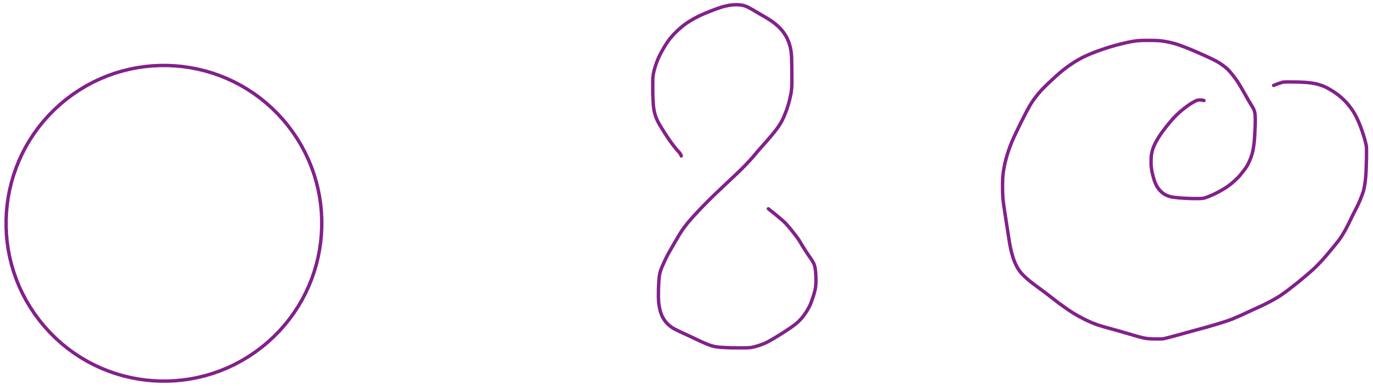
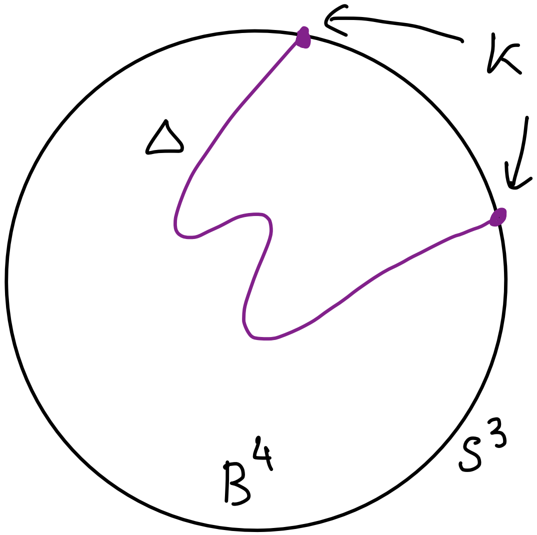
Building bridges seminar
February 17, 2021

