EXERCISE SHEET IV
– TOPOLOGY OF MANIFOLDS –
ETH ZÜRICH, FALL 2022

EXERCISE 1. Show that every closed orientable 3-manifold has a Heegaard splitting, i.e. $N \cong H_g \cup_\phi H_g$ for $H_g \cong \natural^g \mathbb{S}^1 \times \mathbb{D}^2$ and ϕ a diffeomorphism of the genus g surface $\#^g \mathbb{S}^1 \times \mathbb{S}^1$.

For every $g \geq 0$ describe a genus g Heagaard splitting of $\mathbb{S}^3$.

EXERCISE 2. Assume $n \geq 5$. Then every homotopy n-sphere is a twisted sphere and every twisted sphere is an exotic sphere.

EXERCISE 3. Check that every operation on matrices, defining the Whitehead group, can be realised geometrically on handles.

EXERCISE 4. Which steps in the proof of the s-cobordism theorem go through for $\dim W = 5$ and which do not?

EXERCISE 5. Assume N is an oriented n-manifold and $f_i \colon M_i \hookrightarrow N$ are smooth embeddings of complimentary dimension $\dim M_1 = k$, $\dim M_2 = n-k$.

(i) Show that
$$\widetilde{I}(f_2 \pitchfork f_1) = (-1)^{k(n-k)} \overline{\widetilde{I}(f_1 \pitchfork f_2)},$$
where $\overline{\cdot} \colon \mathbb{Z}[\pi_1 N] \to \mathbb{Z}[\pi_1 N]$ is the involution $\sum_i \varepsilon_i g_i \mapsto \sum_i \varepsilon_i g_i^{-1}$.

(ii) Show that
$$eval(\widetilde{I}(f_2 \pitchfork f_1)) = I(f_1 \pitchfork f_2).$$
where $eval \colon \mathbb{Z}[\pi_1 N] \to \mathbb{Z}$ is defined by $\sum_i \varepsilon_i g_i \mapsto \sum_i \varepsilon_i$.

(iii) For any group π and $z \in \mathbb{Z}[\pi]$ construct N, N_1, N_2 with $\widetilde{I}(f_1 \pitchfork f_2) = z$.

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