TOPOLOGY ON MANIFOLDS - EXERCISE SHEET II

Exercise 1.

i). Let $\{U_\alpha, \varphi_\alpha\}$ be a collection of charts and let $\varphi_\alpha(U_\alpha) = V_\alpha \subseteq \mathbb{R}^n$. Suppose that $x \in U_\alpha \cap U_\beta$ and denote by $\tilde{V}_\alpha = \varphi_\alpha(U_\alpha \cap U_\beta)$ and $\tilde{V}_\beta = \varphi_\beta(U_\alpha \cap U_\beta)$. We have the following long commutative diagram of isomorphisms

$$
\begin{array}{ccccccc}
 & & H_n(U_\alpha, U_\alpha \setminus x) & \xrightarrow{\varphi_\alpha} & H_n(V_\alpha, V_\alpha \setminus \varphi_\alpha(x)) & \xrightarrow{\sim} & H_n(\tilde{V}_\alpha, \tilde{V}_\alpha \setminus \varphi_\alpha(x)) \\
 & \overset{\sim}{\nearrow} & & & & & \downarrow{\scriptstyle \varphi_\beta \circ \varphi_\alpha^{-1}} \\
H_n(N, N \setminus x) & & & & & & \\
 & \underset{\sim}{\searrow} & & & & & \\
 & & H_n(U_\beta, U_\beta \setminus x) & \xrightarrow{\varphi_\beta} & H_n(V_\beta, V_\beta \setminus \varphi_\beta(x)) & \xrightarrow{\sim} & H_n(\tilde{V}_\beta, \tilde{V}_\beta \setminus \varphi_\beta(x))
\end{array}
$$

Let us start by proving the implication $\Leftarrow$. Let $\{U_\alpha, \varphi_\alpha\}$ be a collection of orientation compatible charts. We define an orientation on N as follows: given $x \in N$ let α be such that $x \in U_\alpha$ and let $[N]_x$ be defined so that it corresponds to the positive generator of $H_{n-1}(S^{n-1})$ under the isomorphism

$$H_n(N, N \setminus x) \cong H_n(V_\alpha, V_\alpha \setminus \varphi_\alpha(x)) \cong H_{n-1}(S^{n-1})$$

where the last isomorphism is the one explained in the lecture. By the commutative diagram above and the fact that $\varphi_\beta \circ \varphi_\alpha^{-1}$ are orientation preserving, $[N]_x$ is well defined, i.e. doesn't depend on the choice of α.

To show that this indeed defines an orientation we have to find for every x an open set U satisfying the conditions in the definition. Given x, let α be such that $x \in U_\alpha$ and let U be such that $D = \varphi_\alpha(U) \subseteq V_\alpha$ is a disk with closure (in $\mathbb{R}^n$) contained in V_α. Then we choose $[N]_U$ corresponding to the positive generator of $H^{n-1}(S^{n-1})$ under the isomorphism

$$H^n(N, N \backslash U) \cong H^n(U_\alpha, U_\alpha \backslash U) \cong H^n(V_\alpha, V_\alpha \backslash D) \cong H^{n-1}(\mathbb{R}^n, \mathbb{R}^n \backslash D) \cong H^{n-1}(S^{n-1}) \,.$$

The fact that the restriction of $[N]_U$ to x is $[N]_x$ is straightforward to check.

For the implication $\Rightarrow$ we consider any collection of charts $\{U_\alpha, \varphi_\alpha\}$. An orientation on N restricts to orientations on U_α. We define orientations on V_α so that they agree with the orientations on U_α via the isomorphism φ_α. Then the transition maps are orientation preserving since they are the composition of orientation preserving maps.

Finally we want to show that a smooth map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is orientation preserving if and only if $\det(D(f)) > 0$; note that this statement means that a smooth manifold is orientable if and only if its tangent bundle $TM \to M$ is an orientable vector bundle as $D(f)$ are precisely the transition functions of the bundle TM. Without loss of generality we will assume that $f(0) = 0$. Let $S^{n-1} = S^{n-1}_\varepsilon$ be the sphere of radius ε and let $r\colon \mathbb{R}^n \setminus 0 \to S^{n-1}$ be the radial retraction. By definition, the map f is orientation preserving if and only if the composition

$$r \circ f\colon S^{n-1}_\varepsilon \to \mathbb{R}^n \setminus 0 \to S^{n-1}_\varepsilon$$

acts as the indentity in $H^{n-1}(S^{n-1}) \cong \mathbb{Z}$. We claim that $r \circ f$ is homotopic to $r \circ D(f)$ where $D(f)$ denotes the linear map of multiplication by the Jacobian matrix

