# ON HOMOTOPY GROUPS OF SPACES OF EMBEDDINGS OF AN ARC OR A CIRCLE: THE DAX INVARIANT

DANICA KOSANOVIĆ

Abstract. We compute in many classes of examples the first potentially interesting homotopy group of the space of embeddings of either an arc or a circle into a manifold $M$ of dimension $d \geq 4$. In particular, if $M$ is a simply connected 4-manifold the fundamental group of both of these embedding spaces is isomorphic to the second homology group of $M$, answering a question posed by Arone and Szymik. The case $d = 3$ gives isotopy invariants of knots in a 3-manifold, that are universal of Vassiliev type $\leq 1$, and reduce to Schneiderman's concordance invariant.

## 1. Introduction

Classical knot theory studies path components of the space of embeddings of a circle into a 3-manifold, and 2-knot theory is concerned with embeddings of a surface into a 4-manifold. One can also study codimension two knotting phenomena in higher dimensions, and in fact even more generally – there exist smooth embeddings of codimension higher than two which are not mutually isotopic, as shown by Haefliger [Hae66].

In this and the forthcoming paper [Kos] we consider a generalisation of knot theory in another direction: instead of studying only the set of components of a space of smooth embeddings, we study its *homotopy groups*, when the source manifold is a single arc or a single circle, and the target is a compact smooth manifold of dimension $d \geq 3$ (with or without boundary, and the arc is embedded neatly). For $d \geq 4$ these groups give another notion of *knottedness*, as their nontrivial elements are represented by multi-parameter families of embeddings which cannot be trivialised through such families. Interestingly, they also show useful for problems in low-dimensional topology, for example, in the recent work of Budney and Gabai [BG19] and ours with Teichner [KT21].

The present work is concerned with the lowest homotopy group potentially distinguishing embeddings from immersions, namely, "knotted" classes in degree $d-3$ for the ambient manifold of dimension $d$. We state our results for the respective cases of arcs and circles in Sections 1.1 and 1.2. In Section 2 we discuss applications, examples and the case $d = 3$.

1.1. **Arcs.** For $\mathbb{D}^1 = [-1, 1]$ and an oriented smooth $d$-dimensional manifold $X$ consider the spaces

$$\begin{aligned} \operatorname{Emb}_\partial(\mathbb{D}^1, X) &= \{K\colon \mathbb{D}^1 \hookrightarrow X \mid K(-1) = x_-,\, K(1) = x_+\}, \\ \operatorname{Imm}_\partial(\mathbb{D}^1, X) &= \{K\colon \mathbb{D}^1 \looparrowright X \mid K(-1) = x_-,\, K(1) = x_+\}, \end{aligned} \tag{1}$$

of embedded and immersed arcs respectively, which are neat and fixed on the boundary (i.e. $K$ is transverse to $\partial X$ and $K \cap \partial X = \{x_-, x_+\}$). Let us fix an arbitrary basepoint $\mathsf{u} \in \operatorname{Emb}_\partial(\mathbb{D}^1, X)$.

For immersions we have a map

$$p_{\mathsf{u}}\colon \operatorname{Imm}_\partial(\mathbb{D}^1, X) \to \Omega X, \tag{2}$$

which by Smale [Sma58] induces isomorphisms on homotopy groups in degrees $n \leq d-3$:

$$\pi_n p_{\mathsf{u}}\colon \pi_n(\operatorname{Imm}_\partial(\mathbb{D}^1, X), \mathsf{u}) \cong \pi_n(\Omega X, \mathsf{const}_{x_-}) \cong \pi_{n+1}(X, x_-).$$

For $F\colon \mathbb{S}^n \to \operatorname{Imm}_\partial(\mathbb{D}^1, X)$ the map $p_{\mathsf{u}} \circ F\colon \mathbb{S}^n \to \Omega X$ sends $\vec{t}$ to the loop $F(\vec{t}) \cdot \mathsf{u}^{-1}$ based at $x_-$, where the inverse denotes the reverse of a path and the dot stands for the concatenation of paths.

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Following [Dax72] and [Gab21] we described in [KT21] a range of homotopy groups of $\mathrm{Emb}_\partial(\mathbb{D}^1, X)$.

**Theorem 1.1** ([KT21])**.** *Assume $d \geq 4$. Then there is a bijection $\pi_0 \mathrm{Emb}_\partial(\mathbb{D}^1, X) \cong \pi_1 X$, and for any basepoint $\mathsf{u} \in \mathrm{Emb}_\partial(\mathbb{D}^1, X)$ there are isomorphisms*

$$\pi_n(\mathrm{Emb}_\partial(\mathbb{D}^1, X), \mathsf{u}) \xrightarrow[\cong]{\pi_{d-3}(\mathfrak{i}_X, \mathsf{u})} \pi_n(\mathrm{Imm}_\partial(\mathbb{D}^1, X), \mathsf{u}) \xrightarrow[\cong]{\pi_{d-3} p_{\mathsf{u}}} \pi_{n+1} X$$

*where $1 \leq n \leq d-4$, and a group extension*

$$\mathbb{Z}[\pi_1 X \setminus 1] \big/ \mathsf{dax}_{\mathsf{u}}(\pi_{d-1} X) \underset{\mathsf{Dax}}{\overset{\partial\mathfrak{r}}{\rightleftarrows}} \pi_{d-3}\big(\mathrm{Emb}_\partial(\mathbb{D}^1, X), \mathsf{u}\big) \xrightarrow{\pi_{d-3} p_{\mathsf{u}}} \pi_{d-2} X. \tag{3}$$

*For $d = 4$ this extension is central, with an explicit commutator pairing.*

Let us describe the maps appearing in the theorem, referring to Section 3 for details. Firstly, $p_{\mathsf{u}}$ in (3) is obtained from (2) by precomposing with the inclusion $\mathfrak{i}_X \colon \mathrm{Emb}_\partial(\mathbb{D}^1, X) \hookrightarrow \mathrm{Imm}_\partial(\mathbb{D}^1, X)$. The homomorphism $\partial\mathfrak{r}$ is is an explicit *geometric realization*: for $g \in \pi_1 X \setminus 1$ the family $\partial\mathfrak{r}(g)$ has a piece of $u$ dragged around an embedded loop representing $g$, so that it comes back close to a meridian sphere to $\mathsf{u}$ and then swings around it; see Figure 1 and (16). The *Dax invariant* $\mathsf{Dax}$ is the inverse of $\partial\mathfrak{r}$ on the subgroup $\ker(\pi_{d-3} p_{\mathsf{u}})$, and is defined by picking a path through immersed arcs from the given $(d-3)$-parameter family of embedded arcs to the constant one $\mathsf{const}_{\mathsf{u}}$, and then counting double points that occur, together with associated loops; see (15) for the precise formula. Finally, computing $\mathsf{Dax}$ on self-homotopies of the trivial family gives the homomorphism

$$\mathsf{dax}_{\mathsf{u}} \colon \pi_{d-1} X \longrightarrow \mathbb{Z}[\pi_1 X \setminus 1]. \tag{4}$$

Namely, given $a \in \pi_{d-1} X$ we pick $F_A \colon \mathbb{S}^{d-2} \to \mathrm{Imm}_\partial(\mathbb{D}^1, X)$ representing it, that is $p_{\mathsf{u}}(F_A) = a$, and view $F_A$ as a self-homotopy of $\mathsf{const}_{\mathsf{u}}$, for which we can compute $\mathsf{dax}_{\mathsf{u}}(a) := \mathsf{Dax}(F_A)$; see (19). See Remark 2.2 for a relation of $\mathsf{Dax}$ and $\mathsf{dax}_{\mathsf{u}}$ to the Goodwillie–Weiss embedding calculus.

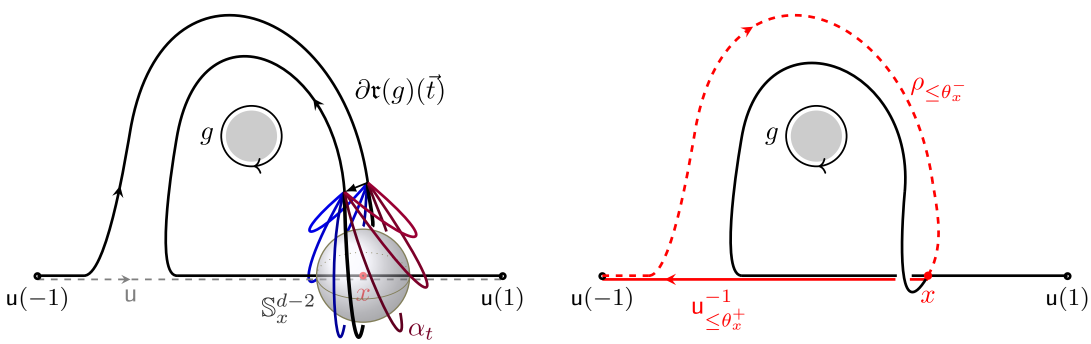


**Figure 1.** *Left.* The family $\partial\mathfrak{r}(g)(\vec{t}) \in \mathrm{Emb}_\partial(\mathbb{D}^1, X)$ for several values $\vec{t} \in \mathbb{S}^{d-3}$ and $d = 4$ (coloured arcs are in past or future). *Right.* The single immersed arc $\rho$ in a homotopy from $\partial\mathfrak{r}(g)$ to $\mathsf{const}_{\mathsf{u}}$ has one double point $x = \rho_{\theta_x^-} = \rho_{\theta_x^+}$, with sign $+1$ and loop $\rho_{\leq\theta_x^-} \cdot \rho^{-1}_{\leq\theta_x^+} \simeq \rho_{\leq\theta_x^-} \cdot \mathsf{u}^{-1}_{\leq\theta_x^+} \simeq g$, so $\mathsf{Dax}(\partial\mathfrak{r}(g)) = g$.

Note that if $\pi_1 X = 1$ we have the isomorphism $\pi_{d-3} p_{\mathsf{u}} \colon \pi_{d-3}(\mathrm{Emb}_\partial(\mathbb{D}^1, X), \mathsf{u}) \xrightarrow{\cong} \pi_{d-2} X$, so the first potentially interesting group (i.e. one which would distinguish embeddings from immersions) turns out *not* to be so interesting. On the other hand, for $\pi_1 X \neq 1$ it remains to understand the image of (4) and also describe the extension (3). For the latter see Questions 2.20 and 2.21 below, while the former is studied in Theorem A, which gives simplifying formulae for $\mathsf{dax}_{\mathsf{u}}$. We use them to compute $\ker(\pi_{d-3} p_{\mathsf{u}})$ for several classes of target manifold $X$ in Sections 2.1 and 2.2.

In Theorem A we consider $\mathsf{dax}_{\mathsf{u}}$ actually for any $d \geq 3$, and compare it to the homomorphism

$$\mathsf{dax} := \mathsf{dax}_{\mathsf{u}_-}$$

for a fixed arc $\mathsf{u}_- \colon \mathbb{D}^1 \hookrightarrow X$ with endpoints $\mathsf{u}_-(-1) = x_-$ and $\mathsf{u}_-(1) = x'_-$, a point close to $x_-$ (so $\mathsf{u}$ and $\mathsf{u}_-$ live in different spaces $\mathrm{Emb}_\partial(\mathbb{D}^1, X)$), and so that $\mathsf{u}_-$ *is isotopic into $\partial X$ rel. endpoints.*

We need to fix some more notation. Let $ga \in \pi_n X$ denote the usual action of $g \in \pi_1 X \coloneqq \pi_1(X, x_-)$ on $a \in \pi_n X \coloneqq \pi_n(X, x_-)$ for $n \geq 1$, and $g\boldsymbol{k} \in \pi_1(X, \partial X)$ the action of $g \in \pi_1 X$ on the set of arcs $\boldsymbol{k} \in \pi_1(X, \partial X) \coloneqq \pi_1(X, \partial X, x_-)$, by precomposition at $x_-$. We will express $\mathsf{dax}_{\mathsf{u}}$ in terms of $\mathsf{dax}$ and the well-known algebraic invariant, the *equivariant intersection pairing*:

$$\lambda\colon \pi_{d-1} X \times \pi_1(X, \partial X) \to \mathbb{Z}[\pi_1 X].$$

This is an intersection pairing on the homology of the universal cover, recalled in Section 3.3. In fact, we use $\lambdabar$, defined as $\lambda$ minus its term at $1 \in \pi_1 X$. Finally, we write $\lambda(\boldsymbol{k}, a) \coloneqq (-1)^{d-1}\overline{\lambda(a, \boldsymbol{k})}$ for $a \in \pi_{d-1} X$ and $\boldsymbol{k} \in \pi_1(X, \partial X)$, using the involution on the group ring $\mathbb{Z}[\pi_1 X]$ linearly extending $\overline{g} \coloneqq g^{-1}$. The following results will be proven in Section 3.3, and applied in Section 2.

**Theorem A.** *Assume $d \geq 3$ and denote by $\mathbf{u}$ the homotopy class of $\mathsf{u}$ in $\pi_1(X, \partial X)$. For any $a \in \pi_{d-1} X$ and $g \in \pi_1 X$ we have*

*(I)* $\mathsf{dax}_{\mathsf{u}}(a) = \mathsf{dax}(a) + \lambdabar(a, \mathbf{u})$,

*(II)* $\mathsf{dax}(ga) = g\mathsf{dax}(a)\overline{g} - \lambdabar(ga, g) + \lambdabar(g, ga)$.

**Corollary B.** *(i) If $a$ has an embedded representative, then* $\mathsf{dax}_{\mathsf{u}}(a) = \lambdabar(a, \mathbf{u})$.

