Operad formality and rational homology of embedding spaces

Building bridges seminar

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December 2, 2020

Some manifold categories that we work with

Definition

Fix a dimension m. Consider the following sequence of categories of m-dimensional manifolds and "inclusions" between them

$$\mathtt{Disk}_1 \subset \mathtt{Disk}_2 \subset \cdots \subset \mathtt{Disk}_k \subset \cdots \subset \mathtt{Disk} \subset \mathcal{M}\mathit{fld}$$

Where

- $\mathcal{M}\mathit{fld}$ be the category of (smooth, not necessarily closed) m-dimensional manifolds and embeddings between them.
- $\mathtt{Disk} \subset \mathcal{M}\mathit{fld}$ is the full subcategory consisting of finite disjoint unions of copies of D^m - the open unit ball in $\mathbb{R}^m$.
- $\mathtt{Disk}_k \subset \mathtt{Disk}$ is the full subcategory consisting of unions of at most k copies of D^m.

Basic idea: extrapolating from $\mathtt{Disk}_k$ to $\mathcal{M}fld$

Suppose we have a contravariant functor $F\colon {\mathcal{M}_m}^{\mathrm{op}} \to \mathrm{Top}$ that we are interested in. Such as $M \mapsto \mathrm{emb(M, N)}$.

Fix a $k \geq 1$. There is a map, natural in M

$$F(M) \to \underset{U \in \mathtt{Disk}_k}{\mathrm{hNat}}(\mathrm{emb}(U, M), F(U)).$$

Definition

We denote that target of this map by $T_k F(M)$.

- $T_k F$ is the "k-th Taylor polynomial of F".
- $T_k F$ is the extrapolation of F from $\mathtt{Disk}_k$ to $\mathcal{M}fld$.
- From this point of view, one might like to think of $T_k F$ as analogous to the interpolation polynomial that agrees with f at the points $0, 1, \ldots, k$.

Reduction to framed embeddings

Definition

Recall that $D^m \subset \mathbb{R}^m$ is the open unit ball. A *standard embedding* $D^m \to D^m$ is an embedding that is the composition of multiplication by a positive scalar and translation. A standard embedding is determined by its image.

Definition

Let $\mathtt{sDisk} \subset \mathtt{Disk}$ be the category consisting of the same objects, but whose morphisms are embeddings that are standard on each disk. Let $\mathrm{semb}(U, M)$ denote the space of standard embeddings from U to M. This is well-defined if U is a union of copies of D^m and $M \subset \mathbb{R}^m$ is an open subset.

Lemma

Suppose $M \subset \mathbb{R}^m$ is an open subset. Then there is an equivalence

$$T_k F(M) \simeq \underset{U \in \mathtt{sDisk}_k}{\mathrm{hNat}} (\mathrm{semb}(U, M), F(U))$$

A favorite example: embeddings modulo immersions

Definition

Suppose we have a manifold M. Define

$$\overline{\text{emb}}(M, \mathbb{R}^n) := \text{hofiber}(\text{emb}(M, \mathbb{R}^n) \to \text{imm}(M, \mathbb{R}^n))$$

- $\overline{\text{emb}}$ measures the difference between the space of embeddings and the space of immersion. In some sense it is the "topological" part of the space of embeddings.
- For the space $\overline{\text{emb}}(M, \mathbb{R}^n)$ to be defined, one needs to fix at least an immersion $M \looparrowright \mathbb{R}^n$. Therefore $\overline{\text{emb}}(-, \mathbb{R}^n)$ is not a well-defined presheaf on $\mathcal{M}\mathit{fld}$. One can fix an immersed manifold $M \hookrightarrow \mathbb{R}^n$ and consider $\overline{\text{emb}}(-, \mathbb{R}^n)$ to be a presheaf on M. But it is not "context-free".
- Nevertheless, if we have inclusions $M \overset{\mathit{open}}{\hookrightarrow} \mathbb{R}^m \overset{\mathit{linear}}{\hookrightarrow} \mathbb{R}^n$, then there is an equivalence of presheaves on M

$$\overline{\text{emb}}(-, \mathbb{R}^n) \simeq \text{semb}(-, \mathbb{R}^n).$$

As a result, we get the following theorem.