$D(f)$. By Taylor's Theorem, there is a constant C such that $|f(x) - D(f)x| \leq C|x|^2$ for every $|x| \leq 1$. It follows that for sufficiently small ε we have

$$f_t(x) = tf(x) + (1-t)D(f)x \neq 0$$

for every $|x| \leq \varepsilon$ and $0 \leq t \leq 1$. Then $r \circ f_t$ defines a homotopy between $r \circ f$ and $r \circ D(f)$. Now $GL(n)$ has two connected components according to whether the determinant is positive or negative. If $\det(D(f)) > 0$ there is a path in $GL(n)$ connecting $D(f)$ and the identity, thus defining a homotopy $r \circ f \simeq r \circ D(f) \simeq \mathrm{id}$, which proves that if $\det(D(f)) > 0$ then f is orientation preserving. On the other hand, if $\det(D(f)) < 0$ then $r \circ f$ is homotopic to the map $S^{n-1} \to S^{n-1}$ sending

$$(x_1, x_2, \ldots, x_n) \mapsto (-x_1, x_2, \ldots, x_n).$$

Since this map acts as -1 in $H_{n-1}(S^{n-1})$ this finishes the proof.

ii). Suppose that N is orientable. We will prove that for any open set $A \subseteq N$ such that $\overline{A} \subseteq \mathrm{int}(N)$ we have an isomorphism $H^n(N, N \setminus A) \cong \mathbb{Z}$ which is compatible with the orientations, i.e. the restriction maps $\mathbb{Z} \cong H^n(N, N \setminus A) \to H^n(N, N \setminus x)$ are isomorphisms sending 1 to $[N]_x$. Since $\overline{A}$ is compact we can find finitely many open sets $U_1, \ldots, U_l$ homeomorphic to disks covering $\overline{A}$ which have orientation classes $[N]_{U_i}$. We show by induction on j that $H^n(N, N \setminus U_1 \cup \ldots \cup U_j) \cong \mathbb{Z}$. The relative Mayer-Vietoris sequence gives

$$\begin{aligned} 0 &\to H^n(N, N \setminus (U_1 \cap U_2)) \to H^n(N, N \setminus U_1) \oplus H^n(N, N \setminus U_2) \to \\ &\to H^n(N, N \setminus (U_1 \cup U_2)) \to H^{n-1}(N, N \setminus (U_1 \cap U_2)) = 0 \end{aligned}$$

The choice of orientation specifices isomorphisms of all the groups $H^n(N, N \setminus (U_1 \cap U_2))$,$H^n(N, N \setminus U_1)$, $H^n(N, N \setminus U_2)$ with $\mathbb{Z}$. Under these isomorphisms, the first map looks like $d \mapsto (d, -d)$ to it follows that $H^n(N, N \setminus (U_1 \cup U_2)) \cong \mathbb{Z}$ and this isomorphism is also compatible with orientations. Iterating this procedure shows that $H^n(N, N \setminus U_1 \cup \ldots \cup U_l)$ is isomorphic to $\mathbb{Z}$ as well. We then have restriction maps for any $x \in A$

$$H^n(N, N \setminus (U_1 \cup \ldots \cup U_l)) \to H^n(N, N \setminus A) \to H^n(N, N \setminus x).$$

Each of the maps is an injection by the respective long exact sequence of triples and their composition is an isomorphism, so each of them must be an isomorphism, showing that $H^n(N, N \setminus A) \cong \mathbb{Z}$. To finish the proof we use the fact that ∂M has a collar; we will prove this in the next exercise in the smooth case and in the topological setting it is proven in Proposition 3.42 in Hatcher. Letting $\partial M \subseteq B$ be a collar and $A = N \setminus B$ we get

$$H^n(N, \partial N) \cong H^n(N, B) = H^n(N, N \setminus A) \cong \mathbb{Z}.$$

For the implication $\Leftarrow$ we fix a generator of $H^n(N, \partial N)$. The inclusion $\partial N \subseteq N \setminus x$ induces a restriction map $H^n(N, \partial N) \to H^n(N, N \setminus x) \cong \mathbb{Z}$. Reversing the induction procedure tor the other implication we can show that such restriction map must be an isomorphism, so we may define an orientation class $[N]_x$ as the image of the generator we fixed on $H^n(N, \partial N)$. It is clear that this defines an orientation since we can take $U = N \setminus \partial N$ is the definition of orientation.

iii). The first two terms always vanish by dimension reasons. The term $H_n(N, \partial N)$ is isomorphic to $\mathbb{Z}$ if and only if N is compact and orientable and $H_{n-1}(\partial N)$ is isomorphic to $\mathbb{Z}$ if and only if ∂N is compact and orientable. Note that ∂N has no boundary and if N is compact/orientable then so is ∂N.

