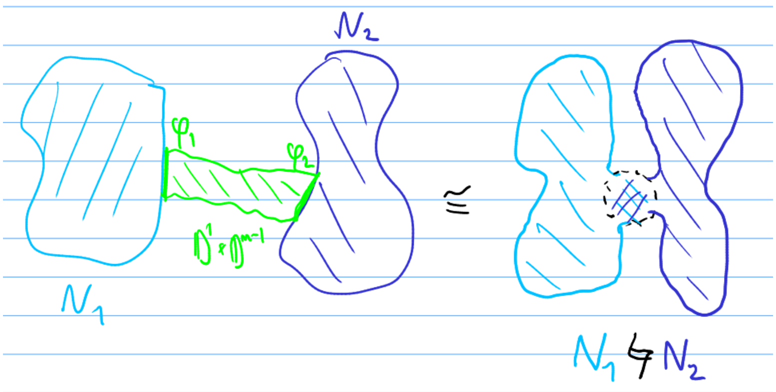
TOPOLOGY ON MANIFOLDS - EXERCISE SHEET II

Exercise 1.

i). Attaching a 1-handle as in the exercise produces precisely the boundary connected sum $N_1 \natural N_2$. Indeed, the gluing map $\varphi\colon S^0 \times \mathbb{D}^{n-1} \hookrightarrow \partial N$ decomposes into inclusions of two disks in the boundaries $\varphi_i\colon \mathbb{D}^{n-1} \hookrightarrow \partial N_i$. So $N \cup_\varphi \mathbb{D}^1 \times \mathbb{D}^{n-1}$ is obtained by gluing N_1 to $\mathbb{D}^1 \times \mathbb{D}^{n-1}$ along φ_1 and then gluing N_2 via φ_2. But that is the same as removing a disk $\mathbb{D}^{n-1}$ from each of the boundaries ∂N_i and gluing along their boundaries, which is precisely what $N_1 \natural N_2$ is (see the picture below).



ii).

- When $k = 0$ the gluing morphism $\varphi\colon S^{-1} \times \mathbb{D}^n \to \partial \mathbb{D}^n$ is by definition empty, so the result is just a disjoint union $\mathbb{D}^n \sqcup \mathbb{D}^n$.
- When $k = 1$ the gluing map is $\varphi\colon S^0 \times \mathbb{D}^{n-1} \hookrightarrow \partial \mathbb{D}^n$. Note that $S^0 \times \mathbb{D}^{n-1}$ is the disjoint union of two disks $\varphi_i\colon \mathbb{D}^{n-1} \hookrightarrow \partial \mathbb{D}^n$. Since $\partial \mathbb{D}^n$ is connected we may apply the unknot lemma from class to conclude that $\mathbb{D}^n \cup_\varphi \mathbb{D}^1 \times \mathbb{D}^{n-1}$ is a $\mathbb{D}^{n-1}$-bundle over S^1.[1] But there are only 2 such bundles over S^1, the trivial bundle $S^1 \times \mathbb{D}^{n-1}$ or the (higher dimensional) Möbius strip $M \times \mathbb{D}^{n-2}$ where M is the usual 2-dimensional Möbius strip.

  Indeed, by Palais disk theorem any two disks $\mathbb{D}^{n-1} \hookrightarrow \partial \mathbb{D}^n$ are isotopic as long as both are orientation preserving or orientation reversing. If both φ_1, φ_2 are orientation preserving or both are orientation reversing we get $S^1 \times \mathbb{D}^{n-1}$, but if one is preserving and the other reversing we get $M \times \mathbb{D}^{n-2}$.
- Let's take $k = 2$ now. In this case the problem becomes much more subtle, especially in the case $n = 4$. We treat $n \geq 5$ and $n = 3$ first.

  If $n \geq 5$ or $n = 3$ then any embedding $S^1 \hookrightarrow S^{n-1}$ bounds a disk (and any two such embedding are isotopic), so we can apply the unknot lemma to conclude that the result is a $\mathbb{D}^{n-2}$-bundle over S^2. Such bundles can be classified. Giving a $\mathbb{D}^{n-2}$-bundle is the same as giving a vector bundle of rank $n-2$ (if you are given a vector bundle, it's always possible to put

[1] There is a typo in the statement in the notes, the conclusion should be that N is a bundle over S^k, not S^n

a Riemannian metric on it and define an associated disk bundle). Vector bundles of rank $n-2$ over S^2 are classified by

$$\pi_1(GL(n-2)) = \begin{cases} 0 & n=3 \\ \mathbb{Z} & n=4 \\ \mathbb{Z}/2 & n\geq 5 \end{cases}$$

So for $n=3$ there is only one possibility which is the trivial bundle $S^2\times\mathbb{D}^1$, and for $n\geq 5$ there are always 2 possible smooth structures.

