Homotopy limits for the working low-dimensional/differential topologist

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Why this talk ?

Goodwillie-Klein-Weiss theory expresses the space of smooth embeddings between two manifolds as some homotopy limit of a diagram:

Theorem (Goodwillie-Klein)

If $\dim(W) > \dim(M) + 2$ then

$$\mathrm{Emb}(M, W) \simeq \mathrm{holim}(O \mapsto \mathrm{Emb}(O, W))$$

where in this homotopy limit O runs over all open subsets of M diffeomorphic to a disjoint union of finitely many copies of $\mathbf{R}^m$.

This is usefull: the terms in the holim are

$$\mathrm{Emb}(O, W) \cong \mathrm{Emb}(\coprod_{i=1}^{k} \mathbf{R}^m, W) \simeq \mathrm{Conf}(k, W) \tilde{\times} (\mathrm{St}_m(\mathbf{R}^{\dim(W)}))^k$$

which are pretty computable spaces.

The goal of the talk is to give some grip on the notion of homotopy limits (and hocolim, derived mapping spaces,. . .)

The slogan

Homotopy limits (or colimits; or derived mapping spaces) are a variation of usual limits (or colimits; or mapping spaces) which preserves (weak) homotopy equivalences.

Eventually this will be achieved by replacing the diagram of the limit by a homotopy equivalent fibrant one (or cofibrant; or a cofibrant source and fibrant target)
We will explain this at the level of topological spaces. There are analogous notions in all categories in which homotopy quivalences make sense i.e. Quillen categories, e.g categories of chain complexes, or of operads in chain complexes, or of CDGAs, or . . .

A classical colimit: the pushout

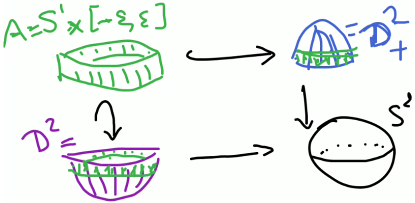
Consider a diagram of topological spaces as follows

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \\ Y & & \end{array}.$$

Its colimit colim($X \xleftarrow{f} A \xrightarrow{g} Y$) is the pushout

$$X \cup_A Y := \frac{X \amalg Y}{\simeq} \quad \text{with } f(a) \simeq g(a) \text{ for } a \in A$$

Example: $X = D^2_+$, $Y = D^2_-$ two disks and $A = S^1 \times [-\varepsilon, \varepsilon]$ a cylinder that maps to the disks as inclusion of a collar of the boundary circles.
Then $D^2_+ \cup_A D^2_- \cong S^2$.
We get a "push-out square"



A classical limit: the pullback

Consider a diagram of topological spaces as follows

$$\begin{array}{ccc} & & E \\ & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & B. \end{array}$$

Its limit $\lim(X \xrightarrow{f} B \xleftarrow{p} E)$ is the pullback

$$X \times_B E := \{(x,e) \in X \times E \,:\, f(x) = p(e)\}$$

An application to differential topology: the sphere eversion

Theorem ("Eversion of the sphere", Smale, 1957)

The standard embedding $i\colon S^2 \hookrightarrow \mathbf{R}^3$ and the reversed embedding $R \circ i\colon S^2 \hookrightarrow \mathbf{R}^3$, where $R(x,y,z) = (x,y,-z)$, are connected by a path of immersions.

An explicit construction of the actual path of immersions between the standard sphere and the everted is pretty complicated to describe...
But the clever Smale's strategy of proof is more elegant

Smale's strategy of proof of the eversion theorem

The eversion of the sphere is an immediate corollary of:

Theorem (Smale 1957)

The space $\mathrm{Imm}(S^2, \mathbf{R}^3) := \{f : S^2 \looparrowright \mathbf{R}^3\}$ is path-connected

Strategy of proof:

1. write $\mathrm{Imm}(S^2, \mathbf{R}^3)$ as a pullback
2. show that the terms of the pullback are homotopy equivalent to spaces easier to understand
3. use standard formula that computes the homotopy groups of a pullback from the homotopy groups of its terms (dual to the Mayer-Vietoris formula for the homology of a push-out)
4. deduce that $\pi_0(\mathrm{Imm}(S^2, \mathbf{R}^3)) = \{*\}$