Filtrations of the knot concordance group

Slice knots

A knot $K \subseteq S^3$ is trivial iff it bounds an embedded disc in $S3$



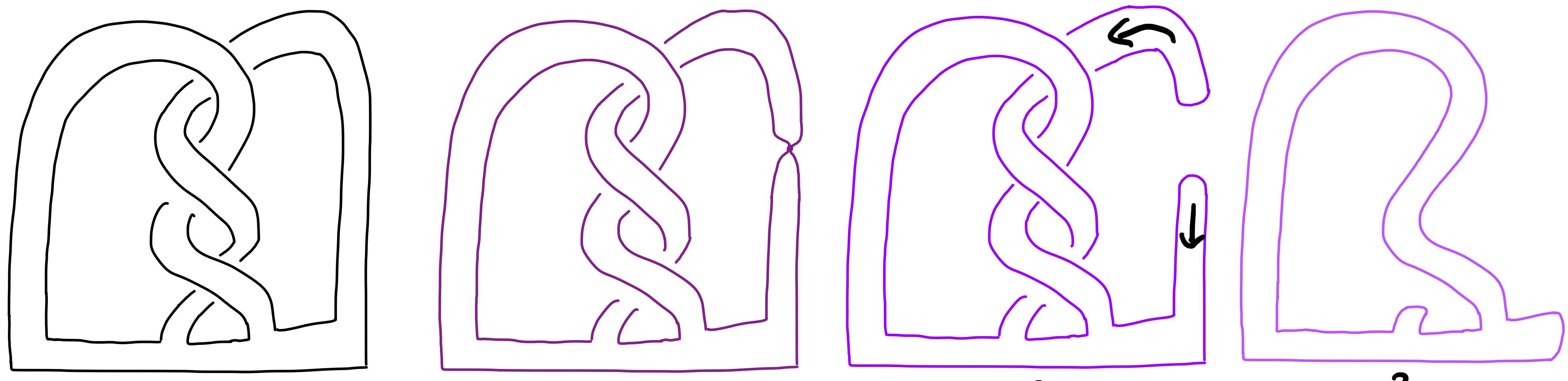


Consider $K \subseteq S^3$ bounding an embedded disc $\Delta \subseteq B^4$

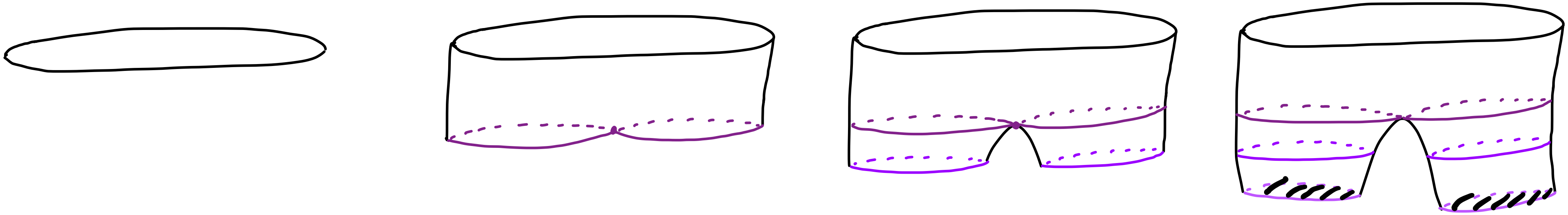
- if Δ is smooth, K is smoothly slice
- if Δ is flat, K is topologically slice
  ↳ has a normal bundle

$$\text{Trivial} \;\underset{\not\Longleftarrow}{\Longrightarrow}\; \text{slice}$$

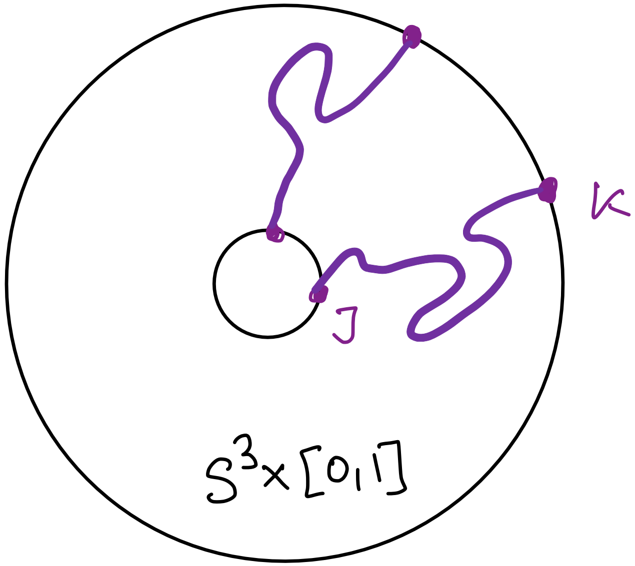
Examples of slice knots



$R \in S^3_1$ $\quad$ $S^3_{1-\varepsilon}$ $\quad$ $S^3_{1-2\varepsilon}$ $\quad$ $S^3_{1-3\varepsilon}$



Concordance of knots



K and J are TOP / sm concordant if cobound a flat / sm annulus in $S^3 \times [0,1]$

$$K \simeq J \iff K \# r\overline{J} \text{ slice}$$

r: reverse arrow

$\overline{J}$: mirror image (change all crossings)

$\mathcal{C} := \left\{ \begin{matrix} \text{conc. classes} \\ \text{of knots} \end{matrix} \right\}$ is a group under $\#$

The knot concordance group

Obstructions to sliceness

Not all knots are $\overset{\text{TOP}}{\wedge}$ slice

e.g. $\overset{\text{TOP}}{\wedge}$ slice $\Longrightarrow$ Arf $= 0$

Tristram-Levine sign. vanish

algebraically slice

Casson-Gordon invts vanish.

Goal: organise these systematically

Solvable filtration of $\mathcal{C}$ (Cochran-Orr-Teichner 2003)

$$\left\{ \begin{matrix} \text{TOP} \\ \text{slice} \\ \text{knots} \end{matrix} \right\} \subseteq \bigcap \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}_{n.5} \subseteq \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}_{0.5} \subseteq \mathcal{F}_0 \subseteq \mathcal{C}$$

Some properties

$$\left\{\begin{matrix}\text{slice}\\ \text{knots}\end{matrix}\right\} \subseteq \bigcap^{\infty} \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}_{n.5} \subseteq \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}_{0.5} \subseteq \mathcal{F}_0 \subseteq \mathcal{C}$$

$$\mathcal{F}_0 = \{ K \mid \mathbf{Arf}(K) = 0 \}$$

$$\mathcal{F}_{0.5} = \{ K \mid \textbf{K alg slice} \}$$

$$\mathcal{F}_{1.5} \subseteq \{ K \mid \textbf{CG invts vanish} \}$$

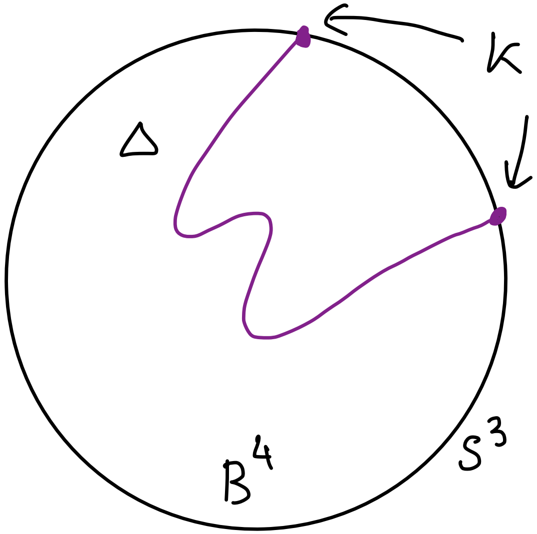
Jiang
Livingston
Cochran-Orr-Teichner
Cochran-Teichner
Cochran-Harvey-Leidy

$$\exists\, \mathbb{Z}^{\infty} \oplus \mathbb{Z}/2^{\infty} \subseteq \mathcal{F}_n / \mathcal{F}_{n.5} \quad \forall n \in \mathbb{N}$$

see also Cha, Davis-Park-R.

