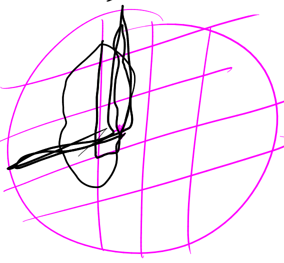
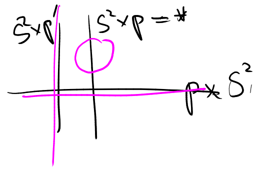
Diffeomorphisms of S^4, Cerf theory and Montesinos twins

$\mathcal{S}_n \subset (\mathcal{S}_n) \supset \mathcal{S}_n^0$: transverse to $\amalg^n p \times S^2$

$\mathcal{S}_n$ = transverse to $\amalg^n S^2 \times p'$, $p' \neq p$





$\mathcal{S}_n$, $* = \amalg^n S^2 \times p$

$\pi_1\left(\mathrm{Emb}(\amalg^n S^2, \#^n S^2 \times S^2)\right)$

$\pi_1(\mathcal{S}_n)$ → $\pi_1(\mathcal{S}_n^0)$

surjective as $n \to \infty$

$\mathrm{Im}(\pi_1(\mathcal{S}_n)) \subset \pi_0(\mathrm{Diff}^+(S^4))$

0

$\| \quad \subset ?$

$\mathrm{Im}(\pi_1(\mathcal{C}_n)) = \mathrm{Im}(\pi_1(\mathcal{C}_1))$

image for n is the same as image for $n=1$

$\mathcal{C}_n = \pi_1\left(\mathrm{Emb}(\amalg^n S^1, \#^n S^1 \times S^3)\right)$

$\supset$ {twist along $S^1 \times \Sigma_g \hookrightarrow S^4$} generated by

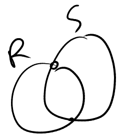
$[-1,1]\times S^1 \times \Sigma_g$ — Dehn twist, id

$\cup$

$\{S^1 \times T^2 \hookrightarrow S^4\}$ R

$\cup$

{boundary of nbhd of Montesinos twin $R, S: S^2 \hookrightarrow S^4$ intersect twice}



$W = (R,S)$

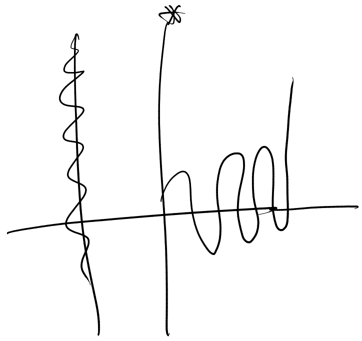
τ_W = twist along W

$\cup$ relations

$\{\tau_W \mid W = (R,S),\ S = \text{unknotted}\}$

$\not\cong\ \|$

$\mathrm{gen\ by}\ \{\tau_{W(i)} \mid i = 1, 2, \ldots\}$



$S^2 \hookrightarrow S^2 \times B^2 \to S^2 \times S^2$

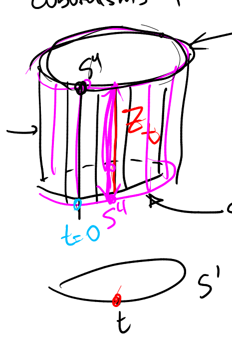
$\pi_1\left(\mathrm{Emb}(\amalg^n S^2, \#^n S^2 \times S^2)\right) \xrightarrow{\mathcal{H}} \pi_0(\mathrm{Diff}^+(S^4))$ $\|$ $\times$

$\alpha_t : \amalg^n S^2 \hookrightarrow \#^n S^2 \times S^2$

build a bundle over S^1 of cobordisms from S^4 to $X^4 \cong S^4$

$S^4 \times_\varphi S^1$ $\quad \varphi : S^4 \to S^4$

$[\varphi] := \mathcal{H}([\alpha_t])$



$Z_t = [0,1] \times S^4$ (vertical), add n 5-dim'l 2-handles in fixed std way,

top $\partial = \#^n S^2 \times S^2$, attach n 5-dim'l 3-handles using α_t

$Z_0 = [0,1] \times S^4$

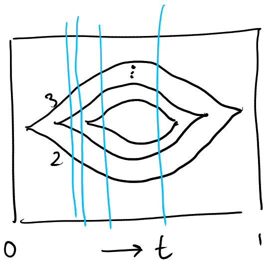
Surjectivity: $\forall\, \varphi : S^4 \to S^4$ $\exists\, \Phi : [0,1]\times S^4 \to [0,1] \times S^4$

$\Phi(0,p) = (0,p)$

$\Phi(1,p) = (1, \varphi(p))$

$f_0 : [0,1] \times S^4 \to [0,1]$ = projection

$f_1 = \Phi^* f_0$ $\qquad$ f_t = generic homotopy of Morse functions



Turning 2/3 pairs into 1/2 pairs

<u>in 4-d</u> $\quad$ (dotted circle) = add a 1-handle = remove D^2 after pushing into B^4

<u>in dim = 5</u> $\quad$ adding a 2-handle (which could be cancelled) = dotted circle (dotted disk)

adding a 1-handle = dotted sphere (dotted 3-ball)

2/3 pair = dotted S^1 + attaching 2-sphere

(switch dots)

1/2 pair = dotted 2-sphere + attaching circle

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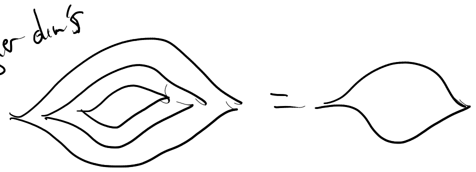
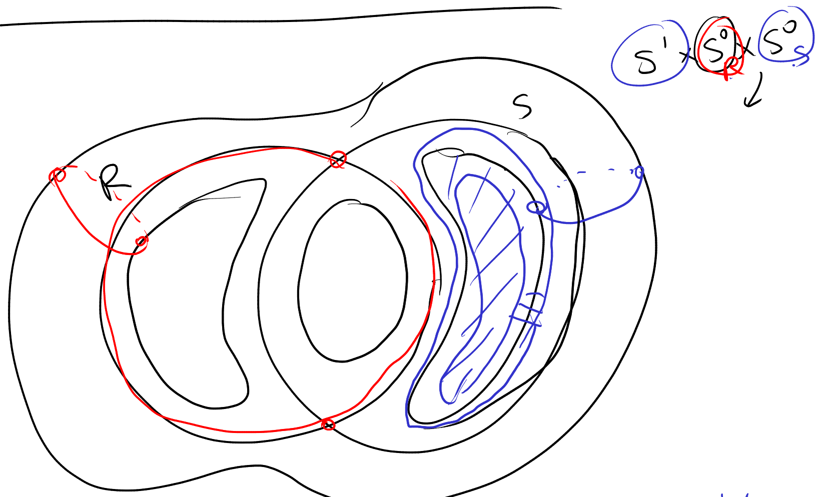
in higher dim's



if 1,2 pairs in dim 5 ✓



$W = (R, S)$

$\partial \nu W = S^1_\lambda \times S^1_R \times S_S$

$(\partial \nu W) = [-1,1] \times S^1_\lambda \times S^1_R \times S^1_W$

τ_W = Dehn twist × id

Given a torus (R, S) with S unknotted

& surgery along S ⟶

$(R, S) \to$ torus in $S^1 \times S^3$

surgery on S turns S^4 into $S^1 \times S^3$

turns R into $S^1 \times S^1$