Theorem

Suppose we have inclusions $M \overset{open}{\hookrightarrow} \mathbb{R}^m \overset{linear}{\hookrightarrow} \mathbb{R}^n$*. Then there is an equivalence, for each* k

$$T_k\overline{\mathrm{emb}}(M, \mathbb{R}^n) \simeq \underset{U \in \mathtt{sDisk}_k}{\mathrm{hNat}}(\mathrm{semb}(U, M), \mathrm{semb}(U, \mathbb{R}^n)).$$

This description of $T_k\overline{\mathrm{emb}}(M, \mathbb{R}^n)$ can be interpreted in terms of the *unframed* little disks operad E_m.
The category `sDisk` is the PROP of the operad E_m.

Presheaves on `sDisk` are the same thing as *right modules* over E_m.

In particular, the presheaves $\mathrm{semb}(-, M)$, $\mathrm{semb}(-, \mathbb{R}^n)$ can be thought of as right modules over E_m, and the space of derived natural transformations between them is the same as the derived mapping spaces in the category of right E_m-modules.

Variation: space of (high dimensional) long knots

Definition

Fix a linear inclusion $\mathbb{R}^m \subset \mathbb{R}^n$. Let $B^m \subset \mathbb{R}^m$ be the closed unit ball. Let $\mathrm{emb}_\partial(B^m, B^n)$ be the space of smooth embeddings that agree with the inclusion near the boundary.

In a similar way define $\mathrm{imm}_\partial(B^m, B^n) \simeq \Omega^m O(n)/O(n-m)$, and

$$\overline{\mathrm{emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) = \mathrm{hofiber}(\mathrm{emb}_\partial(B^m, B^n) \to \mathrm{imm}_\partial(B^m, B^n)).$$

There is a version of the Taylor tower for functors such as this. It turns out that these Taylor towers also can be expressed in operadic terms. In particular, we have the following description of the Taylor tower of $\overline{\mathrm{emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n)$

Theorem (A. - Turchin)

$T_\infty \overline{\mathrm{emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n)$ is equivalent to the derived mapping space from E_m to E_n in the category of infinitesimal bimodules over E_m.

A homological version of Taylor towers

There is a homological version of these functors and polynomial approximations. For example, we have the functor $M \mapsto C_*\overline{\mathrm{emb}}(M, \mathbb{R}^n)$. If M is an open subset of a vector subspace $\mathbb{R}^m \subset \mathbb{R}^n$ then its Taylor tower can be described as follows.

$$T_\infty C_*\overline{\mathrm{emb}}(M, \mathbb{R}^n) \simeq \mathrm{hRightMod}_{C_* E_m}(C_* \,\mathrm{semb}(-, M), C_* E_n)$$

Similarly, we have the following model for the singular chains on the space of long knots.

$$T_\infty C_*\overline{\mathrm{emb}}_\partial(B^m, B^n) \simeq \mathrm{hIBimod}_{C_* E_m}(C_* E_m, C_* E_n).$$

Remark

Homological towers have weaker convergence properties than homotopical towers. While the Taylor tower of functors like $\mathrm{emb}(M, N)$ *converge when* $\dim(N) - \dim(M) \geq 3$*, the Taylor towers of functors like* $C_*\overline{\mathrm{emb}}_\partial(B^m, B^n)$ *are only known to converge when* $2m + 1 \leq n$*.*

The formality of E_m

The advantage of the operadic viewpoint, and in particular of expressing the Taylor tower in terms of modules over E_m, is that it allows us utilize the *formality* of E_m. This is a deep fact about the little disks operad, that has rather far reaching consequences regarding the rational homotopy type of spaces like $\overline{\mathrm{emb}}(M, \mathbb{R}^n)$ and $\overline{\mathrm{emb}}_\partial(B^m, B^n)$

Theorem (Kontsevich, Lambrechts-Volic, Turching-Willwacher, Fresse-Willwacher, . . .)

