EXERCISE SHEET II
– TOPOLOGY OF MANIFOLDS –
ETH ZÜRICH, FALL 2022

EXERCISE 1.

(i) A topological manifold is orientable if and only if it can be covered by a collection of orientation compatible charts, i.e. such that the transition maps $\varphi_\beta \circ \varphi_\alpha^{-1}$ are orientation preserving. Moreover, if this is a transition map in a smooth structure, then it is orientation preserving if and only if the determinant of its Jacobian matrix is positive.

(ii) A compact topological n-manifold N is orientable if and only if $H_n(N, \partial N; \mathbb{Z}) \cong \mathbb{Z}$. Moreover, a choice of an orientation is equivalent to a choice of a generator

$$[N, \partial N] \in H_n(N, \partial N; \mathbb{Z}).$$

(iii) For any topological n-manifold N describe how the following portion of the long exact sequence can look like.

$$H_{n+1}(N, \partial N; \mathbb{Z}) \longrightarrow H_n(\partial N; \mathbb{Z}) \longrightarrow H_n(N, \mathbb{Z}) \longrightarrow H_n(N, \partial N; \mathbb{Z}) \longrightarrow H_{n-1}(\partial N; \mathbb{Z})$$

(iv) A smooth manifold is orientable if and only if every embedded circle has trivial normal bundle.

EXERCISE 2.

(i) The boundary of a smooth n-manifold is a smooth $(n-1)$-manifold without boundary.

(ii) The total space of the tangent bundle of an (orientable) smooth n-manifold is an orientable smooth $2n$-manifold.

(iii) Show that every smooth manifold admits a smooth vector field which is nowhere tangent to ∂N.

(iv) Prove that the boundary of a smooth manifold N has a collar: an open neighbourhood diffeomorphic to $\partial N \times [0, 1)$.

EXERCISE 3. Let M be a compact smooth manifold. Prove that there is no continuous retraction $r\colon M \to \partial M$, i.e. r is a continuous map and $r(x) = x$ for $x \in \partial M$. Give two proofs: one using algebraic topology, one using the fact that a continuous map that is smooth on a subset is homotopic to a smooth map that agrees with it on that subset.

EXERCISE 4.

(i) Describe an immersion $T^2 \setminus \{pt\} \looparrowright \mathbb{R}^2$. [Harder: Describe an immersion $T^n \setminus \{pt\} \looparrowright \mathbb{R}^n$ for $T^n := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ the n-dimensional torus.]

(ii) Describe an embedding of $\mathbb{R}P^2$ in $\mathbb{R}^4$. Show that $\mathbb{R}P^{2n}$ does not embed into $\mathbb{R}^{2n+1}$ for any $n \geq 1$.

(iii) For every smooth embedding $K\colon \mathbb{D}^1 \hookrightarrow \mathbb{D}^3$ (called a long knot), there exists a smooth map $f_K\colon \mathbb{D}^3 \to \mathbb{S}^2$ such that $K = f_K^{-1}(0)$.

EXERCISE 5. If a smooth function $f\colon M \to \mathbb{R}$ on a smooth n-manifold M has exactly 2 critical points, then M is diffeomorphic to the n-sphere, $M \cong \mathbb{S}^n$.

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