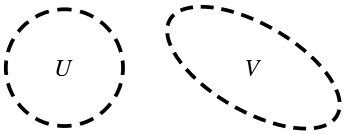


Fig. A.8. A disconnected space.

Proposition A.43. *A subset A of a topological space S is disconnected if and only if there are two open subsets U and V in S such that*

(i) $U \cap A \neq \varnothing$, $V \cap A \neq \varnothing$,
(ii) $U \cap V \cap A = \varnothing$,
(iii) $A \subset U \cup V$.

A pair of open sets in S with these properties is called a separation *of A (Figure* A.9).

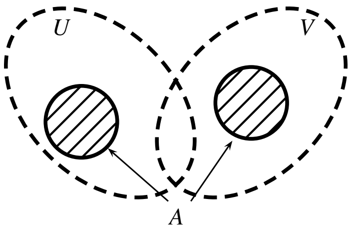


Fig. A.9. A separation of A.

Proof. Problem A.10. □

Proposition A.44. *The image of a connected space X under a continuous map $f : X \to Y$ is connected.*

Proof. Since $f : X \to Y$ is continuous if and only if $f : X \to f(X)$ is continuous, we may assume that $f : X \to Y$ is surjective. Now suppose $Y = f(X)$ is not connected. Then $f(X) = U \cup V$ for disjoint nonempty open subsets U and V. It follows that

$$X = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V).$$

By the continuity of f, $f^{-1}(U)$ and $f^{-1}(V)$ are open. They are clearly nonempty and disjoint, contradicting the connectedness of X. Hence, $f(X)$ must be connected. $\square$

Proposition A.45. *In a topological space S the union of a collection of connected subsets A_α having a point p in common is connected.*

Proof. Suppose $\bigcup_\alpha A_\alpha = U \cup V$, where U and V are disjoint nonempty subsets of $\bigcup_\alpha A_\alpha$. The point p belongs to either U or V. Assume without loss of generality that $p \in U$.

For each α,

$$A_\alpha = A_\alpha \cap (U \cup V) = (A_\alpha \cap U) \cup (A_\alpha \cap V).$$

The two open sets $A_\alpha \cap U$ and $A_\alpha \cap V$ of A_α are clearly disjoint. Since $p \in A_\alpha \cap U$, $A_\alpha \cap U$ is nonempty. By the connectedness of A_α, $A_\alpha \cap V$ must be empty for all α. Hence,

$$V = \left(\bigcup_\alpha A_\alpha\right) \cap V = \bigcup_\alpha (A_\alpha \cap V)$$

is empty, a contradiction. So $\bigcup_\alpha A_\alpha$ must be connected. $\square$

A.10 Connected Components

Definition A.46. In a topological space S, the *connected component* C_x of a point x is the largest connected subset of S containing x.

By Proposition A.45, the connected component of x is the union of all the connected subsets of S containing x.

Remark A.47. For any two points $x, y \in S$, the connected components C_x and C_y are either disjoint or they coincide, for if C_x and C_y have a point p in common, then by Proposition A.45, their union $C_x \cup C_y$ would be a connected set containing both x and y. Hence, $C_x \cup C_y \subset C_x$, from which it follows that $C_x = C_x \cup C_y$. Similarly, $C_y = C_x \cup C_y$.

A *connected component* of S is the connected component of a point in S. By the remark above, the connected components of S partition S into a disjoint subsets.

Proposition A.48. *Suppose C is a connected component of a topological space S. Then a connected subset A of S is either disjoint from C or is contained entirely in C.*

Proof. If A and C have a point p in common, then C is the connected component C_p of p and $A \subset C_p$. $\square$

A.11 Closure

Let S be a topological space and A a subset of S.

Definition A.49. The *closure* of A in S, denoted $\overline{A}$ or $\mathrm{cl}(A)$, is defined to be the intersection of all the closed sets containing A.

As an intersection of closed sets, $\overline{A}$ is a closed set. It is the smallest closed set containing A in the sense that any closed set containing A contains $\overline{A}$.

Definition A.50. A point p in S is an *accumulation point* of A if every neighborhood of p in S contains a point of A other than p. The set of all accumulation points of A is denoted $\mathrm{ac}(A)$.

If U is a neighborhood of p in S, we call $U - \{p\}$ a *deleted neighborhood* of p. An equivalent condition for p to be an accumulation point of A is to require that every deleted neighborhood of p in S contain a point of A. In some books an accumulation point is called a *limit point*.

Example A.51. If $A = [0, 1) \cup \{2\}$ in $\mathbb{R}$, then the set of accumulation points of A is the closed interval $[0, 1]$.

Proposition A.52. *Let A be a subset of a topological space S. Then*

$$\mathrm{cl}(A) = A \cup \mathrm{ac}(A).$$

Proof.
($\supset$) Suppose $p \notin \mathrm{cl}(A)$. Then $p \notin$ some closed set F containing A. So $S - F$ is a neighborhood of p that contains no points of A. Hence, $p \notin \mathrm{ac}(A)$. This proves that $\mathrm{ac}(A) \subset \mathrm{cl}(A)$. By definition, $A \subset \mathrm{cl}(A)$. Therefore, $A \cup \mathrm{ac}(A) \subset \mathrm{cl}(A)$.

($\subset$) Suppose $p \notin A \cup \mathrm{ac}(A)$. Then $p \notin A$ and $p \notin \mathrm{ac}(A)$. Since $p \notin \mathrm{ac}(A)$, it has a neighborhood U that contains no points of A other than p. Since $p \notin A$, in fact U contains no points of A. Therefore, $F := S - U$ is a closed set containing A. Since $p \notin F$, $p \notin \mathrm{cl}(A)$. This proves that $\mathrm{cl}(A) \subset A \cup \mathrm{ac}(A)$. $\square$

Proposition A.53. *A set A is closed if and only if $A = \overline{A}$.*

Proof.
($\Leftarrow$) If $A = \overline{A}$, then A is closed because $\overline{A}$ is closed.

($\Rightarrow$) Suppose A is closed. Then A is a closed set containing A so that $\overline{A} \subset A$. Because $A \subset \overline{A}$, equality holds. $\square$

Proposition A.54. *If $A \subset B$ in a topological space S, then $\overline{A} \subset \overline{B}$.*

Proof. Since $\overline{B}$ contains B, it also contains A. As a closed subset of S containing A, it contains $\overline{A}$ by definition. $\square$

Exercise A.55 (Closure of a finite union or finite intersection). Let A and B be subsets of a topological space S. Prove the following:

(a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(b) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

The example $A = (-\infty, 0)$ and $B = (0, \infty)$ in the real line shows that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

A.12 Convergence

Let S be a topological space. A *sequence* in S is a map from the set $\mathbb{Z}^+$ of positive integers to S. We write a sequence as

$$\langle x_i \rangle \quad \text{or} \quad x_1, x_2, x_3, \ldots .$$

Definition A.56. The sequence $\langle x_i \rangle$ *converges* to p if for every neighborhood U of p, there is a positive integer N such that for all $i \geq N$, $x_i \in U$. In this case we say that p is a *limit* of the sequence $\langle x_i \rangle$ and write $x_i \to p$ or $\lim_{i\to\infty} x_i = p$.

Proposition A.57 (Uniqueness of the limit). *In a Hausdorff space S, if a sequence $\langle x_i \rangle$ converges to p and to q, then $p = q$.*

Proof. Problem A.14. □

Thus, in a Hausdorff space we may speak of *the* limit of a convergent sequence.

Definition A.58. Let S be a topological space and p a point in S. A *basis of neighborhoods at p* is a collection $\mathcal{B} = \{B_\alpha\}$ of neighborhoods of p such that for any neighborhood U of p, there is a $B_\alpha \in \mathcal{B}$ such that $p \in B_\alpha \subset U$. A topological space S is *first countable* if it has a countable basis of neighborhoods at every point $p \in S$.

Example A.59. For $p \in \mathbb{R}^n$, let $B(p, 1/n)$ be the open ball of center p and radius $1/n$ in $\mathbb{R}^n$. Then $\{B(p, 1/n)\}_{n=1}^{\infty}$ is a basis at p. Thus, $\mathbb{R}^n$ is first countable.

Example A.60. An uncountable discrete space is first countable but not second countable.

Proposition A.61 (The sequence lemma). *Let S be a topological space and A a subset of S. If there is a sequence $\langle a_i \rangle$ in A that converges to p, then $p \in \overline{A}$. The converse is true if S is first countable.*

Proof.
($\Rightarrow$) Suppose $a_i \to p$. If $p \in A$, then $p \in \overline{A}$ and there is nothing to prove. So suppose $p \notin A$. By the definition of convergence, every neighborhood U of p contains all but finitely many of the points a_i. In particular, U contains one point of A and this point is not p, since $p \notin A$. Therefore, p is an accumulation point of A. By Proposition A.52, $p \in \operatorname{ac}(A) \subset \overline{A}$.

($\Leftarrow$) Suppose $p \in \overline{A}$. If $p \in A$, then the constant sequence $p, p, p, \ldots$ is a sequence in A that converges to p. So we may assume that $p \notin A$. By Proposition A.52, p is an accumulation point of A. Since S is first countable, we can find a countable basis of neighborhoods $\{U_n\}$ at p such that

$$U_1 \supset U_2 \supset \ldots .$$

In each U_i, choose a point $a_i \in A$. We claim that the sequence $\langle a_i \rangle$ converges to p. If U is any neighborhood of p, then by the definition of a basis of neighborhoods at p, there is a U_N such that $U_N \subset U$. For all $i \geq N$, we then have

$$U_i \subset U_N \subset U,$$

Therefore, for all $i \geq N$,

$$a_i \in U_i \subset U.$$

This proves that $\langle a_i \rangle$ converges to p. $\square$

Problems

A.1. Set theory
If U_1 and U_2 are subsets of a set X, and V_1 and V_2 are subsets of a set Y, prove that

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2).$$

A.2. Union and intersection
Suppose U_i and V_i are disjoint for $i = 1, 2$. Show that the intersection $U_1 \cap U_2$ is disjoint from the union $V_1 \cup V_2$. (*Hint*: Use the distributive property of an intersection over a union.)

A.3. Closed sets
Let S be a topological space. Prove:

(a) If $\{F_i\}_{i=1}^n$ is a finite collection of closed sets in S, then $\bigcup_{i=1}^n F_i$ is closed.
(b) if $\{F_\alpha\}_{\alpha \in A}$ is an arbitrary collection of closed sets in S, then $\bigcap_\alpha F_\alpha$ is closed.

A.4. Projection
A map $f: S \to T$ of topological spaces is said to be *open* if for every open set V in S, the subset $f(V)$ is open in T. Prove that if X and Y are topological spaces, then the projection $\pi : X \times Y \to X$,

$$\pi(x, y) = x,$$

is an open map.

A.5. Closed map
A map $f: S \to T$ of topological spaces is said to be *closed* if for every closed set A in S, the subset $f(A)$ is open in T. Prove that a continuous map from a compact space to a Hausdorff space is closed.

A.6. Continuity in terms of closed sets
Prove Proposition A.31.

A.7. Homeomorphism
Prove that if a continuous bijection $f: S \to T$ is a closed map, then it is a homeomorphism.

A.8.* The Lindelöf condition
Show that if a topological space is second countable, then it is Lindelöf, i.e., every open cover has a countable subcover.

A.9. Compactness
Prove that a finite union of compact sets in a topological space S is compact.

A.10.* Disconnected subset in terms of a separation
Prove Proposition A.43.

A.11. Local connectedness
A topological space S is said to be *locally connected at* $p \in S$ if for every neighborhood U of p, there is a connected neighborhood V of p such that $V \subset U$. The space S is *locally connected* if it is locally connected at every point. Prove that if S is locally connected, then the connected components of S are open.

A.12. Closure
Let U be an open subset and A an arbitrary subset of a topological space S. Prove that $U \cap \bar{A} \neq \varnothing$ iff $U \cap A \neq \varnothing$.

A.13. Countability
Prove that every second countable space is first countable.

A.14.* Uniqueness of the limit
Prove Proposition A.57.

A.15.* Closure in a product
Let S, Y be topological spaces and $A \subset S$. Prove that

$$\mathrm{cl}(A \times Y) = \mathrm{cl}(A) \times Y$$

in the product space $S \times Y$.

B

The Inverse Function Theorem on $\mathbb{R}^n$ and Related Results

This appendix reviews three logically equivalent theorems from real analysis, the inverse function theorem, the implicit function theorem, and the constant rank theorem, which describe the local behavior of a C^∞ map from $\mathbb{R}^n$ to $\mathbb{R}^m$. We will assume the inverse function theorem and deduce the other two, in the simplest cases, from the inverse function theorem. In Chapter 11 these theorems are applied to manifolds in order to clarify the local behavior of a C^∞ map when the rank of the map is maximal at a point or constant in a neighborhood.

B.1 The Inverse Function Theorem

A C^∞ map $f: U \to \mathbb{R}^n$ defined on an open subset U of $\mathbb{R}^n$ is *locally invertible* or a *local diffeomorphism* at a point p in U if f has a C^∞ inverse in some neighborhood of p. The inverse function theorem gives a criterion for a map to be locally invertible. We call the matrix $Jf = [\partial f^i/\partial x^j]$ of partial derivatives of f the *Jacobian matrix* of f and its determinant $\det[\partial f^i/\partial x^j]$ the *Jacobian determinant* of f.

Theorem B.1 (Inverse function theorem). *Let $f: U \to \mathbb{R}^n$ be a C^∞ map defined on an open subset U of $\mathbb{R}^n$. At any point p in U, the map f is invertible in some neighborhood of p if and only if the Jacobian determinant* $\det[\partial f^i/\partial x^j(p)]$ *is not zero.*

This theorem is usually proved in an undergraduate course in real analysis (see, for example, [12, Chapter 7, p. 422]). Although it apparently reduces the invertibility of f on an open set to a single number at p, because the Jacobian determinant is a continuous function, the nonvanishing of the Jacobian determinant at p is equivalent to its nonvanishing in a neighborhood of p.

Since the linear map represented by the Jacobian matrix $Jf(p)$ is the best linear approximation to f at p, it is plausible that f is invertible in a neighborhood of p if and only if $Jf(p)$ is also, i.e., if an only if $\det(Jf(p)) \neq 0$.

B.2 The Implicit Function Theorem

In an equation such as $f(x, y) = 0$, it is often impossible to solve explicitly for one of the variables in terms of the other. The implicit function theorem provides a sufficient condition on a system of equations $f^i(x^1, \ldots, x^n) = 0$, $i = 1, \ldots, m$, under which *locally* a set of variables can be solved implicitly as C^∞ functions of the other variables.

Example B.2. Consider the equation

$$f(x, y) = x^2 + y^2 - 1 = 0.$$

The solution set is the unit circle in the xy-plane.

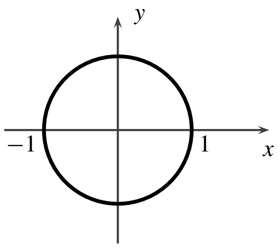


Fig. B.1. The unit circle.

From the picture we see that in a neighborhood of any point other than $(\pm 1, 0)$, y is a function of x. Indeed,

$$y = \pm\sqrt{1 - x^2},$$

and either function is C^∞ as long as $x \neq \pm 1$. At $(\pm 1, 0)$, there is no neighborhood on which y is a function of x.

We will deduce the implicit function theorem from the inverse function theorem for the special case of a C^∞ function f on an open subset of $\mathbb{R}^2$.

Theorem B.3 (Implicit function theorem for $\mathbb{R}^2$). *Let $f\colon U \subset \mathbb{R}^2 \to \mathbb{R}$ be a C^∞ function on an open subset U of $\mathbb{R}^2$. At a point $(a, b) \in U$ where $f(a, b) = 0$ and $\partial f/\partial y\,(a, b) \neq 0$, there are a neighborhood $A \times B$ of (a, b) in U and a unique function $h\colon A \to B$ such that in $A \times B$,*

$$f(x, y) = 0 \quad \textit{iff} \quad y = h(x).$$

Moreover, h is C^∞.

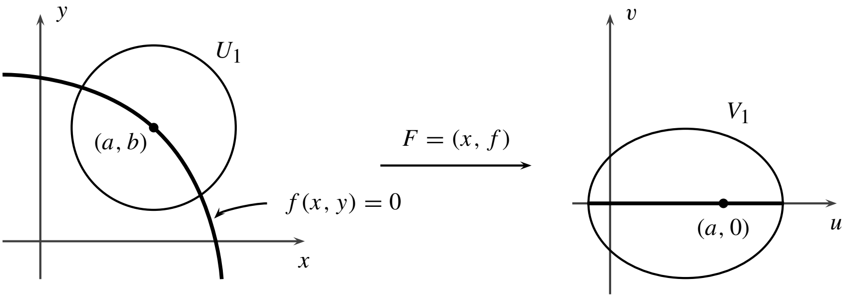


Fig. B.2. F^{-1} maps the u-axis to the zero set of f.