*(ii)* $\mathsf{dax}_{g\mathsf{u}}(a) - \mathsf{dax}_{\mathsf{u}}(a) = \lambdabar(a, g\mathbf{u}) - \lambdabar(a, \mathbf{u})$.

*(iii)* $\mathsf{dax}_{g\mathsf{u}}(ga) - g\mathsf{dax}_{\mathsf{u}}(a)\overline{g} = \lambdabar(g, ga)$.

*(iv)* $\mathsf{dax}_{\mathsf{u}}(ga) - g\mathsf{dax}_{\mathsf{u}}(a)\overline{g} = \lambdabar(ga, \mathbf{u}) - \lambdabar(ga, g\mathbf{u}) + \lambdabar(g, ga)$.

*(v) If $a$ has an embedded representative, then* $\mathsf{dax}_{\mathsf{u}}(ga) = \lambdabar(ga, \mathbf{u}) - \lambdabar(ga, g) + \lambdabar(g, ga)$.

**Remark 1.2.** We have $\pi_0 \operatorname{Emb}_\partial(\mathbb{D}^1, X) \cong \pi_0 \operatorname{Map}_\partial(\mathbb{D}^1, X) = \{f\colon \mathbb{D}^1 \to X \mid f(\partial \mathbb{D}^1) = \{x_\pm\}\}/\simeq$ ($\cong \pi_1 X$ via $p_{\mathsf{u}}$) if $d \geq 4$, so $\mathsf{Dax}$ and $\mathsf{dax}_{\mathsf{u}}$ depend only on the homotopy class $\mathbf{u} \in \pi_0 \operatorname{Map}_\partial(\mathbb{D}^1, X)$. However, we define $\mathsf{Dax}$ also for $d = 3$, when it can depend on $\mathsf{u}$ itself, see Section 2.3. △

**Remark 1.3.** In [KT21] we exhibit for $d = 4$ a relation between $\mathsf{dax}$ and Wall's self-intersection invariant $\mu_3$, also important for computations; for example, using it we showed that every finitely generated abelian group is realised as $\ker(\pi_1 p_{\mathsf{u}})$. See also Remark 2.11 and Theorem 2.17 below. △

1.2. **Circles.** For a $d$-dimensional manifold $N$ (with or without boundary) we study the space $\operatorname{Emb}(\mathbb{S}^1, N)$ of knots in $N$, by comparing it to the space of immersions via the inclusion

$$\mathfrak{i}_N\colon \operatorname{Emb}(\mathbb{S}^1, N) \hookrightarrow \operatorname{Imm}(\mathbb{S}^1, N).$$

Given an arbitrary basepoint $\mathsf{s}\colon \mathbb{S}^1 \hookrightarrow N$, pick a small open ball $D^d = \operatorname{int}(\mathbb{D}^d) \subseteq N$ around the point $\mathsf{s}(e)$. Let $\mathsf{u}\colon \mathbb{D}^1 \hookrightarrow N \setminus D^d$ be the neatly embedded arc obtained by restricting $\mathsf{s}$ so that $\mathsf{u}(\mathbb{D}^1) = \mathsf{s}(\mathbb{S}^1) \cap (N \setminus D^d)$, see Figure 2. In Section 4 we will relate the homotopy type of $\operatorname{Emb}(\mathbb{S}^1, N)$ based at $\mathsf{s}$ to the space $\operatorname{Emb}_\partial(\mathbb{D}^1, N \setminus D^d)$ based at $\mathsf{u}$, and use it to prove the following (the first part seems to be well known, but we provide a proof for the sake of completeness, see Section 4.1).

**Theorem C.** *Assume $d \geq 4$ and denote by $\mathbf{s} \in \pi_1 N = \pi_1(N, \mathsf{s}(e))$ the homotopy class of $\mathsf{s}$.*

*(I) There is a bijection of $\pi_0 \operatorname{Imm}(\mathbb{S}^1, N)$ with the set of conjugacy classes of $\pi_1 N$, and for any basepoint $\mathsf{s} \in \operatorname{Imm}(\mathbb{S}^1, N)$ and $1 \leq n \leq d-3$ there is a group extension*

$$\pi_{n+1}(N)\big/\langle a - \mathbf{s}\, a\rangle \rightarrowtail \pi_n\big(\operatorname{Imm}(\mathbb{S}^1, N), \mathsf{s}\big) \twoheadrightarrow \big\{b \in \pi_n N \mid b = \mathbf{s}\, b\big\}. \tag{5}$$

*Moreover, if $\mathsf{s}$ is nullhomotopic, this extension* splits.

*(II) For $0 \leq n \leq d-4$ the map $\mathfrak{i}_N$ induces isomorphisms $\pi_n(\operatorname{Emb}(\mathbb{S}^1, N), \mathsf{s}) \cong \pi_n(\operatorname{Imm}(\mathbb{S}^1, N), \mathsf{s})$ while for $n = d-3$ there is a group extension*

$$\mathbb{Z}[\pi_1 N \setminus 1]\big/ rel_{\mathsf{s}} \underset{\mathsf{Dax}}{\overset{\partial\mathfrak{r}}{\rightleftarrows}} \pi_{d-3}\big(\operatorname{Emb}(\mathbb{S}^1, N), \mathsf{s}\big) \xrightarrow{\pi_{d-3}(\mathfrak{i}_N, \mathsf{s})} \pi_{d-3}\big(\operatorname{Imm}(\mathbb{S}^1, N), \mathsf{s}\big), \tag{6}$$

*for $rel_{\mathsf{s}} \coloneqq \mathsf{dax}_{\mathsf{u}}(\pi_{d-1}(N \setminus D^d)) \oplus \big\{\mathsf{Dax}(\delta^{whisk}_{\mathsf{s}}(c)) \mid c = \mathbf{s}\, c \in \pi_{d-2} N\big\}$, where the family of embedded arcs $\delta^{whisk}_{\mathsf{s}}(c) \in \ker \pi_{d-3}(\mathfrak{i}_{N \setminus D^d}, \mathsf{u})$ is obtained from $\mathsf{s}$ by isotoping $\mathsf{s}(e)$ around $c$. Moreover, if $\mathsf{s}$ is nullhomotopic, then $\delta^{whisk}_{\mathsf{s}}(c)$ vanishes for all $c$.*

We can use Corollary B to identify a big class of relations in $rel_{\mathsf{s}}$.

**Corollary 1.4.** *In the above setting, let $\Phi\colon \mathbb{S}^{d-1} = \partial\mathbb{D}^d \hookrightarrow N\setminus D^d$ be a parametrization of the boundary of the removed ball, and $\mathbf{\Phi}\in\pi_{d-1}(N\setminus D^d)$ its homotopy class. Then for any $g\in\pi_1 N$ we have $\mathsf{dax}_{\mathsf{u}}(g\,\mathbf{\Phi}) = (-1)^{d-1}\overline{g} - g\bar{\mathsf{s}}$ (mod 1). In particular, these expressions belong to $rel_{\mathsf{s}}$.*

*Proof.* Since $\mathbf{\Phi}$ has an embedded representative, Corollary B(v) gives

$$\mathsf{dax}_{\mathsf{u}}(g\mathbf{\Phi}) = \lambdabar(g\mathbf{\Phi},\mathbf{u}) - \lambdabar(g\mathbf{\Phi},g) + \lambdabar(g,g\mathbf{\Phi}). \tag{7}$$

We claim that $\lambda(\mathbf{\Phi},\mathbf{u}) = 1-\bar{\mathbf{s}}$, so $\lambda(g\mathbf{\Phi},\mathbf{u}) = g\lambda(\mathbf{\Phi},\mathbf{u}) = g-g\bar{\mathbf{s}}$, by a property of $\lambda$ (see Lemma 3.11). Indeed, there are two intersection points of a pushoff $\Phi'\colon\mathbb{S}^{d-1}\hookrightarrow N\setminus D^d$ and $\mathbf{u}$ as in Figure 2. The first point is close to $\mathsf{u}(-1)$, with the group element clearly $1\in\pi_1 N$ and positive sign, and the other one is close to $\mathsf{u}(+1)$, with the negative sign (only the orientation of $\mathsf{u}$ changed) and the group element given by the dashed arc followed by $\mathsf{u}^{-1}$, so homotopic to $(\mathsf{s}\cap D^d)^{-1}\cdot\mathsf{u}^{-1} = \mathsf{s}^{-1}$.

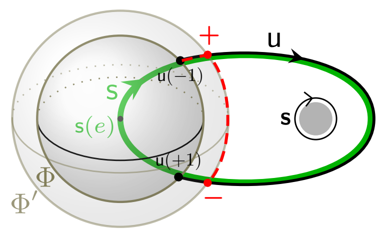


**Figure 2.** The setting of Theorem C and Corollary 1.4. Removing a $d$-ball from a manifold $N$, leaves a boundary component $\Phi$ and turns an embedded circle $\mathsf{s}$ into a neatly embedded arc $\mathsf{u}$. The push-off $\Phi'$ is a $(d-1)$-sphere that intersects $\mathsf{u}$ in two points of opposite sign.

By the same argument we also find $\lambda(g\mathbf{\Phi},g) = g\lambda(\mathbf{\Phi},g) = g(1-\overline{g}) = g-1$, so $\lambdabar(g\mathbf{\Phi},g) = g$ (mod 1), and by definition $\lambdabar(g,g\mathbf{\Phi}) = (-1)^{d-1}\overline{g}$ (mod 1). Plugging this into (7) we have

$$\mathsf{dax}_{\mathsf{u}}(g\mathbf{\Phi}) = g - g\bar{\mathbf{s}} - g + (-1)^{d-1}\overline{g} = (-1)^{d-1}\overline{g} - g\bar{\mathbf{s}} \pmod 1. \qquad \square$$

**Remark 1.5.** See [KT21, Lem.4.10] for a similar computation (when $\mathsf{u}$ is nullhomotopic). Note that the class $g\,\mathbf{\Phi}\in\pi_{d-1}(N\setminus D^d)$ is rarely trivial: in fact, this is the case if and only if $N$ is a simply connected rational homology sphere (in which case $g=\mathbf{s}=1$), see Lemma 4.9. △

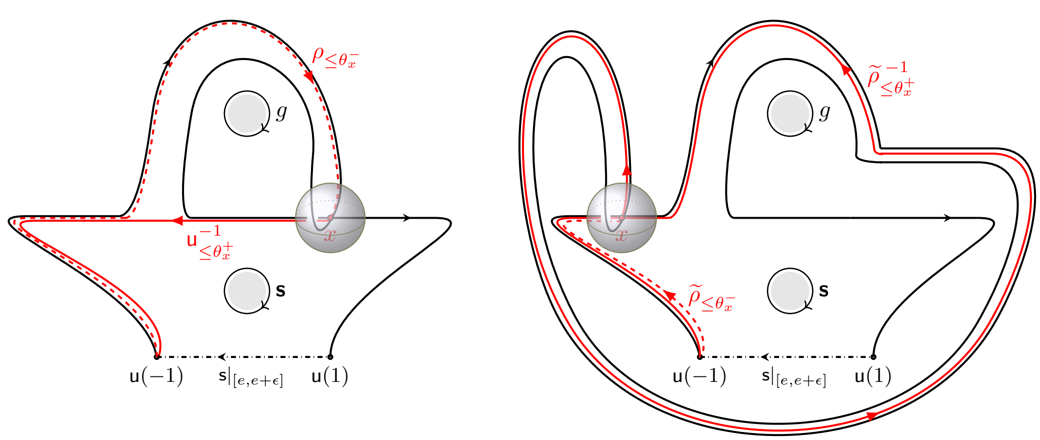


**Figure 3.** *Left.* The $(d-3)$-family of circles $c(g) \coloneqq \partial\mathfrak{r}(g)\cdot\mathsf{s}|_{[e,e+\epsilon]}$ in $N$. *Right.* After an isotopy of $c(g)$ in $N$, the restriction to $N\setminus D^d$ is a family of arcs with $\mathsf{Dax}(\partial\mathfrak{r}(g)^{new}) = (-1)^{d-3}[\widetilde{\rho}_{\leq\theta_x^-}\cdot\widetilde{\rho}^{-1}_{\leq\theta_x^+}] = (-1)^{d-3}\mathbf{s}^{-1}g^{-1}$.

Finally, to understand where the relations $\overline{g} = (-1)^{d-1}g\bar{\mathbf{s}}$ come from, consider $\mathsf{u} = \mathsf{s}|_{\mathbb{S}^1\setminus[e,e+\epsilon]}$ restricted from $\mathsf{s}\colon\mathbb{S}^1\hookrightarrow N$ and the class $\partial\mathfrak{r}(g)$ on $\mathsf{u}$, as in Figure 3. Then in $N$ we can slide the tip of the finger along $\mathsf{s}^{-1}$, to reach the final position as on the left of the figure. Whereas $\mathsf{Dax}(\partial\mathfrak{r}(g)) = g$ as explained in Figure 1, for the newly obtained arc family we have $\mathsf{Dax}(\partial\mathfrak{r}(g)^{new}) = (-1)^{d-3}\overline{\mathsf{s}g}$. Thus, for $\mathsf{Dax}$ for circles to be well defined we must have $(-1)^{d-3}\overline{\mathsf{s}g} - g = 0$, or equivalently $(-1)^{d-1}\overline{h} - h\overline{\mathsf{s}} = 0$ for all $h \in \pi_1 N$.

**Outline.** Sections 2.1 and 2.2 contain examples for $d \geq 4$, while Section 2.3 studies $d = 3$, the relation to finite type and concordance invariants, and states Theorem D. In Section 3 we discuss arcs, proving Theorem A and Corollary B. In Section 4 we discuss circles, proving Theorems C and D.