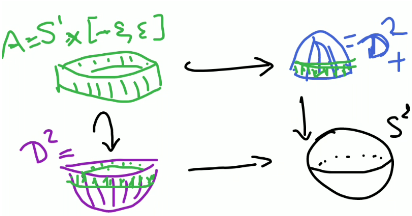
There is a pitfall after step 2: we would like that pullback preserves homotopy equivalence (which it doesn't)

Smale's proof - step 1: $\mathrm{Imm}(S^2, \mathbf{R}^3)$ as a pullback

The 1st step is the standard technique in algebraic topology: cut the space (here S^2) into simpler pieces:

Recall that S^2 is a push out

$$(\star)\qquad \begin{array}{ccc} A = S^1 \times [-\varepsilon, \varepsilon] & \xhookrightarrow{i_+} & D^2_+ \\ {\scriptstyle i_-}\big\downarrow & p.o. & \big\downarrow{\scriptstyle j_+} \\ D^2_- & \xrightarrow{j_-} & S^2 \end{array}$$



Apply the contravariant functor $\mathcal{I} := \mathrm{Imm}(-, \mathbf{R}^3)$ to diagram $(\star)$:

$$\mathcal{I}(\star): \qquad \begin{array}{ccc} \mathcal{I}(A) & \xleftarrow{i_+^*} & \mathcal{I}(D^2_+) \\ {\scriptstyle i_-^*}\big\uparrow & p.b. & \big\uparrow{\scriptstyle j_+^*} \\ \mathcal{I}(D^2_-) & \xleftarrow{j_-^*} & \mathcal{I}(S^2) \end{array}$$

$\mathcal{I}(\star)$ is a pullback square because

$$\begin{aligned} &\mathcal{I}(D^2_+) \times_{\mathcal{I}(A)} \mathcal{I}(D^2_-) \\ =&\{(f_+ : D^2_+ \looparrowright \mathbf{R}^3, f_- : D^2_- \looparrowright \mathbf{R}^3) \text{ s.t.} \\ &\qquad\qquad \text{s.t. } f_+|A = f_-|A\} \\ =&\,\mathrm{Imm}(S^2, \mathbf{R}^3) = \mathcal{I}(S^2) \end{aligned}$$

Smale's proof - step 2: homotopy equivalent pieces in the pb

Lemma

$\mathcal{I}(D^2_\pm) \simeq SO(3)$ and
$\mathcal{I}(A) \simeq \mathrm{map}(S^1, SO(3))$

$$\mathcal{I}(\star): \quad \begin{array}{ccc} \mathcal{I}(A) & \xleftarrow{i_+^*} & \mathcal{I}(D^2_+) \\ \uparrow i_-^* & p.b. & \uparrow j_+^* \\ \mathcal{I}(D^2_-) & \xleftarrow{j_-^*} & \mathcal{I}(S^2) \end{array}$$

We explain the first homotopy equivalence. We have a map

$$\Phi\colon\ \mathrm{Imm}(D^2,\mathbf{R}^3) \xrightarrow{\simeq} \mathrm{St}_2(\mathbf{R}^3)\,,\ f \longmapsto \left(df(0)(\frac{\partial}{\partial x}), df(0)(\frac{\partial}{\partial y})\right)$$

where $\mathrm{St}_2(\mathbf{R}^3)$ is the Stiefel manifold of 2-frames in $\mathbf{R}^3$:

$$\mathrm{St}_2(\mathbf{R}^3) := \left\{(v,w) \in \mathbf{R}^3\ :\ v \text{ and } w \text{ are lin. indep.}\right\} \overset{\text{Gram-Schmidt}}{\simeq} SO(3)$$

and its easy to see that Φ is a homotopy equivalence.

