TOPOLOGY ON MANIFOLDS - EXERCISE SHEET I

Exercise 1.

i). Let $f\colon U \to \mathbb{R}^n$ be the composition of the homeomorphism $U \approx V$ with the inclusion $V \hookrightarrow \mathbb{R}^n$. Since f is the composition of injective continuous maps, f is also injective and continuous. By the Invariance of Domain theorem its image $f(U) = V \subseteq \mathbb{R}^n$ is open.

ii). Suppose that N is a topological n-manifold and m-manifold at the same time for $m \neq n$. Without loss of generality, $m > n$. Let x be a point in the interior of N. By hypothesis, there is an open neighborhood of x that is homeomorphic to an open set of $\mathbb{R}^n$ and an open neighborhood homeomorphic to and open set of $\mathbb{R}^m$. By intersecting the two, we find an open neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$ and of $\mathbb{R}^m$ at the same time.

We show that this is not possible. Suppose that $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are homeomorphic open subsets. Then

$$U' = U \times \{\underbrace{(0,\dots,0)}_{m-n}\} \subseteq \mathbb{R}^m$$

is a subset of $\mathbb{R}^m$ also homeomorphic to V. By i), it is an open subset of $\mathbb{R}^m$; but that is a clear contradiction since $U' \subseteq \mathbb{R}^n \times \{0\}$ has empty interior.

iii). We begin by showing the hint. By definition, every point $x \in M$ satisfies one of the following:

(1) there is an open neighborhood $x \in V$ and an embedding $g\colon V \to \mathbb{R}^n$;
(2) there is an open neighborhood $x \in U$ and an embedding $f\colon U \to \mathbb{R}^n_+$ such that $f(x) \in \mathbb{R}^{n-1} \times \{0\}$.

By definition, the points x satisfying (1) form the interior. We claim that (1) and (2) cannot happen at the same time. If that were the case, then $f(U \cap V)$ would be an open subset of $\mathbb{R}^n_+ \subseteq \mathbb{R}^n$ containing $f(x) \in \mathbb{R}^{n-1} \times \{0\}$ that is homeomorphic to an open subset $g(U \cap V) \subseteq \mathbb{R}^n$. By i) it would follow that $f(U \cap V)$ is open in $\mathbb{R}^n$; but an open subset of $\mathbb{R}^n$ containing $f(x) \in \mathbb{R}^{n-1} \times \{0\}$ cannot be contained in $\mathbb{R}^n_+$, a contradiction. We conclude that the points in the boundary are precisely the ones satisfying (2).

Now we show that ∂M is a $(n-1)$-manifold with no boundary. Since M is metrisable, $\partial M \subseteq M$ is also metrisable, hence it is Hausdorff and paracompact.[1] Let $x \in \partial M$ and consider an open neighborhood U as in (2). Then the restriction of f defines an embedding

$$f_{|U \cap \partial M}\colon U \cap \partial M \to \mathbb{R}^{n-1} \times \{0\} \approx \mathbb{R}^{n-1}.$$

[1] This is just one possible way to argue. Note that a subspace of a Hausdorff space is always Hausdorff, but a subspace of a paracompact space is not always paracompact. However, a closed subspace of a paracompact space is paracompact, so you could also show that ∂M is closed.

iv). Let f be the embedding. We claim that f is an open map, i.e. it sends open subsets to open subsets; to show this, it is enough to prove that $f(U)$ is open for some basis of open sets $\{U\}$. Let $x \in M$ and let U be a sufficiently small open neighborhood of x so that both U is homeomorphic to $\mathbb{R}^n$ and $f(U)$ is contained in an open subset homeomorphic to $\mathbb{R}^n$. Since f is an embedding, $f(U) \approx U$ and by (i) it follows that $f(U)$ is an open subset.

So we conclude that $f(M)$ is an open subset of N. Since M is compact, its image $f(M)$ is also compact and hence $f(M) \subseteq N$ is closed.[2] A non-empty open and closed subset of a connected space must be the entire space, hence $f(M) = N$.

Exercise 2. The issue in the argument is in the applications of the Seifert-van Kampen theorem and the Mayer-Vietoris sequence. Seifert-van Kampen assumes that the subsets A and B are open and Mayer-Vietoris allows them to be closed but asks that the union of their interiors (and not just their union) covers X. These conditions are not satisfied in this case since A and B are defined as the closures of the two components. The application of Whitehead's theorem (or more precisely of Corollary 4.33 in Hatcher's book `https://pi.math.cornell.edu/~hatcher/AT/AT.pdf`) is also problematic since B is not a CW complex.