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## 2. Applications

### 2.1. Simply connected examples.

The part (i) of the next corollary is immediate from Theorem 1.1. The part (ii) follows from (i) using Corollary (i) and the fact from Theorem C that the sequence (5) splits if $\mathsf{s}$ is nullhomotopic.

**Corollary 2.1.** *Let $X$ and $N$ be $d$-manifolds with $d \geq 4$ and $\partial X \neq \emptyset$.*

*(i) If $\pi_1 X = 1$ then embedded and immersed arcs in $X$ cannot be distinguished by $(d-3)$-parameter families: $\pi_{d-3}\operatorname{Emb}_\partial(\mathbb{D}^1, X) \cong \pi_{d-3}\operatorname{Imm}_\partial(\mathbb{D}^1, X) \cong \pi_{d-2}X$.*

*(ii) Similarly, if $\pi_1 N = 1$ then the natural inclusion induces an isomorphism*

$$\pi_{d-3}\operatorname{Emb}(\mathbb{S}^1, N) \cong \pi_{d-3}\operatorname{Imm}(\mathbb{S}^1, N) \cong \pi_{d-3}N \oplus \pi_{d-2}N.$$

For example, if $d \geq 4$ then $\pi_{d-3}\operatorname{Emb}_\partial(\mathbb{D}^1, \mathbb{D}^d) \cong 0$. In fact, Budney shows in [Bud08, Prop. 3.9] that the first homotopy group that distinguishes $\operatorname{Emb}_\partial(\mathbb{D}^1, \mathbb{D}^d)$ and $\operatorname{Imm}_\partial(\mathbb{D}^1, \mathbb{D}^d)$ is in degree $2(d-3)$, namely $\pi_{2(d-3)}\operatorname{Emb}_\partial(\mathbb{D}^1, \mathbb{D}^d) \cong \mathbb{Z}$, while $\pi_{2(d-3)}\operatorname{Imm}_\partial(\mathbb{D}^1, \mathbb{D}^d) \cong \pi_{2d-5}\mathbb{S}^{d-1}$ is finite.

**Remark 2.2.** Whereas Budney describes the generator in the corresponding homology group $H_{2(d-3)}(\operatorname{Emb}_\partial(\mathbb{D}^1, \mathbb{D}^d); \mathbb{Z}) \cong \mathbb{Z}$ (which was computed by Turchin), we can directly write down a generating map

$$\mathbb{S}^{2(d-3)} = \mathbb{S}^{d-3} \wedge \mathbb{S}^{d-3} \to \operatorname{Emb}_\partial(\mathbb{D}^1, \mathbb{D}^d).$$

It sends $(\vec{t_1}, \vec{t_2})$ for $\vec{t_i} \in \mathbb{S}^{d-3}$ to the knot obtained by replacing a subinterval of the basepoint arc $\mathsf{u}$ with the arc obtained as the embedded commutator $[\mu_1(\vec{t_1}), \mu_2(\vec{t_2})]$. Here $\mu_i \colon \mathbb{S}^{d-2} \hookrightarrow \mathbb{D}^d$ are two different meridian spheres for $\mathsf{u}$, and $\mu_i(\vec{t_i})$ are the arcs foliating them.

This is based on constructions from [Kos20] and will appear in [Kos], where we more generally use *gropes* of degree $n \geq 1$ to give generators of the kernel of the surjection $\pi_{n(d-3)}\mathsf{ev}_n$, for the maps $\mathsf{ev}_n \colon \operatorname{Emb}_\partial(\mathbb{D}^1, X) \to T_n \operatorname{Emb}_\partial(\mathbb{D}^1, X)$ to the Goodwillie–Weiss tower. The present example is the case $n = 2$, while the realisation map $\mathfrak{r}$ from Theorem 1.1 is precisely degree $n = 1$.

In fact, $\mathsf{Dax}$ is precisely the isomorphism induced by $\mathsf{ev}_2$, from $\ker(\pi_{d-3}\mathsf{ev}_1)$ to $\ker(\pi_{d-3}p_2)$, where $p_2 \colon T_2 \to T_1 = \operatorname{Imm}_\partial(\mathbb{D}^1, X)$. The latter is isomorphic to the quotient of $\pi_{d-3}\operatorname{fib}(p_2) \cong \mathbb{Z}[\pi_1 X]$ by the image of the connecting map $\Omega T_1 \to \operatorname{fib}(p_2)$, which agrees with $1 \oplus \mathsf{dax}_\mathsf{u}$; cf. Remark 2.15. $\triangle$

If $N$ is simply connected, Corollary (ii) says that the lowest homotopy group that could distinguish embedded from immersed circles in $N$, actually does *not*. In particular, if $N$ is a simply connected 4-manifold, we have $\pi_0\operatorname{Emb}(\mathbb{S}^1, N) = 1$ and $\pi_1\operatorname{Emb}(\mathbb{S}^1, N) \cong \pi_2(N) \cong H_2(N; \mathbb{Z})$.

This answers in negative a question of Arone and Szymik [AS20] whether for a simply connected 4-manifold the inclusion of embedded into immersed circles can fail to be injective on $\pi_1$. Injectivity was shown for a certain class of 4-manifolds by [Mor20].

We thank Daniel Ruberman for pointing out the following application of this result. It is a classical theorem of Wall [Wal64] that for $N$ a closed simply connected oriented 4-manifold with indefinite $\lambda_N$, every automorphism of the intersection form $(H_2(N\#\mathbb{S}^2\times\mathbb{S}^2),\lambda)$ is *realised by a diffeomorphism* of $N\#\mathbb{S}^2\times\mathbb{S}^2$. To prove this Wall takes an unknot $\mathsf{s}\colon\mathbb{S}^1\hookrightarrow N$ and a tubular neighbourhood $\nu\mathsf{s}\colon\mathbb{S}^1\times\mathbb{D}^3\hookrightarrow N$ and considers the composite $\rho$ of the connecting map

$$\pi_1(\mathrm{Emb}(\mathbb{S}^1\times\mathbb{D}^3,N),\nu\mathsf{s})\to\pi_0\,\mathrm{Diff}_\partial(N\setminus\nu\mathsf{s})$$

for the fibration sequence $\mathrm{Diff}_\partial(N\setminus\nu\mathsf{s})\to\mathrm{Diff}(N)\to\mathrm{Emb}(\mathbb{S}^1\times\mathbb{D}^3,N)$, and the map

$$\pi_0\,\mathrm{Diff}_\partial(N\setminus\nu\mathsf{s})\to\pi_0\,\mathrm{Diff}(N\#\mathbb{S}^2\times\mathbb{S}^2),$$

defined using the inclusion $N\setminus\nu\mathsf{s}\hookrightarrow(N\setminus\nu\mathsf{s})\cup\mathbb{D}^2\times\mathbb{S}^2=N\#\mathbb{S}^2\times\mathbb{S}^2$ and extending by the identity. Wall then picks lifts $f_w\in\pi_1(\mathrm{Emb}(\mathbb{S}^1\times\mathbb{D}^3,N),\nu\mathsf{s})$ of classes $w\in\pi_2N$ under the forgetful maps

$$\pi_1(\mathrm{Emb}(\mathbb{S}^1\times\mathbb{D}^3,N),\nu\mathsf{s})\to\pi_1(\mathrm{Emb}(\mathbb{S}^1,N),\mathsf{s})\to\pi_2N. \tag{8}$$

More explicitly, each class $f_w$ is given as a self-isotopy of $\mathsf{s}$ obtained by foliating a generically immersed 2-sphere $S_w\colon\mathbb{S}^2\looparrowright N$ representing $w$. Namely, keep $\mathsf{s}$ fixed near $e\in\mathbb{S}^1$ while the rest of $\mathbb{S}^1$ performs a "lasso" move around $S_w$ (so that in every moment of time the lasso is embedded).

However, $w\in\pi_2N\cong H_2(N;\mathbb{Z})\cong H^2(N;\mathbb{Z})$ does lift to such a class $f_w$ if and only if it satisfies $w^2=0\in\mathbb{Z}/2\cong H^4(N;\mathbb{Z}/2)$ (this can be seen using the long exact sequence for the fibration $\mathrm{Emb}(\mathbb{S}^1\times\mathbb{D}^3,N)\to\mathrm{Emb}(\mathbb{S}^1,N)$ whose fibre is homotopy equivalent to $\Omega SO(3)$). In fact, if $w^2=1$ then we instead obtain a diffeomorphism $\rho(f_w)\colon N\#\mathbb{S}^2\times\mathbb{S}^2\to N\#\mathbb{S}^2\widetilde{\times}\mathbb{S}^2$, where $\mathbb{S}^2\widetilde{\times}\mathbb{S}^2$ is the nontrivial $\mathbb{S}^2$ bundle over $\mathbb{S}^2$ (conversely, if such a diffeomorphism exists then $\lambda_N$ is odd).

Thus, for every $w$ of even square Wall obtains a diffeomorphism $\rho(f_w)$ of $N\#\mathbb{S}^2\times\mathbb{S}^2$, which he then uses to generate all automorphisms of $\lambda_{N\#\mathbb{S}^2\times\mathbb{S}^2}$. On the other hand, we note that in (8) the first map is injective since $\pi_2SO(3)=0$, and the second map is an isomorphism by Corollary 2.1(ii). Therefore, we deduce the following observation about Wall's construction.

**Corollary 2.3.** *For a closed simply connected oriented 4-manifold $N$, we have the identification $\pi_1(\mathrm{Emb}(\mathbb{S}^1\times\mathbb{D}^3,N),\nu\mathsf{s})=\{f_w\mid w\in\pi_2N,\ w^2=0\pmod 2\}$, so the image of Wall's homomorphism $\rho\colon\pi_1(\mathrm{Emb}(\mathbb{S}^1\times\mathbb{D}^3,N),\nu\mathsf{s})\to\pi_0\,\mathrm{Diff}(N\#\mathbb{S}^2\times\mathbb{S}^2)$ consists precisely of the classes $\rho(f_w)$.*

2.2. **Further examples.** The following corollaries are immediate from Theorems 1.1 and C. For the first we use that the set of relations $rel_{\mathsf{s}}$ contains $\mathsf{dax}_{\mathsf{u}}(\pi_{d-1}(N\setminus D^d))$ and $\pi_1(N\setminus D^d)=\pi_1N$. Recall that we denote $\overline{g}\coloneqq g^{-1}$.

**Corollary 2.4.**
*(i) For the inclusion $\mathsf{i}_N$ of embedded into immersed circles in $N$ to induce an isomorphism $\pi_{d-3}(\mathsf{i}_N,\mathsf{s})$, it suffices that the analogous inclusion $\mathsf{i}_{N\setminus D^d}$ for arcs in $N\setminus D^d$ induces an isomorphism $\pi_{d-3}(\mathsf{i}_{N\setminus D^d},\mathsf{u})$.*
*(ii) For the group $\ker\pi_{d-3}(\mathsf{i}_N,\mathsf{s})$ to be isomorphic to $\ker\pi_{d-3}(\mathsf{i}_{N\setminus D^d},\mathsf{u})=\ker\pi_{d-3}p_{\mathsf{u}}$ it suffices that $\mathsf{s}$ is nullhomotopic or that $\{c\in\pi_{d-2}N\mid c=\mathsf{s}\,c\}=\{0\}$.*

**Corollary 2.5.** *Let $X$ and $N$ be $d$-manifolds with $d\geq 4$ and $\partial X\neq\emptyset$.*
*(i) If $\pi_{d-1}X=0$ then there is a short exact sequence*

$$\mathbb{Z}[\pi_1X\setminus 1]\hookrightarrow\pi_{d-3}(\mathrm{Emb}_\partial(\mathbb{D}^1,X),\mathsf{u})\twoheadrightarrow\pi_{d-2}X.$$

*(ii) If $\pi_{d-2}N=\pi_{d-1}N=0$ then there is a short exact sequence*

$$\mathbb{Z}[\pi_1N]\big/\langle 1\rangle\oplus\langle\overline{g}-(-1)^{d-1}g\overline{\mathsf{s}}\mid g\in\pi_1N\rangle\hookrightarrow\pi_{d-3}(\mathrm{Emb}(\mathbb{S}^1,N),\mathsf{s})\twoheadrightarrow\pi_{d-3}N.$$

In particular, if $X$ is aspherical, i.e. $\pi_{*>1}X=0$, then $\pi_{d-3}\,\mathrm{Emb}_\partial(\mathbb{D}^1,X)\cong\mathbb{Z}[\pi_1X\setminus 1]$. For example,

$$\pi_{d-3}(\mathrm{Emb}_\partial(\mathbb{D}^1,\mathbb{S}^1\times\mathbb{D}^{d-1}),\mathsf{u})\cong\mathbb{Z}[\mathbb{Z}\setminus 0],\quad\text{for any }\mathsf{u}.$$

Similarly, in the case of circles we can take an aspherical (closed or not) manifold $N$, for example

$$\pi_{d-3}(\mathrm{Emb}(\mathbb{S}^1,\mathbb{S}^1\times\mathbb{D}^{d-1}),t^{k_0})\cong\mathbb{Z}[t,t^{-1}]\big/\langle t^0,t^{-k}-(-1)^{d-1}t^{k-k_0}\mid k\in\mathbb{Z}\rangle.$$

As another example, a compact irreducible 3-manifold $Y$ with infinite $\pi_1 Y$ is aspherical (by the Hurewicz theorem applied to its (noncompact) universal cover), so $N := \mathbb{S}^1 \times Y$ is as well and

$$\pi_1(\mathrm{Emb}(\mathbb{S}^1, \mathbb{S}^1 \times Y), \mathsf{s}) \cong \mathbb{Z}[\mathbb{Z} \times \pi_1 Y] \big/ \langle 1 \rangle \oplus \langle \overline{g} - (-1)^{d-1} g\bar{\mathsf{s}} \mid g \in \mathbb{Z} \times \pi_1 Y \rangle.$$

The following is a next simplest case, also computed in [BG19] by different means.