The second h.e. of the lemma is by applying Smale'strategy again cutting the cylinder A into two overlapping disks whose intersection is a disjoint union of two disks (exercise)

Pitfall in the proof: p.b. do not preserve homotopy equiv.

$$\mathcal{I}(\star): \quad \begin{array}{ccc} \mathcal{I}(A) \simeq \mathrm{map}(S^1, SO(3)) & \longleftarrow & \mathcal{I}(D^2_+) \simeq SO(3) \\ \uparrow & p.b. & \uparrow \\ \mathcal{I}(D^2_-) \simeq SO(3) & \longleftarrow & \mathcal{I}(S^2) = \mathrm{Imm}(S^2, \mathbf{R}^3) \end{array}$$

This suggests that $\mathrm{Imm}(S^2, \mathbf{R}^3)$ is homotopy equivalent the pullback

$$(\star) \quad SO(3) \longrightarrow \mathrm{map}(S^1, SO(3)) \longleftarrow SO(3)$$

But pullbacks do not preserve homotopy equivalences:

$$\begin{array}{ccccc|c} & & \text{diagram} & & & \text{pullback} \\ \{0\} & \hookrightarrow & [0,1] & \hookleftarrow & \{1\} & \emptyset \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & \downarrow \not\simeq \\ \{*\} & \xrightarrow{=} & \{*\} & \xleftarrow{=} & \{*\} & \{*\} \end{array}$$

Actually $pb(\star) = SO(3) \times_{\mathrm{map}(S^1, SO(3))} SO(3) = SO(3) \not\simeq \mathrm{Imm}(S^2, \mathbf{R}^3)$

Pullbacks of fibrant diagrams preserve weak equiv.

Proposition

Consider a commutative diagram of topological spaces where vertical maps are weak homotopy equivalence and horizontal maps are Serre fibrations

$$\begin{array}{ccccc} X & \xrightarrow{f} & B & \xleftarrow{p} & E \\ \simeq\downarrow\alpha & & \simeq\downarrow\beta & & \simeq\downarrow\gamma \\ X' & \xrightarrow{f'} & B' & \xleftarrow{p'} & E' \end{array}$$

then the pullbacks of the horizontal lines are weakly homotopy equivalent:

$$\alpha \times_\beta \gamma \colon X \times_B E \xrightarrow{\simeq} X' \times_{B'} E'$$

Proof: use the long exact sequence of homotopy groups π_* for pull-backs (dual to Mayer-Vietoris) and five-lemma.

Definition

A map $p \colon E \twoheadrightarrow B$ is a (Serre) fibration if it has the homotopy lifting property with respect to $Z = [0,1]^k$ for every $k \geq 0$:

$$\begin{array}{ccc} Z \times \{0\} & \xrightarrow{\forall H_0} & E \\ \downarrow & \overset{\exists H}{\nearrow} & \downarrow p \\ Z \times [0,1] & \xrightarrow{\forall h} & B \end{array}$$

Converting a map into a fibration

Proposition (Serre)

Every map $f: X \to Y$ can be (functorially) factored as $f = p \circ w$

$$f : X \xrightarrow[\simeq]{w} \hat{X} \overset{p}{\twoheadrightarrow} Y$$

where w is a weak equivalence and p is a Serre fibration

Proof Set $\hat{X} := \{(x, \eta) \in X \times Y^{[0,1]} : f(x) = \eta(0)\}$, $w: X \to \hat{X}, x \mapsto (x, \mathrm{ctt}_x)$ and $p : \hat{X} \to Y, (x, \eta) \mapsto \eta(1)$.

For example the map

$$\mathrm{ctt}: SO(3) \to \mathrm{map}(S^1, SO(3)),\ u \mapsto (\mathrm{ctt}_u : \vartheta \mapsto u)$$

is not a fibration but it factors as fibration $\circ$ equiv:

$$SO(3) \xrightarrow[\simeq]{ctt} \mathrm{map}(D^2, SO(3)) \overset{p=i^*}{\twoheadrightarrow} \mathrm{map}(S^1, SO(3))$$

Key lemma of Smale and end of proof of the eversion

Smale proves that $\text{Imm}(D^2, \mathbf{R}^3) \twoheadrightarrow \text{Imm}(A, \mathbf{R}^3)$ is a Serre fibration:

Lemma (Smale)

For $d > p$, $i^*: \text{Imm}(D^p, \mathbf{R}^d) \twoheadrightarrow \text{Imm}(S^{p-1} \times [0, \epsilon], \mathbf{R}^d)$ is a Serre fibration.

Proof: Not elementary. See M. Weiss "Immersion theory for homotopy theorists" for an accessible proof.