Definition of $\{\mathcal{F}_n\}$

Motivation: wish to approximate sliceness



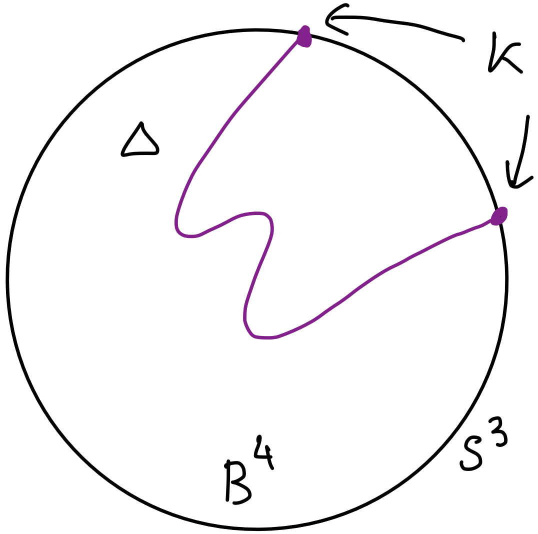
K is slice if it bounds a disc inside B^4 ← **Option 1 approx this**

K TOP slice $\Longleftrightarrow$ K bounds a **emb** disc in TOP W^4 with $\partial W^4 = S^3$

s.t. $\pi_1 W = 1$

$H_2 W = 0$

$\Longleftarrow$ **Freedman** **$W \cong B^4$**

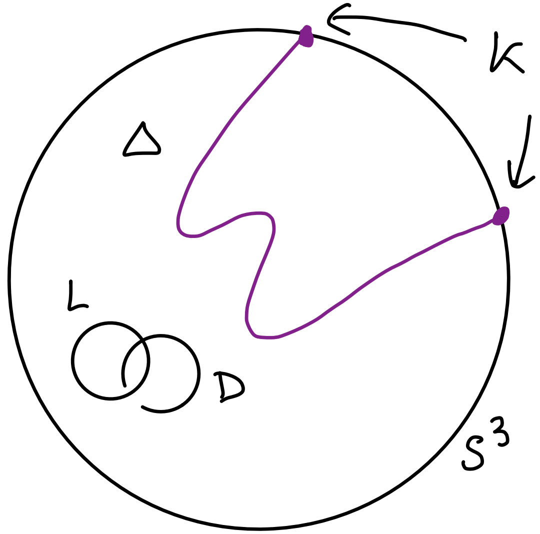


K TOP slice

$$\xrightarrow{\# S^2 \times S^2}$$

$$\xleftarrow{\text{surgery on } L}$$

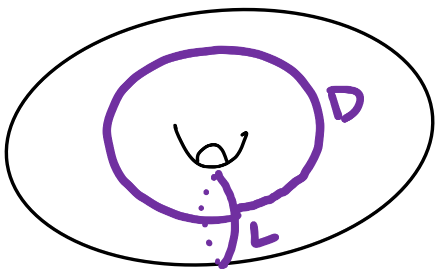
$\setminus L \times D^2$
$\cup\, D^3 \times S^1$



K bounds a disc Δ
in TOP W^4
s.t. $\pi_1 W = 1$

$H_2(W)$ gen by
emb spheres
$\{L, D\}$ w. trivial
normal bundle
and $L \pitchfork D = pt$

$L, D \cap \Delta = \emptyset$



$S^2 \times S^2$

Definition: K is n-solvable, denoted $K \in \mathcal{F}_n$, if it bounds a disc Δ in a TOP W^4 s.t. $\partial W = S^3$

1. $H_1(W) = 0$

2. $H_2(W)$ gen by embedded surfaces $\{L_i, D_i\}$, $L_i, D_i \subseteq W \setminus \Delta$ w. trivial normal bundle s.t. $L_i \pitchfork D_j = \delta_{ij}$,

$L_i \cap L_j = 0 = D_i \cap D_j$

i.e. int form $\bigoplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

3. $i_*(\pi_1(L_i)) \subseteq \pi_1(W \setminus \Delta)^{(n)}$

$i_*(\pi_1(D_i)) \subseteq \pi_1(W \setminus \Delta)^{(n)}$

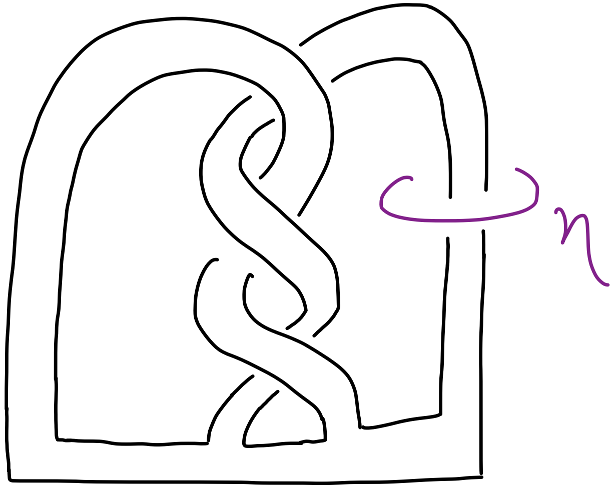
Recall: $G^{(1)} = [G,G]$, $G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$

if in addition $\pi_1(L_i) \subseteq \pi_1(W \setminus \Delta)^{(n+1)}$, then K is n.5 solvable denoted $K \in \mathcal{F}_{n.5}$