Proof. Define $F\colon U \to \mathbb{R}^2$ by $(u, v) = F(x, y) = (x, f(x, y))$. The Jacobian matrix of F is

$$JF = \begin{bmatrix} 1 & 0 \\ \partial f/\partial x & \partial f/\partial y \end{bmatrix}.$$

At (a, b),

$$\det JF(a, b) = \frac{\partial f}{\partial y}(a, b) \neq 0.$$

By the inverse function theorem, there are neighborhoods U_1 of (a, b) and V_1 of $F(a, b) = (a, 0)$ in $\mathbb{R}^2$ such that $F\colon U_1 \to V_1$ is a diffeomorphism, with C^∞ inverse F^{-1}. Since $F\colon U_1 \to V_1$ is defined by

$$\begin{aligned} u &= x, \\ v &= f(x, y), \end{aligned}$$

the inverse map $F^{-1}\colon V_1 \to U_1$ must be of the form

$$\begin{aligned} x &= u, \\ y &= g(u, v) \end{aligned}$$

for some C^∞ function $g\colon V_1 \to \mathbb{R}$.

The two compositions $F^{-1} \circ F$ and $F \circ F^{-1}$ give

$$y = g(x, f(x, y)) \quad \text{for all } (x, y) \in U_1; \tag{B.1}$$

$$v = f(u, g(u, v)) \quad \text{for all } (u, v) \in V_1. \tag{B.2}$$

For $(u, 0) \in V_1 \cap u$-axis, define

$$h(u) = g(u, 0).$$

Claim. For $(x, y) \in U_1$ and $(x, 0) \in V_1$,

$$f(x, y) = 0 \quad \text{iff} \quad y = h(x).$$

Proof (*of claim*).
($\Rightarrow$) From (B.1), if $f(x, y) = 0$, then

$$y = g(x, f(x, y)) = g(x, 0) = h(x). \tag{B.3}$$

($\Leftarrow$) If $y = h(x)$ and in (B.2) we set $(u, v) = (x, 0)$, then

$$0 = f(x, g(x, 0)) = f(x, h(x)) = f(x, y). \qquad \square$$

Since $(a, b) \in U_1$ and $f(a, b) = 0$, we have

$$F(a, b) = (a, f(a, b)) = (a, 0) \in V_1.$$

By the claim, in some neighborhood of $(a, b) \in U_1$, the zero set of $f(x, y)$ is precisely the graph of h. To find a product neighborhood of (a, b) as in the statement of the theorem, let $A_1 \times B$ be a neighborhood of (a, b) contained in U_1. Since h is continuous, there is a neighborhood A of a such that $A \subset h^{-1}(B) \cap A_1$ and $A \times \{0\} \subset V_1$. Then $h(A) \subset B$,

$$A \times B \subset A_1 \times B \subset U_1, \quad \text{and} \quad A \times \{0\} \subset V_1.$$

By the claim, in $A \times B$,

$$f(x, y) = 0 \quad \text{iff} \quad y = h(x).$$

Equation (B.3) proves the uniqueness of h. Because g is C^∞, so is h. $\square$

Replacing a partial derivative such as $\partial f/\partial y$ with a Jacobian matrix $[\partial f^i/\partial y^j]$, we can prove the general case of the implicit function theorem in exactly the same way.

Theorem B.4 (Implicit function theorem). *Let U be an open set in $\mathbb{R}^n \times \mathbb{R}^m$ and $f\colon U \to \mathbb{R}^m$ a C^∞ function. Write $(x, y) = (x^1, \dots, x^n, y^1, \dots, y^m)$ for a point in U. Suppose $[\partial f^i/\partial y^j\ (a, b)]$ is nonsingular at a point (a, b) in the zero set of f in U. Then a neighborhood $A \times B$ of (a, b) in U and a unique function $h\colon A \to B$ exist such that in $A \times B \subset U \subset \mathbb{R}^n \times \mathbb{R}^m$,*

$$f(x, y) = 0 \quad \textit{iff} \quad y = h(x).$$

Moreover, h is C^∞.

Of course, $y^1, \dots, y^m$ need not be the last m coordinates in $\mathbb{R}^{n+m}$; they can be any set of m coordinates in $\mathbb{R}^{n+m}$.

Theorem B.5. *The implicit function theorem is equivalent to the inverse function theorem.*

Proof (*for $m = 2$ and $n = 2$*). We have already shown, for one typical case, that the inverse function theorem implies the implicit function theorem. We now prove the reverse implication, again for one typical case.

So assume the implicit function theorem, and suppose that $f: U \to \mathbb{R}^2$ is a C^∞ function defined on an open subset U of $\mathbb{R}^2$, $a = (a^1, a^2) \in U$, and the Jacobian determinant $\det[\partial f^i/\partial x^j]$ is nonzero at a. Let

$$z = (z^1, z^2) = (f^1(x^1, x^2), f^2(x^1, x^2)) = f(x).$$

Consider the C^∞ function $F: U \times \mathbb{R}^2 \to \mathbb{R}^2$

$$F(x, z) = f(x) - z.$$

Note that $[\partial F^i/\partial x^j(a, f(a))] = [\partial f^i/\partial x^j\,(a)]$ is nonsingular. By the implicit function theorem, there is a neighborhood $V \times W$ of $(a, f(a))$ in $U \times \mathbb{R}^n$ so that in $V \times W$, $F(x, z) = 0$ implies that x is a C^∞ function of z. This says precisely that the function $z = f(x)$ is invertible for $x \in V$ and $z \in W$. $\square$

B.3 Constant Rank Theorem

Every C^∞ map $f: U \to \mathbb{R}^m$ on an open set U of $\mathbb{R}^n$ has a *rank* at each point p in U, namely the rank of its Jacobian matrix $[\partial f^i/\partial x^j\,(p)]$.

Theorem B.6 (Constant rank theorem). *If $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ has constant rank k in a neighborhood of a point $p \in U$, then after a change of coordinates near p in U and $f(p)$ in $\mathbb{R}^m$, the map f assumes the form*

$$(x^1, \dots, x^n) \mapsto (x^1, \dots, x^k, 0, \dots, 0).$$

More precisely, there are diffeomorphisms G of a neighborhood of p in U and F of a neighborhood of $f(p)$ in $\mathbb{R}^m$ such that

$$F \circ f \circ G^{-1}(x^1, \dots, x^n) = (x^1, \dots, x^k, 0, \dots, 0).$$

Proof (*for $n = m = 2$, $k = 1$*). Suppose $f = (f^1, f^2): U \subset \mathbb{R}^2 \to \mathbb{R}^2$ has constant rank 1 in a neighborhood of $p \in U$. By reordering the functions f^1, f^2 or the variables x, y, we may assume that $\partial f^1/\partial x(p) \neq 0$. (Here we are using the fact that f has rank ≥ 1 at p.) Define $G: U \to \mathbb{R}^2$ by

$$(u, v) = G(x, y) = (f^1(x, y), y).$$

The Jacobian matrix of G is

$$JG = \begin{bmatrix} \partial f^1/\partial x & \partial f^1/\partial y \\ 0 & 1 \end{bmatrix}.$$

Since $\det JG(p) = \partial f^1/\partial x(p) \neq 0$, by the inverse function theorem there are neighborhoods U_1 of $p \in \mathbb{R}^2$ and V_1 of $G(p) \in \mathbb{R}^2$ such that $G: U_1 \to V_1$ is a diffeomorphism. By making U_1 a sufficiently small neighborhood of p, we may assume that f has constant rank 1 on U_1.

On V_1,

$$(u, v) = G \circ G^{-1}(u, v) = (f^1 \circ G^{-1}, y \circ G^{-1})(u, v).$$

Comparing the first components gives $u = f^1 \circ G^{-1}(u, v)$. Hence,

$$\begin{aligned} f \circ G^{-1}(u, v) &= (f^1 \circ G^{-1}, f^2 \circ G^{-1})(u, v) \\ &= (u, f^2 \circ G^{-1}(u, v)) \\ &= (u, h(u, v)), \end{aligned}$$

where we set $h = f^2 \circ G^{-1}$. Because G^{-1} is a diffeomorphism and f has constant rank on V_1, the composite $f \circ G^{-1}$ has constant rank 1 on V_1. Its Jacobian matrix is

$$J(f \circ G^{-1}) = \begin{bmatrix} 1 & 0 \\ \partial h/\partial u & \partial h/\partial v \end{bmatrix}.$$

For this matrix to have constant rank 1, $\partial h/\partial v$ must be identically zero on V_1. (Here we are using the fact that f has rank ≤ 1 in a neighborhood of p). Thus, h is a function of u alone and we may write

$$f \circ G^{-1}(u, v) = (u, h(u)).$$

Finally, let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the change of coordinates $F(x, y) = (x, y - h(x))$. Then

$$(F \circ f \circ G^{-1})(u, v) = F(u, h(u)) = (u, h(u) - h(u)) = (u, 0). \qquad \square$$

Problems

B.1.* The rank of a matrix

The *rank* of a matrix A, denoted rk A, is defined to be the number of linearly independent columns of A. By a theorem in linear algebra, it is also the number of linearly independent rows of A. Prove the following lemma.

Let A be an $m \times n$ matrix (not necessarily square), and k a positive integer. Then rk $A \ge k$ *if and only if A has a nonsingular $k \times k$ submatrix. Equivalently,* rk $A \le k - 1$ *if and only if all $k \times k$ minors of A vanish. (A $k \times k$* minor *of a matrix A is the determinant of a $k \times k$ submatrix of A.)*

B.2.* Matrices of rank at most r

For an integer $r \ge 0$, define D_r to be the subset of $\mathbb{R}^{m\times n}$ consisting of all $m \times n$ real matrices of rank at most r. Show that D_r is a closed subset of $\mathbb{R}^{m\times n}$. (*Hint*: Use Problem B.1.)

B.3.* Maximal rank

We say that the rank of an $m \times n$ matrix A is *maximal* if rk $A = \min(m, n)$. Define $D_{\max}$ to be the subset of $\mathbb{R}^{m\times n}$ consisting of all $m \times n$ matrices of maximal rank $r := \min(m, n)$. Show that $D_{\max}$ is an open subset of $\mathbb{R}^{m\times n}$. (*Hint*: Suppose $n \le m$. Then $D_{\max} = \mathbb{R}^{m\times n} - D_{n-1}$. Apply Problem B.2.)

B.4.* Degeneracy loci and maximal rank locus of a map
Let $F\colon S \to \mathbb{R}^{m\times n}$ be a continuous map from a topological space S to the space $\mathbb{R}^{m\times n}$. The *degeneracy locus of rank* r of F is defined to be

$$D_r(F) := \{x \in S \mid \operatorname{rk} F(x) \le r\}.$$

(a) Show that the degeneracy locus $D_r(F)$ is a closed subset of S. (*Hint*: $D_r(F) = F^{-1}(D_r)$, where D_r was defined in Problem B.2.)
(b) Show that the *maximal rank locus* of F,

$$D_{\max}(F) := \{x \in S \mid \operatorname{rk} F(x) \text{ is maximal}\},$$

is an open subset of S.

B.5. Rank of a composition of linear maps
Suppose V, W, V', W' are finite-dimensional vector spaces.

(a) Prove that if the linear map $L\colon V \to W$ is surjective, then for any linear map $f\colon W \to W'$, $\operatorname{rk}(f \circ L) = \operatorname{rk} f$.
(b) Prove that if the linear map $L\colon V \to W$ is injective, then for any linear map $g\colon V' \to V$, $\operatorname{rk}(L \circ g) = \operatorname{rk} g$.

B.6. Constant rank theorem
Prove that the constant rank theorem (Theorem B.6) implies the inverse function theorem (Theorem B.1). Hence, the two theorems are equivalent.

C

Existence of a Partition of Unity in General

This appendix contains a proof of Theorem 13.10 on the existence of a C^∞ partition of unity on a general manifold.

Lemma C.1. *Every manifold M has a countable basis all of whose elements have compact closure.*

Remark. Recall that a collection of open sets $\mathcal{B} = \{B_\alpha\}$ in a topological space X is a *basis* if, given any open set U in X and any $x \in U$, there is an open set $B_\alpha \in \mathcal{B}$ with $x \in B_\alpha \subset U$.

NOTATION. If A is a subset of a topological space X, the notation $\overline{A}$ denotes the closure of A in X.

Proof (*of Lemma C.1*). Start with a countable basis $\mathcal{B}$ for M and consider the subcollection $\mathcal{S}$ of open sets of $\mathcal{B}$ that have compact closure. We claim that $\mathcal{S}$ is again a basis. Given any open set U in M and any point $p \in U$, choose a neighborhood V of p such that $V \subset U$ and V has compact closure. This is always possible since M is locally Euclidean.

Since $\mathcal{B}$ is a basis, there is an open set $B \in \mathcal{B}$ such that

$$p \in B \subset V \subset U.$$

Then $\overline{B} \subset \overline{V}$. Because $\overline{V}$ is compact, so is the closed subset $\overline{B}$. Hence, $B \in \mathcal{S}$. Given any open set U and any $p \in U$, we have found a set $B \in \mathcal{S}$ such that $p \in B \subset U$. This proves that $\mathcal{S}$ is a basis. □

Proposition C.2. *Every manifold M has a countable sequence of subsets*

$$V_1 \subset \overline{V_1} \subset V_2 \subset \overline{V_2} \subset \dots,$$

with each V_i open and $\overline{V_i}$ compact, such that M is the union of the V_i's (*Figure* C.1).

Proof. By Lemma C.1, M has a countable basis $\{B_i\}_{i=1}^{\infty}$ with each $\overline{B_i}$ compact. Set $V_1 = B_1$, and define i_1 to be the smallest integer ≥ 2 such that

$$\overline{V_1} \subset B_1 \cup B_2 \cup \cdots \cup B_{i_1}.$$

Suppose $V_1, \ldots, V_m$ have been defined. If i_m is the smallest integer $\geq m+1$ and $\geq i_{m-1}$ such that

$$\overline{V_m} \subset B_1 \cup B_2 \cup \cdots \cup B_{i_m},$$

then we set

$$V_{m+1} = B_1 \cup B_2 \cup \cdots \cup B_{i_m}.$$

Since a finite union of compact sets is compact and

$$\overline{V_{m+1}} \subset \overline{B_1} \cup \overline{B_2} \cup \cdots \cup \overline{B_{i_m}}$$

is a closed subset of a compact set, $\overline{V_{m+1}}$ is compact. Since $i_m \geq m+1$, $B_{m+1} \subset V_{m+1}$. Thus,

$$M = \cup B_i \subset \cup V_i \subset M.$$

This proves that $M = \cup_{i=1}^{\infty} V_i$. $\square$

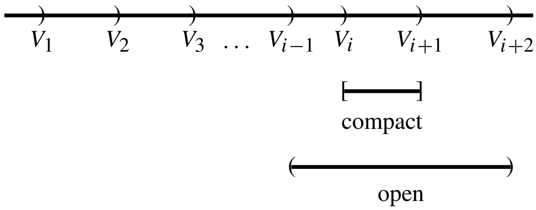


Fig. C.1. A nested open cover.

Define V_0 to be the empty set. For each $i \geq 1$, because $\overline{V_{i+1}} - V_i$ is a closed subset of the compact $\overline{V_{i+1}}$, it is compact. Moreover, it is contained in the open set $V_{i+2} - \overline{V_{i-1}}$.

Theorem 13.10 (Existence of a C^∞ partition of unity). *Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of a manifold M.*

(i) *Then there is a C^∞ partition of unity $\{\varphi_k\}_{k=1}^{\infty}$ with compact support such that for each k,* $\operatorname{supp}\varphi_k \subset U_\alpha$ *for some $\alpha \in A$.*
(ii) *If we do not require compact support, then there is a C^∞ partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$.*

Proof.
(i) Let $\{V_i\}_{i=0}^{\infty}$ be an open cover of M as in Proposition C.2, with V_0 being the empty set. Fix an integer $i \geq 1$. For each $p \in \overline{V_{i+1}} - V_i$, choose an open set U_α containing p from the open cover $\{U_\alpha\}$. Then p is in the open set

$$U_\alpha \cap (V_{i+2} - \overline{V_{i-1}}).$$

Let ψ_p be a C^∞ bump function on M that is positive on a neighborhood W_p of p and has support in $U_\alpha \cap (V_{i+2} - \overline{V_{i-1}})$. Since $\operatorname{supp} \psi_p$ is a closed set contained in the compact set $\overline{V_{i+2}}$, it is compact.