We can finish Smale eversion theorem proof:

$$\begin{array}{ccccc} \mathcal{I}(D^2_+) & \twoheadrightarrow & \mathcal{I}(A) & \twoheadleftarrow & \mathcal{I}(D^2_-) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ \text{map}(D^2_+, SO(3)) & \twoheadrightarrow & map(S^1, SO(3)) & \twoheadleftarrow & \text{map}(D^2_-, SO(3)) \end{array} \quad \Longrightarrow \quad \begin{array}{c} \text{Imm}(S^2, \mathbf{R}^3) \\ \downarrow \simeq \\ \text{map}(S^2, SO(3)) \end{array}$$

Therefore

$$\pi_0(\text{Imm}(S^2, \mathbf{R}^3)) \cong \pi_0(\text{map}(S^2, SO(3)) \cong \pi_0(SO(3)) \times \pi_2(SO(3)) = \{*\}$$

Some terminology around diagrams and their limits

A diagram $\mathbb{D}: \; X \xrightarrow{f} B \xleftarrow{p} E$ can be seen as a functor $\mathbb{D}\colon S_{pb} \to \mathrm{Top}$ where $S = S_{pb}$ is the category $\{\{0\} \hookrightarrow \{0,1\} \hookleftarrow \{1\}\}$ with 3 objects and 2 non identity morphisms with the same target. S is the shape of $\mathbb{D}$.
A morphism of diagrams $\alpha\colon \mathbb{D} \to \mathbb{D}'$ is a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & B & \xleftarrow{p} & E \\ \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_{01}} & & \downarrow{\scriptstyle \alpha_1} \\ X' & \xrightarrow{f'} & B' & \xleftarrow{p'} & E', \end{array}$$

in other words a natural transformation between the functors $\mathbb{D}$ and $\mathbb{D}'$.
This defines the category Top^S of diagrams of spaces of shape S.
The morphism α is a weak equivalence if each map α_i is a weak equivalence (i.e. induce bijection on all π_k).
The pullback of such a diagram is its limit which is a functor

$$\lim\colon \; \mathrm{Top}^S \to \mathrm{Top}\,, \; \mathbb{D} \mapsto \lim \mathbb{D} = X \times_B E$$

Pullbacks do not preserve weak equivalence:

$$\alpha\colon \mathbb{D} \xrightarrow{\simeq} \mathbb{D}' \quad \not\Longrightarrow \quad \lim \alpha\colon \; \lim \mathbb{D} \xrightarrow{\simeq} \lim \mathbb{D}'$$

The pullback as a space of natural transformations

Let $\mathbb{D}: \; X \xrightarrow{f} B \xleftarrow{p} E$ be a diagram of shape S_{pb}.
Its limit is the pullback

$$\begin{aligned}
X \times_B E &= \{(x,e) \in X \times E \colon f(x) = p(e)\} \\
&\cong \{(x,b,e) \in X \times B \times E \colon f(x) = b = p(x)\} \\
&\cong \left\{ \begin{array}{ccccc} \{*\} & \xrightarrow{=} & \{*\} & \xleftarrow{=} & \{*\} \\ \downarrow x & & \downarrow b & & \downarrow e \\ X & \xrightarrow{f} & B & \xleftarrow{p} & E, \end{array} \right\} \\
&= \mathrm{Nat}\left(\mathbb{D}_0 := \left(\{*\} \xrightarrow{=} \{*\} \xleftarrow{=} \{*\} \right), \mathbb{D}\right)
\end{aligned}$$

Thus

$$\lim \mathbb{D} \cong \mathrm{Nat}(\mathbb{D}_0, \mathbb{D})$$

where $\mathbb{D}_0$ is the constant diagram of one-point spaces $\{*\}$.

Definition and key properties of the homotopy pullback

Let $\mathbb{D}: \; X \xrightarrow{f} B \xleftarrow{p} E$ be a diagram of shape S_{pb}.

Set $\mathbb{D}_0 := \left(\{*\} \xrightarrow{=} \{*\} \xleftarrow{=} \{*\} \right)$. Then $\lim \mathbb{D} \cong \mathrm{Nat}(\mathbb{D}_0, \mathbb{D})$

Set ${\mathbb{D}_0}^\flat := \left(\{0\} \hookrightarrow [0,1] \hookleftarrow \{1\} \right)$.