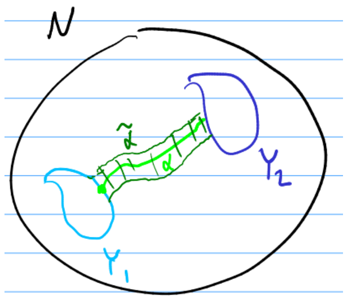
In the $n=4$ case things get much more complicated due to the existence of non-trivial knots $S^1\to S^3$. If the knot is trivial (it is the boundary of a disk) by what was explained before we get a disk bundle for each integer, that corresponds to the complex line bundle $\mathcal{O}_{\mathbb{CP}^1}(n)$ over $\mathbb{CP}^1\cong S^2$.

More generally, the gluing map $\varphi\colon S^1\times\mathbb{D}^2\hookrightarrow S^3$ is specified up to homotopy by the knot $K\colon S^1=S^1\times 0\hookrightarrow S^3$ and a choice of framing. Fixing the knot, the choices of framing are classified by $\pi_1(GL(2))\cong\mathbb{Z}$, so for each knot K and integer n we get a 4-manifold denoted by $X_n(K)$, called the knot trace. The integer n is related to the number of "twists" between $\varphi(S^1\times 0)$ and $\varphi(S^1\times 1)$.

iii). Choose points $p_i\in Y_i\subseteq N$. By Thom transversality theorem we can find a curve $\alpha\colon[0,1]\to N$ which has $\alpha(0)=p_1,\alpha(1)=p_2$ and $\alpha_{(0,1)}$ is transverse to $Y_1\cup Y_2$. But because Y_i has codimension at least 2 in N, transversality means that $\alpha(t)\notin Y_1\cup Y_2$ for every t. We can then pick a tube around α, i.e. $\widetilde{\alpha}\colon[0,1]\times\mathbb{D}^m\hookrightarrow\mathbb{D}^n$ such that $\widetilde{\alpha}(0\times\mathbb{D}^m)\subseteq Y_1$ and $\widetilde{\alpha}(1\times\mathbb{D}^m)\subseteq Y_2$. Then the union of Y_1, Y_2 minus the disks above with the cylinder

$$\big(Y_1\setminus\widetilde{\alpha}(0\times\mathbb{D}^m)\big)\cup\widetilde{\alpha}([0,1]\times\partial\mathbb{D}^m)\cup\big(Y_2\setminus\widetilde{\alpha}(1\times\mathbb{D}^m)\big)$$

is homeomorphic to the connected sum. By smoothing it out we get a smooth embedding.



iv). By taking a neighborhood diffeomorphic to $\mathbb{R}^n_+$ of the point in the boundary of N along which we are gluing, it is enough to prove that $\mathbb{R}^n_+\natural\mathbb{D}^n\cong\mathbb{R}^n_+$. Recall that $\mathbb{R}^n_+\natural\mathbb{D}^n$ is defined as $\mathbb{R}^n_+\,{}_{\nu_1}\#_{\nu_2}\mathbb{D}^n$ for embedding $\nu_1\colon\mathbb{R}^n_+\hookrightarrow\mathbb{R}^n_+$ and $\nu_2\colon\mathbb{R}^n_+\hookrightarrow\mathbb{D}^n$. The definition does not depend on the choice of ν_1,ν_2 so we can take ν_1 to be the identity and ν_2 to be the inclusion of the half disk

$$\mathbb{R}^n_+\cong\mathbb{D}^n\cap(\mathbb{R}^{n-1}\times\mathbb{R}_+)\subseteq\mathbb{D}^n$$

Then by definition

$$\mathbb{R}^n_+\,{}_{\nu_1}\#_{\nu_2}\mathbb{D}^n=\frac{\mathbb{R}^n_+\setminus\nu_1(0)\sqcup\mathbb{D}^n\setminus\nu_2(0)}{\nu_1(v)=\nu_2\big(\mathrm{rev}(|v|)v/|v|\big)}=\mathbb{D}^n\setminus\nu_2(0)\,.$$

The last equality is simply because ν_1 is the identity, so every point of $\mathbb{R}^n_+$ is identified with some point of the disk $\mathbb{D}^n$. But the disk minus a boundary point $\nu_2(0)$ is diffeomorphic to $\mathbb{R}^n_+$, so we are done. An explicit diffeomorphism can be written as follows: without loss of generality assume that $\nu_2(0) = (0, \ldots, 0, 1)$ and define

$$\begin{aligned} \mathbb{D}^n \setminus \{(0, \ldots, 0, 1)\} &\to \mathbb{R}^{n-1} \times \mathbb{R}_+ = \mathbb{R}^n_+ \\ (v, t) &\mapsto \frac{1}{|v|^2 + (1-t)^2}\big(2v, 1 - |v|^2 - t^2\big) \end{aligned}$$

