EXERCISE SHEET I

– TOPOLOGY OF MANIFOLDS –

ETH ZÜRICH, FALL 2022

EXERCISE 1. Use the Invariance of Domain to prove the following claims.

(i) If $U, V \subseteq \mathbb{R}^n$ and $U \cong V$ and U open, then V is also open.

(ii) A topological n-manifold is not a topological m-manifold for any $m \neq n$. That is, *dimension of a topological manifold is well defined.*

(iii) If M is a topological m-manifold, then ∂M is a topological $(m-1)$-manifold without boundary. *Hint: Show that*

$$\partial M = \left\{ x \in M : \exists U \subseteq M, x \in U, f \colon \mathbb{R}^{n-1} \times [0,\infty) \xrightarrow{\cong} U \text{ such that } x \in f(\mathbb{R}^{n-1} \times \{0\}) \right\}.$$

(iv) Let M and N be topological n-manifolds, and assume M is closed and N is connected. Then any embedding $M \hookrightarrow N$ must be surjective (so a homeomorphism).

EXERCISE 2. What is wrong in the following argument?

Let S denote the Alexander Horned Sphere, so the image of Alexander's embedding $\mathbb{S}^2 \hookrightarrow \mathbb{S}^3$. The complement $\mathbb{S}^3 \setminus S$ has two components. Let A denote the closure of the part interior to S (inside of the handles), and B the closure of the complement, so $A \cap B = S$. Then A is homeomorphic to $\mathbb{D}^3$, so $\pi_1 A = 1$. If we had $\pi_1 B \neq 1$, then by Seifert–Van Kampen Theorem we would have

$$\pi_1 \mathbb{S}^3 = \pi_1(A \cup B) \cong \frac{\pi_1 A * \pi_1 B}{(i_A)_* \pi_1(A \cap B) = (i_B)_* \pi_1(A \cap B)} \cong \pi_1 B \neq 1,$$

which is a contradiction. Thus, $\pi_1 B = 1$.

Moreover, the Meyer–Vietoris sequence for $A \cup B = \mathbb{S}^3$ implies that B has trivial homology. Therefore, by the Whitehead Theorem, B is homotopy equivalent to $\mathbb{D}^3$.

EXERCISE 3. Can $[0,\infty)$ be given a structure of a smooth manifold? What about $[0,\infty) \times [0,\infty)$? For smooth manifolds M and N show that $M \times N$ is a topological manifold, and determine when $M \times N$ has a smooth structure. What is the boundary of $M \times N$?

EXERCISE 4. Give a definition of a fibre bundle with fibre F and structure group $G \trianglelefteq Homeo(F)$.

Which fibre bundles are called covering spaces? When do we say that a fibre bundle is a vector bundle? What are smooth vector bundles? Define orientable vector bundles and a choice of an orientation.

What is the classifying map of a fibre bundle?

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