**Corollary 2.6.** *Let $\mathsf{s} = t^{W_0} \in \mathbb{Z}\{t\} \cong \pi_1(\mathbb{S}^1 \times \mathbb{S}^{d-1})$ with $W_0 \geq 0$. Then we have for $d \geq 5$:*

$$\pi_{d-3}(\mathrm{Emb}(\mathbb{S}^1, \mathbb{S}^1 \times \mathbb{S}^{d-1}), \mathsf{s}) \cong \mathbb{Z}[t, t^{-1}] \big/ \langle t^0, t^{-1} + \cdots + t^{-(W_0-1)},\ t^{-k} - (-1)^{d-1} t^{k-W_0} \mid k \in \mathbb{Z} \rangle \tag{9}$$

*while for $d = 4$ the group $\pi_1(\mathrm{Emb}(\mathbb{S}^1, \mathbb{S}^1 \times \mathbb{S}^3), \mathsf{s})$ is the sum of the displayed group with $\mathbb{Z}$.*

*Proof.* Denote $N := \mathbb{S}^1 \times \mathbb{S}^{d-1}$ and $X := N \setminus D^d$. As $\pi_n(N) = 0$ for $2 \leq n \leq d-2$, Theorem C implies $\pi_n(\mathrm{Imm}(\mathbb{S}^1, N), \mathsf{s}) = 0$ for $0 \leq n \leq d-3$, except that $\pi_1(\mathrm{Imm}(\mathbb{S}^1, N), \mathsf{s}) \cong \pi_1 N$.

Moreover, $\pi_{d-1} X \cong \pi_{d-1}(\mathbb{S}^1 \vee \mathbb{S}^{d-1}) \cong \mathbb{Z}[t, t^{-1}]$, generated by $i_2 \colon \{pt\} \times \mathbb{S}^{d-1} \subseteq X \subseteq N$. However, a more convenient generator is $\Phi \colon \partial D^d \hookrightarrow X$, and the long exact sequence of the pair $(N, X)$ yields

$$\pi_{d-1} X \cong \pi_{d-1} N \oplus \partial_* \pi_d(N, X) \cong \mathbb{Z}\langle \boldsymbol{i_2} \rangle \ \oplus\ \mathbb{Z}[t, t^{-1}]\langle \boldsymbol{\Phi} \rangle.$$

Thus, if $d - 3 > 1$ then Theorem C implies (using $\pi_{d-3}(\mathrm{Imm}(\mathbb{S}^1, N), \mathsf{s}) = 0$):

$$\pi_{d-3} \mathrm{Emb}(\mathbb{S}^1, N) \cong \langle t^k, t^{-k} \mid k > 0 \rangle \big/ \langle \mathsf{dax_u}(\boldsymbol{i_2}), \mathsf{dax_u}(t^k \boldsymbol{\Phi}) \mid k \in \mathbb{Z} \rangle$$

while for $d - 3 = 1$ we have this plus the group $\mathbb{Z} \cong \pi_1(\mathrm{Imm}(\mathbb{S}^1, N), \mathsf{s})$. It is straightforward to compute $\lambda(\boldsymbol{i_2}, t^{W_0}) = 1 + t^{-1} + \cdots + t^{-(W_0-1)}$, so $\mathsf{dax_u}(\boldsymbol{i_2}) = \lambda\!\!\!\!-(\boldsymbol{i_2}, t^{W_0}) = t^{-1} + \cdots + t^{-(W_0-1)}$ by Corollary B, while Corollary 1.4 gives $\mathsf{dax_u}(t^k \boldsymbol{\Phi}) = (-1)^{d-1} t^{-k} - t^k \mathsf{s}^{-1} = (-1)^{d-1} t^{-k} - t^{k-W_0}$. $\square$

**Remark 2.7.** Note that the resulting abelian group in (9) is simply a $\mathbb{Z}^\infty$, unless $d$ and $W_0$ are both odd, when it is $\mathbb{Z}/2 \oplus \mathbb{Z}^\infty$. Namely, the generators $t^{-k}$ are redundant for $k > W_0/2$ by the second relation, which also gives $(-1)^{d-1} t^{-W_0/2} = t^{-W_0/2}$ if $W_0$ even. This is empty unless $d$ is also even, in which case it says that $t^{-W_0/2}$ is 2-torsion. However, if $d$ and $W_0$ are even $t^{-W_0/2}$ vanishes by the first relation (for $W_0$ odd it is empty). On the other hand, if $d$ is odd, the first relation says that for $W_0$ even $t^{-W_0/2}$ is redundant, whereas for $W_0$ odd $t^{-1} + \cdots + t^{-(W_0-1)/2}$ is 2-torsion. We just mention that one can get a similar result for $W_0 < 0$ (using $\lambda(\boldsymbol{i_2}, t^{W_0}) = -(t + \cdots + t^{-W_0})$).

Finally, let us observe that $\pi_{d-2}(\mathbb{S}^1 \times \mathbb{S}^{d-1}) = 0$, so Corollary (ii) applies, giving an isomorphism: $\pi_{d-3}(\mathrm{Emb}(\mathbb{S}^1, \mathbb{S}^1 \times \mathbb{S}^{d-1} \setminus \mathbb{D}^d), \mathsf{u}) \cong \pi_{d-3}(\mathrm{Emb}(\mathbb{S}^1, \mathbb{S}^1 \times \mathbb{S}^{d-1}), \mathsf{s})$. $\triangle$

**Remark 2.8.** Let us compare Corollary 2.6 to the computation of Budney and Gabai in [BG19, Sec.2]. Their group $\pi_{d-3}(\mathrm{Emb}(\mathbb{S}^1, \mathbb{S}^1 \times \mathbb{S}^{d-1}), t^{W_0}) \cong \Lambda_n^{W_0}$ is described as the quotient of the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$ by a different set of relations, namely $\langle t^k - (-1)^d t^{W_0-1-k}, t^0, t^1 \rangle$. However, $\varphi(t^k) = t^{k+W_0} - t^{k+W_0-1}$ is a well-defined map from our group to $\Lambda_n^{W_0}$, and an explicit inverse $\psi$ is given by: $\psi(t^k) := t^{k-W_0} + \cdots + t^1 + t^0$ for $k \geq W_0$, $\psi(t^k) := -t^{k-(W_0-1)} - \cdots - t^{-1} - t^0$ for $0 \leq k \leq W_0 - 1$, and $\psi(t^k) := (-1)^d \psi(t^{W_0-1-k})$ for $k \leq 0$.

Moreover, their class $\theta_{t^{W_0},k} \in \pi_{d-3}(\mathrm{Emb}(\mathbb{S}^1, \mathbb{S}^1 \times \mathbb{S}^{d-1}), t^{W_0})$ is related to our realization map: the class $\partial\mathfrak{r}(g)^{new}$ on the right of Figure 3 is up to a sign (not fixed in [BG19]) the class $\theta_{t^{W_0},k}$ for $s = t^{W_0}$ and $t^k = gs$ (the guiding arc for the finger), so $g = t^{k-W_0}$. Indeed, in their family the double point occurs *before* the root of the finger. Whereas the cocircular method gives $W_2(\theta_{t^{W_0},k}) = t^k - t^{k-1}$, we have $\mathsf{Dax}(\partial\mathfrak{r}(t^{k-W_0})^{new}) = (-1)^{d-3} t^{-k} = t^{k-W_0}$ as explained in that figure. Since $\varphi(t^{k-W_0}) = t^k - t^{k-1}$, the two computations agree. $\triangle$

**Remark 2.9.** As a further example, consider the "self-referential" family from [Gab21, Fig.13-15]. Firstly, Gabai's $\tau_g$ is precisely our $\partial\mathfrak{r}_u(g)$, and the foliation $\mathsf{selfref}_u(g)$ of his self-referential disk $D_g$ is the family depicted in Figure 4. This is given similarly as $\partial\mathfrak{r}(g)$, except that after following a guiding arc representing $g$, we come back and swing around the sphere $S$ linking the root of the finger, instead of just the basepoint arc. Therefore, the homotopy which pulls back using the ball that $S$ bounds produces two double points. The associated loops are computed in Figure 4: $g$ at $x_2$ as before, and $-(-1)^{d-1}g^{-1}$ at $x_1$ since not only the guiding arc is reversed (giving $-1$) but also now the second sheet is the one moving (giving $(-1)^{d-1}$), cf. the proof of Corollary 1.4. Thus,

$$\mathsf{Dax}(\mathsf{selfref}_u(g)) = g - (-1)^{d-1}\overline{g}.$$

Indeed, Gabai has $d=4$ and $\mathsf{Dax}(\mathsf{selfref}_u(g)) = g + \overline{g}$.

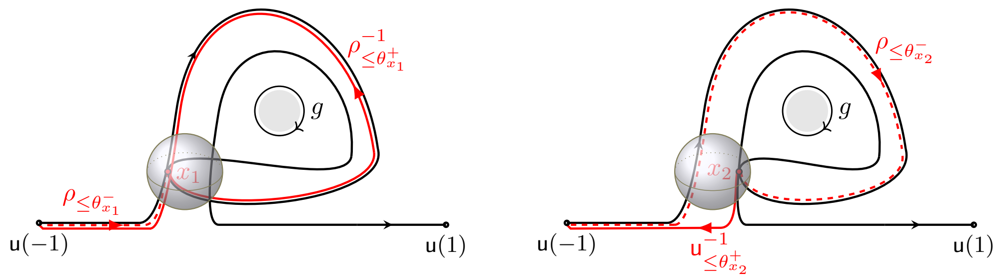


**Figure 4.** *Left.* The double point loops $g_{x_1} = [\rho_{\le\theta^-_{x_1}} \cdot \rho^{-1}_{\le\theta^+_{x_1}}]$ and $g_{x_2} = [\rho_{\le\theta^-_{x_2}} \cdot \rho^{-1}_{\le\theta^+_{x_2}}] = [\rho_{\le\theta^-_{x_2}} \cdot \mathsf{u}^{-1}_{\le\theta^+_{x_2}}]$.

In particular, for $X = \mathbb{S}^1 \times \mathbb{S}^{d-1} \setminus D^d$ the previous remark gives $\varphi\mathsf{Dax}(\mathsf{selfref}_{t^{W_0}}(t^k)) = \varphi(t^k - (-1)^{d-1}t^{-k}) = t^{k+W_0} - t^{k+W_0-1} + (-1)^d(t^{-k+W_0} - (-1)^d t^{-k+W_0-1}) = t^{k+W_0} - t^{k+W_0-1} + t^{k-1} - t^k$. Compare to a similar family $\alpha_{t^{W_0},k} \in \pi_{d-3}(\mathrm{Emb}(\mathbb{S}^1, \mathbb{S}^1 \times \mathbb{S}^{d-1}), t^{W_0})$ in [BG19, Fig.4, Prop.2.6]. △

For the next example, observe that $\lambda(a,g)$ vanishes if $X = \mathbb{D}^k \times Y$ for $1 \le k \le d$.

**Corollary 2.10.** *Given a $(d-k)$-dimensional manifold $Y$ for $k \ge 1$, let $\mathcal{S}(Y)$ denote a collection of $\mathbb{Z}[\pi_1 Y]$-generators for $\pi_{d-1}Y$. Then*

$$\mathbb{Z}[\pi_1 Y \setminus 1]\big/\langle g\mathsf{dax}(a)\overline{g} \mid a \in \mathcal{S}(Y), g \in \pi_1 Y\rangle \hookrightarrow \pi_{d-3}\big(\mathrm{Emb}_\partial(\mathbb{D}^1, \mathbb{D}^k \times Y), 1\big) \twoheadrightarrow \pi_{d-2}Y.$$

**Remark 2.11.** For a 4-manifold $X = \mathbb{D}^k \times Y$ the cases $k = 2,3,4$ are completely computed by the discussion so far. For $k=1$ we can say more thanks to [KT21, Thm.B and Cor.1.6]. If a 3-manifold $Y$ has $H_3\widetilde{Y} = 0$, then every $a \in \pi_3 X$ is the composite of the Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$ and a map $S\colon \mathbb{S}^2 \to X$, and we have $\mathsf{dax}(a) = \lambda(S,S)$. This vanishes since any $S$ can be pushed off itself in the $\mathbb{D}^1$-direction. Therefore, for $X = \mathbb{D}^1 \times Y$ we have $\mathsf{dax} = 0$ and the exact sequence as in Corollary 2.5(i), unless $Y$ is closed with finite fundamental group. In that case, we have $\pi_3 X \cong H_3\widetilde{Y} \cong \mathbb{Z}$ generated by the map $a_Y\colon \mathbb{S}^3 = \widetilde{Y} \to \{pt\} \times Y \subseteq X$, and computing $\mathsf{dax}(a_Y)$ remains an interesting problem! △

2.3. **Arcs in dimension three.** For $d=3$ we consider the induced map on sets of components $\pi_0\mathsf{i}_X\colon \pi_0\,\mathrm{Emb}_\partial(\mathbb{D}^1, X) \to \pi_0\,\mathrm{Imm}_\partial(\mathbb{D}^1, X) \cong \pi_1 X$, so $\ker(\pi_{d-3}(\mathsf{i}_X, \mathsf{u}))$ translates into the set

$$\mathbb{K}(X, \mathbf{u}) := (\pi_0\mathsf{i}_X)^{-1}(\mathbf{u}) \subseteq \pi_0\,\mathrm{Emb}_\partial(\mathbb{D}^1, X),$$

of those isotopy classes of (long) knots $K\colon \mathbb{D}^1 \hookrightarrow X$ in the 3-manifold $X$ homotopic to the fixed knot $\mathsf{u}\colon \mathbb{D}^1 \hookrightarrow X$. Counting double point loops during a generic homotopy from $K$ to $\mathsf{u}$ still gives

$$\mathsf{Dax} = \mathsf{Dax}_{\mathsf{u}}\colon \mathbb{K}(X, \mathbf{u}) \twoheadrightarrow \mathbb{Z}[\pi_1 X]\big/\mathsf{dax}_{\mathsf{u}}(\pi_2 X) \tag{10}$$

which is now only a map of sets, see (15). However, it is still surjective (perform crossing changes along given group elements) and the related map $\mathsf{dax}_{\mathsf{u}}\colon \pi_2 X \to \mathbb{Z}[\pi_1 X]$, which applies $\mathsf{Dax}_{\mathsf{u}}$ to self-homotopies, is a homomorphism of abelian groups (see Lemma 3.5). Additionally, in this 3-dimensional setting Corollary B shows particularly useful thanks to the following classical result.