Exercise 2.

i). The inclusion

$$\iota \colon S^{k-1} \times \mathbb{D}^{n-k} \cup \mathbb{D}^k \times 0 \hookrightarrow \mathbb{D}^k \times \mathbb{D}^{n-k}$$

defines a homotopy equivalence. This can be shows from the fact that the both $\mathbb{D}^k \times 0 \hookrightarrow S^{k-1} \times \mathbb{D}^{n-k}$ and the composition

$$\mathbb{D}^k \times 0 \hookrightarrow S^{k-1} \times \mathbb{D}^{n-k} \cup \mathbb{D}^k \times 0 \hookrightarrow \mathbb{D}^k \times \mathbb{D}^{n-k}$$

are homotopy equivalences.

Then it follows that

$$N \cup_\varphi h^k \simeq N \cup_\varphi \big(S^{k-1} \times \mathbb{D}^{n-k} \cup \mathbb{D}^k \times 0\big) = N \cup_{\varphi_{|S^{k-1} \times 0}} \mathbb{D}^k .$$

ii). By the previous exercise we want to describe the fundamental group of N after attaching a k-cell $\mathbb{D}^k$ via the map $\varphi \colon S^{k-1} \to N$. We may apply Seifert-van Kampen to (neighborhoods of) N and S^{k-1} to get

$$\pi_1(N \cup_\varphi \mathbb{D}^k) = \pi_1(N) *_{\pi_1(S^{k-1})} \pi_1(\mathbb{D}^k)$$

if $k > 0$.

(1) If $k = 0$ then Seifert-van Kampen cannot be applied because the intersection is not connected. If N has two connected components $N_1 \sqcup N_2$ and $\varphi(S^0)$ lands on the two connected components, then the fundamental group of the new connected component is $\pi_1(N_1) * \pi_2(N_2)$. Since $N \cup_\varphi \mathbb{D}^k$ only depends up to homotopy on the homotopy class of φ, if $\varphi(S^0)$ is contained in one of the connected components we may assume that $\varphi(S^0) = \{p\}$, in which case
$$\pi_1(N \cup_\varphi \mathbb{D}^1) \cong \pi_1(N \vee S^1) = \pi_1(N) * \mathbb{Z} .$$

(2) If $k = 1$ then $\pi_1(N \cup_\varphi \mathbb{D}^k) \cong \pi_1(N)/\mathrm{im}(\varphi_*)$ where φ_* is the induced map on fundamental groups
$$\varphi_* \colon \mathbb{Z} \cong \pi_1(S^1) \to \pi_1(N) .$$

(3) If $k > 1$ then $\pi_1(N \cup_\varphi \mathbb{D}^k) \cong \pi_1(N)$ as $\pi_1(S^{k-1}) = 0$ in that case.

iii). Recall that if $A \subseteq X$ is a contractible subset then the contraction X/A is homotopy equivalent to X. The boundary connected sum of M_1, M_2 can be topologically described as follows: we pick $p_i \in A_i \subseteq M_i$ where $A_i \cong \mathbb{R}^+_n$ is an open neighborhood of $p_i \in \partial M_i$, take the union $M_1 \setminus p_1 \sqcup M_2 \setminus p_2$ and identify $A_1 \setminus p_1$ with $A_2 \setminus p_2$. Let $\widetilde{A} \subseteq M_1 \natural M_2$ be the image of $A_i \setminus p_i$. Note that $\widetilde{A} \cong \mathbb{R}^+_n \setminus 0$ is contractible, so

$$M_1 \natural M_2 \simeq \big(M_1 \natural M_2\big)/\widetilde{A} = (M_1/A_1) \vee (M_2/A_2) \simeq M_1 \vee M_2 .$$

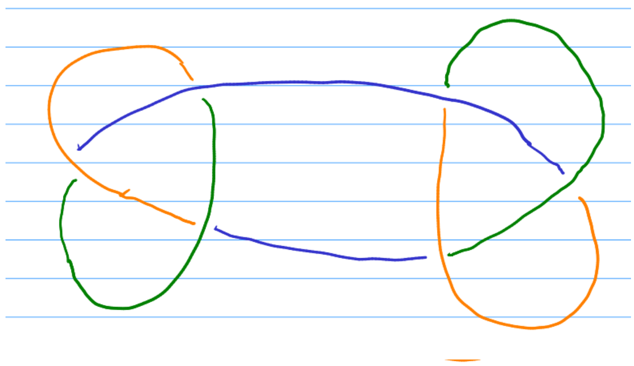
Here $\vee$ denotes the wedge sum obtained by taking the union of two spaces and identifying one point in each.