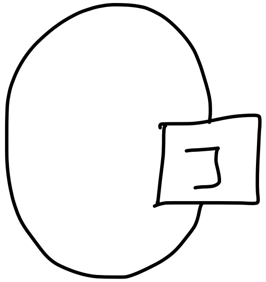
Examples

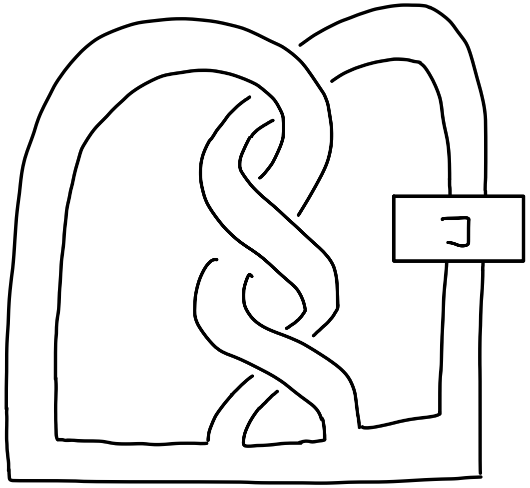
Infection/satellite operation: $\overbrace{S^3 \setminus \eta\times D^2}^{\text{solid torus} \cong R} \cup S^3 \setminus N(J) = S^3$

$R \longmapsto R_\eta(J)$



R





$R_\eta(J)$

if $R \in \mathcal{F}_n$ & $\eta \in \pi_1(S^3 \setminus R)^{(k)}$ then $R_\eta(\mathcal{F}_{n-k}) \subseteq \mathcal{F}_n$

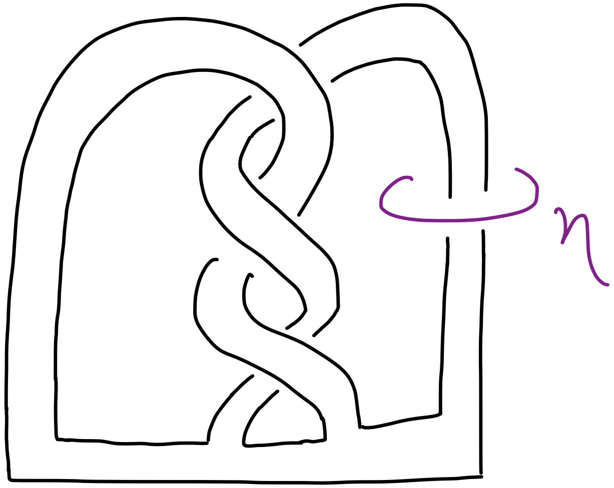
E.g. let R ribbon ($\Rightarrow R \in \mathcal{F}_n\ \forall n$), $lk(\eta, R) = 0$ ($\Rightarrow \eta \in \pi_1(S^3 \setminus R)^{(1)}$)

and $J \in \mathcal{F}_0$ ($\Leftrightarrow \mathrm{Arf}(J) = 0$)

Then $R_\eta(J) \in \mathcal{F}_1$, $R_\eta(R_\eta(J)) \in \mathcal{F}_2$, ... $R_\eta^n(J) \in \mathcal{F}_n$

For lin. indep, change J, R (COT, CT, CHL), or take cables (DPR)

Proof: if $R \in \mathcal{F}^U_n$, $\eta \in \pi_1(S^3 \setminus R)^{(k)}$ then $R_\eta(\mathcal{F}^U_{n-k}) \subseteq \mathcal{F}^U_n$



R

Let $J \in \mathcal{F}^U_{n-k}$ $\quad R \subseteq S^3 \setminus \eta \times D^2$