**Theorem 2.12** (Corollary of the Sphere Theorem, see [Hat07, Prop. 3.12])**.** *In a compact orientable 3-manifold $X$ there exists a finite collection $\mathcal{S}(X)$ of disjointly embedded 2-spheres generating $\pi_2 X$ as a $\mathbb{Z}[\pi_1 X]$-module. Each sphere in the collection is either a connected sum sphere in the prime decomposition of $X$, or coming from a $\mathbb{S}^1 \times \mathbb{S}^2$ factor, or it is parallel to a sphere in $\partial X$.*

**Corollary 2.13.** *For a 3-manifold $X$ with $\mathcal{S}(X)$ as in the theorem, there is a surjection of sets*

$$\mathsf{Dax}_{\mathsf{u}}\colon \mathbb{K}(X,\mathbf{u}) \twoheadrightarrow \mathbb{Z}[\pi_1X\setminus 1]\big/\langle \lambda(ga,\mathbf{u}) - \lambda(ga,g) + \overline{\lambda(ga,g)} \mid g\in\pi_1X,\ a\in\mathcal{S}(X)\rangle \tag{11}$$

*In particular, $\mathsf{dax}_{\mathsf{u}}(\pi_2X)$ depends only on the homotopy class $\mathbf{u}$ (but $\mathsf{Dax}_{\mathsf{u}}$ could depend on $\mathsf{u}$).*

*Proof.* This follows immediately from Lemma 3.5, Corollary B(v) and Theorem 2.12. □

**Theorem 2.14.** *The map $\mathsf{Dax}_{\mathsf{u}}$ is a* universal *Vassiliev type $\leq 1$ invariant of knotted arcs in $X$.*

*Proof.* An invariant $v\colon \mathbb{K}(X,\mathbf{u})\to A$, with values in an abelian group $A$, is of type $\leq 1$ if and only if for an arc $J$ with a single double point $v(res(J)) \coloneqq v(J^+) - v(J^-)$ depends only on the isotopy class of $J$, where $J_\pm$ are two knots obtained by resolving the double point. In particular, this is true for $v=\mathsf{Dax}_{\mathsf{u}}$ since $\mathsf{Dax}_{\mathsf{u}}(res(J))$ is the homotopy class of the double point loop.

To show that $\mathsf{Dax}_{\mathsf{u}}$ is universal, we will find for any $v$ as above a homomorphism $w_v\colon \mathrm{im}(\mathsf{Dax}_{\mathsf{u}})\to A$ such that $v(K)=v(\mathsf{u})+w_v\circ\mathsf{Dax}_{\mathsf{u}}(K)$ for all $K\in\mathbb{K}(X,\mathbf{u})$. We simply define $w_v$ by linear extension, $w_v(g)\coloneqq v(res(J_g))$ for an immersed arc $J_g$ with a single double point and loop $g=\mathsf{Dax}_{\mathsf{u}}(J_g^+)$. A homotopy from $K$ to $\mathsf{u}$ is a sequence of crossing changes, so $K-\mathsf{u}=\sum_i res(J_i)$, where $J_i$ is the immersed arc occurring during the $i$-th crossing change. Thus, $v(K)-v(\mathsf{u})=\sum_i v(res(J_i)) = w_v\circ\mathsf{Dax}_{\mathsf{u}}(K)$. To see that $w_v$ vanishes on relations $R_1-R_2=0$ defining the target of $\mathsf{Dax}_{\mathsf{u}}$, i.e. $R_1-R_2\in\mathsf{dax}_{\mathsf{u}}(\pi_2X)$, recall that each such relation arises from having two different homotopies $h_i$ from $K$ to $\mathsf{u}$, with $\mathsf{Dax}_{\mathsf{u}}(h_i)=R_i$, so $w_v(R_1)=w_v\circ\mathsf{Dax}_{\mathsf{u}}(K)=w_v(R_2)$. □

**Remark 2.15.** We saw in Remark 2.2 that $\mathsf{Dax}$ agrees with the second Goodwillie–Weiss knot invariant $\pi_0\mathsf{ev}_2$ (and $\mathsf{ev}_1=\mathfrak{i}_X$). In fact, $\pi_0\mathsf{ev}_n$ is invariant under the $(n-1)$-equivalence relation of knots due to Gusarov and Habiro [Kos20]. For $M=\mathbb{D}^3$ this is equivalent to being of Vassiliev type $\leq n-1$, but in general 3-manifolds the analogous connection is an open problem for $n\geq 3$. △

Finally, we study the isotopy invariant $\mathsf{Dax}_{\mathsf{u}}$ modulo the relation of concordance of knotted arcs.

**Definition 2.16.** Two knots $K_0,K_1\in\mathrm{Emb}_\partial(\mathbb{D}^1,X)$ are *concordant* if there exists a neat embedding $Q\colon\mathbb{D}^1\times[0,1]\hookrightarrow X\times[0,1]$ with $Q|_{\mathbb{D}^1\times\{i\}}=K_i$ and $Q(\partial\mathbb{D}^1\times\{t\})=\mathsf{u}(\partial\mathbb{D}^1)\times\{t\}\subseteq\partial X\times\{t\}$. Let $\mathcal{C}(X,\mathbf{u})$ be the set of concordance classes of knotted arcs in $X$ which are *homotopic* to $\mathsf{u}$. △

The next theorem says that for $\mathsf{u}\in\mathrm{Emb}_\partial(\mathbb{D}^1,X)$ homotopic into $\partial X$ rel. endpoints, $\mathsf{Dax}_{\mathsf{u}}$ reduces to Schneiderman's [Sch03] concordance invariant

$$\mu_2\colon\mathcal{C}(X,\mathbf{u})\to\mathbb{Z}[\pi_1X\setminus 1]\big/\langle\overline{g}-g\mid g\in\pi_1X\rangle.$$

For a concordance class $[K]\in\mathcal{C}(X,\mathbf{u})$ this is defined as Wall's self-intersection invariant $\mu_2(H)$ of any *immersed* concordance $H\colon\mathbb{D}^1\times[0,1]\looparrowright X\times[0,1]$ from $K$ to $\mathsf{u}$, e.g. take for $H$ the trace of any homotopy $h$ from $K$ to $\mathsf{u}$. The count of double point loops in $H$ is well defined only modulo $\overline{g}-g$, as there is no preference between the two intersecting sheets of $H$. For a proof that $\mu_2$ is well defined see [Sch03, Thm.1], where the analogous invariant was defined for *nullmotopic circles*.

**Theorem 2.17.** *For a compact orientable 3-manifold $X$,* if $\mathsf{u}\in\mathrm{Emb}_\partial(\mathbb{D}^1,X)$ is homotopic into $\partial X$ rel. endpoints, *then there is a commutative diagram of sets*

$$\begin{tikzcd}
\mathbb{K}(X,\mathbf{u}) \arrow[rr, two heads] \arrow[d, two heads, "\mathsf{Dax}_{\mathsf{u}}"] & & \mathcal{C}(X,\mathbf{u}) \arrow[d, two heads, "\mu_2"] \\
\mathbb{Z}[\pi_1X\setminus 1]\big/\langle\overline{\lambda(ga,g)}-\lambda(ga,g)\mid g\in\pi_1X, a\in\mathcal{S}\rangle \arrow[rr, two heads, "q"] & & \mathbb{Z}[\pi_1X\setminus 1]\big/\langle\overline{g}-g\mid g\in\pi_1X\rangle
\end{tikzcd}$$

*Proof of Theorem 2.17.* The proof is the same as in [KT21, Thm.5.11]: for $K \in \mathrm{Emb}_\partial(\mathbb{D}^1, X)$ and a homotopy $h$ from $K$ to $\mathsf{u}$, we defined the class $\mathsf{Dax}_\mathsf{u}(K) = \mathsf{Dax}_\mathsf{u}(h) \in \mathbb{Z}[\pi_1 X]$ as the sum of signed double point loops occurring in $h$. This is equal to $\mu_2(H)$ for one choice of sheets: this is the sum of signed double point loops occurring in $h$ when seen as an immersed annulus $H$ in $X \times [0,1]$. □

In other words, $\lambda(ga, g)$ becomes zero if calculated in $X \times [0,1]$ since we can push $ga$ and $g$ off each other using the $[0,1]$-direction; however, in this setting we have to mod out $g - \overline{g}$ since in a concordance we have forgotten the foliations by arcs. As an example, in $X = N \setminus \mathbb{D}^3$ the type $\leq 1$ invariant $\mathsf{Dax}_\mathsf{u}$ simply agrees with $\mu_2$, since we have $\Phi\colon \mathbb{S}^2 = \partial\mathbb{D}^3 \hookrightarrow X$ for which $\lambda(g\boldsymbol{\Phi}, g) = g$ (see Corollary 1.4). However, if $X$ is an irreducible 3-manifold then the target of $\mathsf{Dax}_\mathsf{u}$ is $\mathbb{Z}[\pi_1 X \setminus 1]$, so it is an isotopy invariant which is in general strictly stronger than the concordance invariant $\mu_2$.

2.4. **Circles in dimension three.** Consider now the set $\mathbb{K}(N, \mathbf{s}) \coloneqq (\pi_0 \mathfrak{i}_N)^{-1}(\mathbf{s})$ consisting of isotopy classes of knots $\mathbb{S}^1 \hookrightarrow N$ which are in the free homotopy class $\mathbf{s}$ of a fixed knot $\mathsf{s}$. Denote by $\zeta(\mathbf{s}) \coloneqq \{b \in \pi_1 N \mid b = \mathbf{s}\, b\}$ the centralizer of $\mathbf{s} \in \pi_1 N = \pi_1(N, \mathsf{s}(e))$.

**Theorem D.** *There is a map $\mathsf{Dax}_\mathsf{s}$ from the set $\mathbb{K}(N, \mathbf{s})$ to the quotient of the group*

$$\mathbb{Z}[\pi_1 N \setminus 1]\Big/\Big\langle \overline{g} - g\overline{\mathbf{s}},\ \lambda(ga, \mathsf{s}) - \lambda(ga, g) + \overline{\lambda(ga, g)} \mid g \in \pi_1 N, a \in \mathcal{S}(N \setminus \mathbb{D}^3)\ \Big\rangle$$

*by the following set-theoretic action of $\zeta(\mathbf{s})$: let $b \in \zeta(\mathbf{s})$ act by $r \mapsto b \cdot r \cdot b^{-1} + \mathsf{Dax}_\mathsf{s}(\delta^{whisk}_\mathsf{s}(b))$.*

**Remark 2.18.** The proof of Theorem 2.14 together with Remark 4.10 implies that the invariant $\mathsf{Dax}_\mathsf{s}$ of from Theorem D is a *universal Vassiliev invariant of type* $\leq 1$ *of knots in* $N$. For example, the type $\leq 1$ invariants from [KL98] factor through $\mathsf{Dax}$, but one can now construct many more! △

Recall from Theorem C that $\delta^{whisk}_\mathsf{s}(b)$ is the knot obtained by dragging a piece of $\mathsf{s}$ around a loop $\beta$ representing $b$. Thus, $\delta^{whisk}_\mathsf{s}(b) \simeq \beta\mathsf{s}\beta^{-1} \simeq \mathsf{s}$ and $\mathsf{Dax}(\delta^{whisk}_\mathsf{s}(b))$ counts double points in a based homotopy witnessing this, i.e. in a disk bounded by (an embedded version of) the commutator $[\beta, \mathsf{s}]$.

In particular, if $\mathsf{s}$ is freely nullhomotopic then all $\delta^{whisk}_\mathsf{s}(b)$ and $\lambda(ga, \mathsf{s})$ vanish, so we are only modding out by $\overline{g} - g$ and conjugation. The same argument as in Theorem 2.17 then shows that $\mathsf{Dax}_\mathsf{s}$ (itself!) agrees with Schneiderman's invariant $\mu_2$ for $\mathcal{C}(N, \mathbf{s})$ in this case. One could also prove that $\mathsf{Dax}_\mathsf{s}$ reduces to $\mu_2$ modulo concordance for any $\mathsf{s}$, but we omit this as the target of $\mu_2$ is in general somewhat complicated, see [Sch03]. We just note that $\mathsf{Dax}_\mathsf{s}(\delta^{whisk}_\mathsf{s}(\mathbf{s}^n)) = 0$ for all $n$.