(1) If $\partial N = \emptyset$ and N is compact and oriented then

$$0 \to 0 \to \mathbb{Z} \xrightarrow{\sim} \mathbb{Z} \to 0$$

(2) If $\partial N \neq \emptyset$ and N is compact and oriented then

$$0 \to 0 \to 0 \to \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}$$

(3) If N is not compact and oriented but ∂N is then

$$0 \to 0 \to 0 \to 0 \to \mathbb{Z}$$

(4) In every other case the sequence vanishes.

iv). We will use the fact that there are exactly 2 rank r vector bundles over S^1 up to isomorphism, one orientable and the other non-orientable. A sketch of the proof is the following: write S^1 as the union of two open intervals whose intersection is a disjoint union $A \cup B$ of two smaller open intervals. Since any bundle over the open interval is trivial, to specify the isomorphism class of a bundle over S^1 it is enough to specify the gluing data $A \cup B \to GL(r)$ up to homotopy. Since $GL(r)$ has two connected components defined by $\det(M) > 0$ or $\det(M) < 0$ and A, B are both contractible we can show the claim.

Let N be the normal bundle and consider the implication $\Rightarrow$. By definition

$$TS^1 \to TM_{|S^1} \to N\,.$$

Since both S^1 and M are orientable, the vector bundles $TS^1 \to TM_{|S^1}$ are both orientable, so N is orientable as well and thus trivial.

For the implication $\Leftarrow$, by the same argument it follows that $TM_{|S^1}$ is a trivial bundle for every embedded circle S^1. We define an orientation as follows: fix $x_0 \in N$ and an orientation $[N]_{x_0}$. Such an orientation as equivalent to choosing an orientation of $T_{x_0}N$ as explained in *i).* Now given any $x \in N$ pick a path $\gamma\colon [0,1] \to N$ from x_0 to x. The restriction of TM to $[0,1]$ must be trivial, so we get an identification between $T_xN \cong T_{x_0}N$ depending on the path γ. We define the orientation $[N]_x$ so that it corresponds to $[N]_{x_0}$ under this isomorphism. By the condition that TM is trivial over any embedded circle, this is well defined, i.e. the orientation $[N]_x$ does not depend on the choice of path γ. We leave the details of showing that this is indeed an orientation to the reader.

Exercise 2.

i). We showed last time that the boundary of a topological manifold is a topological manifold without boundary. Suppose now that N is smooth and let $\{U_\alpha, \varphi_\alpha\}$ be a smooth structure. Let $V_\alpha = U_\alpha \cap \partial N \subseteq \partial N$. We showed in the previous exercise sheet that V_α if V_α is non-empty then it is precisely the pre-image of $\mathbb{R}^{n-1} \times 0 \subseteq \mathbb{R}^n_+$ via φ_α. The restriction $\psi_\alpha := (\varphi_\alpha)_{|V_\alpha}\colon V_\alpha \to \mathbb{R}^{n-1}$ now defines a chart. Since the transition maps $\varphi_\beta \circ \varphi_\alpha^{-1}$ are smooth, their restrictions $\psi_\beta \circ \psi_\alpha^{-1}$ are also smooth so $\{V_\alpha, \psi_\alpha\}$ forms a smooth structure.

ii). Let $\{U_\alpha, \varphi_\alpha\}$ be a smooth structure on N and let $V_\alpha = \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$. Denote by $\pi\colon TN \to N$ the projection. We have a map

$$d\varphi_\alpha\colon \pi^{-1}(U_\alpha) \cong TU_\alpha \to TV_\alpha \cong V_\alpha \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}\,.$$

Hence, $\{\pi^{-1}(U_\alpha), d\varphi_\alpha\}$ defines a smooth structure on TM.