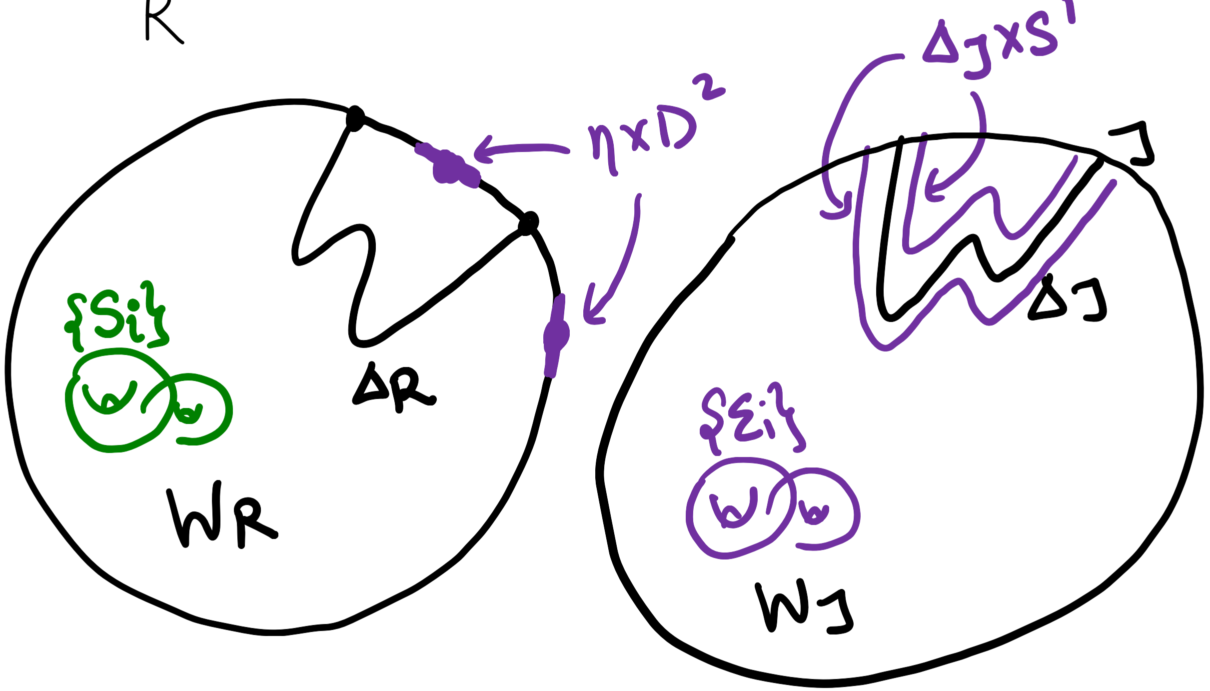
$R = \partial \Delta_R$ in some W_R n-solution

$J = \partial \Delta_J$ in some W_J $(n-k)$-solution

$$W_{R_\eta(J)} = W_R \cup W_J \setminus (\Delta_J \times D^2)$$

$\eta \times D^2 = \Delta_J \times S^1$

$\eta \mapsto \mu_J$



Check: $W_{R_\eta(J)}$ is an n-solution

Image of Δ_R in $W_{R_\eta(J)}$ is bounded by $R_\eta(J)$

$\{S_i\} \cup \{\Sigma_i\}$ are n-surfaces

Obstructions

M^3 closed, oriented, Γ discrete group

define $\rho(M, \varphi : \pi_1 M \longrightarrow \Gamma) := \sigma_\Gamma^{(2)}(W, \psi) - \sigma(W)$

where $M = \partial W^4$. W compact, oriented

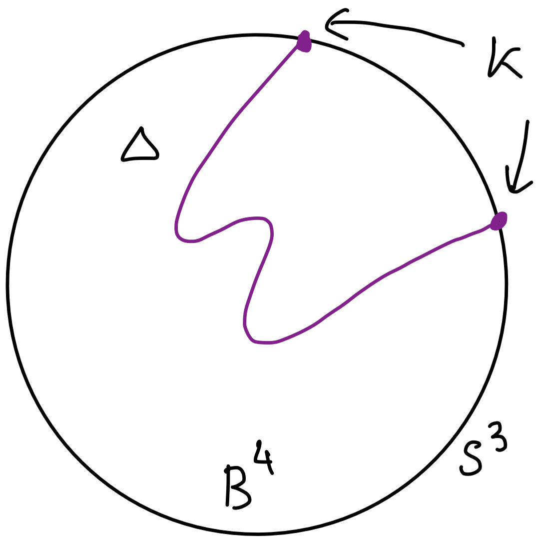
$$\begin{tikzcd}
\pi_1 M \arrow[r, "\varphi"] \arrow[d, "i_*"'] & \Gamma \\
\pi_1 W \arrow[ur, "\psi"'] &
\end{tikzcd}$$

[Cochran-Orr-Teichner] $K \in \mathcal{F}_{n.5}$ and Γ is PTFA with $\Gamma^{(n+1)} = 0$

then $\rho(S^3_0(K), \varphi : \pi_1(S^3_0(K)) \longrightarrow \Gamma) = 0$

Other approximations?

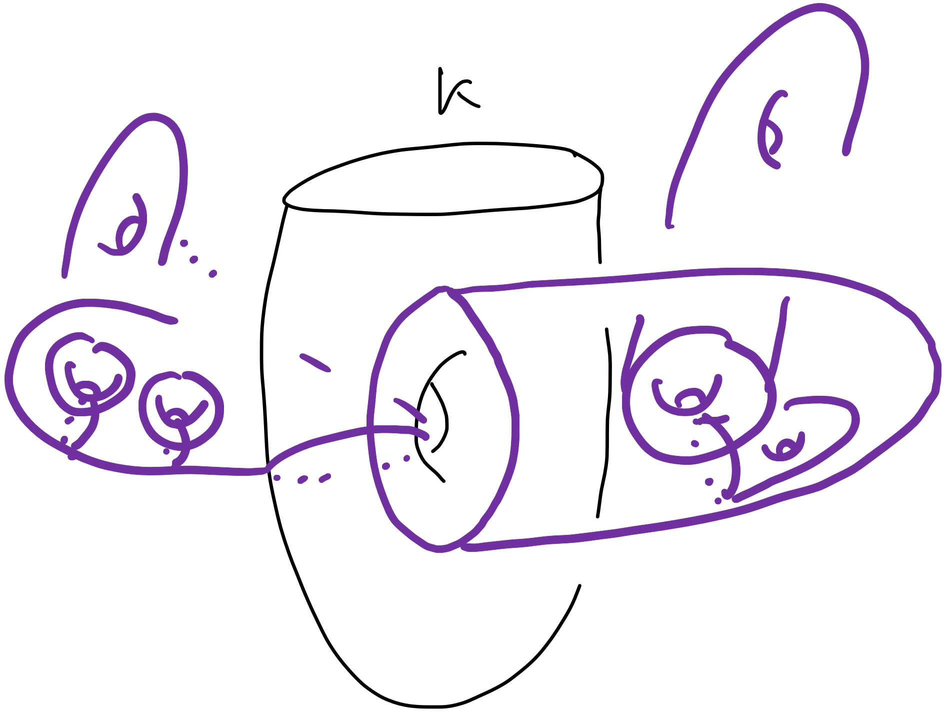
Motivation: wish to approximate sliceness