The collection $\{W_p \mid p \in \overline{V_{i+1}} - V_i\}$ is an open cover of the compact set $\overline{V_{i+1}} - V_i$ and so there is a finite subcover $\{W_{p_1}, \dots, W_{p_m}\}$, with associated bump functions $\psi_{p_1}, \dots, \psi_{p_m}$. Since m, W_{p_j}, and ψ_{p_j} all depend on i, we relabel them as $m(i)$, $W_1^i, \dots, W_{m(i)}^i$ and $\psi_1^i, \dots, \psi_{m(i)}^i$.

In summary, for each $i \geq 1$, we have found finitely many open sets $W_1^i, \dots, W_{m(i)}^i$ and finitely many C^∞ bump functions $\psi_1^i, \dots, \psi_{m(i)}^i$ such that

(1) $\psi_j^i > 0$ on W_j^i for $j = 1, \dots, m(i)$;
(2) $W_1^i, \dots, W_{m(i)}^i$ cover $\overline{V_{i+1}} - V_i$;
(3) $\operatorname{supp} \psi_j^i \subset U_{\alpha_{ij}} \cap (V_{i+2} - \overline{V_{i-1}})$ for some $\alpha_{ij} \in A$;
(4) $\operatorname{supp} \psi_j^i$ is compact.

As i runs from 1 to ∞, we obtain countably many bump functions $\{\psi_j^i\}$. The collection of their supports, $\{\operatorname{supp} \psi_j^i\}$ is locally finite, since only finitely many of these sets intersect any V_i. Indeed, since

$$\operatorname{supp} \psi_j^\ell \subset V_{\ell+2} - \overline{V_{\ell-1}}$$

for all ℓ, as soon as $\ell \geq i+1$,

$$(\operatorname{supp} \psi_j^\ell) \cap V_i = \varnothing, \text{ the empty set.}$$

Any point $p \in M$ is contained in $\overline{V_{i+1}} - V_i$ for some i and therefore, $p \in W_j^i$ for some j. For this (i, j), $\psi_j^i(p) > 0$. Hence, the sum $\psi := \sum_{i,j} \psi_j^i$ is locally finite and everywhere positive. To simplify the notation, we now relabel the countable set $\{\psi_j^i\}$ as $\{\psi_1, \psi_2, \psi_3, \dots\}$. Define

$$\varphi_k = \frac{\psi_k}{\psi}.$$

Then $\sum \varphi_k = 1$ and

$$\operatorname{supp} \varphi_k = \operatorname{supp} \psi_k \subset U_\alpha$$

for some $\alpha \in A$. So $\{\varphi_k\}$ is a partition of unity with compact support such that for each k, $\operatorname{supp} \varphi_k \subset U_\alpha$ for some $\alpha \in A$.

(ii) For each $k = 1, 2, \ldots,$ let $\tau(k)$ be an index in A such that

$$\operatorname{supp} \varphi_k \subset U_{\tau(k)}$$

as in the preceding paragraph. Group the collection $\{\varphi_k\}$ according to $\tau(k)$ and define

$$\rho_\alpha = \sum_{\tau(k)=\alpha} \varphi_k$$

if there is a k with $\tau(k) = \alpha$; otherwise, set $\rho_\alpha = 0$. Then

$$\sum_{\alpha \in A} \rho_\alpha = \sum_{\alpha \in A} \sum_{\tau(k)=\alpha} \varphi_k = \sum_{k=1}^{\infty} \varphi_k = 1.$$

By Problem 13.6,

$$\operatorname{supp} \rho_\alpha \subset \bigcup_{\tau(k)=\alpha} \operatorname{supp} \varphi_k \subset U_\alpha.$$

Hence, $\{\rho_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$. $\square$

D

Linear Algebra

This appendix collects a few facts from linear algebra that are used throughout the book, especially in Chapters 23 and 24.

D.1 Linear Transformations

Let V and W be vector spaces over $\mathbb{R}$. A map $f\colon V \to W$ is called a *linear transformation*, a *vector space homomorphism*, a *linear operator*, or a *linear map* over $\mathbb{R}$ if for all $u, v \in V$ and $r \in \mathbb{R}$,

$$\begin{aligned} f(u+v) &= f(u) + f(v), \\ f(ru) &= rf(u). \end{aligned}$$

The *kernel* of f is

$$\ker f = \{v \in V \mid f(v) = 0\}$$

and the *image* of f is

$$\operatorname{im} f = \{f(v) \in W \mid v \in V\}.$$

The kernel of f is a subspace of V and the image of f is a subspace of W. Hence, one can form the quotient spaces $V/\ker f$ and $W/\operatorname{im} f$. This latter space $W/\operatorname{im} f$, denoted $\operatorname{coker} f$, is called the *cokernel* of the linear map $f\colon V \to W$.

For now, denote by K the kernel of f. The linear map $f\colon V \to W$ induces a linear map $\bar{f}\colon V/K \to \operatorname{im} f$, by

$$\bar{f}(v + K) = f(v).$$

It is easy to check that $\bar{f}$ is bijective. This gives the following fundamental result of linear algebra.

Theorem D.1 (The first isomorphism theorem). *Let $f\colon V \to W$ be a homomorphism of vector spaces. Then f induces an isomorphism*

$$\bar{f}\colon \frac{V}{\ker f} \xrightarrow{\sim} \operatorname{im} f.$$

It follows from the first isomorphism theorem that

$$\dim V - \dim \ker f = \dim \operatorname{im} f.$$

Since the dimension is the only isomorphism invariant of a real vector space, we therefore have the following corollary.

Corollary D.2. *Under the hypotheses of the first isomorphism theorem,*

$$V \simeq \ker f \oplus \operatorname{im} f.$$

(The right-hand side is an external direct sum because $\ker f$ and $\operatorname{im} f$ are not subspaces of the same vector space.)

D.2 Quotient Vector Spaces

If V is a vector space and W is a subspace of V, a *coset* of W in V is a subset of the form

$$v + W = \{v + w \mid w \in W\}$$

for some $v \in V$.

Two cosets $v + W$ and $v' + W$ are equal if and only if $v' = v + w$ for some $w \in W$, or equivalently, if and only if $v' - v \in W$. This introduces an equivalence relation on V:

$$v \sim v' \quad \text{iff} \quad v' - v \in W \quad \text{iff} \quad v + W = v' + W.$$

A coset of W in V is simply an equivalence class under this equivalence relation. Any element of $v + W$ is called a *representative* of the coset $v + W$.

The set V/W of all cosets of W in V is again a vector space, with addition and scalar multiplication defined by

$$\begin{aligned}(u + W) + (v + W) &= (u + v) + W,\\ \lambda(v + W) &= \lambda v + W\end{aligned}$$

for $u, v \in V$ and $\lambda \in \mathbb{R}$. We call V/W the *quotient vector space* or *the quotient space of V by W*.

Example D.3. For $V = \mathbb{R}^2$ and W a line through the origin in $\mathbb{R}^2$, a coset of W in $\mathbb{R}^2$ is a line in $\mathbb{R}^2$ parallel to W. (For the purpose of this discussion, two lines are *parallel* if and only if the nonzero vectors they contain are scalar multiples of one another. Accordingly, a line is parallel to itself.) The quotient space $\mathbb{R}^2/W$ is this collection of parallel lines (Figure D.1).

If L is a line through the origin not parallel to W, then L will intersect each parallel line $\in \mathbb{R}^2/W$ in one and only one point. This one-to-one correspondence

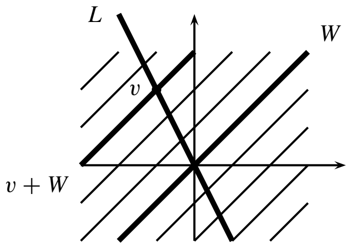


Fig. D.1. Quotient vector space of $\mathbb{R}^2$ by W.

$$\begin{aligned} L &\to \mathbb{R}^2/W \\ v &\mapsto v + W \end{aligned}$$

preserves addition and scalar multiplication, and so is an isomorphism of vector spaces. Thus in this example the quotient space $\mathbb{R}^2/W$ can be identified with the line L.

The *sum* of two subspaces A and B of a vector space V is the subspace

$$A + B = \{a + b \in V \mid a \in A,\ b \in B\}.$$

This sum is called an *internal direct sum* and written $A \oplus B$ if $A \cap B = \{0\}$. In an internal direct sum $A \oplus B$, every element has a representation as $a + b$ for a unique $a \in A$ and a unique $b \in B$. Indeed, if $a + b = a' + b' \in A \oplus B$, then

$$a - a' = b' - b \in A \cap B = \{0\}.$$

Hence, $a = a'$ and $b = b'$.

If $V = A \oplus B$, then A is called *complementary subspace to* B in V. In the example above, the line L is a complementary subspace to W, and we may identify the quotient vector space $\mathbb{R}^2/W$ with any complementary subspace to W.

In general, if W is a subspace of a vector space V and W' is a complementary subspace to W, then there is a linear map

$$\begin{aligned} \varphi\colon W' &\to V/W \\ w' &\mapsto w' + W. \end{aligned}$$

Exercise D.4. Show that $\varphi\colon W' \to V/W$ is an isomorphism of vector spaces.

Thus, the quotient space V/W may be identified with any complementary subspace to W in V. This identification is not canonical, for there are many complementary subspaces to a given subspace W and there is no reason to single out any one of them. However, when V has an inner product, one can single out a canonical complementary subspace, the *orthogonal complement* of W:

$$W^\perp = \{v \in V \mid \langle v, w\rangle = 0 \text{ for all } w \in W\}.$$

Exercise D.5. Check that $W^{\perp}$ is a complementary subspace to W.

In this case, there is a canonical identification $W^{\perp} \xrightarrow{\sim} V/W$.

Remark D.6. If A and B are two vector spaces, not necessarily subspaces of the same vector space, then their *direct product* $A \times B$ is the set of all ordered pairs (a, b):

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\},$$

with the obvious addition and scalar multiplication: for $(a, b), (a', b') \in A \times B$ and $\lambda \in \mathbb{R}$,

$$\begin{aligned}(a, b) + (a', b') &= (a + a', b + b'),\\ \lambda(a, b) &= (\lambda a, \lambda b).\end{aligned}$$

The direct product $A \times B$ is also called the *external direct sum* and written $A \oplus B$.

This means that when A and B are subspaces of the same vector space, the notation $A \oplus B$ is ambiguous; it could be either the internal direct sum or the external direct sum. Fortunately, in this case, there is an isomorphism between the internal direct sum $A \oplus B$ and the direct product $A \times B$, so that the confusion of notations is not serious.

Exercise D.7 (Direct sums). Assume A and B are subspaces of a vector space V. For now, let $A \oplus B$ denote the internal direct sum. Show that the map

$$\begin{aligned}\varphi\colon A \times B &\to A \oplus B\\ (a, b) &\mapsto a + b\end{aligned}$$

is a linear isomorphism of vector spaces.

Solutions to Selected Exercises Within the Text

3.14 The symmetrizing operator
A k-linear function $h\colon V \to \mathbb{R}$ is symmetric iff $\tau h = h$ for all $\tau \in S_k$. Now

$$\tau(Sf) = \tau \sum_{\sigma \in S_k} \sigma f = \sum_{\sigma \in S_k} (\tau\sigma) f.$$

As σ runs over all elements of the permutation groups S_k, so does $\tau\sigma$. Hence,

$$\sum_{\sigma \in S_k} (\tau\sigma) f = \sum_{\tau\sigma \in S_k} (\tau\sigma) f = Sf.$$

This proves that $\tau(Sf) = Sf$. ◇

3.16 The alternating operator
$f(v_1, v_2, v_3) - f(v_1, v_3, v_2) + f(v_2, v_3, v_1) - f(v_2, v_1, v_3) + f(v_3, v_1, v_2) - f(v_3, v_2, v_1)$. ◇

3.21 Wedge product of two 2-covectors

$$\begin{aligned}
&(f \wedge g)(v_1, v_2, v_3, v_4) \\
&\quad = f(v_1, v_2)g(v_3, v_4) - f(v_1, v_3)g(v_2, v_4) + f(v_1, v_4)g(v_2, v_3) \\
&\qquad + f(v_2, v_3)g(v_1, v_4) - f(v_2, v_4)g(v_1, v_3) + f(v_3, v_4)g(v_1, v_2).
\end{aligned}$$ ◇

3.23 The sign of a permutation
We can achieve the permutation τ from the initial configuration $1, 2, \ldots, k + \ell$ in k steps.

(1) First, move the element k to the very end across the ℓ elements $k + 1, \ldots, k + \ell$. This requires ℓ transpositions.
(2) Next, move the element $k - 1$ across the ℓ elements $k + 1, \ldots, k + \ell$.
(3) Then move the element $k - 2$ across the same ℓ elements, and so on.

Each of the k steps requires ℓ transpositions. In the end we achieve τ from the identity using ℓk transpositions.

Alternatively, one can count the number of inversions in the permutation τ. There are k inversions starting with $k+1$, namely, $(k+1, 1), \dots, (k+1, k)$. Indeed, for each $i = 1, \dots, \ell$, there are k inversions starting with $k+i$. Hence, the total number of inversions in τ is $k\ell$. By Proposition 3.6, $\operatorname{sgn}(\tau) = (-1)^{k\ell}$. $\diamond$

4.6 A basis for 3-covectors

By Proposition 3.29, a basis for $A_3(T_p(\mathbb{R}^4))$ is $dx^1 \wedge dx^2 \wedge dx^3$, $dx^1 \wedge dx^2 \wedge dx^4$, $dx^1 \wedge dx^3 \wedge dx^4$, $dx^2 \wedge dx^3 \wedge dx^4$. $\diamond$

4.7 Wedge product of a 2-form with a 1-form

The (2, 1)-shuffles are $(1 < 2, 3)$, $(1 < 3, 2)$, $(2 < 3, 1)$, with respective signs $+$, $-$, $+$. By Equation (3.3),

$$(\omega \wedge \tau)(X, Y, Z) = \omega(X, Y)\tau(Z) - \omega(X, Z)\tau(Y) + \omega(Y, Z)\tau(X). \qquad \diamond$$

7.11 Projective space as a quotient of a sphere

Define $\bar{f} \colon \mathbb{R}P^n \to S^n/{\sim}$ by $\bar{f}([x]) = [\frac{x}{|x|}] \in S^n/{\sim}$. This map is well defined because $\bar{f}([tx]) = [\frac{tx}{|tx|}] = [\pm\frac{x}{|x|}] = [\frac{x}{|x|}]$. Note that if $\pi_1 \colon \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}P^n$ and $\pi_2 \colon S^n \to S^n/{\sim}$ are the projection maps, then there is a commutative diagram

$$\begin{tikzcd}
\mathbb{R}^n - \{0\} \arrow[r, "f"] \arrow[d, "\pi_1"'] & S^n \arrow[d, "\pi_2"] \\
\mathbb{R}P^n \arrow[r, "\bar{f}"'] & S^n/{\sim}.
\end{tikzcd}$$

By Proposition 7.1, $\bar{f}$ is continuous because $\pi_2 \circ f$ is continuous.

Next define $g \colon S^n \to \mathbb{R}^{n+1} - \{0\}$ by $g(x) = x$. This map induces a map $\bar{g} \colon S^n/{\sim} \to \mathbb{R}P^n$, $\bar{g}([x]) = [x]$. By the same argument as above, $\bar{g}$ is well defined and continuous. Moreover,

$$\begin{aligned}
\bar{g} \circ \bar{f}([x]) &= \left[\frac{x}{|x|}\right] = [x], \\
\bar{f} \circ \bar{g}([x]) &= [x],
\end{aligned}$$

so $\bar{f}$ and $\bar{g}$ are inverses to each other. $\diamond$

13.2 Let (V, ϕ) be a chart centered at q so that V is diffeomorphic to an open ball $B(0, r)$. Choose real numbers a and b so that

$$\overline{B}(0, a) \subset B(0, b) \subset \overline{B}(0, b) \subset B(0, r).$$

With the σ given in (13.2), the function $\sigma \circ \phi$, extended by zero to U, is a desired bump function.