We now show that M is orientable (even if M is not!). Consider the smooth structure defined above and let

$$f = \varphi_\beta \circ \varphi_\alpha^{-1}\colon V_1 \to V_2$$

be a transition function where $V_1 = \varphi_\alpha(U_\alpha \cap U_\beta)$, $V_2 = \varphi_\beta(U_\alpha \cap U_\beta)$ are open subsets of $\mathbb{R}^n$. The induced transition function

$$df\colon V_1 \times \mathbb{R}^n \cong TV_1 \to TV_2 \cong V_2 \times \mathbb{R}^n$$

is given by $(df)(p,v) = (f(p), (D(f))_p v)$. Hence the Jaconian $D(df)$ is given by the block matrix

$$D(df)_{(p,v)} = \begin{bmatrix} D(f)_p & 0 \\ 0 & D(f)_p \end{bmatrix}.$$

Its determinant is equal to $\det(D(f)_p)^2 > 0$, proving that TM is orientable.

iii). Choose a locally finite atlas $\{U_\alpha, \varphi_\alpha\}$ and a corresponding partition of unity φ_α. Let X_α be a vector field on U_α defined as follows: if $\varphi_\alpha \colon U_\alpha \to \mathbb{R}^n$ then $X_\alpha = 0$ and if $\varphi_\alpha \colon U_\alpha \to \mathbb{R}^n_+$ then we define X_α to be the pullback of the constant vector field $(0, \dots, 0, 1)$ in $\mathbb{R}^n_+$. The vector fields X_α are inwards pointing in U_α. Thus the vector

$$X = \sum_\alpha \varphi_\alpha X_\alpha$$

is also inwards pointing, so it is nowhere tangent to ∂N.

iv). Consider the vector field X as before and its induced (positive) flow

$$\gamma \colon \{(p,t) \in \partial N \times [0, +\infty) \colon t \leq \epsilon_p\} \to N$$

Note that $(d\gamma)_{(p,0)}$ is a invertible for every $p \in \partial N$. Indeed, writing it as a block matrix with respect to the decomposition $T_p\partial N \oplus \mathbb{R} \to T_p N$ we get

$$(d\gamma)_{(p,0)} = \begin{bmatrix} (d\iota)_p & X_p \end{bmatrix}$$

which is invertible due to the fact that X_p is not tangent to ∂N. By the inverse function theorem there is a neighborhood of $(p, 0)$ where γ is an embedding. By possibly making ϵ_p smaller we may assume that γ is an embedding. If ∂N were compact, ϵ_p would have a minimum ϵ and we would then get an embedding $\partial N \times [0, \epsilon] \hookrightarrow N$. If ∂N is not compact we instead use the trick of reparametrizing X. Without loss of generality assume that ϵ is smooth (there is always a smooth map bounded by ϵ, so we may replace ϵ by that smooth map). We define the reparametrized flow map

$$\widetilde{\gamma} \colon [0,1] \to M$$

by $\widetilde{\gamma}_p(t) = \gamma_p(\epsilon_p t)$ which is still an embedding, and thus its image is a collar of N.

Exercise 3.

Algebraic topology proof. Suppose that such r exists and let $\iota \colon \partial N \to N$ be the inclusion. Since $\iota \circ r$ is the identity, the composition

$$H_{n-1}(\partial M; \mathbb{Z}/2) \xrightarrow{\iota_*} H_{n-1}(M; \mathbb{Z}/2) \xrightarrow{r_*} H_{n-1}(\partial M; \mathbb{Z}/2)$$

must be the identity, so ι_* is injective. On the other hand we have the exact sequence

$$0 \to H_n(N, \partial M; \mathbb{Z}/2) \to H_{n-1}(\partial M; \mathbb{Z}/2) \xrightarrow{\iota_*} H_{n-1}(M; \mathbb{Z}/2)$$

so we conclude that $H_n(M, \partial M; \mathbb{Z}/2)$ vanishes. But for any compact manifold M we have $H_n(M, \partial M; \mathbb{Z}/2) \cong \mathbb{Z}/2$. This can be shows exactly as in 1.b) using the fact that $H_n(M, M \setminus x; \mathbb{Z}/2) \cong \mathbb{Z}/2$ has a unique generator (i.e. every manifold is $\mathbb{Z}/2$-orientable).