These conditions are often fine to ignore, but not in this case! Sometimes, it is possible to enlarge the sets A, B to open subsets $A \subseteq \widetilde{A}$, $B \subseteq \widetilde{B}$ that have the same topology as A and B (that deformation retract to A, B). This would be possible if S admitted a "bicollar", i.e. if there was an embedding $S^2 \times (-1, 1) \hookrightarrow S^3$ that restricts to S on $S^2 \times \{0\}$. We will see later that the Alexander Horned Sphere does not admit a bicollar.

Exercise 3. Note that $[0, +\infty)$ is precisely $\mathbb{R}^1_+$, by definition, so it has a smooth structure with just one chart

$$\{([0, +\infty), \mathrm{id})\}.$$

The topological manifold $[0, +\infty) \times [0, +\infty)$ can be given the structure of a smooth manifold as follows: there is a homeomorphism $\varphi\colon [0, +\infty) \times [0, +\infty) \approx \mathbb{R}^2_+$, so we define again a smooth structure with one chart

$$\{([0, +\infty) \times [0, +\infty), \varphi)\}.$$

Note that this smooth structure makes the topological embedding $[0, +\infty) \times [0, +\infty) \hookrightarrow \mathbb{R}^2$ not smooth due to the corner at $(0, 0)$ (i.e. φ is not smooth at $(0, 0)$).

The product of two topological manifolds $M \times N$ is always a topological manifold. First, the product of Hausforff paracompact spaces is always Hausdorff and paracompact. If $(x, y) \in M \times N$ then there are open sets $x \in U$ and $y \in V$ with $U \approx \mathbb{R}^n$ or $U \approx \mathbb{R}^n_+$ and $V \approx \mathbb{R}^m$ or $V \approx \mathbb{R}^m_+$. In each of the 4 possible situations, $U \times V$ is homeomorphic to either

$$U \times V \approx \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m} \text{ or } U \times V \approx \mathbb{R}^n \times \mathbb{R}^m_+ \approx \mathbb{R}^n_+ \times \mathbb{R}^m \approx \mathbb{R}^n_+ \times \mathbb{R}^m_+ \approx \mathbb{R}^{n+m}_+.$$

In particular, (x, y) is in the boundary if and only if either x or y are in the boundary, so

$$\partial(M \times N) = M \times \partial(N) \cup \partial(N) \cup M.$$

[2] Compact subsets of Hausdorff spaces are closed.

If $\partial M = \emptyset$ or $\partial N = \emptyset$ then $M \times N$ has a natural smooth structure given by the product of the smooth structures and the identifications

$$\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m \text{ and } \mathbb{R}^n_+ \times \mathbb{R}^m = \mathbb{R}^{n+m}_+ = \mathbb{R}^n \times \mathbb{R}^m_+.$$

If $\partial M, \partial N$ are both non-empty then the product of the smooth structures on M and N is not a smooth structure on $M \times N$. This is due to the fact that $\mathbb{R}^n_+ \times \mathbb{R}^m_+$ is not diffeomorphic to $\mathbb{R}^{n+m}_+$ (in the sense of existing a smooth map $\mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ restricting to a homeomorphism $\mathbb{R}^{n+m}_+ \to \mathbb{R}^n_+ \times \mathbb{R}^m_+$ with non-vanishing derivative) because of the corner $(0,0) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+$.

Exercise 4. A fiber bundle with fiber F and structure group G is a continuous map $p\colon E \to B$ (from a space E called the total space to a space B, called the base) with fibers homeomorphic to F that is locally trivial – i.e. locally looks like the projection $B \times F \to B$ – and whose transition functions between different charts are in G. More precisely, there is a family of local trivializations $\{(U_\alpha, \varphi_\alpha)\}$ where $U_\alpha \subseteq B$ are open sets covering B and

$$\varphi_\alpha \colon p^{-1}(U_\alpha) \to U_\alpha \times F$$

are homeomorphisms compatible with the projection to B, i.e. such that the diagram

$$\begin{array}{ccc} p^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times F \\ \big\downarrow p & \swarrow \scriptstyle{\mathrm{pr}_1} & \\ U_\alpha & & \end{array}$$

commutes, where pr_1 is the projection onto the first component. The condition that p has structure group G means that for every α, β the composition

$$(U_\alpha \cap U_\beta) \times F \xrightarrow{\varphi_\alpha^{-1}} p^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\varphi_\beta} (U_\alpha \cap U_\beta) \times F \tag{1}$$

is of the form

$$(x, v) \mapsto (p, \phi_{\alpha,\beta}(x)(v))$$

for a continuous map $\phi_{\alpha,\beta} \colon U_\alpha \cap U_\beta \to G$, called a transition map.