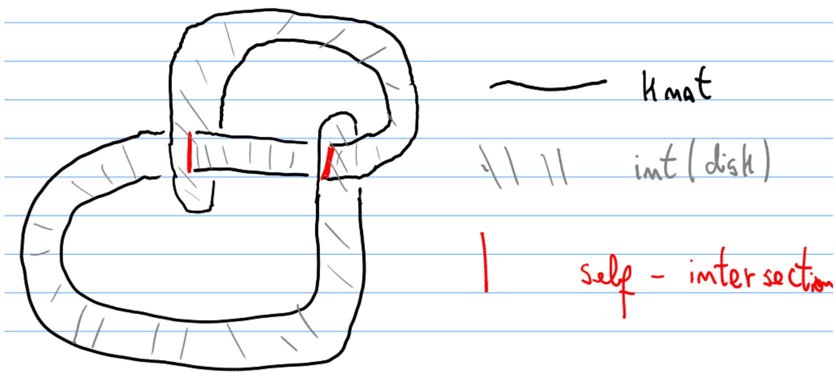
Exercise 3. We will focus on the case $k = 1$, $n = 3$ and in the end will say a few words about the general situation.

When $k = 1$ and $n = 3$ the 3 first conditions are equivalent. First of all, it is clear that (i) implies (iii): the unknot bounds the standard disk $\mathbb{D}^{k+1}$, so if there is such a diffeomorphism $F^{-1}(\mathbb{D}^{k+1})$ is bounded by K. The implication from (iii) to (i) holds by Palais disk theorem: if K bounds a disk D, then there is a isotopy sending D to the standard disk $\mathbb{D}^k$ and hence sending K to U. For the implication (ii) implies (i) one has to use the the Cerf-Palais isotopy extension theorem. It states that any isotopy $K_t\colon S^1 \to S^3$ actually lifts to an ambient space isotopy F_t such that $F_t \circ K_0 = K_t$. But then F_1 is a diffeomorphism sending K to U.

The last condition (iv) is not equivalent to the previous 3. Indeed, there are knots that bound a disk in $\mathbb{D}^4$ but are not trivial, called sliced knots. The simplest example is the square knot, which is the connected sum of two trefoil knots (with reverse orientations) as in the picture. This knot is non-trivial by the 3-colorability criterion (see `https://en.wikipedia.org/wiki/Tricolorability`). In the picture, we can see that the square knot bounds a *immersed* disk in $\mathbb{R}^3$. By including $\mathbb{R}^3$ into $\mathbb{D}^4$ we can bump the disk slightly at the self-intersection areas, using the additional dimension, to make it an embedding.

A few remarks about other dimensions. In the topological (or PL) category, non-trivial knots exist only in codimension 2, i.e. $m - k = 2$. This is no longer true in the smooth category, e.g. the Haefliger trefoil knot $S^3 \to S^6$. Indeed, there are $\mathbb{Z}$-many knots $S^3 \to S^6$ which are not smoothly isotopic. In codimension ≥ 3 condition (iv) is equivalent to the other 3 (Smale-Cerf). Every knot $S^2 \to S^4$ is slice (i.e. satisfies (iv)) but there are many non-trivial such knots (Kervaire).





Exercise 5.

i). We assume that $n > 1$ since for $n = 1$ the only connected compact 1-manifold is S^1. Recall that the a homotopy sphere is a compact smooth n-manifold M homotopic to a sphere S^n. By Withehead's theorem and the fact that there always exists a degree 1 map from M to S^n (essentially obtained by collapsing the complement of a disk in M) a smooth compact manifold M is a homotopy sphere if and only if $\pi_1(M) = 0$ and $H_j(M) = 0$ for $j = 1, \ldots, n-1$.

Suppose now that $M = M_1 \# M_2$ is a homotopy sphere. By the construction of M, we can write M as the union $M = U_1 \cup U_2$ where U_i are open sets homeomorphic to $M_i \setminus p_i$ and with intersection $U_1 \cap U_2$ homeomorphic to the cylinder $S^{n-1} \times [0, 1]$. Since $U_1 \cap U_2$ is connected, by SvK $0 = \pi_1(M) = \pi_1(U_1) * \pi_1(U_2)$ so $\pi_1(U_i)$ are both trivial. But that implies that $\pi_1(M_i)$ is also trivial by using again SvK on $M_i = M \setminus p_i \cup D_i$ where D_i is a disk around p_i.