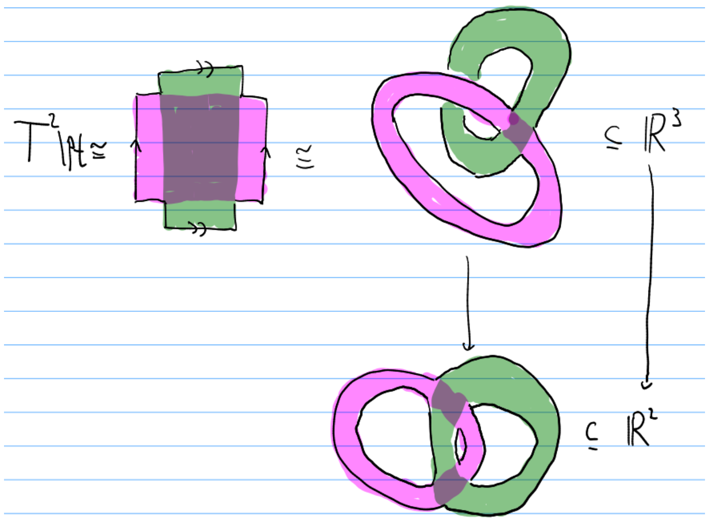
Differential geometry proof. We can assume without loss of generality that r is smooth since a continuous r such that $r_{|\partial M} = \mathrm{id}_{\partial M}$ can always be approximated by a smooth r with the same property. By Sard's theorem there is at least 1 regular value $y \in \partial M$. Then $r^{-1}(y)$ is a 1-dimensional compact manifold with boundary

$$\partial r^{-1}(y) = r^{-1}(y) \cap \partial M = \{y\}\,.$$

But this is not possible since a 1-dimensional compact manifold always has an even number of boundary points (since it must be a disjoint union of closed intervals and circles).

Exercise 4.

i). A torus minus a point is homeomorphic to a torus minus a disk, which can be obtained as the union of two cylinders $S^1 \times [0, 1]$ along a square as explained in the picture below. By regarding $T^2 \setminus pt$ we can easily embed it into $\mathbb{R}^3$ as the picture suggests. Then projectint onto $\mathbb{R}^2$ gives the desired immersion.



ii). Recall that $\mathbb{R}P^2$ can be obtained as a sphere S^2 modulo the equivalence relation $p \sim -p$. Define a map $f : S^2 \to \mathbb{R}^4$ by

$$(x, y, z) \mapsto (xy, yz, zx, x^2 + 2y^2 + 3z^2)\,.$$

Clearly $f(p) = f(-p)$, so this map induces $f : \mathbb{R}P^2 \to \mathbb{R}^4$. We show first that the map is injective. Suppose that $f(p_1) = f(p_2)$ with $p_i = (x_i, y_i, z_i)$. If x_1, y_1, z_1 are all non-zero then the equations $x_1y_1 = x_2y_2$, $y_1z_1 = y_2z_2$, $z_1x_1 = z_2x_2$ imply that $p_1 = \pm p_2$. If for example $x_1 = 0$ then x_2 must also be zero and we have

$$y_1^2 + z_1^2 = 1 = y_2^2 + z_2^2\,, \quad 2y_1^2 + 3z_1^2 = 2y_2^2 + 3z_2^2$$

These equations force $y_1 = \pm y_2$ and $z_1 = \pm z_2$; together with $y_1z_1 = y_2z_2$ this implies again that $p' = \pm p$. It remains to check that $D(f)_{(x,y,z)}$ has rank 2 for every (x, y, z); this can be achieved again by dividing into cases according to which variables x, y, z are 0 and we leave the details to the reader.

We now prove that $\mathbb{R}P^{2n}$ cannot be embedded into $\mathbb{R}^{2n+1}$. Suppose it does. Since $\mathbb{R}P^{2n}$ is compact, we can apply Corollary 3.45 in Hatcher to conclude that $H_{2n-1}(\mathbb{R}P^{2n})$ is torsion free. But that is a contradiction with $H_{2n-1}(\mathbb{R}P^{2n}) \cong \mathbb{Z}/2$ (see Example 2.42 in Hatcher).

iii). Since $\mathbb{D}^1$ is contractible, every vector bundle over $\mathbb{D}^1$ is trivial. In particular, its normal bundle in $\mathbb{D}^3$ is isomorphic to $\mathbb{D}^1 \times \mathbb{R}^2$ so there is a tubular neighborhood $\mathbb{D}^1 \subseteq U \subseteq \mathbb{D}^3$ diffeomorphic to $\mathbb{D}^1 \times D$ where $D \subseteq \mathbb{R}^2$ is a disk. Let $f_K : U \to D$ be the composition of the diffeomorphism with the projection onto the disk. Regarding the sphere as $S^2 = D \cup \infty$ the union of a disk and a point at infinity we can extend f_K to a continuous map $\mathbb{D}^3 \to S^2$ by setting $f_K(x) = \infty$ for all $x \notin U$. Clearly $f_K^{-1}(0) = \mathbb{D}^1$. Since the restriction of f_K to U is smooth, we can perturb f_K away from an open set around $\mathbb{D}^1$ to make it smooth with the desired property.

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