**Remark 2.19.** Our set of relations should be compared to Schneiderman's generating set of relations $\Phi(\mathsf{s})$ from [Sch03, Sec.4.7], and our action of the centralizer $\zeta(\mathbf{s})$ with his in [Sch03, Sec.4.7.2]. By [Sch03, Prop.5.4.1] the set $\Phi(\mathsf{s})$ has a part coming from spheres (corresponding to our relations coming from $\mathcal{S}(X)$), and a "toroidal" part (note that if $b \in \zeta(\mathbf{s})$ then there is an immersed torus $\mathbb{S}^1 \times \mathbb{S}^1 \looparrowright N$ with $\mathsf{s} = \mathbb{S}^1 \times pt$). As observed there, it is only this latter part that is sensitive to the isotopy class of $\mathsf{s}$, while the rest depends only on its homotopy class $\mathbf{s}$. This is also the case in our relations, where only $\mathsf{Dax}_\mathsf{s}(\delta^{whisk}_\mathsf{s}(b))$ can depend on $\mathsf{s}$. △

2.5. **Open problems.** There are some natural questions that to our knowledge remain open.

**Question 2.20.** *Does the image of $\mathsf{dax}_\mathsf{u}$ depend only on the homotopy type of a manifold $X$?*

**Question 2.21.** *What is the isomorphism class of the central extension (3)? Does it depend only on the homotopy type of a 4-manifold $X$?*

We saw in Corollary 2.13 that the answer to the first question is affirmative for $d = 3$. However, we suspect that there is an example of a pair of homotopy equivalent nondiffeomorphic 4-manifolds $X, X'$ for which the extensions (3) (or (6) for the circle case) are distinct. Namely, the Dax invariant is intimately related to the configuration space of two points in $X$ (i.e. the second stage of the embedding calculus tower, see Remark 2.2), and a result of Longoni and Salvatore [LS05] shows that those spaces can distinguish nondiffeomorphic (3-)manifolds. Let us point out, however, that by [AS20] these extensions agree for homeomorphic smooth manifolds.

## 3. Knotted arcs

Throughout this section $X$ is a connected oriented smooth manifold of dimension $d \geq 3$ with nonempty boundary. We pick $x_-, x_+ \in \partial X$ and let $x_-$ be the basepoint. For $\mathbb{D}^1 = [-1,1]$ we consider spaces $\mathrm{Emb}_\partial(\mathbb{D}^1, X)$ and $\mathrm{Imm}_\partial(\mathbb{D}^1, X)$ of embedded and immersed arcs as in (1), with boundary condition $\{x_-, x_+\}$.

Note that $\pi_0 \mathrm{Emb}_\partial(\mathbb{D}^1, \mathbb{D}^3) \cong \pi_0 \mathrm{Emb}(\mathbb{S}^1, \mathbb{D}^3) \cong \pi_0 \mathrm{Emb}(\mathbb{S}^1, \mathbb{S}^3)$ is the set of isotopy classes of classical knots. On the other hand, if $d \geq 4$ we simply have bijections $\pi_0 \mathrm{Emb}_\partial(\mathbb{D}^1, X) \cong \pi_1 X \cong \pi_0 \mathrm{Imm}_\partial(\mathbb{D}^1, X)$, so it is natural to try to determine the lowest homotopy group distinguishing embeddings from immersions. In other words, we are asking about the connectivity of the inclusion

$$\mathfrak{i}_X \colon \mathrm{Emb}_\partial(\mathbb{D}^1, X) \hookrightarrow \mathrm{Imm}_\partial(\mathbb{D}^1, X). \tag{12}$$

By general position the map $\mathfrak{i}_X$ is $(d-4)$-connected, i.e. an isomorphism in the so-called *stable* range $[0, d-3)$. A range of nontrivial relative homotopy groups that comes after the stable range has been studied by Dax [Dax72], following the work of Haefliger and Hatcher–Quinn, and is called the *metastable* range $[d-3, 2d-6)$. This was then further developed into a theory of embedding calculus by Goodwillie, Klein and Weiss [GKW01], where the stable and metastable range only correspond to respectively the first and second stage of a whole tower of spaces. Remarkably, when $d \geq 4$ this tower completely describes the homotopy type of $\mathrm{Emb}_\partial(\mathbb{D}^1, X)$.

In joint work [KT21] we used these ideas to describe explicitly the first potential difference in homotopy groups, obtaining Theorem 1.1; we next provide the outline of its proof in order to fix notation. In Proposition 3.5 we will also extend to the case $d = 3$, not discussed in [KT21], which was stated in (10) above.

Using the long exact sequence of the pair (12), for $\mathsf{u} \in \mathrm{Emb}_\partial(\mathbb{D}^1, X)$ the kernel of the surjection $\pi_{d-3}(\mathfrak{i}_X, \mathsf{u}) \colon \pi_{d-3}(\mathrm{Emb}_\partial(\mathbb{D}^1, X), \mathsf{u}) \twoheadrightarrow \pi_{d-3}(\mathrm{Imm}_\partial(\mathbb{D}^1, X), \mathsf{u})$ is given as the cokernel of

$$\delta_{\mathfrak{i}_X} \colon \pi_{d-2}(\mathrm{Imm}_\partial(\mathbb{D}^1, X), \mathsf{u}) \to \pi_{d-2}^{Imm,Emb}(X, \mathsf{u}) \coloneqq \pi_{d-2}(\mathrm{Imm}_\partial(\mathbb{D}^1, X), \mathrm{Emb}_\partial(\mathbb{D}^1, X), \mathsf{u}). \tag{13}$$

Note that for $d = 3$ the kernel of $\pi_0(\mathfrak{i}_X, \mathsf{u})$ is interpreted as the preimage set $\mathbb{K}(X, \mathbf{u}) \coloneqq (\pi_0 \mathfrak{i}_X)^{-1}(\mathbf{u})$, while the mentioned cokernel is the set of orbits of the action of the group $\pi_1(\mathrm{Imm}_\partial(\mathbb{D}^1, X), \mathsf{u})$ on the set $\pi_1^{Imm,Emb}(X)$ via $\delta_{\mathfrak{i}_X}$ (postconcatenate a loop to a path).

Therefore, we have two tasks: first to compute the relative homotopy group $\pi_{d-2}^{Imm,Emb}(X, \mathsf{u})$ (which will be done using the Dax invariant) and then the connecting map $\delta_{\mathfrak{i}_X}$ and its image.

### 3.1. The Dax invariant.

Let $\mathbb{I} = [0,1]$. A class in the relative homotopy group $\pi_{d-2}^{Imm,Emb}(X, \mathsf{u})$ is represented by a map $F \colon \mathbb{I}^{d-2} \to \mathrm{Imm}_\partial(\mathbb{D}^1, X)$ which takes $\mathbb{I}^{d-3} \times \{0\}$ into the subspace $\mathrm{Emb}_\partial(\mathbb{D}^1, X)$, and has constant value $\mathsf{u}$ on $\partial \mathbb{I}^{d-3} \times \mathbb{I} \cup \mathbb{I}^{d-3} \times \{1\}$, the rest of the boundary of the cube $\mathbb{I}^{d-2}$. For the proof of the following lemma, see for example [Dax72, p. 329] (and [Dax72, p. 331] for the relative case).

**Lemma 3.1.** *After an arbitrary small deformation of $F$ preserving the boundary conditions, we can assume that the map $\widetilde{F} \colon \mathbb{I}^{d-2} \times \mathbb{D}^1 \to \mathbb{I}^{d-2} \times X$, given by $(\vec{t}, \theta) \mapsto (\vec{t}, F(\vec{t})(\theta))$ is an immersion with only isolated transverse double points.*

The double points of $\widetilde{F}$ must be of the form $(\vec{t}_i, x_i) \in \mathbb{I}^{d-2} \times X$, $i = 1, \dots, N$, for some $\vec{t}_i \in \mathbb{I}^{d-2}$ and $x_i = F(\vec{t}_i)(\theta_i^-) = F(\vec{t}_i)(\theta_i^+)$, for some $\theta_i^- < \theta_i^+ \in \mathbb{D}^1$. Define the double point loop at $x_i$ by

$$g_{x_i} \coloneqq \left[ F(\vec{t}_i)_{\leq \theta_i^-} \cdot F(\vec{t}_i)^{-1}_{\leq \theta_i^+} \right] \quad \in \pi_1 X \coloneqq \pi_1(X, x_-), \tag{14}$$

where for an arc $\alpha \colon \mathbb{D}^1 \looparrowright X$ and $\theta \in \mathbb{D}^1$ we use the notation $\alpha_{\leq \theta} \coloneqq \alpha|_{[-1,\theta]}$. See Figure 5. Moreover, let $\varepsilon_{x_i} \in \{-1, 1\}$ be the local orientation of $\widetilde{F}$ at $x_i$, obtained by comparing orientations of tangent bundles to both sheets, namely the image of the derivatives $d\widetilde{F}|_{(\vec{t}_i, \theta_i^-)} \oplus d\widetilde{F}|_{(\vec{t}_i, \theta_i^+)}$ with the tangent space of $\mathbb{I}^{d-2} \times X$ at $(\vec{t}_i, x_i)$.

Following [Dax72; Gab21] define for $d \geq 3$ the *Dax invariant* by

$$\mathsf{Dax}\colon \pi_{d-2}^{Imm,Emb}(X,\mathsf{u}) \to \mathbb{Z}[\pi_1 X], \quad \mathsf{Dax}(\boldsymbol{F}) := \sum_{i=1}^{N} \varepsilon_{x_i} g_{x_i} \in \mathbb{Z}[\pi_1 X]. \tag{15}$$

This is well defined and does not depend on the choice of a perturbation of the map $F$, but only on its homotopy class $\boldsymbol{F}$. For $d \geq 4$ the left hand side is a group and $\mathsf{Dax}$ is clearly a group homomorphism since the corresponding maps $F$ get stacked in the $\mathbb{I}^{d-2}$-direction.

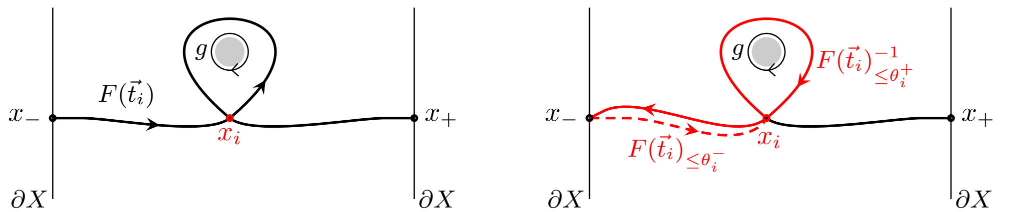


**Figure 5.** *Left.* An immersed arc $F(\vec{t}_i)$ with a double point $x_i = F(\vec{t}_i)(\theta_i^-) = F(\vec{t}_i)(\theta_i^+)$. *Right.* The red double point loop $g_{x_i}$ is homotopic to $g$ (first follow the dashed arc, then the solid red arc).

Let us describe an explicit (partial) inverse

$$\mathfrak{r}\colon \mathbb{Z}[\pi_1 X] \to \pi_{d-2}^{Imm,Emb}(X,\mathsf{u}). \tag{16}$$

The class $\mathfrak{r}(g)$ is represented by the map $\mathfrak{r}(g)\colon \mathbb{I}^{d-2} = \mathbb{I}^{d-3}\times\mathbb{I} \to \mathrm{Imm}_\partial(\mathbb{D}^1, X)$ given as the $(d-3)$-parameter family of embedded arcs $\mathfrak{r}(g)(\vec{t},0) \in \mathrm{Emb}_\partial(\mathbb{D}^1,X)$ for $\vec{t}\in\mathbb{I}^{d-3}$, obtained as the "family finger move" of $\mathsf{u}$ along $g$, together with paths through immersed arcs $\mathfrak{r}(g)(\vec{t},t)$ from this family $\mathfrak{r}(g)(\vec{t},0)$ to the constant family $\mathsf{const}_\mathsf{u}$ equal to $\mathsf{u}$, using the meridian ball as in Figure 1. In more detail, we pick a meridian sphere $\mathbb{S}^{d-2}_x$ for $\mathsf{u}$, and foliate it by arcs $\alpha_{\vec{t}}$ using its description as the suspension of a $(d-3)$-sphere. We fix $\theta_L < \theta_R \in \mathbb{D}^1$ and drag the interval $\mathsf{u}([\theta_L,\theta_R])$ around $g$, then connect sum it into the arc $\alpha_{\vec{t}}$. For the immersed arcs $\mathfrak{r}(g)(\vec{t},t)$ we instead connect sum into an arc foliating the meridian ball that $\mathbb{S}^{d-2}_x$ bounds.

For $d \geq 4$ we extend $\mathfrak{r}$ to the group ring $\mathbb{Z}[\pi_1 X]$ linearly. Namely, subdivide the $s \in \mathbb{I}$ factor into $N$ parts, and define $\mathfrak{r}(\sum_{i=1}^N g_i)$ by applying the above foliated finger move along $g_i$ for the corresponding values of $s$. For $d = 4$ one needs to checks that this is independent of the order of indices $i$, so that $\mathfrak{r}(\sum_{i=1}^N g_i)$ is a well-defined class in $\pi_2^{Imm,Emb}(X,\mathsf{u})$, which is not necessarily abelian; this follows by an Eckmann–Hilton argument, see [KT21].

However, for $d = 3$ we can only define a set-theoretic map

$$\mathfrak{r}\colon \mathbb{Z}[\pi_1 X] \to \pi_1^{Imm,Emb}(X,\mathsf{u}) = \pi_1(\mathrm{Imm}_\partial(\mathbb{D}^1,X),\mathrm{Emb}_\partial(\mathbb{D}^1,X),\mathsf{u}). \tag{17}$$

Namely, for $r \in \mathbb{Z}[\pi_1 X]$ we first write $r = \sum_{i=1}^N g_i$ for *some order* $i = 1,2,\dots,N$. Subdivide $[\theta_L,\theta_R] = [\theta_L^1,\theta_R^1]\cup\dots\cup[\theta_L^N,\theta_R^N]$ and also pick some points $x_i \in \mathbb{D}^1$ so that $\theta_R < x_1 < \dots < x_N$. Then define $\mathfrak{r}(r)\colon (\mathbb{I},\{0\},\{1\}) \to (\mathrm{Imm}_\partial(\mathbb{D}^1,X),\mathrm{Emb}_\partial(\mathbb{D}^1,X),\mathsf{u})$ as the path which, as $s \in \mathbb{I}$ decreases, performs one by one finger move of $[\theta_L^i,\theta_R^i]$ along $g_i$ across the meridian disk to $\mathsf{u}$ at $x_i$.