The homotopy pullback or homotopy limit of $\mathbb{D}$ is

$$\begin{aligned} \mathrm{holim}\,\mathbb{D} &:= \mathrm{Nat}({\mathbb{D}_0}^\flat, \mathbb{D}) \\ &= \left\{ (x, \beta, e) \in X \times B^{[0,1]} \times E \; : \; f(x) = \beta(0), \beta(1) = p(e) \right\} \end{aligned}$$

1. holim is a functor $\mathrm{Top}^{S_{pb}} \to \mathrm{Top}$
2. holim preserves weak equiv.: $\alpha: \mathbb{D} \xrightarrow{\simeq} \mathbb{D}' \Longrightarrow \mathrm{holim}\,\mathbb{D} \xrightarrow{\simeq} \mathrm{holim}\,\mathbb{D}'$
3. there is a natural map η: $\lim \mathbb{D} \to \mathrm{holim}\,\mathbb{D}$ as $\eta := \mathrm{Nat}({\mathbb{D}_0}^\flat \to \mathbb{D}_0, \mathbb{D})$
4. if $\mathbb{D}$ is $_{inj}$–fibrant (i.e.f and p are Serre fibration) then η is a weak equiv.
5. every diagram $\mathbb{D}$ has a $_{inj}$–fibrant replacement $\mathbb{D} \xrightarrow{\simeq} \mathbb{D}^\sharp$

These properties are quite easy to prove.

(4) and (5) gives another way to compute the holim: $\mathrm{holim}\,\mathbb{D} \simeq \lim \mathbb{D}^\sharp$

Why $\mathbb{D}_0^\flat$?

holim $\mathbb{D} := \mathrm{Nat}(\mathbb{D}_0^\flat, \mathbb{D})$. Why do we take $\mathbb{D}_0^\flat = (\{0\} \hookrightarrow [0,1] \hookleftarrow \{1\})$?

- $\mathbb{D}_0^\flat \in \mathrm{Top}^{S_{pb}}$ is a $_{proj}-$cofibrant diagram
- $\mathbb{D}_0^\flat \xrightarrow{\simeq} \mathbb{D}_0$ is a weak equivalence
- any $\mathbb{D}$ is $\mathit{proj}-$fibrant

In other words the key fact is that $\mathbb{D}_0^\flat$ is a cofibrant replacement of $\mathbb{D}_0$

Second approach: $\mathrm{holim}'\mathbb{D} := \mathrm{Nat}(\mathbb{D}_0, \mathbb{D}^\sharp) = \lim \mathbb{D}^\sharp$

- $\mathbb{D}^\sharp \in \mathrm{Top}^{S_{pb}}$ is a $_{inj}-$fibrant diagram
- $\mathbb{D}^\sharp \xrightarrow{\simeq} \mathbb{D}$ is a weak equivalence
- $\mathbb{D}_0$ is $\mathit{inj}-$cofibrant because the one-point spaces * are CW-complexes

In other words the key fact is that $\mathbb{D}^\sharp$ is a fibrant replacement of $\mathbb{D}$

Advantage of the second approch holim $\mathbb{D} = \lim \mathbb{D}^\sharp$: It make sense in categories where there is no internal objects of natural transformations, e.g. if $\mathbb{D} = (A \to B \leftarrow C)$ is a diagram of CDGA then we can set

$$\mathrm{holim}'\mathbb{D} = \lim((\ A \longrightarrow B \longleftarrow C\ )^\sharp) = \lim(\ \hat{A} \twoheadrightarrow B \twoheadleftarrow \hat{C}\ )$$

Holim for more general diagrams

Let S be any small category (giving the shape of diagrams).
$\mathbb{D}\colon S \to \mathrm{Top}$ a diagram of spaces
$\mathbb{D}_0\colon S \to \mathrm{Top}$ the diagram of one-point spaces
$\lim \mathbb{D} = \lim_{s\in S} \mathbb{D}(s) \cong \mathrm{Nat}(\mathbb{D}_0, \mathbb{D})$
There exists a "$_{proj-}$cofibrant replacement" $\mathbb{D}_0^\flat \xrightarrow{\simeq} \mathbb{D}_0$ and we set

$$\operatorname*{holim}_{s\in S} \mathbb{D}(s) := \mathrm{Nat}(\mathbb{D}_0^\flat, \mathbb{D}).$$

We get a functor holim: $\mathrm{Top}^S \to \mathrm{Top}$ with properties (1)-(5).