18.2 Transition functions for a 2-form

$$\begin{aligned} a_{ij} &= \omega(\partial/\partial x^i, \partial/\partial x^j) = \sum_{k,\ell} b_{k\ell} dy^k \wedge dy^\ell(\partial/\partial x^i, \partial/\partial x^j) \\ &= \sum_{k,\ell} (dy^k(\partial/\partial x^i) dy^\ell(\partial/\partial x^j) - dy^k(\partial/\partial x^j) dy^\ell(\partial/\partial x^i \\ &= \sum_{k,\ell} b_{k\ell} \left(\frac{\partial y^k}{\partial x^i} \frac{\partial y^\ell}{\partial x^j} - \frac{\partial y^k}{\partial x^j} \frac{\partial y^\ell}{\partial x^i} \right). \end{aligned} \qquad \diamond$$

21.2 Smooth functions on a nonopen set

By definition, for each p in S there is an ope set U_p in $\mathbb{R}^n$ and a C^∞ function $\tilde{f}_p : U_p \to \mathbb{R}^n$ such that $f = \tilde{f}_p$ on $U_p \cap S$. Let $U = \bigcup_{p \in S} U_p$. Choose a partition of unity $\{\sigma_p\}_{p \in S}$ on U subordinate to the open cover $\{U_p\}_{p \in S}$ of U and form the function $\tilde{f} : U \to \mathbb{R}^m$ by

$$\tilde{f} = \sum_{p \in S} \sigma_p \tilde{f}_p. \tag{$*$}$$

Since this is a locally finite sum, $\tilde{f}$ is well defined and C^∞ for the usual reason. (Every point $q \in U$ has a neighborhood W_q that intersects finitely many of $\operatorname{supp} \sigma_p$. Hence, the sum $(*)$ is a finite sum on W_q.)

If $q \in S \cap U_p$, then $\tilde{f}_p(q) = f(q)$. Thus, for $q \in S$,

$$\tilde{f}(q) = \sum_{p \in S} \sigma_p(q) \tilde{f}_p(q) = \sum_{p \in S} \sigma_p(q) f_p(q) = f(q). \qquad \diamond$$

23.5 A nowhere-vanishing 1-form on the circle

Although θ is a multi-valued function on the circle, $\cos\theta$ and $\sin\theta$ are well-defined single-valued functions, since any two branches of θ differ by a multiple of 2π. The notation $\cos\theta$ means the function obtained by taking cos of any branch of θ on any open interval of the circle. Thus, on the circle, $x = \cos\theta$ and $y = \sin\theta$. On the open set $U_x = \{x \neq 0\}$,

$$\frac{dy}{x} = \frac{d(\sin\theta)}{\cos\theta} = d\theta.$$

Similarly, on the open set $U_y = \{y \neq 0\}$,

$$-\frac{dx}{y} = -\frac{d(\cos\theta)}{\sin\theta} = d\theta.$$

Hence, $\omega = d\theta$ everywhere on the circle. $\diamond$

24.6 Connecting homomorphism

Suppose $b, b' \in B^k$ both map to c under j. Then $j(b - b') = jb - jb' = c - c = 0$. By the exactness at B^k, $b - b' = i(a'')$ for some $a'' \in A^k$.

With the choice of b, the element $d^*[c]$ is represented by a cocycle $a \in A^{k+1}$ such that $i(a) = db$. Similarly, with the choice of b' the element $d^*[c]$ is represented by a cocycle $a' \in A^{k+1}$ such that $i(a') = d(b')$. Then $i(a - a') = d(b - b') = di(a'') = id(a'')$. Since i is injective, $a - a' = d(a'')$, and thus $[a] = [a']$. This proves that $d^*[c]$ is independent of the choice of b. We summarize the proof by the commutative diagram

$$\begin{tikzcd}
d(a'') = a - a' \arrow[r, "i"] & db - d(b') & \\
a'' \arrow[u, "d"] \arrow[r, "i"'] & b - b' \arrow[u, "d"'] \arrow[r] & 0.
\end{tikzcd}$$

Next suppose $[c] = [c'] \in H^k(\mathcal{C})$. Then $c - c' = d(c'')$ for some $c'' \in C^{k-1}$. By the surjectivity of $j\colon B^{k-1} \to C^{k-1}$, there is a $b'' \in B^{k-1}$ such that $j(b'') = c''$. Choose $b \in B^k$ such that $j(b) = c$ and choose $b' \in B^k$ such that $b - b' = d(b'')$. Then $j(b') = j(b) - jd(b'') = c - dj(b'') = c - d(c'') = c'$. With the choice of b, the element $d^*[c]$ is represented by a cocycle $a \in A^{k+1}$ such that $i(a) = db$. With the choice of b', the element $d^*[c]$ is represented by a cocycle $a' \in A^{k+1}$ such that $i(a') = d(b')$. Then

$$i(a - a') = d(b - b') = dd(b'') = 0.$$

By the injectivity of i, $a = a'$ and $[a] = [a']$. This shows that $d^*[c]$ is independent of the choice of c in the cohomology class $[c]$. See the commutative diagram below:

$$\begin{tikzcd}
a - a' \arrow[r] & db - d(b') = 0 & & \\
& b - b' \arrow[u, "d"] \arrow[r, "j"] & c - c' & \\
& b'' \arrow[u, "d"] \arrow[r, "j"'] & c'' \arrow[u, "d"'] \arrow[r] & 0.
\end{tikzcd}$$ ◇

A.37 Compact Hausdorff space

Let S be a compact Hausdorff space, and A, B two closed subsets of S. By Proposition A.34, A and B are compact. By Proposition A.35, for any $a \in A$ there are disjoint open sets $U_a \ni a$ and $V_a \supset B$. Since A is compact, the open cover $\{U_a\}_{a \in A}$ for A has a finite subcover $\{U_{a_i}\}_{i=1}^n$. Let $U = \bigcup_{i=1}^n U_{a_i}$ and $V = \bigcap_{i=1}^n V_{a_i}$. Then $A \subset U$ and $B \subset V$. The open sets U and V are disjoint because if $x \in U \cap V$, then $x \in U_{a_i}$ for some i and $x \in V_{a_i}$ for the same i, contradicting the fact that $U_{a_i} \cap V_{a_i} = \varnothing$. ◇

Hints and Solutions to Selected End-of-Chapter Problems

Problems with complete solutions are starred (*). Equations are numbered consecutively within each problem.

1.1 Let $h(x) = \int_0^x g(t)\,dt$, where $g\colon \mathbb{R} \to \mathbb{R}$ is the function in Example 1.2.

1.2* A C^∞ function very flat at 0
(a) Assume $x > 0$. For $k = 1$, $f'(x) = (1/x^2)\,e^{-1/x}$. With $p_2(y) = y^2$, this verifies the claim. Now suppose $f^{(k)}(x) = p_{2k}(1/x)\,e^{-1/x}$. By the product rule and the chain rule,

$$\begin{aligned}
f^{(k+1)}(x) &= p_{2k-1}\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right)e^{-\frac{1}{x}} + p_{2k}\left(\frac{1}{x}\right)\cdot\frac{1}{x^2}e^{-\frac{1}{x}}\\
&= \left(q_{2k+1}\left(\frac{1}{x}\right) + q_{2k+2}\left(\frac{1}{x}\right)\right)e^{-\frac{1}{x}}\\
&= p_{2k+2}\left(\frac{1}{x}\right)e^{-\frac{1}{x}},
\end{aligned}$$

where $q_n(y)$ and $p_n(y)$ are polynomials of degree n in y. By induction, the claim is true for all $k \geq 1$. It is trivially true for $k = 0$ also.

(b) For $x > 0$, the formula in (a) shows that $f(x)$ is C^∞. For $x < 0$, $f(x) \equiv 0$, which is trivially C^∞. It remains to show that $f^{(k)}(x)$ is defined and continuous at $x = 0$ for all k.

Suppose $f^{(k)}(0) = 0$. By the definition of the derivative,

$$f^{(k+1)}(0) = \lim_{x\to 0}\frac{f^{(k)}(x) - f^{(k)}(0)}{x} = \lim_{x\to 0}\frac{f^{(k)}(x)}{x}.$$

The limit from the left is clearly 0. So it suffices to compute the limit from the right:

$$\lim_{x \to 0^+} \frac{f^{(k)}(x)}{x} = \lim_{x \to 0^+} \frac{p_{2k}\left(\frac{1}{x}\right) e^{-\frac{1}{x}}}{x} = \lim_{x \to 0^+} p_{2k+1}\left(\frac{1}{x}\right) e^{-\frac{1}{x}} \tag{1.2.1}$$
$$= \lim_{y \to \infty} \frac{p_{2k+1}(y)}{e^y} \quad \left(\text{replacing } \frac{1}{x} \text{ by } y\right).$$

Applying l'Hôpital's rule $2k+1$ times, we reduce this limit to 0. Hence, $f^{(k+1)}(0) = 0$. By induction, $f^{(k)}(0) = 0$ for all $k \geq 0$.

A similar computation as (1.2.1) for $\lim_{x \to 0} f^{(k)}(x) = 0$ proves that $f^{(k)}(x)$ is continuous at $x = 0$. ◇

1.3 (b) $h(t) = (\pi/(b-a))(t-a) - (\pi/2)$.

1.4

(a) Let $f(x)$ be the function of Example 1.3. Then $h(x) = f(x)g(x)$. Since both $f(x)$ and $g(x) = \sec x$ are strictly increasing on $[0, \pi/2)$ and C^∞ on $(-\pi/2, \pi/2)$, so is their product. We have already established that the derivatives $f^{(k)}(0) = 0$ for all $k \geq 0$. By the product rule,

$$h^{(k)}(0) = \sum_{i=0}^{k} \binom{k}{i} f^{(i)}(0) g^{(k-i)}(0) = 0.$$

(b) The inverse map to F is $G\colon \mathbb{R}^n \to B(0, \pi/2)$,

$$G(y) = \begin{cases} h^{-1}(|y|)\dfrac{y}{|y|} & \text{for } y \neq 0, \\ 0 & \text{for } y = 0. \end{cases}$$

1.5* Taylor's theorem with remainder to order 2

To simplify the notation, we write 0 for (0, 0). By Taylor's theorem with remainder, there exist C^∞ functions g_1, g_2 such that

$$f(x, y) = f(0) + x g_1(x, y) + y g_2(x, y). \tag{1.5.1}$$

Applying the theorem again, but to g_1 and g_2, we obtain

$$g_1(x, y) = g_1(0) + x g_{11}(x, y) + y g_{12}(x, y), \tag{1.5.2}$$
$$g_2(x, y) = g_2(0) + x g_{21}(x, y) + y g_{22}(x, y). \tag{1.5.3}$$

Since $g_1(0) = \partial f/\partial x(0)$ and $g_2(0) = \partial f/\partial y(0)$, substituting (1.5.2) and (1.5.3) into (1.5.1) gives the result. ◇

1.6* A function with a removable singularity

In Problem 1.5, set $x = t$ and $y = tu$. We obtain

$$f(t, tu) = f(0) + t\frac{\partial f}{\partial x}(0) + tu\frac{\partial f}{\partial y}(0) + t^2(\cdots),$$

where

$$(\cdots) = f_{11}(t, tu) + u f_{12}(t, tu) + u^2 f_{22}(t, tu)$$

is a C^∞ function of t and u. Since $f(0) = 0$,

$$\frac{f(t, tu)}{t} = \frac{\partial f}{\partial x}(0) + u\frac{\partial f}{\partial y}(0) + t(\cdots),$$

which is clearly C^∞. ◇

3.1 $f = \sum g_{ij}\alpha^i \otimes \alpha^j$.

3.2

(a) Use the formula $\dim\ker f + \dim\operatorname{im} f = \dim V$.

(b) Choose a basis $e_1, \dots, e_{n-1}$ for $\ker f$, and extend it to a basis $e_1, \dots, e_{n-1}, e_n$ for V. Let $\alpha^1, \dots, \alpha^n$ be the dual basis for V^*. Write both f and g in terms of this dual basis.

3.3 We write temporarily α^I for $\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}$ and e_J for $(e_{j_1}, \dots, e_{j_k})$.

(a) Prove that $f = \sum f(e_I)\alpha^I$ by showing that both sides agree on all (e_J). This proves that the set $\{\alpha^I\}$ spans.

(b) Suppose $\sum c_I \alpha^I = 0$. Applying both sides to e_J gives $c_J = \sum c_I \alpha^I(e_J) = 0$. This proves that the set $\{\alpha^I\}$ is linearly independent.

3.9* Linear independence of covectors

($\Rightarrow$) If $\alpha^1, \dots, \alpha^k$ are linearly dependent, then one of them is a linear combination of the others. Without loss of generality, we may assume that

$$\alpha^k = \sum_{i=1}^{k-1} c_i \alpha^i.$$

In the wedge product $\alpha^1 \wedge \cdots \wedge \alpha^{k-1} \wedge (\sum_{i=1}^{k-1} c_i\alpha^i)$, every term has a repeated α^i. Hence, $\alpha^1 \wedge \cdots \wedge \alpha^k = 0$.

($\Leftarrow$) Suppose $\alpha^1, \dots, \alpha^k$ are linearly independent. Then they can be extended to a basis $\alpha^1, \dots, \alpha^k, \dots, \alpha^n$ for V^*. Let $v_1, \dots, v_n$ be the dual basis for V. By Proposition 3.28,

$$(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \dots, v_k) = \det[\alpha^i(v_j)] = \det[\delta^i_j] = 1.$$

Hence, $\alpha^1 \wedge \cdots \wedge \alpha^k \neq 0$. ◇

3.10* Exterior multiplication

($\Leftarrow$) Clear because $\alpha \wedge \alpha = 0$.

($\Rightarrow$) Suppose $\alpha \wedge \omega = 0$. Extend α to a basis $\alpha^1, \dots, \alpha^n$ for V^*, with $\alpha^1 = \alpha$. Write $\omega = \sum c_J \alpha^J$. In the sum $\alpha \wedge \omega = \sum c_J \alpha \wedge \alpha^J$, all the terms $\alpha \wedge \alpha^J$ with $j_1 = 1$ vanish since $\alpha = \alpha^1$. Hence,

$$0 = \alpha \wedge \omega = \sum_{j_1 \neq 1} c_J \alpha \wedge \alpha^J.$$

Since $\{\alpha \wedge \alpha^J\}_{j_1 \neq 1}$ is a subset of a basis for $A_{k+1}(V)$, it is linearly independent and so all $c_J = 0$ if $j_1 \neq 1$. Thus,

$$\omega = \sum_{j_1=1} c_J \alpha^J = \alpha \wedge \left(\sum_{j_1=1} c_J \alpha^{j_2} \wedge \cdots \wedge \alpha^{j_k} \right). \qquad \diamond$$

3.11 Let $e_1, \ldots, e_n$ be a basis for V and $\alpha^1, \ldots, \alpha^n$ the dual basis for V^*. Then a basis for $A_n(V)$ is $\alpha^1 \wedge \cdots \wedge \alpha^n$ and $L^*(\alpha^1 \wedge \cdots \wedge \alpha^n) = c\alpha^1 \wedge \cdots \wedge \alpha^n$ for some constant c. Suppose $L(e_j) = \sum_i a^i_j e_i$. Compute c in terms of a^i_j.

4.1 $\omega(X) = yz$, $d\omega = -dx \wedge dz$.

4.2 $\omega_p = p^3\, dx^1 \wedge dx^2$.

4.3 $dx = \cos\theta\, dr - r\sin\theta\, d\theta$, $dy = \sin\theta,\, dr + r\cos\theta\, d\theta$, $dx \wedge dy = r\, dr \wedge d\theta$.

4.4 $dx \wedge dy \wedge dz = \rho^2 \sin\phi\, d\rho \wedge d\phi \wedge d\theta$.

4.5 $\alpha \wedge \beta = (a_1 b_1 - a_2 b_2 + a_3 b_3)\, dx^1 \wedge dx^2 \wedge dx^3$.

4.7* Interior multiplication

$$\begin{aligned}
&(\iota_v(\alpha^1 \wedge \cdots \wedge \alpha^k))(v_2, \ldots, v_k) \\
&\quad = \alpha^1 \wedge \cdots \wedge \alpha^k(v, v_2, \ldots, v_k) \\
&\quad = \det \begin{bmatrix} \alpha^1(v) & \alpha^1(v_2) & \cdots & \alpha^1(v_k) \\ \alpha^2(v) & \alpha^2(v_2) & \cdots & \alpha^2(v_k) \\ \vdots & \vdots & & \vdots \\ \alpha^k(v) & \alpha^k(v_2) & \cdots & \alpha^k(v_k) \end{bmatrix} && \text{(Proposition 3.28)} \\
&\quad = \sum_{i=1}^k (-1)^{i+1} \alpha^i(v) \det[\alpha^\ell(v_j)]_{\substack{1 \le \ell \le k, \ell \neq i \\ 2 \le j \le k}} && \text{(expansion along first column)} \\
&\quad = \sum_{i=1}^k (-1)^{i+1} \alpha^i(v) \alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k(v_2, \ldots, v_k). && \diamond
\end{aligned}$$

4.8* Interior multiplication

(a) By the definition of interior multiplication,

$$\begin{aligned}
(\iota_v \circ \iota_v \omega)(v_3, \ldots, v_k) &= (\iota_v \omega)(v, v_3, \ldots, v_k) \\
&= \omega(v, v, v_3, \ldots, v_k) = 0,
\end{aligned}$$

because of the repeated variable v.