**Remark 3.2.** For every $d \geq 3$ an Eckmann–Hilton argument also shows that $\mathfrak{r}(\sum_{i=1}^N g_i)$ is homotopic to the map which does the finger moves "in parallel", i.e. maps $\vec{t}\in\mathbb{I}^{d-3}$ to the *concatenation of the respective arcs* for each subinterval $[\theta_L^i,\theta_R^i]$. This uses that the adjoint is defined on $\mathbb{I}^{d-3}\times\mathbb{I}\times\mathbb{D}^1$ and that finger moves can be chosen to be contained in disjoint $d$-balls in $X$. △

In the family $\mathfrak{r}(g)$ there is only one immersed arc, and it has a unique double point, with the loop $+g$, see Figure 1 for the loop and [KT21, Thm.4.5] for the sign. This implies that $\mathsf{Dax}\circ\mathfrak{r} = \mathrm{Id}$, so $\mathsf{Dax}$ is surjective for all $d \geq 3$. For $d \geq 4$ the results of Dax imply that $\mathfrak{r}\circ\mathsf{Dax} = \mathrm{Id}$ as well, so:

**Theorem 3.3** ([Dax72], [Gab21], [KT21])**.** *For $d \geq 4$ the map $\mathsf{Dax}$ is an isomorphism of groups; for $d = 3$ it is a surjection of sets.*

**Remark 3.4.** One can define the relative Dax invariant $\mathsf{Dax}(f_0, f_1)$, if $f_i \in \pi_{d-3}\operatorname{Emb}_\partial(\mathbb{D}^1, N\setminus D^d)$ are such that $p_{\mathsf{u}}(f_0) = p_{\mathsf{u}}(f_1)$. Then $\mathsf{Dax}(f_0, f_1) = \mathsf{Dax}(f_1^{-1}\cdot f_0)$, see [KT21, Sec.5.1.1]. △

3.2. **The connecting map.** Let us now study the composite

$$\pi_{d-2}(\operatorname{Imm}_\partial(\mathbb{D}^1, X), \mathsf{u}) \xrightarrow{\delta_{\mathsf{i}_X}} \pi_{d-2}^{Imm,Emb}(X, \mathsf{u}) \xrightarrow{\mathsf{Dax}} \mathbb{Z}[\pi_1 X].$$

**Lemma 3.5.** *The map* $\mathsf{Dax}\circ\delta_{\mathsf{i}_X}$ *is a homomorphism of groups also for* $d = 3$.

*Proof.* The argument is the same as for showing that $\mathsf{Dax}$ is a homomorphism when $d \geq 4$: given $a_i\colon (\mathbb{I}, \partial\mathbb{I}) \to (\operatorname{Imm}_\partial(\mathbb{D}^1, X), \mathsf{u})$ for $i = 1, 2$, the loop $a_1 \cdot a_2$ is obtained by concatenating in the $\mathbb{I}$-direction, so $\mathsf{Dax}\circ\delta_{\mathsf{i}_X}(a_1\cdot a_2)$ is computed as first counting double point loops in the family $F_{a_1}$, then in the family $F_{a_2}$, so it equals $\mathsf{Dax}\circ\delta_{\mathsf{i}_X}(a_1) + \mathsf{Dax}\circ\delta_{\mathsf{i}_X}(a_2)$. □

For any $d \geq 3$ recall from (2) that the map $p_{\mathsf{u}}\colon \operatorname{Imm}_\partial(\mathbb{D}^1, X) \to \Omega X$, defined by $p_{\mathsf{u}}(\alpha) = \alpha\cdot\mathsf{u}^{-1}$, induces isomorphisms on $\pi_n$ for $n \leq d-3$, by Smale. Moreover, for $n = d-2$ we have

$$\mathbb{Z} \rightarrowtail \pi_{d-2}(\operatorname{Imm}_\partial(\mathbb{D}^1, X), \mathsf{u}) \xrightarrow{\pi_{d-2}p_{\mathsf{u}}} \pi_{d-1}X, \tag{18}$$

a short exact sequence with the first map sending $1 \in \mathbb{Z}$ to the "interior twist", see [KT21]. There we also show that the value of $\mathsf{Dax}\circ\delta_{\mathsf{i}_X}$ on the interior twist is $1 \in \mathbb{Z}[\pi_1 X]$. That was for $d \geq 4$, but the proof goes through for $d = 3$ and is easy: a crossing change on $\mathsf{u}$ along a trivial group element can be isotoped to $\mathsf{u}$.

Therefore, picking any set-theoretic section of $\pi_{d-2}p_{\mathsf{u}}$ (*we will fix a particular choice soon*) and composing it with $\mathsf{Dax}\circ\delta_{\mathsf{i}_X}$ we obtain a well-defined homomorphism

$$\mathsf{dax}_{\mathsf{u}}\colon \pi_{d-1}X \to \mathbb{Z}[\pi_1 X\setminus 1]. \tag{19}$$

In other words, two sections $l_1, l_2$ differ by an element of $\mathbb{Z}$, so $\mathsf{dax}_{\mathsf{u}}(a) = \mathsf{Dax}\circ\delta_{\mathsf{i}_X}\circ l_i(a) \pmod 1$ is well defined. We conclude from this, (13) and Theorem 3.3 that there are surjective maps

$$\mathsf{Dax}\colon \ker(\pi_{d-3}(\mathsf{i}_X, \mathsf{u})) \twoheadrightarrow {\mathbb{Z}[\pi_1 X]}\big/{\mathsf{dax}_{\mathsf{u}}(\pi_{d-1}X)}.$$

For $d \geq 4$ they are isomorphisms of groups, while for $d = 3$ they are surjections of sets with the source understood as $\mathbb{K}(X, \mathbf{u}) = (\pi_0\mathsf{i}_X)^{-1}(\mathsf{u})$. This completes our recollection of the proof of Theorem 1.1, and we next turn to computing $\mathsf{dax}_{\mathsf{u}}$ and proving Theorem A.

3.2.1. *Computing* $\mathsf{dax}_{\mathsf{u}}$. By definition, $\mathsf{dax}_{\mathsf{u}}(a) \in \mathbb{Z}[\pi_1 X\setminus 1]$ for $a \in \pi_{d-1}X$ is computed as follows. Firstly, represent $a$ by a map $A\colon \mathbb{S}^{d-2} \to \Omega X$, and pick any lift $F_A\colon (\mathbb{I}^{d-2}, \partial) \to (\operatorname{Imm}_\partial(\mathbb{D}^1, X), \mathsf{u})$ of $A$, meaning that $p_{\mathsf{u}}\circ F_A = A$. Secondly, compute $\mathsf{Dax}(F_A) \in \mathbb{Z}[\pi_1 X]$, and then disregard any potential $g_{x_i} = 1$ in the sum to obtain $\mathsf{dax}_{\mathsf{u}}(a)$. We now describe a construction of a lift $F_A$, by first choosing a convenient map $A$, which is in terms of a so-called $\mathsf{u}$-whiskered representative $A_{\mathsf{u}}$.

Recall that we fix a subinterval $[\theta_L, \theta_R] \subseteq \mathbb{D}^1 = [-1, 1]$, and that for an arc $\alpha\colon [c, d] \hookrightarrow X$ we denote by $\alpha^{-1}$ its time reversal, and for $\theta \in \mathbb{D}^1$ write $\alpha_{\leq\theta} \coloneqq \alpha|_{[c,\theta]}$ and $\alpha_{\geq\theta} \coloneqq \alpha|_{[\theta,d]}$.

**Definition 3.6.** A map $A_{\mathsf{u}}\colon \mathbb{I}^{d-2}\times[\theta_L, \theta_R] \to X$ is called a $\mathsf{u}$-*whiskered* representative of a class $a \in \pi_{d-1}(X; x_-) \cong \pi_{d-2}(\Omega X, \mathsf{u}_{\leq\theta_R}\cdot\mathsf{u}_{\leq\theta_R}^{-1})$ if the following conditions hold (see Figure 6).

(i) Each $A_{\mathsf{u}}(\vec{t})\colon [\theta_L, \theta_R] \hookrightarrow X$ for $\vec{t} \in \mathbb{I}^{d-2}$ is an immersion, whose interior intersects $\mathsf{u}$ transversely and only to the right of $\mathsf{u}(\theta_R)$ on $\mathsf{u}$.

(ii) For $\vec{t} \in \partial\mathbb{I}^{d-2}$ we have $A_{\mathsf{u}}(\vec{t}) = \mathsf{u}|_{[\theta_L,\theta_R]}$.

(iii) The map $(\mathbb{I}^{d-2}, \partial) \to (\Omega X, \mathsf{u}_{\leq\theta_R}\cdot\mathsf{u}_{\leq\theta_R}^{-1})$ defined by the formula $\vec{t} \mapsto \mathsf{u}_{\leq\theta_L}\cdot A_{\mathsf{u}}(\vec{t})\cdot\mathsf{u}_{\leq\theta_R}^{-1}$ is continuous and represents the homotopy class $a$. △

**Remark 3.7.** Any $a \in \pi_{d-1}(X; x_-)$ has a $\mathsf{u}$-whiskered representative $A_\mathsf{u}$. Namely, we first represent $a$ by an immersion $A\colon (\mathbb{I}^{d-1}, \partial\mathbb{I}^{d-1}) \to (X, x_-)$, which intersects $\mathsf{u}$ transversely in finitely many points $\mathsf{u}(\theta_i)$ such that $\theta_i > \uptheta_R$, as in Figure 6 (where $w$ is the whisker to the basepoint $x_-$). Moreover, we can reparametrize $A$ so that writing $\mathbb{I}^{d-1} = \mathbb{I}^{d-2} \times \mathbb{D}^1$ gives adjoint maps $A(\vec{t})\colon \mathbb{D}^1 \to X$ for $\vec{t} \in \mathbb{I}^{d-2}$, each of which is an immersed arc from $x_-$ to itself, cf. Lemma 3.1.

Finally, we precompose the whisker $w$ of $A$ by the loop $\mathsf{u} \cdot \mathsf{u}^{-1}$, then perform a homotopy until each $A(\vec{t})$ agrees with $\mathsf{u}_{\leq\uptheta_L}$ on $[-1, \uptheta_L]$ and with $\mathsf{u}_{\geq\uptheta_R} \cdot \mathsf{u}^{-1}$ on $[\uptheta_R, 1]$. The restriction of this final family to $[\uptheta_L, \uptheta_R]$ is the desired family $A_\mathsf{u}$. △

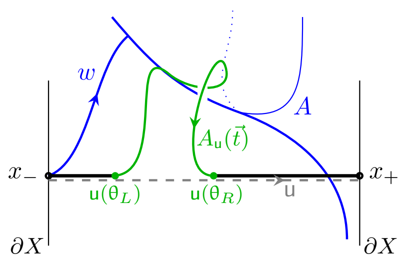


**Figure 6.** A $\mathsf{u}$-whiskered representative $A_\mathsf{u}$ of $a$ for a value $\vec{t}$ is shown in green. The corresponding immersed arc $F_{A_\mathsf{u}}(\vec{t})$ is the union of the green arc $A_\mathsf{u}(\vec{t})$ and the solid horizontal arcs.

**Definition 3.8.** Given a class $a \in \pi_{d-1}X$ and its $\mathsf{u}$-whiskered representative $A_\mathsf{u}$, define

$$F_{A_\mathsf{u}}\colon (\mathbb{I}^{d-2}, \partial) \to (\mathrm{Imm}_\partial(\mathbb{D}^1, X), \mathsf{u}), \quad F_{A_\mathsf{u}}(\vec{t}) \coloneqq \mathsf{u}_{\leq\uptheta_L} \cdot A_\mathsf{u}(\vec{t}) \cdot \mathsf{u}_{\geq\uptheta_R}. \qquad △$$

Strictly speaking, $F_{A_\mathsf{u}}(\vec{t})$ might not be smooth at the points $\mathsf{u}(\uptheta_L)$ and $\mathsf{u}(\uptheta_R)$, but we can assume that $A_\mathsf{u}$ is chosen so that this is the case, cf. Figure 6. Moreover, the condition (ii) for $A_\mathsf{u}$ ensures that $F_{A_\mathsf{u}}(\vec{t}) = \mathsf{u}$ for $\vec{t} \in \partial\mathbb{I}^{d-2}$, so $F_{A_\mathsf{u}}$ indeed represents a class in $\pi_{d-2}(\mathrm{Imm}_\partial(\mathbb{D}^1, X), \mathsf{u})$.