An alternative approach which works in more general categories than Top (Quillen model categories): there is a notion of "$_{inj-}$fibrant replacement" $\mathbb{D} \xrightarrow{\simeq} \mathbb{D}^\sharp$ and we can define

$$\mathrm{holim}'\mathbb{D} := \lim \mathbb{D}^\sharp$$

with properties (1)-(5) .

Homotopy colimits

Consider a diagram of spaces $\mathbb{E} := (\ X \xleftarrow{f} A \xrightarrow{g} Y\ )$ and its colimit

$$\operatorname{colim} \mathbb{E} = X \cup_A Y := \frac{X \amalg A \amalg Y}{\sim} \text{ with } f(a) \sim a \sim g(a)$$

Note that $\mathbb{E} \colon (S_{pb})^{op} \to \mathrm{Top}$ and recall $\mathbb{D}_0 \colon S_{pb} \to \mathrm{Top}$. Then

$$\operatorname{colim} \mathbb{E} \cong \mathbb{E} \otimes_{S_{pb}} \mathbb{D}_0 := \frac{\amalg_{s \in S} \mathbb{E}(s) \times \mathbb{D}_0(s)}{\equiv}$$

where $(\mathbb{E}\phi(e'), d) \equiv (e', \mathbb{D}_0\phi(d))$ for $e' \in \mathbb{E}(s'), d \in \mathbb{D}_0(s), \phi \colon s \to s'$ in S
Then

$$\operatorname{hocolim} \mathbb{E} := \mathbb{E} \otimes_{S_{pb}} \mathbb{D}_0^\flat \cong \frac{X \amalg A \times [0,1] \amalg Y}{\equiv} \text{with } f(a) \equiv (a,0), (a,1) \equiv g(a)$$

1. hocolim is a functor $\mathrm{Top}^{(S_{pb})^{op}} \to \mathrm{Top}$
2. hocolim preserves weak equivalences of diagrams
3. there is a natural map $\epsilon \colon \operatorname{hocolim} \mathbb{E} \to \operatorname{colim} \mathbb{E}$
4. if $\mathbb{E}$ is ${}_{proj-}$cofibrant then ϵ is a weak equiv.
5. every diagram $\mathbb{E}$ has a ${}_{proj-}$cofibrant replacement $\mathbb{E}^\flat \xrightarrow{\simeq} \mathbb{E}$

Thus we also have $\operatorname{hocolim} \mathbb{E} \simeq \operatorname{colim} \mathbb{E}^\flat$

Derived mapping spaces

In Top we have the mapping space $\mathrm{map}(X, Y) := \{f : X \to Y\}$.
It behaves well with (co)limits: for diagrams $\mathbb{X}, \mathbb{Y} : S \to \mathrm{Top}$

1. $\mathrm{map}(\mathrm{colim}_{s\in S}\, \mathbb{X}(s), Y) \cong \lim_{s\in S} \mathrm{map}(\mathbb{X}(s), Y)$
2. $\mathrm{map}(X, \lim_{s\in S} \mathbb{Y}(s)) \cong \lim_{s\in S} \mathrm{map}(X, \mathbb{Y}(s))$

The derived mapping space $\mathrm{homap} = \mathrm{map}^h$ is a variation of map which preserves weak equivalences:
if $\alpha : X \xrightarrow{\simeq} X'$ and $\beta : Y \xrightarrow{\simeq} Y'$ are weak equivalences then

$$(\alpha^*, \beta_*) : \mathrm{homap}(X', Y) \xrightarrow{\simeq} \mathrm{homap}(X, Y').$$

Also homap behaves well with homotopy (co)limits: same as (1)-(2) with prefixes "ho" and $\simeq$ instead of $\cong$.
The definition will be

$$\mathrm{homap}(X, Y) := \mathrm{map}(X^\flat, Y^\sharp)$$

where

- $X^\flat \xrightarrow{\simeq} X$ is a cofibrant replacement of the source
- $Y \xrightarrow{\simeq} Y^\sharp$ is a fibrant replacement of the target