K is slice if it bounds a disc inside B^4

Option 2 approx this

Gropes

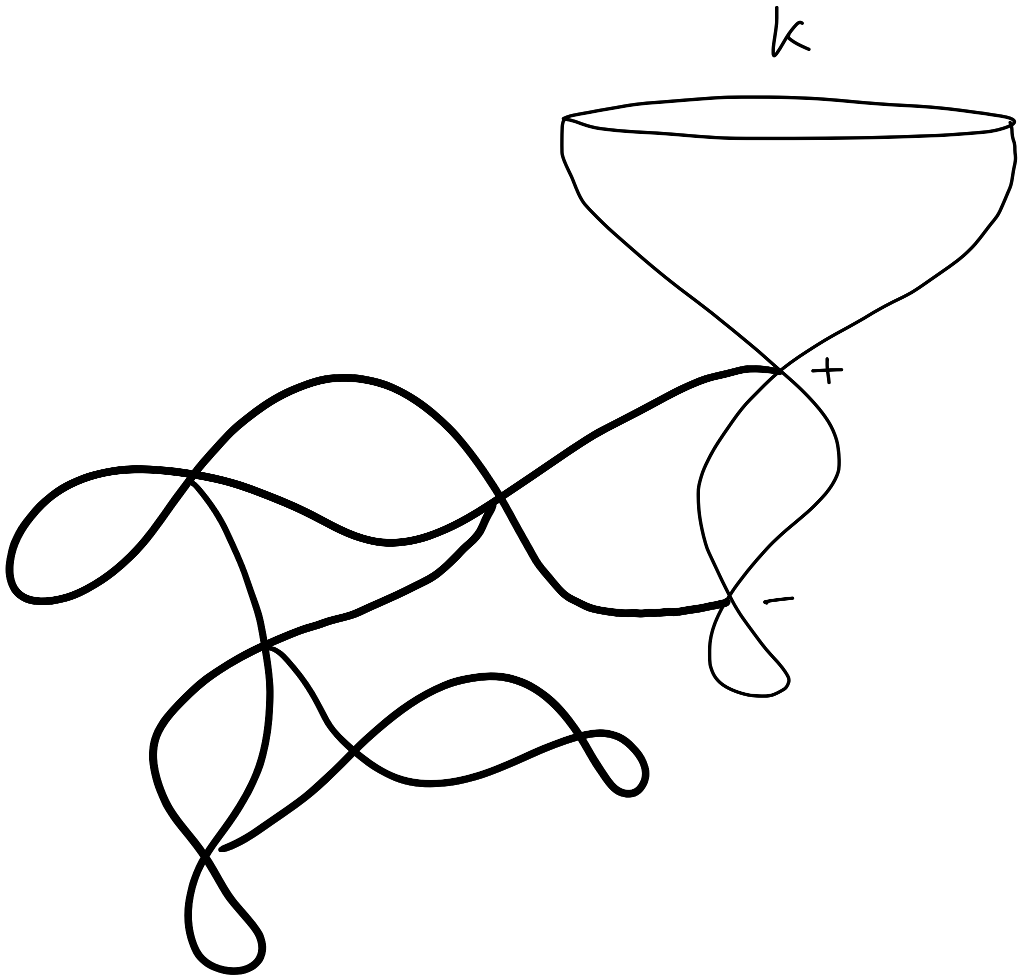


grope of height n has n layers of surfaces

$K \in \mathcal{G}_n$ if it bounds a height n grope in B^4.