(b) Since both sides of the equation are linear in ω and linear in τ, we may assume that

$$\omega = \alpha^1 \wedge \cdots \wedge \alpha^k, \quad \tau = \alpha^{k+1} \wedge \cdots \wedge \alpha^{k+\ell},$$

where the α^i are all 1-covectors. By Problem 4.7,

$$
\begin{aligned}
\iota_v(\omega \wedge \tau) &= \iota_v(\alpha^1 \wedge \cdots \wedge \alpha^{k+\ell}) \\
&= \left(\sum_{i=1}^{k} (-1)^{i+1} \alpha^i(v) \alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k \right) \\
&\quad \wedge \alpha^{k+1} \wedge \cdots \wedge \alpha^{k+\ell} + (-1)^k \alpha^1 \wedge \cdots \wedge \alpha^k \\
&\quad \wedge \sum_{i=1}^{k} (-1)^{i+1} \alpha^{k+i}(v) \alpha^{k+1} \wedge \cdots \wedge \widehat{\alpha^{k+i}} \wedge \cdots \wedge \alpha^{k+\ell} \\
&= (\iota_v \omega) \wedge \tau + (-1)^k \omega \wedge \iota_v \tau. \qquad \diamond
\end{aligned}
$$

5.3 The image $\phi_4(U_{14}) = \{(x, z) \mid -1 < z < 1,\ 0 < x < \sqrt{1 - z^2}\}$. The transition function $\phi_1 \circ \phi_4^{-1}(x, z) = \phi_1(x, y, z) = (y, z) = (-\sqrt{1 - x^2 - z^2}, z)$ is a C^∞ function of x, z.

6.4* Coordinate maps are C^∞
For any $p \in U$, choose the charts (U, ϕ) about p and $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ about $\phi(p)$. Since $1_{\mathbb{R}^n} \circ \phi \circ \phi^{-1} \colon \phi(U) \to \mathbb{R}^n$ is C^∞ at $\phi(p)$, ϕ is C^∞ at p. $\diamond$

6.7 See Example 15.2.

7.1* Quotient space by a group
Let U be an open subset of S. For each $g \in G$, since left multiplication by g is a homeomorphism: $S \to S$, the set gU is open. But

$$\pi^{-1}(\pi(U)) = \cup_{g \in G}\, gU,$$

which is a union of open sets, hence is open. By the definition of the quotient topology, $\pi(U)$ is open. $\diamond$

7.3* The real projective space
By Exercise 7.11 there is a continuous surjective map $\pi\colon S^n \to \mathbb{R}P^n$. Since the sphere S^n is compact, and the continuous image of a compact set is compact (Proposition A.38), $\mathbb{R}P^n$ is compact. $\diamond$

8.1* Differential of a map
To determine the coefficient a in $F_*(\partial/\partial x) = a\, \partial/\partial u + b\, \partial/\partial v + c\, \partial/\partial w$, we apply both sides to u to get

$$F_*\left(\frac{\partial}{\partial x}\right) u = \left(a \frac{\partial}{\partial u} + b \frac{\partial}{\partial v} + c \frac{\partial}{\partial w}\right) u = a.$$

Hence,

$$a = F_*\left(\frac{\partial}{\partial x}\right) u = \frac{\partial}{\partial x}(u \circ F) = \frac{\partial}{\partial x}(x) = 1.$$

Similarly,

$$b = F_*\left(\frac{\partial}{\partial x}\right) v = \frac{\partial}{\partial x}(v \circ F) = \frac{\partial}{\partial x}(y) = 0$$

and

$$c = F_*\left(\frac{\partial}{\partial x}\right) w = \frac{\partial}{\partial x}(w \circ F) = \frac{\partial}{\partial x}(xy) = y.$$

So $F_*(\partial/\partial x) = \partial/\partial u + y\,\partial/\partial w$. ◇

8.4* Velocity of a curve in local coordinates

We know that $c'(t) = \sum a^j \partial/\partial x^j$. To compute a^i, evaluate both sides on x^i:

$$a^i = \left(\sum a^j \frac{\partial}{\partial x^j}\right) x^i = c'(t)x^i = c_*\left(\frac{d}{dt}\right) x^i = \frac{d}{dt}(x^i \circ c) = \frac{d}{dt}c^i = (c^i)'(t).$$

◇

8.7* Tangent space to a product

If $(U, x^1, \ldots, x^m)$ and $(V, y^1, \ldots, y^n)$ are charts centered at p in M and q in N, respectively, then by Proposition 5.17, $(U \times V, x^1, \ldots, x^m, y^1, \ldots, y^n)$ is a chart centered at (p, q) in $M \times N$. In local coordinates the projection maps are

$$\begin{aligned} \pi_1(x^1, \ldots, x^m, y^1, \ldots, y^n) &= (x^1, \ldots, x^m) \\ \pi_2(x^1, \ldots, x^m, y^1, \ldots, y^n) &= (y^1, \ldots, y^n). \end{aligned}$$

If $\pi_{1*}\left(\partial/\partial x^j\right) = \sum a^i_j\, \partial/\partial x^i$, then

$$a^i_j = \pi_{1*}\left(\frac{\partial}{\partial x^j}\right)(x^i) = \frac{\partial}{\partial x^j}\left(x^i \circ \pi_1\right) = \frac{\partial}{\partial x^j} x^i = \delta^i_j.$$

Hence,

$$\pi_{1*}\left(\frac{\partial}{\partial x^j}\right) = \sum_i \delta^i_j \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^j}.$$

This really means that

$$\pi_{1*}\left(\left.\frac{\partial}{\partial x^j}\right|_{(p,q)}\right) = \left.\frac{\partial}{\partial x^j}\right|_p. \tag{8.7.1}$$

Similarly,

$$\pi_{1*}\left(\frac{\partial}{\partial y^j}\right) = 0, \quad \pi_{2*}\left(\frac{\partial}{\partial x^j}\right) = 0, \quad \pi_{2*}\left(\frac{\partial}{\partial y^j}\right) = \frac{\partial}{\partial y^j}. \tag{8.7.2}$$

A basis for $T_{(p,q)}(M \times N)$ is

$$\left.\frac{\partial}{\partial x^1}\right|_{(p,q)}, \ldots, \left.\frac{\partial}{\partial x^m}\right|_{(p,q)}, \left.\frac{\partial}{\partial y^1}\right|_{(p,q)}, \ldots, \left.\frac{\partial}{\partial y^n}\right|_{(p,q)}.$$

A basis for $T_pM \times T_qN$ is

$$\left(\left.\frac{\partial}{\partial x^1}\right|_{(p,q)}, 0\right), \ldots, \left(\left.\frac{\partial}{\partial x^m}\right|_{(p,q)}, 0\right), \left(0, \left.\frac{\partial}{\partial y^1}\right|_{(p,q)}\right), \ldots, \left(0, \left.\frac{\partial}{\partial y^n}\right|_{(p,q)}\right).$$

It follows from (8.7.1) and (8.7.2) that the linear map $\pi_{1*} \times \pi_{2*}$ maps a basis of $T_{(p,q)}(M \times N)$ to a basis of $T_pM \times T_qN$ and is therefore an isomorphism. $\diamond$

8.8 (a) Let $c(t)$ be a curve starting at e in G with $c'(0) = X_e$. Then $\alpha(t) = (c(t), e)$ is a curve starting at (e, e) in $G \times G$ with $\alpha'(0) = (X_e, 0)$. Compute $\mu_{*,(e,e)}$ using $\alpha(t)$.

8.9* Transforming vectors to coordinate vectors
Let $(V, y^1, \dots, y^n)$ be a chart about p. Suppose $(X_j)_p = \sum_i a^i_j \, \partial/\partial y^i|_p$. Since $(X_1)_p, \dots, (X_n)_p$ are linearly independent, the matrix $A = [a^i_j]$ is nonsingular.

Define a new coordinate system $x^1, \dots, x^n$ by

$$y^i = \sum_{j=1}^{n} a^i_j \, x^j \quad \text{for } i = 1, \dots, n. \tag{8.9.1}$$

By the chain rule,

$$\frac{\partial}{\partial x^j} = \sum_i \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i} = \sum a^i_j \frac{\partial}{\partial y^i}.$$

At the point p,

$$\left.\frac{\partial}{\partial x^j}\right|_p = \sum a^i_j \left.\frac{\partial}{\partial y^i}\right|_p = (X_i)_p.$$

In matrix notation, $[y^1 \cdots y^n] = [x^1 \cdots x^n]A$, so $[x^1 \cdots x^n] = [y^1 \cdots y^n]A^{-1}$. This means that (8.9.1) is equivalent to $x^j = \sum_{i=1}^n \left(A^{-1}\right)^j_i \, y^i$. $\diamond$

9.1 $c \in \mathbb{R} - \{0, -108\}$.

9.2 Yes, because it is a regular level set of the function $f(x, y, z, w) = x^5 + y^5 + z^5 + w^5$.

9.3 Yes, see Example 9.15.

9.4* Regular submanifolds
Let $p \in S$. By hypothesis there is an open set U in $\mathbb{R}^2$ so that on $U \cap S$ one of the coordinates is a C^∞ function of the other. Without loss of generality, we assume that $y = f(x)$ for some C^∞ function $f\colon A \subset \mathbb{R} \to B \subset \mathbb{R}$, where A and B are open sets in $\mathbb{R}$ and $V := A \times B \subset U$. Let $F\colon V \to \mathbb{R}^2$ be given by $F(x, y) = (x, y - f(x))$. Since F is a diffeomorphism onto its image, it can be used as a coordinate map. In the chart $(V, x, y - f(x))$, $V \cap S$ is defined by the vanishing of the coordinate $y - f(x)$. This proves that S is a regular submanifold of $\mathbb{R}^2$. $\diamond$

9.5 $(\mathbb{R}^3, x, y, z - f(x, y))$ is an adapted chart for $\mathbb{R}^3$ relative to $\Gamma(f)$.

9.6 Differentiate (9.3) with respect to t.

9.9* The transversality theorem

(a) $f^{-1}(U) \cap f^{-1}(S) = f^{-1}(U \cap S) = f^{-1}(g^{-1}(0)) = (g \circ f)^{-1}(0)$.

(b) Let $p \in f^{-1}(U) \cap f^{-1}(S) = f^{-1}(U \cap S)$. Then $f(p) \in U \cap S$. Because S is a fiber of g, the pushforward $g_*(T_{f(p)}S) = 0$. Because $g\colon U \to \mathbb{R}^k$ is a projection, $g_*(T_{f(p)}M) = T_0(\mathbb{R}^k)$. Applying g_* to the transversality equation (9.4), we get
$$g_* f_*(T_p N) = g_*(T_{f(p)}M) = T_0(\mathbb{R}^k).$$
Hence, $g \circ f\colon f^{-1}(U) \to \mathbb{R}^k$ is a submersion at p. Since p is an arbitrary point of $f^{-1}(U) \cap f^{-1}(S) = (g \circ f)^{-1}(0)$, this set is a regular level set of $g \circ f$.

(c) By the regular level set theorem, $f^{-1}(U) \cap f^{-1}(S)$ is a regular submanifold of $f^{-1}(U) \subset N$. Thus every point $p \in f^{-1}(S)$ has an adapted chart relative to $f^{-1}(S)$ in N. ◇

11.1 Let $c(t) = (x^1(t), \dots, x^{n+1}(t))$ be a curve in S^n with $c(0) = p$ and $c'(0) = X_p$. Differentiate $\sum_i (x^i)^2(t) = 1$ with respect to t.

11.2* Critical points of a smooth map on a compact manifold
If $f\colon N \to \mathbb{R}^m$ has no critical points, then the differential $f_{*,p}$ would be surjective for every $p \in N$. In other words, f would be a submersion. Since a submersion is an open map (Corollary 11.9), the image $f(N)$ would be open in $\mathbb{R}^m$. But the continuous image of a compact set is compact and a compact subset of $\mathbb{R}^n$ is closed and bounded. Hence, $f(N)$ is a nonempty proper closed subset of $\mathbb{R}^m$. This is a contradiction, because being connected, $\mathbb{R}^m$ cannot have a nonempty proper subset that is both open and closed. ◇

11.3 At $p = (a, b, c)$, $i_*(\partial/\partial u|_p) = \partial/\partial x - (a/c)\partial/\partial z$, and $i_*(\partial/\partial v|_p) = \partial/\partial y - (b/c)\partial/\partial z$.

11.4 Use Problem A.5 to show that f is a closed map. Then apply Problem A.7 and Theorem 11.17.

12.1* The Hausdorff condition on the tangent bundle
Let (p, X) and (q, Y) be distinct points of the tangent bundle TM.

Case 1: $p \neq q$. Because M is Hausdorff, p and q can be separated by disjoint neighborhoods U and V. Then TU and TV are disjoint open subsets of TM containing (p, X) and (q, Y), respectively.

Case 2: $p = q$. Let U be a coordinate neighborhood of p. Then (p, X) and (p, Y) are distinct points in the open set $TU \simeq U \times \mathbb{R}^n$, which is Hausdorff. So (p, X) and (p, Y) can be separated with open sets in TU. ◇

13.1* Support of a finite sum
Let A be the set where $\sum \rho_i$ is not zero and A_i the set where ρ_i is not zero:
$$A = \left\{x \in M \mid \sum \rho_i(x) \neq 0\right\}, \quad A_i = \{x \in M \mid \rho_i(x) \neq 0\}.$$
If $\sum \rho_i(x) \neq 0$, then at least one $\rho_i(x)$ must be nonzero. This implies that $A \subset \cup A_i$. Taking closure of both sides gives $\overline{A} \subset \overline{\cup A_i}$. For a finite union, $\overline{\cup A_i} = \cup \overline{A_i}$ (Exercise A.55). Hence,

$$\operatorname{supp}\left(\sum \rho_i\right) = \overline{A} \subset \overline{\cup A_i} = \cup \overline{A_i} = \cup \operatorname{supp} \rho_i. \qquad \diamond$$

13.2* Locally finite family and compact set
For each $p \in K$, let W_p be a neighborhood of p that intersects only finitely many of the sets A_α. The collection $\{W_p\}_{p \in K}$ is an open cover of K. By compactness, K has a finite subcover $\{W_{p_i}\}_{i=1}^r$. Since each W_{p_i} intersects only finitely many of the A_α, the finite union $W := \cup_{i=1}^r W_{p_i}$ intersects only finitely many of the A_α. $\diamond$

13.3 Take $f = \rho_{M-B}$.

13.4* Support of the pullback of a function
Let $A = \{p \in M \mid f(p) \neq 0\}$. Then $\operatorname{supp} f = \operatorname{cl}(A)$. Remark that

$$(\pi^* f)(p, q) \neq 0 \quad \text{iff } f(p) \neq 0 \quad \text{iff } p \in A.$$

Hence,

$$\{(p, q) \in M \times N \mid (\pi^* f)(p, q) \neq 0\} = A \times N.$$

So

$$\operatorname{supp}(\pi^* f) = \operatorname{cl}(A \times N) = \operatorname{cl}(A) \times N = (\operatorname{supp} f) \times N$$

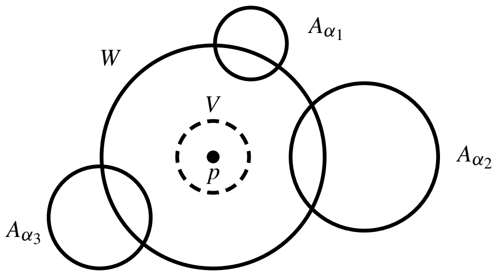
by Problem A.15. $\diamond$

13.6* Closure of a locally finite union
($\supset$) Since $A_\alpha \subset \cup A_\alpha$, taking the closure of both sides gives

$$\overline{A_\alpha} \subset \overline{\cup A_\alpha}.$$

Hence, $\cup \overline{A_\alpha} \subset \overline{\cup A_\alpha}$.

($\subset$) Let $p \in \overline{\cup A_\alpha}$. By local finiteness, p has a neighborhood W that intersects only finitely many of the A_α's, say $A_{\alpha_1}, \ldots, A_{\alpha_m}$. Suppose $p \notin \cup \overline{A_\alpha}$. A fortiori, $p \notin \cup_{i=1}^m \overline{A_{\alpha_i}}$. Since $\cup_{i=1}^m \overline{A_{\alpha_i}}$ is closed, there is a neighborhood V of p in W such that $V \subset W - \cup_{i=1}^m \overline{A_{\alpha_i}}$ (see the figure below).