Map does not preserve weak equivalence in Top

Weak equivalences in Top are maps $f\colon X \xrightarrow{\simeq} Y$ such that $\pi_i(f)$ are isomorphisms for every $i \geq 0$ and every basepoint.
Consider the Warsaw interval $W = \{(0,0)\} \cup \{(t, \sin(\pi/t)) : 0 < t \leq 1\}$ and the 0-sphere $S^0 \cong \{(0,0),(0,1)\}$.
The inclusion $\alpha\colon S^0 \hookrightarrow W$ is a weak equivalence but α^* is not:

$$\begin{array}{lll} \mathrm{map}(W,S^0) & = \{\text{constant maps } W \to S^0\} \cong S^0 \\ \quad \alpha^* \downarrow \not\simeq & \\ \mathrm{map}(S^0,S^0) & \cong S^0 \times S^0 \end{array}$$

Reasons:

- weak equivalences are detected by π_*, hence are recognized by maps out of spheres S^{n-1} and disks D^n
- W cannot be built using spheres and disks, i.e. it is not a CW-complex constructed by iterated pushouts with $S^{n-1} \hookrightarrow D^n$ ("attachments of cells")

Proposition

If $\alpha\colon X \xrightarrow{\simeq} X'$ is a weak equivalences between CW-complexes then $\alpha^*\colon \mathrm{map}(X,Y) \xrightarrow{\simeq} \mathrm{map}(X',Y)$ is a weak equivalence.

Some Quillen model category terminology

Categories like Top (or Top^S, Chains,CDGA,$\text{CDGA}^S, \ldots$) admits a/many Quillen model structures determined by three classes of maps:

- weak equivalences $\alpha\colon X \xrightarrow{\simeq} X'$
- cofibrations $i\colon A \rightarrowtail X$ obtained as iterated attachment of "cells"
- fibration $p\colon E \to B$ which are maps that have the right lifting property with respect to cofibration weak equivalences $\overset{\simeq}{\rightarrowtail}$

that satisfy some axioms like: every map $f\colon X \to Y$ factors as $X \overset{\simeq}{\rightarrowtail} \hat{X} \twoheadrightarrow Y$ and as $X \rightarrowtail \tilde{Y} \overset{\simeq}{\twoheadrightarrow} Y$

- an object C is cofibrant if $\emptyset \rightarrowtail C$
- an object F is fibrant if $F \twoheadrightarrow *$

Each map $\emptyset \to X$ factors as $\emptyset \rightarrowtail X^\flat \xrightarrow{\simeq} X$ which gives a cofibrant replacement of X

Each map $Y \to *$ factors as $Y \xrightarrow{\simeq} Y^\sharp \twoheadrightarrow *$ which gives a fibrant replacement of Y

holim, hocolim, homap usually make sense in such categories

(Serre) Quillen model structure on Top

The classical Quillen structure on Top is given by:

- weak equivalences are $f\colon X \xrightarrow{\simeq} X'$ s.t $\pi_*(f)$ are iso
- cofibrations are relative CW-complexes $X \rightarrowtail X \cup e^{n_1} \cup e^{n_2} \cup \cdots$
- cofibrant objects are CW-complexes and their retracts
- $[0,1]^k \times \{0\} \overset{\simeq}{\rightarrowtail} [0,1]^k \times [0,1]$ are cofibrations weak equivalences
- fibrations are exactly the Serre fibrations
- every space X is fibrant (because $X \to *$ is a Serre fibration)

A cofibrant replacement of a space X is a CW-approximation $X^\flat \xrightarrow{\simeq} X$. For example $\alpha\colon S^0 \xrightarrow{\simeq} W$ is the CW-approximation of the Warsaw interval by the CW-complexe S^0

In Top the derived mapping space is given by

$$\mathrm{homap}(X,Y) := \mathrm{map}(X^\flat, Y)$$

where $X^\flat \xrightarrow{\simeq} X$ is a CW-approximation and we can take $Y^\sharp = Y$ because every space is fibrant.

Projective Quillen model structure on Top^S

We said that ${\mathbb{D}_0}^\flat := (\{0\} \to [0,1] \leftarrow \{1\})$ was a cofibrant replacement of $\mathbb{D}_0 = (* \to * \leftarrow *)$. For which Quillen structure on Top^S (with $S = S_{pb}$) ?
The projective Quillen structure on Top^S:

- weak equivalences $\alpha\colon \mathbb{D} \xrightarrow{\simeq} \mathbb{D}'$ are maps of diagrams such that $\alpha(s)\colon \mathbb{D}(s) \xrightarrow{\simeq} \mathbb{D}'(s)$ for every $s \in S$
- fibration $f\colon \mathbb{X} \twoheadrightarrow \mathbb{Y}$ are maps of diagrams such that $f(s)\colon \mathbb{X}(s) \twoheadrightarrow \mathbb{Y}(s)$ is a Serre fibration for every $s \in S$
- thus every diagram $\mathbb{D}$ is $_{proj-}$fibrant
- cofibrations are more complicated: there are built by cell attachments.