K slice $\Longrightarrow K \in \mathcal{G}_n \ \forall n$

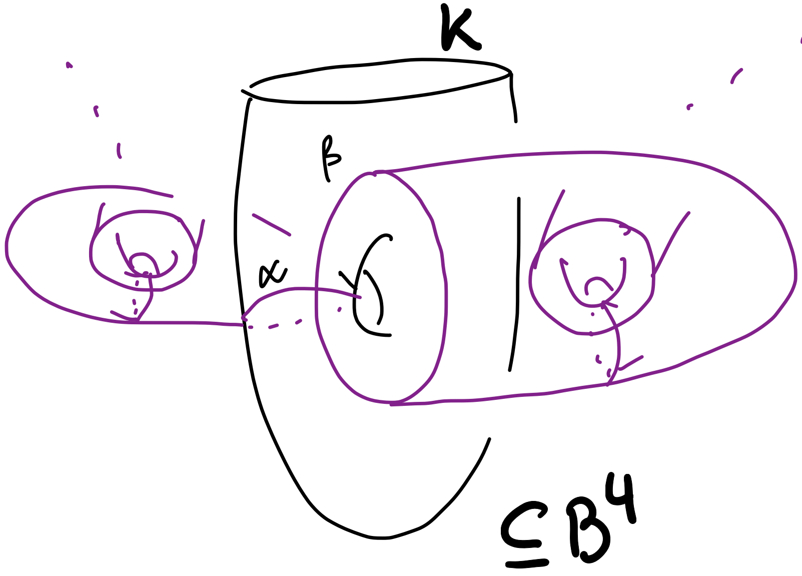
Whitney towers



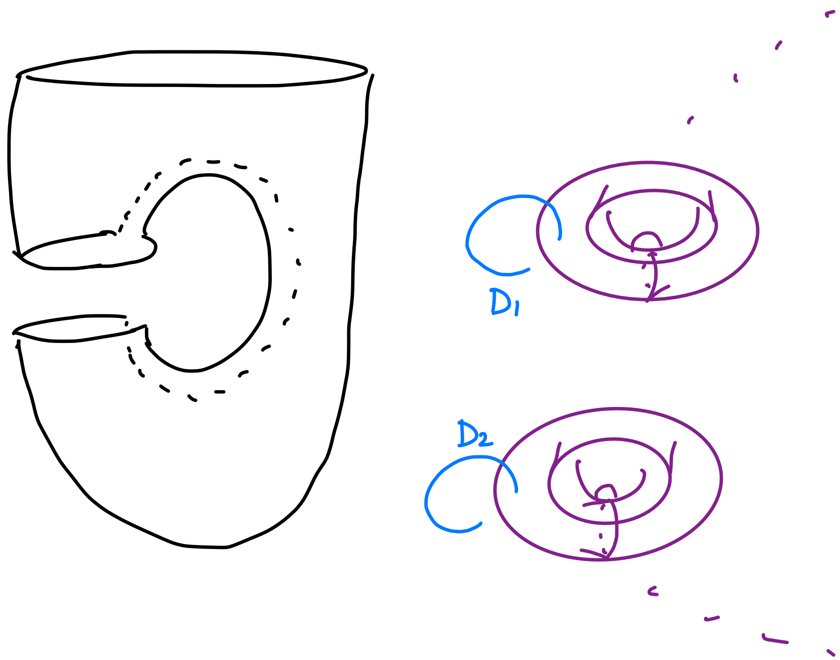
$K \in W_n$ if it bounds a height n Whitney tower in B^4

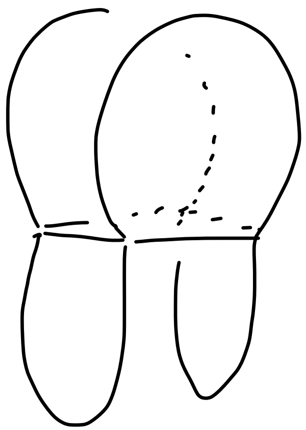
K slice $\Longrightarrow$ $K \in W_n \; \forall n$

$\mathcal{G}_{n+2} \subseteq \mathcal{F}_n \ \forall n$ [Cochran-Orr-Teichner]



$\longrightarrow$ $\setminus S^1 \times D^3 \cup D^2 \times S^2$ along α, β.





Summary

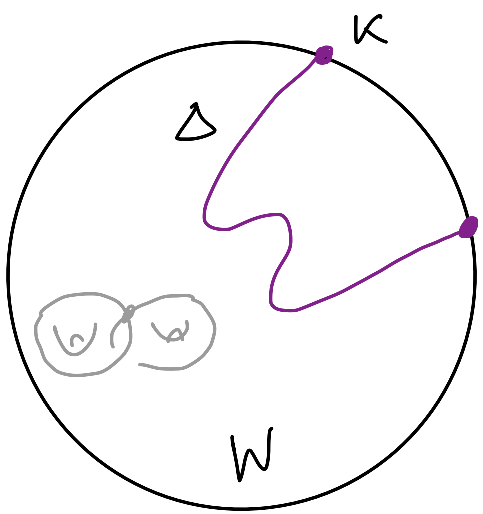
- surger B^4 to $\# S^2 \times S^2$
- surger first stage of grope to a disc bounded by K
- second stage surfaces (and spheres coming from surgery) become the n-surfaces

Similarly $W_{n+2} \subseteq \mathcal{F}_n \ \forall n$

Smooth vs topological concordance

$$\mathcal{F}_n^{sm} / \mathcal{F}_{n.5}^{sm} \cong \mathcal{F}_n^{TOP} / \mathcal{F}_{n.5}^{TOP} \quad \forall n$$

$$K \in \mathcal{F}_n^{sm} \iff K \in \mathcal{F}_n^{TOP}$$



$K = \partial \Delta$ with $\Delta \subseteq W^4$ TOP

Check: $ks(W) = 0$

Positive / negative / bipolar filtrations

$K \in \mathcal{P}_n$ if it bounds a disc Δ in a smooth W^4 s.t. $\partial W = S^3$