Since W is disjoint from A_α for all $\alpha \neq \alpha_i$, V is disjoint from A_α for all α. This proves that $p \notin \overline{\cup A_\alpha}$, a contradiction. Hence, $p \in \cup \overline{A_\alpha}$. $\diamond$

14.1* Equality of vector fields
The implication in the direction ($\Rightarrow$) is obvious. For the converse, let $p \in M$. To show that $X_p = Y_p$, it suffices to show that $X_p[h] = Y_p[h]$ for any germ $[h]$ of C^∞ functions in $C_p^\infty(M)$. Suppose $h\colon U \to \mathbb{R}$ is a C^∞ function that represents the germ $[h]$. We can extend it to a C^∞ function $\tilde{h}\colon M \to \mathbb{R}$ by multiplying it by a C^∞ bump function supported in U that is identically 1 in a neighborhood of p. By hypothesis, $X\tilde{h} = Y\tilde{h}$. Hence,

$$X_p\tilde{h} = (X\tilde{h})_p = (Y\tilde{h})_p = Y_p\tilde{h}. \tag{14.1.1}$$

Because $\tilde{h} = h$ in a neighborhood of p, we have $X_p h = X_p\tilde{h}$ and $Y_p h = Y_p\tilde{h}$. It follows from (14.1.1) that $X_p h = Y_p h$. Thus, $X_p = Y_p$. Since p is an arbitrary point of M, the two vector fields X and Y are equal. $\diamond$

14.7 $c(t) = 1/((1/p) - t)$ on $(-\infty, 1/p)$.

14.9 $c^k = \sum_i \left(a^i \dfrac{\partial b^k}{\partial x^i} - b^i \dfrac{\partial a^k}{\partial x^i} \right)$.

14.10 Show that both sides applied to a C^∞ function h on M are equal. Then use Problem 14.1.

15.2
(a) Apply Proposition A.44.
(b) Apply Proposition A.44.
(c) Apply Problem A.11.
(d) By (a) and (b), the subset C_e is a subgroup of G. By (c), it is an open submanifold.

15.3* Open subgroup of a connected Lie group
For any $g \in G$, left multiplication $\ell_g\colon G \to G$ by g maps the subgroup H to the left coset gH. Since H is open and ℓ_g is a homeomorphism, the coset gH is open. Thus, the set of cosets gH, $g \in G$, partitions G into a disjoint union of open subsets. Since G is connected, there can be only one coset. Therefore, $H = G$. $\diamond$

15.4 Let $c(t)$ be a curve in G with $c(0) = a$, $c'(0) = X_a$. Then $(c(t), b)$ is a curve through (a, b) with initial velocity $(X_a, 0)$. Compute $\mu_{*,(a,b)}(X_a, 0)$ using this curve (Proposition 8.17). Compute similarly $\mu_{*,(a,b)}(0, Y_b)$.

15.6* Differential of the determinant map
Let $c(t) = Ae^{tX}$. Then $c(0) = A$ and $c'(0) = AX$. Using the curve $c(t)$ to calculate the differential,

$$\begin{aligned} \det{}_{A,*}(AX) &= \frac{d}{dt}\bigg|_{t=0} \det(c(t)) = \frac{d}{dt}\bigg|_{t=0} (\det A) \det e^{tX} \\ &= (\det A) \frac{d}{dt}\bigg|_{t=0} e^{t\,\mathrm{tr}\,X} = (\det A)\,\mathrm{tr}\,X. \end{aligned}$$ $\diamond$

15.7* Special linear group
If $\det A = 1$, then Exercise 15.6 gives

$$\det_{*,A}(AX) = \operatorname{tr} X.$$

Since tr X can assume any real value, $\det_{*,A} \colon T_A GL(n, \mathbb{R}) \to \mathbb{R}$ is surjective for all $A \in \det^{-1}(1)$. Hence, 1 is a regular value of det. ◇

15.9
(a) $O(n)$ is defined by polynomial equations.
(b) If $A \in O(n)$, then each column of A has length 1.

15.10 Write out the conditions $A^T A = I$, $\det A = 1$. If $a^2 + b^2 = 1$, then (a, b) is a point on the unit circle, and so $a = \cos\theta$, $b = \sin\theta$ for some $\theta \in [0, 2\pi]$.

15.13

$$\begin{bmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{bmatrix},$$

where $\cosh t = (e^t + e^{-t})/2$ and $\cosh t = (e^t - e^{-t})/2$ are hyperbolic cosine and sine, respectively.

15.14 The correct target space for f is the vector space $K_{2n}(\mathbb{C})$ of $2n \times 2n$ skew-symmetric complex matrices.

16.3 Let $c(t)$ be a curve in $\mathrm{Sp}(n)$ with $c(0) = I$ and $c'(0) = X$. Differentiate $c(t)^T J c(t) = J$ with respect to t.

16.4
(a) Use Problems 16.2 and 16.3.
(b) Show that the derivative of $\varphi(t) = e^{tX^T} J e^{tX}$ is identically zero and hence $\varphi(t)$ is a constant function.
(e) Let $\mathfrak{u}(n)$ be the vector space of $n \times n$ skew-Hermitian matrices and $S_n(\mathbb{C})$ the vector space of $n \times n$ complex symmetric matrices. Then $\dim \mathrm{Sp}(n) = \dim \mathfrak{u}(n) + \dim S_n(\mathbb{C}) = n^2 + (n^2 + n) = 2n^2 + n$.

16.5 Mimic Example 16.3. The left-invariant vector fields on $\mathbb{R}^n$ are the constant vector fields $\sum_{i=1}^n a^i\, \partial/\partial x^i$, where $a^i \in \mathbb{R}$.

16.9 A basis $X_{1,e}, \dots, X_{n,e}$ for the tangent space $T_e(G)$ at the identity gives rise to a frame consisting of left-invariant vector fields $X_1, \dots, X_n$.

16.10 (b) Let $(U, x^1, \dots, x^n)$ be a chart about e in G. Relative to this chart, the differential $c(a)_*$ at e is represented by the Jacobian matrix $[\partial(x^i \circ c(a))/\partial x^j|_e]$. Since $c(a)(x) = axa^{-1}$ is a C^∞ function of x and a, all the partial derivatives $\partial(x^i \circ c(a))/\partial x^j|_e$ are C^∞ and therefore $\mathrm{Ad}(a)$ is a C^∞ function of a.

17.1 $\omega = (x\,dx + y\,dy)/(x^2 + y^2)$.

17.3 $a_j = \sum_i b_i\, \partial y^i/\partial x^j$.

18.3* Vertical plane
Since $ax + by$ is the zero function on the vertical plane, its differential is identically zero:

$$a dx + b dy = 0.$$

Thus, at each point of the plane, dx is a multiple of dy or vice versa. In either case, $dx \wedge dy = 0$. ◇

18.4* Support of a sum or product
(a) If $(\omega + \tau)(p) \neq 0$, then $\omega(p) \neq 0$ and $\tau(p) \neq 0$. Hence,

$$\{p \mid (\omega+\tau)_p \neq 0\} \subset \{p \mid \omega_p \neq 0\} \cup \{p \mid \tau_p \neq 0\}.$$

Taking the closure of both sides gives

$$\operatorname{supp}(\omega+\tau) \subset \operatorname{supp}\omega \cup \operatorname{supp}\tau.$$

(b) Suppose $(\omega \wedge \tau)_p \neq 0$. Then $\omega_p \neq 0$ and $\tau_p \neq 0$. Hence,

$$\{p \mid (\omega \wedge \tau)_p \neq 0\} \subset \{p \mid \omega_p \neq 0\} \cap \{p \mid \tau_p \neq 0\}.$$

Taking the closure of both sides and remembering that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$, we get

$$\operatorname{supp}(\omega \wedge \tau) \subset \overline{\{p \mid \omega_p \neq 0\} \cap \{p \mid \tau_p \neq 0\}} \subset \operatorname{supp}\omega \cap \operatorname{supp}\tau. \qquad \diamond$$

18.5* Locally finite supports
Let $p \in \operatorname{supp}\omega$. Since $\{\operatorname{supp}\rho_\alpha\}$ is locally finite, there is a neighborhood W_p of p in M that intersects only finitely many of the sets $\operatorname{supp}\rho_\alpha$. The collection $\{W_p \mid p \in \operatorname{supp}\omega\}$ covers $\operatorname{supp}\omega$. By the compactness of $\operatorname{supp}\omega$, there is a finite subcover $\{W_{p_1}, \dots, W_{p_m}\}$. Since each W_{p_i} intersects only finitely many $\operatorname{supp}\rho_\alpha$, $\operatorname{supp}\omega$ intersects only finitely many $\operatorname{supp}\rho_\alpha$.

By Problem 18.4,

$$\operatorname{supp}(\rho_\alpha\omega) \subset \operatorname{supp}\rho_\alpha \cap \operatorname{supp}\omega.$$

Thus, for all but finitely many α, $\operatorname{supp}(\rho_\alpha\omega)$ is empty, i.e., $\rho_\alpha\omega \equiv 0$. ◇

18.7* Pullback by a surjective submersion
The fact that $\pi^*\colon \Omega^*(M) \to \Omega^*(\tilde{M})$ is an algebra homomorphism follows from Propositions 18.5 and 18.7.

Suppose $\omega \in \Omega^k(M)$ is a k-form on M for which $\pi^*\omega = 0$ in $\Omega^k(\tilde{M})$. To show that $\omega = 0$, pick an arbitrary point $p = \pi(\tilde{p}) \in M$, and arbitrary vectors $v_1, \dots, v_k \in T_pM$. Since π is a submersion, there exist $\tilde{v}_1, \dots, \tilde{v}_k \in T_{\tilde{p}}\tilde{M}$ such that $\pi_{*,\tilde{p}}\tilde{v}_i = v_i$. Then

$$\begin{aligned} 0 &= (\pi^*\omega)_{\tilde{p}}(\tilde{v}_1, \dots, \tilde{v}_k) && \text{(because } \pi^*\omega = 0) \\ &= \omega_{\pi(\tilde{p})}(\pi_*\tilde{v}_1, \dots, \pi_*\tilde{v}_k) && \text{(definition of } \pi^*\omega) \\ &= \omega_p(v_1, \dots, v_k). \end{aligned}$$

Since $p \in M$ and $v_1, \dots, v_k \in T_pM$ are arbitrary, this proves that $\omega = 0$. Therefore, $\pi^*\colon \Omega^*(M) \to \Omega^*(\tilde{M})$ is injective. ◇

18.8 (c) Because $f(a)$ is induced by $\operatorname{Ad}(a)$, we have $f(a) = \det(\operatorname{Ad}(a))$ by Problem 3.11. According to Problem 16.10, $\operatorname{Ad}(a)$ is a C^∞ function of a.

19.1* Extension of a C^∞ form
Choose a C^∞ bump function ρ at p supported in U. For any $q \in M$, define

$$\tilde{\tau}(q) = \begin{cases} \rho(q)\tau(q) & \text{for } q \in U; \\ 0 & \text{for } q \notin U. \end{cases}$$

On U, the form $\tilde{\tau}$ is clearly C^∞, since it is the product of two C^∞ forms ρ and τ. Suppose $q \notin U$. Since τ is supported in U, $q \notin \operatorname{supp} \tau$. Because $\operatorname{supp}\tau$ is a closed set, there is an open neighborhood U_q of q such that $U_q \cap \operatorname{supp}\tau = \phi$. Thus, $\tilde{\tau} \equiv 0$ on U_q and is trivially C^∞ at q. ◇

19.3 $F^*(dx \wedge dy \wedge dz) = d(x \circ F) \wedge d(y \circ F) \wedge d(z \circ F)$. Apply Problem 19.2.

19.4 $F^*(u\,du + v\,dv) = (2x^3 + 3xy^2)dx + (3x^2y + 2y^3)dy$.

19.5 $c^*\omega = dt$.

19.6* Coordinates and differential forms
Let $(V, x^1, \dots, x^n)$ be a chart about p. By Problem 19.2,

$$df^1 \wedge \cdots \wedge df^n = \det\left[\frac{\partial f^i}{\partial x^j}\right] dx^1 \wedge \cdots \wedge dx^n.$$

So $(df^1 \wedge \cdots \wedge df^n)_p \neq 0$ if and only if $\det[\partial f^i/\partial x^j(p)] \neq 0$. By the inverse function theorem, this condition is equivalent to the existence of a neighborhood W on which the map $F := (f^1, \dots, f^n)\colon W \to \mathbb{R}^n$ is a C^∞ diffeomorphism unto its image. In other words, $(W, f^1, \dots, f^n)$ is a chart. ◇

19.8 Mimic the proof of Proposition 19.4.

19.9 It is enough to check the formula in a chart $(U, x^1, \dots, x^n)$, so we may assume $\omega = \sum a_i\, dx^i$. Since both sides of the equation are $\mathbb{R}$-linear in ω, we may further assume that $\omega = f\,dh$, where $f, h \in C^\infty(U)$.

19.10
(a) Mimic Example 19.7.
(b) On M, $df = f_x\,dx + f_y\,dy + f_z\,dz \equiv 0$.

19.12 $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$ and $\operatorname{div}\mathbf{B} = 0$.

20.2* Equivalent nowhere-vanishing n-forms
Suppose $\mathfrak{U} = \{(U_\alpha, x_\alpha^1, \dots, x_\alpha^n)\}$ and $\mathfrak{V} = \{(V_\beta, y_\beta^1, \dots, y_\beta^n)\}$ are equivalent oriented atlases. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$, and $\{\sigma_\beta\}$ a partition of unity subordinate to $\{V_\beta\}$. Define

$$\omega_\alpha = \sum \rho_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n \quad \text{and} \quad \tau = \sum \sigma_\beta dy_\beta^1 \wedge \cdots \wedge dy_\beta^n.$$

Being both nowhere-vanishing n-forms, $\tau = h\omega$ for a nowhere-vanishing function h. It suffices to show that h is everywhere positive.

Let $p \in M$. In the proof of Proposition 20.9, it is shown that for any chart $(U, x^1, \dots, x^n)$ containing p in $\mathfrak{U}$,

$$\omega = f_U\, dx^1 \wedge \cdots \wedge dx^n \quad \text{for some function } f_U > 0.$$

Similarly, for any chart $(V, y^1, \dots, y^n)$ containing p in $\mathfrak{V}$,

$$\tau = g_V\, dy^1 \wedge \cdots \wedge dy^n \quad \text{for some function } g_V > 0.$$

On $U \cap V$,

$$\begin{aligned}
\tau &= g_V \det\left[\frac{\partial y^i}{\partial x^j}\right] dx^1 \wedge \cdots \wedge dx^n \\
&= \frac{g_V}{f_U} \det\left[\frac{\partial y^i}{\partial x^j}\right] f_U\, dx^1 \wedge \cdots \wedge dx^n \\
&= \frac{g_V}{f_U} \det\left[\frac{\partial y^i}{\partial x^j}\right] \omega.
\end{aligned}$$

Since $\mathfrak{U}$ and $\mathfrak{V}$ are equivalent oriented atlases, $\det[\partial y^i/\partial x^j] > 0$. Hence,

$$\tau_p = (\text{positive number}) \cdot \omega_p.$$

As p is an arbitrary point of M, the function h is positive everywhere on M. Therefore, ω and τ are equivalent nowhere-vanishing n-forms on M. ◇

20.4 Use Problem 19.10(c).

20.7 See Problem 12.2.

21.1 $\mathrm{bd}(M) = \{0, 1, 2\}$. $\partial M = \{0\}$.

21.2* Boundary orientation of the left half-space
We map M to the upper half-space $\mathbb{H}^n$ by the coordinate map:

$$x^1 = y^2, \dots, x^{n-1} = y^n, x^n = -y^1.$$

Then the orientation form on M is

$$\begin{aligned}
dy^1 \wedge \cdots \wedge dy^n &= -dx^n \wedge dx^1 \wedge \cdots \wedge dx^{n-1} \\
&= (-1)^n dx^1 \wedge \cdots d_X^{n-1} dx^n.
\end{aligned}$$

Hence, the orientation form on ∂M is

$$\begin{aligned}
(-1)^n(-1)^n dx^1 \wedge \cdots \wedge dx^{n-1} &= dx^1 \wedge \cdots \wedge dx^{n-1} \\
&= dy^2 \wedge \cdots \wedge dy^n. \qquad ◇
\end{aligned}$$

21.3* Inward-pointing vectors at the boundary
($\Leftarrow$) Suppose $(U, \phi = (x^1, \dots, x^n))$ is a chart for M centered at p such that $X_p = \sum a^i \partial/\partial x^i|_p$ with $a^n > 0$. Then the curve $c(t) = \phi^{-1}(a^1 t, \dots, a^n t)$ in M satisfies

$$c(0) = p, \qquad c((0,\epsilon)) \subset \int(M), \quad \text{and} \quad c'(0) = X_p. \tag{21.3.1}$$

So X_p is inward-pointing.