To prove that the homology vanishes we use a smiliar argument but with Mayer-Vietoris. We have

$$H_{j+1}(M) \to H_j(S^{n-1}) \to H_j(U_1) \oplus H_j(U_2) \to H_j(M)\,.$$

If $1 \le j \le n-2$ then $H_j(S^{n-1}) = H_j(M) = 0$ by hypothesis, so $H_j(U_i)$ is also trivial. If $j = n-1$ then the boundary map $H_n(M) \to H_n(S^{n-1})$ is an isomorphism, so we can still conclude that $H_{n-1}(U_i) = 0$. Using Mayer-Vietoris on M_i with the same decomposition as above we conclude that $H_j(M_i) = 0$ for $j = 1, \ldots, n-1$.

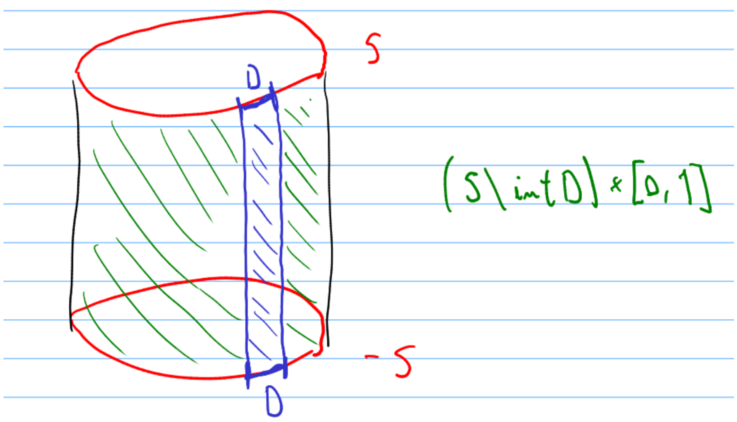
ii). We consider the cylinder $S \times [0, 1]$ and a disk $D \subseteq S$. The connected sum $S\#(-S)$ embeds into $S \times [0, 1]$ as (a smoothing of)

$$(S \setminus D) \times 0 \cup \partial D \times [0, 1] \cup (S \setminus D) \times 1$$

(see the picture). But then the closure of the complement

$$\big(S \setminus \mathrm{int}(D)\big) \times [0, 1]$$

is a contractible set bounded by $S\#(-S)$. To prove that it is contractible, by Withehead theorem we can show that both the fundamental group and the homology groups of $S \setminus \mathrm{int}(D)$ are trivial. This is easily shown using that S is a homotopy sphere and SvK and Mayer-Vietoris theorems; indeed, the argument is exactly the same as in i) taking U_1 to be $\mathrm{int}(D)$ and U_2 to be a disk slightly larger than $\mathrm{int}(D)$.



iii). If f extends to a diffeomorphism $\widetilde{f}\colon \mathbb{D}^k \to \mathbb{D}^k$ then we can define a diffeomorphism

$$\mathbb{D}^k \cup_f \mathbb{D}^k \to \mathbb{D}^k \cup_{\mathrm{id}} \mathbb{D}^k \cong S^k$$

by gluing the maps $\widetilde{f}\colon \mathbb{D}^k \to \mathbb{D}^k$ with $\mathrm{id}\colon \mathbb{D}^k \to \mathbb{D}^k$; since $\widetilde{f}_{|S^{k-1}} = f$ the gluing is well defined and we get a diffeomorphism.

Suppose on the other hand that we have a diffeomorphism $\mathbb{D}^k \cup_f \mathbb{D}^k \to S^k$. By Palais disk theorem, the embedding $\varphi\colon \mathbb{D}^k \to S^k$ is equivalent to the embedding of $\mathbb{D}^k$ into the standard embedding of the northern hemisphere in S^k, so we can assume that φ is obtained as the gluing $\mathbb{D}^k \cup_f \mathbb{D}^k \to \mathbb{D}^k \cup_{\mathrm{id}} \mathbb{D}^k$ of $\varphi_1\colon \mathbb{D}^k \to \mathbb{D}^k$ and $\varphi_2\colon \mathbb{D}^k \to \mathbb{D}^k$ satisfying

$$\varphi_{1|S^{k-1}} = \varphi_{2|S^{k-1}} \circ f\,.$$

But then $\varphi_1 \circ \varphi_2^{-1}$ is the extension required.

Corrections or questions about these solutions can be sent to miguel.moreira@math.ethz.ch.