1. $\pi_1(W) = 0$

2. $H_2(W)$ gen by embedded surfaces $\{S_i\}$, $S_i \subseteq W \setminus \Delta$
$S_i \cap S_j = \emptyset \ \forall i \neq j$
intersection form is positive definite
↳ by Donaldson, int form $= \oplus [1]$
[for $\mathcal{F}_n$, int form $= \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$]

3. $\pi_1(L_i) \subseteq \pi_1(W \setminus \Delta)^{(n)}$
$\pi_1(D_i) \subseteq \pi_1(W \setminus \Delta)^{(n)}$

$K \in \mathcal{N}_n$ if exactly as above, except negative definite

$K \in \mathcal{B}_n$ if $K \in \mathcal{P}_n \cap \mathcal{N}_n =: \mathcal{B}_n$

Smooth vs topological concordance

Let $\mathcal{T} := \{$smooth concordance classes of topologically slice knots$\}$

Define $\mathcal{T}_n := \mathcal{B}_n \cap \mathcal{T}$ $\forall n$

$$\left.\begin{array}{l}\text{Cochran-Harvey-Horn} \\ \text{Cochran-Horn} \\ \text{Cha-Kim} \\ \text{(see also Cha-Powell)}\end{array}\right\} \exists\, \mathbb{Z}^\infty \subseteq \mathcal{T}_n/\mathcal{T}_{n+1} \quad \forall n$$

Miscellaneous results and open questions.

Generalisations

- Links?
  - string link concordance group
  - define $\mathcal{F}_n$, $\mathcal{G}_n$, W_n, P_n, N_n, B_n, T_n similarly.

- Double concordance group
  - analogues for $\mathcal{F}_n$, $\mathcal{G}_n$, W_n. [T. Kim, Cha-Kim]
  - smooth vs TOP?

Nontriviality

- $\exists\, \mathbb{Z}^\infty \oplus \mathbb{Z}/2^\infty \subseteq \mathcal{G}_n/\mathcal{G}_{n+1}$ $\forall n$ [Horn, Jang]

- $\mathcal{F}^m_n / \mathcal{G}^m_{n+2} \neq 0$ for $m \geqslant 2^{n+2}$ [Otto]

  What about for knots?

- $\mathcal{F}^m_{n.5} / \mathcal{F}^m_{n+1} \neq 0$ for $m \geqslant 3 \cdot 2^{n+1}$ [Otto]

  What about for knots?

  Every genus 1 knot in $\mathcal{F}_{0.5}$ is in $\mathcal{F}_1$ [Davis-Martin-Otto-Park]

- $\mathcal{G}_n \subseteq W_n \ \forall n$ [Schneiderman]

  are they equal?

- Geometric analogue for $\mathcal{P}_n, \mathcal{N}_n, \mathcal{B}_n$?
  - in terms of Casson towers [R.]
  - **is there a better version?**

- Does there exist $\mathbb{Z}/2^\infty \subseteq \mathcal{T}_n / \mathcal{T}_{n+1} \ \forall n$?
  - $\mathbb{Z}/2^\infty \subseteq \mathcal{T}_0 / \mathcal{T}_1$ [Chen]

- $\{\text{TOP slice}\} \subseteq \bigcap \mathcal{F}_n$

  are they equal?

  $$\{\textbf{TOP slice}\} \subseteq \bigcap \mathcal{G}_n \subseteq \bigcap \mathcal{F}_n$$

Characterisation

- $\mathcal{F}_0^{\nu} = \{ K \mid \mathrm{Arf}(K) = 0 \}$, [Cochran-Orr-Teichner]
- $\mathcal{F}_{0.5}^{\nu} = \{ K \mid \text{alg slice} \}$

- $\mathcal{F}_0^{\nu m}$ characterised via Milnor invts [Martin]

- $\mathcal{F}_{0.5}^{\nu m}$? P_0, N_0, B_0?
  - $\tilde{P}_0$ in terms of gen. crossing changes [Cochran-Tweedy]

Interaction with other properties

- $\exists ?\ K \in \mathcal{T}_n$ with large g_4?

  $n=2$ [Cha-Miller-Powell]

  smooth version?

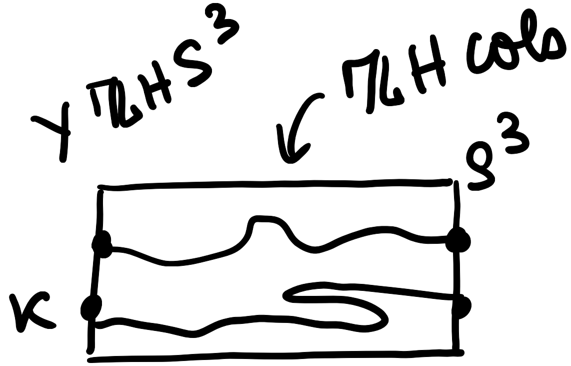
$\mathbb{Z}HS^3$ conc

$(Y,K) \simeq (S^3,J)$

if $\exists$ $\mathbb{Z}H$ cob in which $\exists$ annulus

- $\exists ?\ K \in \mathcal{T}_n,\ K \not\simeq K^r$?

  ⋮



Proxy for sliceness/concordance

- Is every knot in a $\mathbb{Z}HS^3$ TOP conc. to a knot in S^3?

  Yes, "up to solvable filtration" [Davis]

Questions?