There is a quasi-isomorphism of operads

$$C_*(E_m) \otimes \mathbb{Q} \cong H_*(E_m; \mathbb{Q})$$

Moreover, if $\mathbb{R}^m \subset \mathbb{R}^n$ *and* $n - m \neq 1$ *then there is a quasi-isomorphism of pairs of operads*

$$(C_*(E_n) \otimes \mathbb{Q}, C_*(E_m) \otimes \mathbb{Q}) \cong (H_*(E_n; \mathbb{Q}), H_*(E_m; \mathbb{Q}))$$

Consequences of formality

Theorem (A. - Lambrechts - Volic, A. - Turchin)

There are equivalences

$$T_\infty C_* \overline{\mathrm{emb}}(M, \mathbb{R}^n) \otimes \mathbb{Q} \simeq \mathrm{hRightMod}_{C_* E_m}(C_* \,\mathrm{semb}(-, M), H_*(E_n, \mathbb{Q}))$$

$$T_\infty C_* \overline{\mathrm{emb}}_\partial(B^m, B^n) \otimes \mathbb{Q} \simeq \mathrm{hIBimod}_{C_* E_m}(C_* E_m, H_*(E_n, \mathbb{Q})).$$

Here the $C_* E_m$-module structure on $H_*(E_n; \mathbb{Q})$ is essentially trivial. I.e., it factors through the operad map $E_m \to \mathrm{Com}$. It follows that the formulas can be recast in terms of modules over the commutative operad, using a standard "change of ~~rings~~ operads isomorphism".

Corollary

$$T_\infty C_* \overline{\mathrm{emb}}(M, \mathbb{R}^n) \otimes \mathbb{Q} \simeq \mathrm{hRightMod}_{\mathrm{Com}}(C_* M^-, H_*(E_n, \mathbb{Q}))$$

$$T_\infty C_* \overline{\mathrm{emb}}_\partial(B^m, B^n) \otimes \mathbb{Q} \simeq \mathrm{hIBimod}_{\mathrm{Com}}((S^m)^-, H_*(E_n, \mathbb{Q})).$$

In fact, it might be convenient to recast the formula back in terms of natural transformations between functors on the Prop of the commutative operad.

The Prop of the commutative operad is the category of finite sets $\mathcal{F}$.

Right modules over Com are presheaves on $\mathcal{F}$.

It turns out that infinitesimal bimodules over Com are the same thing as presheaves on the category of pointed finite sets Γ.

Corollary

$$T_\infty C_*\overline{\mathrm{emb}}(M, \mathbb{R}^n) \otimes \mathbb{Q} \simeq \mathrm{hNat}_{\mathcal{F}}(C_* M^-, H_*(E_n, \mathbb{Q}))$$

$$T_\infty C_*\overline{\mathrm{emb}}_\partial(B^m, B^n) \otimes \mathbb{Q} \simeq \mathrm{hNat}_\Gamma((S^m)^-, H_*(E_n, \mathbb{Q})).$$

There is one further reduction in the case of $\overline{\mathrm{emb}}_\partial$.

From pointed sets to epimorphisms

Let $\mathcal{E}$ be the category of finite sets and epimorphisms. It turns out that there is an equivalence of functor categories, where Ab can be any abelian category (Pirashvili)

$$\begin{array}{ccc} [\Gamma, \mathrm{Ab}] & \xrightarrow{\cong} & [\mathcal{E}, \mathrm{Ab}] \\ F(-) & \mapsto & cr_{-}F \end{array}$$

Corollary

There is an equivalence

$$T_\infty C_* \overline{\mathrm{emb}}_\partial(B^m, B^n) \otimes \mathbb{Q} \simeq \mathrm{hNat}_{\mathcal{E}}(H_*(S^{m\cdot -}), cr_{-}H_*(E_n, \mathbb{Q})).$$

We also proved an analogous statement about the rational homotopy of $\overline{\mathrm{emb}}_\partial(B^m, B^n)$

Theorem (A. - Turchin)

The groups $\pi_* \left(T_\infty \overline{\mathrm{emb}}_\partial(B^m, B^n) \right) \otimes \mathbb{Q}$ *are isomorphic to the homotopy groups of the following chain complex*

$$\mathrm{hNat}_{\mathcal{E}}(H_*(S^{m\cdot -}), cr_- pi_*(E_n) \otimes \mathbb{Q})$$

The last two theorems give rise to a *double splitting* of $H_*(\overline{\mathrm{emb}}_\partial(B^m, B^n); \mathbb{Q})$, and even of $\pi_*(\overline{\mathrm{emb}}_\partial(B^m, B^n)) \otimes \mathbb{Q}$, as the homology of a direct sum of explicitly defined graph complexes.