($\Rightarrow$) Suppose X_p is inward-pointing. Then $X_p \notin T_p(\partial M)$ and there is a curve $c\colon [0,\epsilon) \to M$ such that (21.3.1) holds. Let $(U, \phi = (x^1, \dots, x^n))$ be a chart centered at p such that $U \cap M$ is defined by $x^n \geq 0$. If $\phi \circ c(t) = (c^1(t), \dots, c^n(t))$, then $c^n(0) = 0$ and $c^n(t) > 0$ for $t > 0$. Therefore,

$$(c^n)'(0) = \lim t \to 0^+ \frac{c^n(t) - c^n(0)}{t} \geq 0.$$

Since $X_p = \sum_{i=1}^n (c^i)'(0)\partial/\partial x^i|_p$, the coefficient of $\partial/\partial x^i|_p$ in X_p is $(c^n)'(0)$. In fact, $(c^n)'(0) > 0$ because if $(c^n)'(0) = 0$, then $X_p \in T_p(\partial M)$. ◇

21.4* Boundary orientation in terms of tangent vectors
Let $(U, \phi = (x^1, \dots, x^n))$ be a chart centered at p such that $U \cap M$ is defined by $x^n \geq 0$. Then an orientation form on $U \cap M$ is $dx^1 \wedge \cdots \wedge dx^n$ and an orientation form on $U \cap \partial M$ is $(-1)^n dx^1 \wedge \cdots \wedge dx^{n-1}$. Note that

$$\begin{aligned}
&dx^1 \wedge \cdots \wedge dx^n(X_p, v_1, \dots, v_{n-1}) \\
&\quad = (-1)^{n-1} dx^n \wedge dx^1 \wedge \cdots \wedge dx^{n-1}(X_p, v_1, \dots, v_{n-1}) \\
&\quad = (-1)^{n-1} dx^n(X_p)(dx^1 \cdots dx^{n-1})(v_1, \dots, v_{n-1}) \\
&\qquad \begin{array}{l} (dx^n(v_i) = 0 \text{ for all } i = 1, \dots, n-1, \\ \text{because } v_1, \dots, v_{n-1} \text{ are contained in the} \\ \text{subspace of } T_p(M) \text{ generated by} \\ \partial/\partial x^1, \dots, \partial/\partial x^{n-1}) \end{array} \\
&\quad = (-1)^n (\text{positive number}) \times dx^1 \wedge \cdots \wedge dx^{n-1}(v_1, \dots, v_{n-1}),
\end{aligned}$$

where in the last equality, $dx^n(X_p) < 0$ because X_p is outward-pointing.

Thus, the ordered basis $(X_p, v_1, \dots, v_{n-1})$ gives the orientation on $T_p(M)$ if and only if the ordered basis $(v_1, \dots, v_{n-1})$ for $T_p(\partial M)$ gives the boundary orientation on $T_p(\partial M)$. ◇

21.5* Orientation form of the boundary orientation
Let $p \in \partial M$ and let $v_1, \dots, v_{n-1}$ be a basis for $T_p(\partial M)$ that gives the boundary orientation on ∂M. By Problem 21.4, the basis $X_p, v_1, \dots, v_{n-1}$ specifies the orientation of $T_p M$. Thus,

$$\omega(X_p, v_1, \dots, v_{n-1}) > 0.$$

But

$$\iota_{X_p}\omega(v_1, \dots, v_{n-1}) = \omega(X_p, v_1, \dots, v_{n-1}) > 0.$$

Hence, $\iota_{X_p}\omega$ is an orientation form for ∂M. ◇

21.6 Viewed from the top, C_1 is clockwise and C_2 is counterclockwise.

22.1 The map F is orientation-preserving.

22.2* Integral under a diffeomorphism

Let $\{(U_\alpha, \phi_\alpha)\}$ be an oriented atlas for M that specifies the orientation of M, and $\{\rho_\alpha\}$ a partition of unity on M subordinate to the open cover $\{U_\alpha\}$. Assume that $F\colon N \to M$ is orientation-preserving. By Problem 20.3, $\{(F^{-1}(U_\alpha), \phi_\alpha \circ F)\}$ is an oriented atlas for N that specifies the orientation of N. By Problem 13.5, $\{F^*\rho_\alpha\}$ is a partition of unity on N subordinate to the open cover $\{F^{-1}(U_\alpha)\}$.

By the definition of the integral,

$$\begin{aligned}
\int_N F^*\omega &= \sum_\alpha \int_{F^{-1}(U_\alpha)} (F^*\rho_\alpha)\,(F^*\omega) \\
&= \sum_\alpha \int_{F^{-1}(U_\alpha)} F^*(\rho_\alpha\omega) \\
&= \sum_\alpha \int_{(\phi_\alpha\circ F)(F^{-1}(U_\alpha))} (\phi_\alpha \circ F)^{-1*} F^*(\rho_\alpha\omega) \\
&= \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^*(\rho_\alpha\omega) \\
&= \sum \int_{U_\alpha} \rho_\alpha\omega = \int_M \omega.
\end{aligned}$$

If $F\colon N \to M$ is orientation-reversing, then $\{(F^{-1}(U_\alpha), \phi_\alpha \circ F)\}$ is an oriented atlas for N that gives the opposite orientation of N. Using this atlas to calculate the integral as above gives $-\int_N F^*\omega$. Hence in this case $\int_M \omega = -\int_N F^*\omega$. ◇

22.3* Stokes' theorem for $\mathbb{R}^n$ and for $\mathbb{H}^n$

An $(n-1)$-form ω with compact support on $\mathbb{R}^n$ or $\mathbb{H}^n$ is a linear combination

$$\omega = \sum_{i=1}^n f_i\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n. \tag{22.3.1}$$

Since both sides of Stokes' theorem are $\mathbb{R}$-linear in ω, it suffices to check the theorem for just one term of the sum (22.3.1). So we may assume

$$\omega = f\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n,$$

where f is a C^∞ function with compact support in $\mathbb{R}^n$ or $\mathbb{H}^n$. Then

$$\begin{aligned}
d\omega &= \frac{\partial f}{\partial x^i}\, dx^i \wedge dx^1 \wedge \cdots \wedge dx^{i-1} \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\
&= (-1)^{i-1} \frac{\partial f}{\partial x^i}\, dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^n.
\end{aligned}$$

Since f has compact support in $\mathbb{R}^n$ or $\mathbb{H}^n$, we may choose $a > 0$ large enough so that $\operatorname{supp} f$ lies in the interior of the cube $[-a, a]^n$.

Stokes' theorem for $\mathbb{R}^n$

By Fubini's theorem, one can first integrate with respect to x^i:

$$\begin{aligned}
\int_{\mathbb{R}^n} d\omega &= \int_{\mathbb{R}^n} (-1)^{i-1} \frac{\partial f}{\partial x^i}\, |dx^1 \cdots dx^n| \\
&= (-1)^{i-1} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \frac{\partial f}{\partial x^i}\, |dx^i|\, |dx^1 \cdots \widehat{dx^i} \cdots dx^n| \\
&= (-1)^{i-1} \int_{\mathbb{R}^{n-1}} \int_{-a}^{a} \frac{\partial f}{\partial x^i}\, |dx^i|\, |dx^1 \cdots \widehat{dx^i} \cdots dx^n|.
\end{aligned}$$

But

$$\begin{aligned}
\int_{-a}^{a} \frac{\partial f}{\partial x^i}\, |dx^i| &= f(\dots, x^{i-1}, a, x^{i+1}, \dots) - f(\dots, x^{i-1}, -a, x^{i+1}, \dots) \\
&= 0 - 0 = 0,
\end{aligned}$$

because the support of f lies in the interior of $[-a, a]^n$. Hence, $\int_{\mathbb{R}^n} d\omega = 0$.

The right-hand side of Stokes' theorem is $\int_{\partial\mathbb{R}^n} \omega = \int_{\varnothing} \omega = 0$, because $\mathbb{R}^n$ has empty boundary. This checks Stokes' theorem for $\mathbb{R}^n$.

Stokes' theorem for $\mathbb{H}^n$

Case 1: $i \neq n$.

$$\begin{aligned}
\int_{\mathbb{H}^n} d\omega &= (-1)^{i-1} \int_{\mathbb{H}^n} \frac{\partial f}{\partial x^i}\, |dx^1 \cdots dx^n| \\
&= (-1)^{i-1} \int_{\mathbb{H}^{n-1}} \int_{-\infty}^{\infty} \frac{\partial f}{\partial x^i}\, |dx^i|\, |dx^1 \cdots \widehat{dx^i} \cdots dx^n| \\
&= (-1)^{i-1} \int_{\mathbb{H}^{n-1}} \int_{-a}^{a} \frac{\partial f}{\partial x^i}\, |dx^i|\, |dx^1 \cdots \widehat{dx^i} \cdots dx^n| \\
&= 0 \quad \text{for the same reason as the case of } \mathbb{R}^n.
\end{aligned}$$

As for $\int_{\partial\mathbb{H}^n} \omega$, note that $\partial\mathbb{H}^n$ is defined by the equation $x^n = 0$. Hence, on $\partial\mathbb{H}^n$, the 1-form $dx^n \equiv 0$. Since $i \neq n$, $\omega = f\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \equiv 0$ on $\partial\mathbb{H}^n$, so $\int_{\partial\mathbb{H}^n} \omega = 0$. Thus Stokes' theorem holds in this case.

Case 2: $i = n$.

$$\begin{aligned}
\int_{\mathbb{H}^n} d\omega &= (-1)^{n-1} \int_{\mathbb{H}^n} \frac{\partial f}{\partial x^n}\, |dx^1 \cdots dx^n| \\
&= (-1)^{n-1} \int_{\mathbb{R}^{n-1}} \int_{0}^{\infty} \frac{\partial f}{\partial x^n}\, |dx^n|\, |dx^1 \cdots dx^{n-1}|.
\end{aligned}$$

In this integral

$$\begin{aligned}
\int_{0}^{\infty} \frac{\partial f}{\partial x^n}\, |dx^n| = \int_{0}^{a} \frac{\partial f}{\partial x^n}\, |dx^n| &= f(x^1, \dots, x^{n-1}, a) - f(x^1, \dots, x^{n-1}, 0) \\
&= -f(x^1, \dots, x^{n-1}, 0).
\end{aligned}$$

Hence,

$$\int_{\mathbb{H}^n} d\omega = (-1)^n \int_{\mathbb{R}^{n-1}} f(x^1, \ldots, x^{n-1}, 0)\, |dx^1 \cdots dx^{n-1}| = \int_{\partial\mathbb{H}^n} \omega$$

because $(-1)^n\mathbb{R}^{n-1}$ is precisely $\partial\mathbb{H}^n$ with its boundary orientation. So Stokes' theorem also holds in this case. ◇

23.1* Locally constant map on a connected space

We first show that for every $y \in Y$, the inverse $f^{-1}(y)$ is an open set. Suppose $p \in f^{-1}(y)$. Then $f(p) = y$. Since f is locally constant, there is a neighborhood U of p such that $f(U) = \{y\}$. Thus, $U \subset f^{-1}(y)$. This proves that $f^{-1}(y)$ is open.

The equality $S = \bigcup_{y\in Y} f^{-1}(y)$ exhibits S as a disjoint union of open sets. Since S is connected, this is possible only if there is just one such open set $S = f^{-1}(y_0)$. Hence, f assumes the constant value y_0 on S. ◇

25.2 The given exact sequence is equivalent to a collection of short exact sequences

$$0 \to \operatorname{im} d_{k-1} \to A^k \xrightarrow{d_k} \operatorname{im} d_k \to 0$$

for all $k = 0, \ldots, m-1$. (Define $d_{-1} = 0$.) By the first isomorphism theorem,

$$\dim A^k = \dim \operatorname{im} d_{k-1} + \dim \operatorname{im} d_k.$$

When we compute the alternating sum of the left-hand side, the right-hand side will cancel to 0. ◇

27.1 Let U be the punctured projective plane $\mathbb{R}P^2 - \{p\}$ and V a small disk containing p. Because U can be deformation retracted to the boundary circle, it has the homotopy type of S^1. Apply the Mayer–Vietoris sequence. The answer is $H^0(\mathbb{R}P^2) = \mathbb{R}$, $H^k(\mathbb{R}P^2) = 0$ for $k > 0$.

27.2 $H^k(S^n) = \mathbb{R}$ for $k = 0, n$, and $H^k(S^n) = 0$ otherwise.

27.3 One way is to apply the Mayer–Vietoris sequence to $U = \mathbb{R}^2 - \{p\}$, $V = \mathbb{R}^2 - \{q\}$.

A.8* The Lindelöf condition

Let $\{B_i\}_{i\in I}$ be a countable basis and $\{U_\alpha\}_{\alpha\in A}$ an open cover of the topological space S. For every $p \in U_\alpha$, there exists a B_i such that

$$p \in B_i \subset U_\alpha.$$

Since this B_i depends on p and α, we write $i = i(p,\alpha)$. Thus,

$$p \in B_{i(p,\alpha)} \subset U_\alpha.$$

Now let J be the set of all indices $j \in J$ such that $j = i(p,\alpha)$ for some p and some α. Then $\bigcup_{j\in J} B_j = M$ because every p in M is contained in some $B_{i(p,\alpha)} = B_j$.

For each $j \in J$, choose an $\alpha(j)$ such that $B_j \subset U_{\alpha(j)}$. Then $M = \bigcup_j B_j \subset U_{\alpha(j)}$. So $\{U_{\alpha(j)}\}_{j \in J}$ is a countable subcover of $\{U_\alpha\}_{\alpha \in A}$. ◇

A.10* Disconnected subset in terms of a separation

($\Rightarrow$) By (iii),

$$A = (U \cap V) \cap A = (U \cap A) \cup (V \cap A).$$

By (i) and (ii), $U \cap A$ and $V \cap A$ are disjoint nonempty open subsets of A. Hence, A is disconnected.

($\Leftarrow$) Suppose A is disconnected in the subspace topology. Then $A = U' \cup V'$, where U' and V' are two disjoint nonempty open subsets of A. By the definition of the subspace topology, $U' = U \cap A$ and $V' = V \cap A$ for some open sets U, V in S.

(i) holds because U' and V' are nonempty.
(ii) holds because U' and V' are disjoint.
(iii) holds because $A = U' \cup V' \subset U \cup V$. ◇

A.14* Uniqueness of the limit

Suppose $p \neq q$. Since S is Hausdorff, there exist disjoint open sets U_p and U_q such that $p \in U_p$ and $q \in U_q$. By the definition of convergence, there are integers N_p and N_q such that for all $i \geq N_p$, $x_i \in U_p$, and for all $i \geq N_q$, $x_i \in U_q$. This is a contradiction since $U_p \cap U_q$ is the empty set. ◇

A.15* Closure in a product

($\subset$) Because $\mathrm{cl}(A) \times Y$ is a closed set containing $A \times Y$, by the definition of closure, $\mathrm{cl}(A \times Y) \subset \mathrm{cl}(A) \times Y$.

($\supset$) Conversely, suppose $(p, y) \in \mathrm{cl}(A) \times Y$. If $p \in A$, then $(p, y) \in A \times N \subset \mathrm{cl}(A \times Y)$. Suppose $p \notin A$. By Proposition A.52, p is an accumulation of A. Let $U \times V$ be any basis open set in $S \times Y$ containing (p, y). Because $p \in \mathrm{ac}(A)$, the open set U contains a point $a \in A$ with $a \neq p$. So $U \times V$ contains the point $(a, y) \in A \times Y$ with $(a, y) \neq (p, y)$. This proves that (p, y) is an accumulation point of $A \times Y$. By Proposition A.52 again, $(p, y) \in \mathrm{ac}(A \times Y) \subset \mathrm{cl}(A \times Y)$. This proves that $\mathrm{cl}(A) \times Y \subset \mathrm{cl}(A \times Y)$. ◇

B.1* The rank of a matrix

($\Rightarrow$) Suppose $\mathrm{rk}\, A \geq k$. Then one can find k linearly independent columns, which we call $a_1, \ldots, a_k$. Since the $m \times k$ matrix $[a_1 \cdots a_k]$ has rank k, it has k linearly independent rows $b^1, \ldots, b^k$. The matrix B whose rows are $b^1, \ldots, b^k$ is a $k \times k$ submatrix of A, and $\mathrm{rk}\, B = k$. In other words, B is nonsingular $k \times k$ submatrix of A.