A covering space is a fiber bundle with discrete fiber F. A vector bundle is a fiber bundle with fiber $\mathbb{R}^n$ and structure group $GL(n) \subseteq \mathrm{Homeo}(\mathbb{R}^n)$. A smooth fiber bundle is a fiber bundle such that B, F are smooth, $G < \mathrm{Diffeo}(F)$ is a Lie group and the transition maps $\phi_{\alpha,\beta}$ are smooth; equivalently, B, E have smooth structures, $p\colon E \to B$ is smooth and the local trivializations φ_α are diffeomorphisms. An orientable vector bundle is a fiber bundle with fibers $\mathbb{R}^n$ and structure group

$$GL^+(n) \subseteq GL(n) \subseteq \mathrm{Homeo}(\mathbb{R}^n)$$

where

$$GL^+(n) = \{A \in M_{n\times n}(\mathbb{R}) \colon \det(A) > 0\}.$$

A choice of orientation for $p\colon E \to B$ consists in a choice of compatible local trivializations $\{(U_\alpha, \varphi_\alpha)\}$ such that the transition maps land in $GL^+(n)$, up to equivalence. Two such choices $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ are equivalent (i.e. give the same choice of orientation) if they are compatible in the sense that the transition functions of the union $\{(U_\alpha, \varphi_\alpha)\} \cup \{(V_\beta, \psi_\beta)\}$ are also in $GL^+(n)$. An orientable vector bundle always admits such a choice; indeed, it admits exactly $2^{\#\text{connected components}}$ such choices.

Given a group G, there is a contractible space EG on which G acts properly and freely. The classifying space BG of G is the quotient EG/G; this space is unique up to (weak) homotopy and the quotient $EG \to BG$ is called the universal principal G-bundle. A principal G bundle is a fiber bundle $P \to B$ with fiber G and structure group G, where G acts on itself by left multiplication; a principal G-bundle always comes with a right G-action on the total space P. The space BG classifies principal G-bundles in the sense that there is a 1:1 correspondence between continuous maps $f\colon B \to BG$ up to homotopy and principal G-bundles. Given such an f, the associated principal G-bundle is the pullback $f^*(EG) \to B$.

If G acts on F then there is a further correspondence between principal G-bundles and fiber bundles with fiber F and structure group G. Given a principal G-bundle $P \to B$ the associated fiber bundle with fiber F is the fiber product

$$(P \times F)/G \to B$$

where G acts on $P \times F$ as $g \cdot (x, v) = (xg^{-1}, gv)$. This construction can also be reversed to give a principal G-bundle from a fiber bundle with structure group F. So we have a triangle of equivalences between fiber bundles, principal G-bundles and continuous maps $B \to BG$ up to homotopy. The map $B \to BG$ associated to a fiber bundles is called its classifying map.

The construction of EG, BG is not completely obvious, neither are the proofs that BG and the classifying maps are unique up to homotopy; more details can be found in many places, for examples `https://math.mit.edu/~mbehrens/18.906spring10/prin.pdf`.

Example 1. We consider the case of vector bundles of rank 1 (line bundles), i.e. $G = GL(1; \mathbb{R}) = \mathbb{R}^\times$. In this case, $EG \simeq \mathbb{R}^\infty \setminus \{0\}$ with the diagonal action of $\mathbb{R}^\times$ and $BG = EG/G \simeq \mathbb{RP}^\infty$ is the infinite real projective space. Giving a line bundle on B is the same as giving (the homotopy class of) a map $B \to \mathbb{RP}^\infty$.

Example 2. An orientable line bundle is always trivial. This is reflected by the fact that $BGL^+(1) = B\mathbb{R}^+ = 1$ is trivial.

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