There are 0-cells $\mathbb{Z}^d$ for every $d \in S := \{\{0\} \to \{0,1\} \leftarrow \{1\}\}$ given by

$$\mathbb{Z}^d : S \to \mathrm{Top}\,,\; s \mapsto \mathbb{Z}^d(s) := \hom_S(d,s) \in \mathrm{Set} \subset \mathrm{Top}$$

So: $\mathbb{Z}^{\{0\}} = (* \to * \leftarrow \emptyset)\,,\; \mathbb{Z}^{\{0,1\}} = (\emptyset \to * \leftarrow \emptyset)\,,\; \mathbb{Z}^{\{1\}} = (\emptyset \to * \leftarrow *)$
Higher cells are just $S^{n-1} \times \mathbb{Z}^d \hookrightarrow D^n \times \mathbb{Z}^d$.
Exercise: ${\mathbb{D}_0}^\flat$ is $_{proj-}$cofibrant i.e. built out of cells, but $\mathbb{D}_0$ is not. (see Dror-Farjoun "Homology and homotopy of diagrams")

Injective Quillen model structure on Top^S

There is another Quillen structure on Top^S: the injective one:

- weak equivalences $\alpha\colon \mathbb{D} \xrightarrow{\simeq} \mathbb{D}'$ same as before
- cofibration $f\colon \mathbb{X} \rightarrowtail \mathbb{Y}$ are maps of diagrams such that $f(s)\colon \mathbb{X}(s) \rightarrowtail \mathbb{Y}(s)$ is a cofibration for every $s \in S$ (relative CW-complex)
- thus a diagram $\mathbb{D}$ is cofibrant if every term $\mathbb{D}(s)$ is a CW-complex
- fibrations and fibrant diagrams are more complicated to describe.

For the special case $S = S_{pb}$ a diagram $\mathbb{D} = (\, X \xrightarrow{f} B \xleftarrow{p} E \,)$ is fibrant if both f and p are Serre fibrations.
For more complicated shapes S it is more delicate to give a criterion for a diagram to be fibrant (see Dwyer-Spalanski)

Homotopy limits as a special case of derived mapping space

In Top^S there is a notion of mapping space between diagram $\mathbb{X}, \mathbb{Y} : S \to \text{Top}$

$$\text{map}(\mathbb{X}, \mathbb{Y}) := \text{map}_{\text{Top}^S}(\mathbb{X}, \mathbb{Y}) = \text{Nat}(\mathbb{X}, \mathbb{Y}).$$

We defined two Quillen model structures on Top^S : projective and injective but both with the same weak equivalences.

For each there is a notion of derived mapping space. Let us compute the derived mapping space $\text{map}^h(\mathbb{D}_0, \mathbb{D}) = \text{homap}(\mathbb{D}_0, \mathbb{D})$ for both:

$$\begin{aligned}
\text{homap}^{proj}(\mathbb{D}_0, \mathbb{D}) &= \text{map}((\mathbb{D}_0)^{\flat_{proj}}, (\mathbb{D})^{\sharp_{proj}}) \\
&= \text{map}(\mathbb{D}_0^\flat, \mathbb{D}) =: \text{holim}\, \mathbb{D} \\
\text{homap}^{inj}(\mathbb{D}_0, \mathbb{D}) &= \text{map}((\mathbb{D}_0)^{\flat_{inj}}, (\mathbb{D})^{\sharp_{inj}}) \\
&= \text{map}(\mathbb{D}_0, \mathbb{D}^\sharp) = \lim \mathbb{D}^\sharp =: \text{holim}'\mathbb{D}
\end{aligned}$$

It turns out that if the Quillen model structures have the same weak equivalences then the derived mapping spaces are equivalent, hence $\text{holim}\, \mathbb{D} \simeq \text{holim}'\mathbb{D}$.