($\Leftarrow$) Suppose A has a nonsingular $k \times k$ submatrix B. Let $a_1, \ldots, a_k$ be the columns of A such that the submatrix $[a_1 \cdots a_k]$ contains B. Since $[a_1 \cdots a_k]$ has k linearly independent rows, it also has k linearly independent columns. Thus, $\mathrm{rk}\, A \geq k$. ◇

B.2* Matrices of rank at most *r*

Let A be an $m \times n$ matrix. By Problem B.1, $\mathrm{rk}\, A \leq r$ iff all $(r+1) \times (r+1)$ minors

$m_1(A), \dots, m_s(A)$ of A vanish. As the common zero set of a finite collection of continuous functions, D_r is closed in $\mathbb{R}^{m\times n}$. ◇

B.3* Maximal rank

For definiteness, suppose $n \le m$. Then the maximal rank is n and every matrix $A \in \mathbb{R}^{m\times n}$ has rank $\le n$. Thus,

$$D_{\max} = \{A \in \mathbb{R}^{m\times n} \mid \operatorname{rk} A = n\} = \mathbb{R}^{m\times n} - D_{n-1}.$$

Since D_{n-1} is a closed subset of $\mathbb{R}^{m\times n}$ (Problem B.2), $D_{\max}$ is open in $\mathbb{R}^{m\times n}$. ◇

B.4* Degeneracy loci and maximal rank locus of a map

(a) Let D_r be the subset of $\mathbb{R}^{m\times n}$ consisting of matrices of rank at most r. The degeneracy locus of rank r of the map $F\colon S \to \mathbb{R}^{m\times n}$ may be described as

$$D_r(F) = \{x \in S \mid F(x) \in D_r\} = F^{-1}(D_r).$$

Since D_r is a closed subset of $\mathbb{R}^{m\times n}$ (Problem B.2) and F is continuous, $F^{-1}(D_r)$ is a closed subset of S.

(b) Let $D_{\max}$ be the subset of $\mathbb{R}^{m\times n}$ consisting of all matrices of maximal rank. Then $D_{\max}(F) = F^{-1}(D_{\max})$. Since $D_{\max}$ is open in $\mathbb{R}^{m\times n}$ (Problem B.3) and F is continuous, $F^{-1}(D_{\max})$ is open in S. ◇

List of Symbols

$\mathbb{R}^n$	Euclidean space of dimension n (p. 5)
$p = (p^1, \ldots, p^n)$	point in $\mathbb{R}^n$ (p. 5)
C^∞	smooth or infinitely differentiable (p. 5)
$\partial f / \partial x^i$	partial derivative with respect to x^i (pp. 5, 60)
$f^{(k)}(x)$	the kth derivative of $f(x)$ (p. 7)
$B(p, r)$	open ball in $\mathbb{R}^n$ with center p and radius r (pp. 9, 281)
$T_p(\mathbb{R}^n)$ or $T_p R^n$	tangent space to $\mathbb{R}^n$ at p (p. 12)
$v = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \langle v^1, \ldots, v^n \rangle$	column vector (p. 12)
$\{e_1, \ldots, e_n\}$	standard basis for $\mathbb{R}^n$ (p. 12)
$D_v f$	directional derivative of f in the direction of v at p (p. 12)
$x \sim y$	equivalence relation (p. 13)
C_p^∞	algebra of germs of C^∞ functions at p in $\mathbb{R}^n$ (p. 13)
$\mathcal{D}_p(\mathbb{R}^n)$	vector space of derivations at p in $\mathbb{R}^n$ (p. 14)
$\mathfrak{X}(U)$	vector space of C^∞ vector fields on U (p. 16)
$\mathrm{Der}(A)$	vector space of derivations of an algebra A (p. 17)
$\mathrm{Hom}(V, W)$	vector space of linear maps $f : V \to W$ (p. 19)
$V^* = \mathrm{Hom}(V, \mathbb{R})$	dual of a vector space (p. 19)
δ^i_j	Kronecker delta (p. 19)
V^k	Cartesian product $V \times \cdots \times V$ of k copies of V (p. 22)

$L_k(V)$	vector space of k-linear functions on V (p. 22)
$(a_1\ a_2\ \cdots\ a_r)$	cyclic permutation, r-cycle (p. 20)
(ab)	transposition (p. 20)
S_k	group of permutations of k objects (p. 21)
$\operatorname{sgn}(\sigma)$ or $\operatorname{sgn}\sigma$	sign of a permutation (p. 21)
$A_k(V)$	vector space of alternating k-linear functions on V (p. 22)
σf	a function f acted on by a permutation σ (p. 23)
$\sigma \cdot x$	left action of σ on x (p. 23)
$x \cdot \sigma$	right action of σ on x (p. 23)
Sf	symmetrizing operator applied to f (p. 24)
Af	alternating operator applied to f (p. 24)
$f \otimes g$	tensor product of multilinear functions f and g (p. 25)
$f \wedge g$	wedge product of multicovectors f and g (p. 25)
$B = [b^i_j]$ or $[b_{ij}]$	matrix whose (i, j)-entry is b^i_j or b_{ij} (p. 29)
$\det [b^i_j]$ or $\det [b_{ij}]$	determinant of the matrix $[b^i_j]$ or $[b_{ij}]$ (p. 29)
$I = (i_1, \dots, i_k)$	multi-index (p. 30)
e_I	k-tuple $(e_{i_1}, \dots, e_{i_k})$ (p. 30)
α^I	k-covector $\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}$ (p. 30)
$T^*_p(\mathbb{R}^n)$ or $T^*_p\mathbb{R}^n$	cotangent space to $\mathbb{R}^n$ (p. 33)
df	differential of a function (pp. 33, 175)
dx^I	$dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ (p. 35)
$\Omega^k(U)$	vector space of C^∞ k-forms on U (p. 35)
$\Omega^*(U)$	direct sum $\bigoplus_{k=0}^n \Omega^k(U)$ (p. 35)
$\omega(X)$	the function $p \mapsto \omega_p(X_p)$ (p. 36)
$\mathcal{F}(U)$ or $C^\infty(U)$	ring of C^∞ functions on U (p. 36)
$d\omega$	exterior derivative of ω (p. 36)
$\bigoplus_{k=0}^\infty A^k$	direct sum of $A^0, A^1, \dots$ (p. 37)
$\operatorname{grad} f$	gradient of a function f (p. 39)
$\operatorname{curl} \mathbf{F}$	curl of a vector field $\mathbf{F}$ (p. 39)

$G(k, n)$	Grassmannian of k-planes in $\mathbb{R}^n$ (p. 73)
$\operatorname{rk} A$	rank of a matrix A (p. 73 (p. 304)
$C_p^\infty(M)$	germs of C^∞ functions at p in M (p. 77)
$T_p(M)$ or T_pM	tangent space to M at p (p. 78)
$\partial/\partial x^i\|_p$	coordinate tangent vector at p (p. 78)
$F_{*,p}$ or F_*	differential of F at p (p. 78)
$c(t)$	curve in a manifold (p. 83)
$c'(t) := c_*(\frac{d}{dt}\big\|_t)$	velocity vector of a curve (p. 83)
ϕ_S	coordinate map on a submanifold S (p. 92)
$f^{-1}(\{c\})$ or $f^{-1}(c)$	level set (p. 94)
$Z(f) = f^{-1}(0)$	zero set (p. 94)
$\mathrm{SL}(n, K)$	special linear group over a field K (p. 97)
m_{ij} or $m_{ij}(A)$	(i, j)-minor of a matrix A (p. 98)
$\mathrm{Hom}(A, B)$	the set of morphisms from A to B (p. 101)
1_A	identity map on A (p. 101)
(M, q)	pointed manifold (p. 102)
$\mathcal{C}, \mathcal{D}$	categories (p. 102)
$\{e_1, \ldots, e_n\}$	basis for a vector space V (p. 103)
$\{\alpha^1, \ldots, \alpha^n\}$	dual basis for V^* (p. 103)
L^*	dual of linear map L (p. 103)
$O(n)$	orthogonal group (p. 107)
A^T	transpose of a matrix A (p. 107)
ℓ_g	left multiplication by g (p. 107)
r_g	right multiplication by g (p. 107)
$D_{\max}(F)$	maximal rank locus of $F\colon S \to \mathbb{R}^{m\times n}$ (pp. 108, 305)
$i\colon N \to M$	inclusion map (p. 113)
TM	tangent bundle (p. 119)
$\coprod_{p\in M} T_pM$	disjoint union (p. 119)
$\tilde{\phi} = (\phi, \phi_*)$	coordinate map on the tangent bundle (p. 119)

$E_p := \pi^{-1}(p)$	fiber at p of a vector bundle (p. 122)
X	vector field (p. 123)
X_p	tangent vector at p (p. 123)
$\Gamma(U, E)$	vector space of C^∞ sections of E over U (p. 125)
$\Gamma(E) := \Gamma(M, E)$	vector space of C^∞ sections of E over M (p. 125)
$\operatorname{supp} f$	support of a function f (p. 127)
$\overline{B}(p, r)$	closed ball in $\mathbb{R}^n$ with center p and radius r (p. 130)
$\overline{A}$ or $\mathrm{cl}(A)$	closure of a set A (p. 134 (p. 295)
$c_t(p)$	integral curve through p (p. 136)
$\mathrm{Diff}(M)$	group of diffeomorphisms of M (p. 137)
$F_t(q) = F(t, q)$	local flow (p. 140)
$[X, Y]$	Lie bracket of vector fields, bracket in a Lie algebra (pp. 141, 142)
S_n	vector space of $n \times n$ real symmetric matrices (p. 150)
$\mathbb{R}^2/\mathbb{Z}^2$	torus (p. 152)
$\exp(X)$ or e^X	exponential of a matrix X (p. 154)
$\mathrm{tr}(X)$	trace (p. 155)
$\mathrm{SO}(n)$	special orthogonal group (p. 159)
$U(n)$	unitary group (p. 159)
$\mathrm{SU}(n)$	special unitary group (p. 159)
I_n	$n \times n$ identity matrix (p. 160)
J	the matrix $\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ (p. 160)
$\mathrm{Sp}(2n, \mathbb{C})$	complex symplectic group (p. 160)
$\mathrm{Sp}(n)$	compact symplectic group (p. 160)
e	identity element of a Lie group (p. 161)
$L(G)$	Lie algebra of left-invariant vector fields on G (p. 163)
$\mathfrak{g}$	Lie algebra (p. 165)
$\mathfrak{h} \subset \mathfrak{g}$	Lie subalgebra (p. 165)
$\tilde{A}$	left-invariant vector field generated by $A \in T_eG$ (p. 165)

$\mathfrak{gl}(n, \mathbb{R})$	Lie algebra of $\mathrm{GL}(n, \mathbb{R})$ (p. 166)
$\mathfrak{sl}(n, \mathbb{R})$	Lie algebra of $\mathrm{SL}(n, \mathbb{R})$ (p. 169)
$\mathfrak{o}(n)$	Lie algebra of $O(n, \mathbb{R})$ (p. 169)
$\mathfrak{u}(n)$	Lie algebra of $U(n, \mathbb{R})$ (p. 169)
$T_p^*(M)$ or T_p^*M	cotangent space at p (p. 175)
T^*M	cotangent bundle (p. 177)
$F^*\colon T_{F(p)}^*M \to T_p^*N$	codifferential (p. 179)
$F^*\omega$	pullback of a differential form ω by F (pp. 179, 184)
$\bigwedge^k(V^*) = A_k(V)$	k-covectors on a vector space V (p. 181)
ω_p	value of a differential form ω at p (p. 181)
$\bigwedge^k(T^*M)$	kth exterior power of the cotangent bundle (p. 183)
$\Omega^k(M)$	vector space of C^∞ k-forms on M (p. 183)
$\Omega^*(M)$	the direct sum $\oplus_{k=0}^n \Omega^k(M)$ (p. 185)
$\Omega^k(G)^G$	left-invariant k-forms on a Lie group G (p. 186)
$\operatorname{supp}\omega$	support of a k-form (p. 187)
$d\omega$	exterior derivative of a differential form ω (p. 192)
$\omega\|_S$	restriction of a differential from ω to a submanifold S (p. 193)
$(v_1, \dots, v_n)$	ordered basis (p. 202)
$[v_1, \dots, v_n]$	ordered basis as a matrix (p. 202)
$(M, [\omega])$	oriented manifold with orientation $[\omega]$ (p. 205)
$\mathbb{H}^n$	closed upper half-space (p. 211)
$\operatorname{int}(\mathbb{H}^n)$	interior of $\mathbb{H}^n$ (p. 211)
$\partial(\mathbb{H}^n)$	boundary of $\mathbb{H}^n$ (p. 211)
$\mathbb{L}^1$	left half-line (p. 213)
∂M	boundary of a manifold (p. 214)
$\mathrm{bd}(A)$	topological boundary of a set A (p. 214)
$\mathbb{H}^1 = [0, \infty)$	half-open interval in $\mathbb{R}$ (p. 218)
$\{p_0, \dots, p_n\}$	partition of a closed interval (p. 221)
$P = \{P_1, \dots, P_n\}$	partition of a closed rectangle (p. 221)

$L(f, P)$	lower sum of f with respect to a partition P (p. 221)
$U(f, P)$	upper sum of f with respect to a partition P (p. 221)
$\overline{\int}_R f$	upper integral of f over a closed rectangle R (p. 222)
$\underline{\int}_R f$	lower integral of f over a closed rectangle R (p. 222)
$\int_R f(x)\,\lvert dx^1 \cdots dx^n \rvert$	Riemann integral of f over a closed rectangle R (p. 222)
$\int_U \omega$	Riemann integral of a differential form ω over U (p. 224)
$v(A)$	volume of a subset A of $\mathbb{R}^n$ (p. 223)
$\mathrm{Dist}(f)$	set of discontinuities of a function f (p. 223)
$\Omega_c^k(U)$	vector space of C^∞ k-forms with compact support on U (p. 225)
$-M$	the oriented manifold having the opposite orientation as M (p. 227)
$Z^k(M)$	vector space of closed k-forms on M (p. 236)
$B^k(M)$	vector space of exact k-forms on M (p. 236)
$H^k(M)$	de Rham cohomology of M in degree k (p. 236)
$[\omega]$	cohomology class of ω (p. 236)
$F^\#$ or F^*	induced map in cohomology (p. 240)
$H^*(M)$	the cohomology ring $\oplus_{k=0}^n H^k(M)$ (p. 241)
$\mathcal{C} = (\{C^k\}_{k\in\mathbb{Z}}, d)$	cochain complex (p. 243)
$(\Omega^*(M), d)$	de Rham complex (p. 243)
$H^k(\mathcal{C})$	kth cohomology of $\mathcal{C}$ (p. 245)
$d^*\colon H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A})$	connecting homomorphism (p. 246)
$i_U : U \to M$	inclusion map of U in M (p. 249)
$j_U : U \cap V \to U$	inclusion map of $U \cap V$ in U (p. 249)
$\chi(M)$	Euler characteristic of M (p. 254)
$f \sim g$	f is homotopic to g (p. 257)
Σ_g	compact orientable surface of genus g (p. 271)
$d(p, q)$	distance between p and q (p. 281)
(a, b)	open interval (p. 281)
$(S, \mathcal{T})$	a set S with a topology $\mathcal{T}$ (p. 282)

$Z(f_1, \ldots, f_r)$	zero set of $f_1, \ldots, f_r$ (p. 282)
$Z(I)$	zero set of all the polynomials in an ideal I (p. 282)
$\mathbb{Q}$	the set of rational numbers (p. 285)
$\mathbb{Q}^+$	the set of positive rational numbers (p. 285)
$A \times B$	Cartesian product of two sets A and B (p. 287)
C_x	connected component of a point x (p. 294)
$\mathrm{ac}(A)$	the set of accumulation points of A (p. 295)
$\mathbb{Z}^+$	the set of positive integers (p. 296)
D_r	the set of matrices of rank $\leq r$ in $\mathbb{R}^{m\times n}$ (p. 304)
$D_{\max}$	the set of matrices of maximal rank in $\mathbb{R}^{m\times n}$ (p. 304)
$D_r(F)$	degeneracy locus of rank r of a map $F\colon S \to \mathbb{R}^{m\times n}$ (p. 305)
$\ker f$	kernel of a homomorphism f (p. 311)
$\mathrm{im}\, f$	image of a map f (p. 311)
$\mathrm{coker}\, f$	cokernel of a homomorphism f (p. 311)
$v + W$	coset of a subspace W (p. 312)
V/W	quotient vector space of V by W (p. 312)
$A + B$	sum of two vector subspaces (p. 313)
$A \oplus B$	internal direct sum, external direct sum (pp. 313, 314)
$W^\perp$	W "perp," orthogonal complement of W (p. 314)
$A \times B$	direct product, external direct sum (p. 314)

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