EXERCISE SHEET III
– TOPOLOGY OF MANIFOLDS –
ETH ZÜRICH, FALL 2022

Exercise 1.

(i) Determine the diffeomorphism type of $N \cup_\varphi \mathbb{D}^1 \times \mathbb{D}^{n-1}$ for N the disjoint union of smooth n-manifolds $N_1 \sqcup N_2$ and the attaching map $\varphi\colon \mathbb{S}^0 \times \mathbb{D}^{n-1} \hookrightarrow \partial N$ has image in both N_i.

(ii) Determine possible diffeomorphism types of $\mathbb{D}^n \cup_\varphi \mathbb{D}^k \times \mathbb{D}^{n-k}$ for $0 \leq k \leq 3$.

(iii) If Y_1, Y_2 smoothly and disjointly embed into an n-manifold N, and $\dim Y_i = m \leq n-2$, show that $Y_1 \# Y_2$ also smoothly embeds into N.

(iv) Show that the boundary connected summing with a disk is diffeomorphic to doing nothing.

Exercise 2.

(i) Show that attaching a k-handle is homotopy equivalent to adding a k-cell.

(ii) Express the fundamental group of $N \cup_\varphi h^k$ in terms of $\pi_1 N$.

(iii) Determine the homotopy type of $M_1 \natural M_2$ (in terms of M_i).

Exercise 3. Let $K\colon \mathbb{S}^k \hookrightarrow \mathbb{S}^n$ be a smooth knot and $\mathrm{U}\colon \mathbb{S}^k \hookrightarrow \mathbb{S}^n$ the unknot (standard inclusion). Which of the following are equivalent? Give some examples and counterexamples.

(i) K is equivalent to U, i.e. there exists a diffeomorphism $F\colon \mathbb{S}^n \to \mathbb{S}^n$ such that $F \circ K = \mathrm{U}$.

(ii) K is isotopic to U, i.e. there exists $K_t\colon \mathbb{S}^k \hookrightarrow \mathbb{S}^n$ with $K_0 = K$ and $K_1 = \mathrm{U}$.

(iii) K bounds a disk in $\mathbb{S}^n$, i.e. there exists $\Delta\colon \mathbb{D}^{k+1} \hookrightarrow \mathbb{S}^n$ with $\partial\Delta = K$.

(iv) K bounds a disk in $\mathbb{D}^{n+1}$, i.e. there exists $\Delta\colon \mathbb{D}^{k+1} \hookrightarrow \mathbb{D}^{n+1}$ with $\partial\Delta = K$.

Exercise 4. Show that every closed orientable 3-manifold has a Heegaard splitting, i.e. $N \cong H_g \cup_\phi H_g$ for $H_g \cong \natural^g \mathbb{S}^1 \times \mathbb{D}^2$ and ϕ a diffeomorphism of the genus g surface $\#^g \mathbb{S}^1 \times \mathbb{S}^1$.

For every $g \geq 0$ describe a genus g Heagaard splitting of $\mathbb{S}^3$.

Exercise 5.

(i) The connected sum of two manifolds is a homotopy sphere if and only if both of them are homotopy spheres.

(ii) Let S be a smooth homotopy sphere. Show that $S \# -S$ bounds a smooth contractible manifold.

(iii) A twisted sphere $S(f) = \mathbb{D}^k \cup_f \mathbb{D}^k$ is diffeomorphic to $\mathbb{S}^k$ if and only if $f\colon \mathbb{S}^{k-1} \to \mathbb{S}^{k-1}$ extends to a diffeomorphism of $\mathbb{D}^k$.

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