**Lemma 3.9.** *The homotopy class of $p_\mathsf{u} \circ F_{A_\mathsf{u}}\colon (\mathbb{I}^{d-2}, \partial) \to (\Omega X, \mathsf{u} \cdot \mathsf{u}^{-1})$ is precisely $a \in \pi_{d-1}X$, so $\mathsf{dax}_\mathsf{u}(a) = \mathsf{Dax}(F_{A_\mathsf{u}})$. Moreover, any self intersection of the arc $F_{A_\mathsf{u}}(\vec{t})$ arises either as a self-intersection $x_i$ of an arc $A_\mathsf{u}(\vec{t}_i)$, or as an intersection $y_j$ of an arc $A_\mathsf{u}(\vec{t}_j)$ with $\mathsf{u}_{\geq\uptheta_R}$, and the corresponding double point loops in the two cases can be calculated as:*

$$\begin{cases} g_{x_i} = \mathsf{u}_{\leq\uptheta_L} \cdot A_\mathsf{u}(\vec{t}_i)_{\leq\theta_i^-} \cdot A_\mathsf{u}(\vec{t}_i)^{-1}_{\leq\theta_i^+} \cdot \mathsf{u}^{-1}_{\leq\uptheta_L} & \text{for } x_i \in A \cap A, \\ g_{y_j} = \mathsf{u}_{\leq\uptheta_L} \cdot A_\mathsf{u}(\vec{t}_j)_{\leq\theta_j^-} \cdot \mathsf{u}^{-1}_{\leq\theta_j^+} & \text{for } y_j \in A \cap \mathsf{u}. \end{cases}$$

*Proof.* By definition of $p_\mathsf{u}$ and $F_{A_\mathsf{u}}$ we have $p_\mathsf{u} \circ F_{A_\mathsf{u}}(\vec{t}) = \mathsf{u}_{\leq\uptheta_L} \cdot A_\mathsf{u}(\vec{t}) \cdot \mathsf{u}_{\geq\uptheta_R} \cdot \mathsf{u}^{-1}$. There is an obvious homotopy $\mathsf{u}_{\geq\uptheta_R} \cdot \mathsf{u}^{-1} \simeq \mathsf{u}_{\leq\uptheta_R}$ rel. endpoints, so the first claim follow from Definition 3.6 (iii). For the second claim, first note that the double points of the arcs $F_{A_\mathsf{u}}(\vec{t})$ can arise either

- as the intersections of $\mathsf{u}_{\leq\uptheta_L}$ with $A_\mathsf{u}(\vec{t}_j)$, or
- as the intersections of $\mathsf{u}_{\leq\uptheta_L}$ with $\mathsf{u}_{\geq\uptheta_R}$, or

(x) as the self-intersections $x_i$ of $A_\mathsf{u}(\vec{t}_i)$ for some $\vec{t}_i$, or

(y) as the intersections $y_j$ of $A_\mathsf{u}(\vec{t}_j)$ with $\mathsf{u}_{\geq\uptheta_R}$.

However, by Definition 3.6(i) points in $A \pitchfork \mathsf{u}$ occur in $\mathsf{u}$ to the right of $\mathsf{u}(\uptheta_R)$, so the first case does not arise. The second case is obviously not possible.

In both remaining cases we have $F_{A_\mathsf{u}}(\vec{t}_i)_{\leq\theta_i^-} = \mathsf{u}_{\leq\uptheta_L} \cdot A_\mathsf{u}(\vec{t}_i)_{\leq\theta_i^-}$. In the $x$-case we similarly have $F_{A_\mathsf{u}}(\vec{t}_i)_{\leq\theta_i^+} = \mathsf{u}_{\leq\uptheta_L} \cdot A_\mathsf{u}(\vec{t}_i)_{\leq\theta_i^+}$, so the formula for $g_{x_i}$ follows from its definition (14). For the $y$-case observe that the arc $F_{A_\mathsf{u}}(\vec{t}_j)_{\leq\theta_j^+}$ is homotopic rel. endpoints to $\mathsf{u}_{\leq\theta_j^+}$ since each $A_\mathsf{u}(\vec{t}_j)$ is homotopic to $A_\mathsf{u}(\vec{t}) = \mathsf{u}|_{[\uptheta_L,\uptheta_R]}$, for $\vec{t} \in \partial\mathbb{I}^{d-2}$. □

3.2.2. *Computing* $\mathsf{dax}$. Right before the statement of Theorem A we have introduced the notation

$$\mathsf{dax} := \mathsf{dax}_{\mathsf{u}_-} : \pi_{d-1}X \to \mathbb{Z}[\pi_1 X \setminus 1]$$

for a fixed arc $\mathsf{u}_- : \mathbb{D}^1 \hookrightarrow X$ isotopic into $\partial X$ rel. endpoints and with $\mathsf{u}_-(-1) = x_-$. We choose

$$\mathsf{u}_- = \mathsf{u}_{\leq \theta_R} \cdot \mathsf{u}'_{\geq \theta_R}, \tag{20}$$

as in the left part of Figure 7. Namely, $\mathsf{u}'_{\geq\theta_R} : [\theta_R, 1] \to X$ is a slight pushoff of the reverse of $\mathsf{u}_{\leq\theta_R}$, so that it goes from $\mathsf{u}(\theta_R)$ to some point $x'_- \in \partial X$ near $x_- \in \partial X$, and so that $\mathsf{u}_-$ is a smooth neat arc from $x_-$ to $x'_-$. By abuse of notation, we consider $\mathsf{u}_-$ also as a loop based at $x_-$.

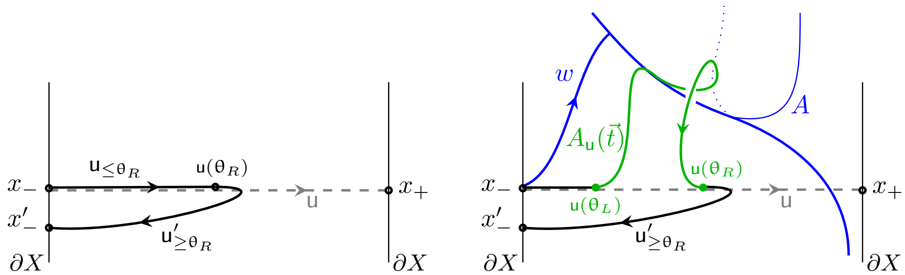


**Figure 7.** *Left.* The arc $\mathsf{u}_-$ goes from $x_-$ to a nearby point $x'_-$. *Right.* A $\mathsf{u}_-$-whiskered representative of $a$.

**Remark 3.10.** Note that if $A_{\mathsf{u}}$ is a $\mathsf{u}$-whiskered representative of $a$, then it is also its $\mathsf{u}_-$-whiskered representative, since all condition of Definition 3.6 are still fulfilled, see the right part of Figure 7. Hence, the map $F_{A_{\mathsf{u}}}(\vec{t}) = \mathsf{u}_{\leq\theta_L} \cdot A_{\mathsf{u}}(\vec{t}) \cdot \mathsf{u}_{\geq\theta_R}$ simply changes to $F_{A_{\mathsf{u}_-}}(\vec{t}) = \mathsf{u}_{\leq\theta_L} \cdot A_{\mathsf{u}}(\vec{t}) \cdot \mathsf{u}'_{\geq\theta_R}$. △

3.3. **Formulae.** In this section we express the values of $\mathsf{dax}_{\mathsf{u}}$ on certain elements in $\pi_{d-1}X$, and prove Theorem A and Corollary B. Let us first fix some notation.

3.3.1. *Notation.* Equip $\pi_{d-1}X = \pi_{d-1}(X, x_-)$ with the usual $\pi_1 X = \pi_1(X, x_-)$-action, and the set

$$\pi_1(X, \partial X) := \pi_1(X, \partial X, x_-) = \{k : \mathbb{D}^1 \to X \mid k(-1) = x_-,\ k(+1) \in \partial X\}/\simeq$$

with the action of $g \in \pi_1 X$ by precomposition, $\boldsymbol{k} \mapsto g\boldsymbol{k}$. Moreover, we have the standard involution on the group ring $\mathbb{Z}[\pi_1 X] := \{\sum \varepsilon_i g_i : \varepsilon_i = \pm 1, g_i \in \pi_1 X\}$, which linearly extends $\overline{g} := g^{-1}$.

Next, we recall (see for example [Ran02]) the usual definition of the *equivariant intersection pairing*

$$\lambda : \pi_{d-1}X \times \pi_1(X, \partial X) \to \mathbb{Z}[\pi_1 X].$$

Given classes $a \in \pi_{d-1}X$ and $\boldsymbol{k} \in \pi_1(X, \partial X)$, pick smooth representatives $A : \mathbb{S}^{d-1} \to X$ and $k : (\mathbb{D}^1, \partial\mathbb{D}^1) \to (X, \partial X)$ that intersect transversely and in the interior of $X$, excluding the point $A(e) = k(-1) = x_- \in \partial X$, see Figure 8. For a transverse intersection point $y \in A(\mathbb{S}^{d-1}) \cap k(\mathbb{D}^1)$ define the double point loop

$$\lambda_y(A, k) := \lambda_y(A) \cdot \lambda_y(k)^{-1}, \tag{21}$$

where paths $\lambda_y(A)$ and $\lambda_y(k)$ go from $x_-$ to $y$, respectively along $A$ and $k$ (the choice of a path along $A$ is irrelevant as $\mathbb{S}^{d-1}$ is simply connected). The sign $\varepsilon_y(A, k)$ is obtained by comparing the orientation of the tangent space $T_y X$ to that of $dA|_y(T\mathbb{S}^{d-1}) \oplus dk|_y(T\mathbb{D}^1)$, and we let

$$\lambda(a, \boldsymbol{k}) := \sum_{y \in (A\cap k)\setminus\{x_-\}} \varepsilon_y(A, k)[\lambda_y(A, k)].$$

Note that we can also compute $\lambda(a, g)$ for $g \in \pi_1 X$, using the canonical map $\pi_1 X \to \pi_1(X, \partial X)$.

That $\lambda$ is a well-defined invariant of homotopy classes follows from its description using the Hurewicz map on the universal cover

$$\pi_{d-1}X \cong \pi_{d-1}\widetilde{X} \to H_{d-1}(\widetilde{X}; \mathbb{Z}) \cong H_{d-1}(X; \mathbb{Z}[\pi_1 X])$$

and Poincaré duality with $\mathbb{Z}[\pi_1 X]$-coefficients $H_{d-1}(X; \mathbb{Z}[\pi_1 X]) \times H_1(X, \partial X; \mathbb{Z}[\pi_1 X]) \to \mathbb{Z}[\pi_1 X]$.

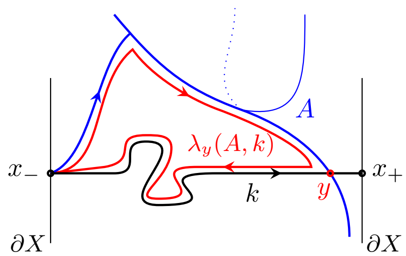


**Figure 8.** Associated to the intersection point $y \in A \cap k$ is the loop $\lambda_y(A,k)$, based at $x_-$.

Finally, one analogously defines a pairing $\lambda\colon \pi_1(X,\partial X)\times\pi_{d-1}X \to \mathbb{Z}[\pi_1X]$, and immediately has

$$\lambda(\boldsymbol{k},a) = (-1)^{d-1}\overline{\lambda(a,\boldsymbol{k})}, \tag{22}$$

since $\lambda_y(k,A) = \lambda_y(k)\cdot\lambda_y(A)^{-1} = \lambda_y(A,k)^{-1}$ and exchanging the order of $dA|_y(T\mathbb{S}^{d-1})$ and $dK|_y(T\mathbb{D}^1)$ changes the orientation by $(-1)^{d-1}$. We use this pairing just as a shorter notation for the right hand side of (22).

The following standard properties of $\lambda$ are easy to check and will be useful in our computations.

**Lemma 3.11.** *For $\boldsymbol{k}\in\pi_1(X,\partial X)$, $g_1,g_2\in\pi_1X$ and $a_1,a_2\in\pi_{d-1}X$ we have*

$$\lambda(g_1a_1+g_2a_2,\ \boldsymbol{k}) = g_1\lambda(a_1,\boldsymbol{k})+g_2\lambda(a_2,\boldsymbol{k}),$$
$$\lambda(a,\ g\boldsymbol{k}) = \lambda(a,g)+\lambda(a,\boldsymbol{k})\overline{g}.$$

In other words, for any $\boldsymbol{k}$ the map $\lambda(-,\boldsymbol{k})\colon \pi_{d-1}X\to\mathbb{Z}[\pi_1X]$ is $\mathbb{Z}[\pi_1X]$-linear, while for any $a$ the map $\lambda(a,-)\colon\pi_1(X,\partial X)\to\mathbb{Z}[\pi_1X]$ is a map of $\pi_1X$-sets which satisfies a Fox derivative rule. Note that taking $\boldsymbol{k}=\overline{g}$ in the second equality, we obtain $0=\lambda(a,1)=\lambda(a,g)+\lambda(a,\overline{g})\overline{g}$.

Recall from the introduction that $\lambdabar(a,\boldsymbol{k})\in\mathbb{Z}[\pi_1X\setminus 1]$ is defined by forgetting the term given by 1 in $\lambda(a,\boldsymbol{k})$. One needs to be careful when applying the formulae from Lemma 3.11, since, for example, $\lambdabar(ga,\boldsymbol{k})\neq g\lambdabar(a,\boldsymbol{k})$. However, one easily checks that the following does hold.

**Lemma 3.12.** *$\lambdabar(ga,\ g\boldsymbol{k}) = \lambdabar(ga,g)+g\lambdabar(a,\boldsymbol{k})\overline{g}$.*

3.3.2. *Proofs of Theorem A and Corollary B.* We now relate homomorphisms $\mathsf{dax}_\mathsf{u}$ and $\mathsf{dax} := \mathsf{dax}_{\mathsf{u}_-}$ from $\pi_{d-1}X$ to $\mathbb{Z}[\pi_1X\setminus 1]$, for the arc $\mathsf{u}_-$ as in (20). That is, we prove the formula from A(I):

$$\mathsf{dax}_\mathsf{u}(a) = \mathsf{dax}(a)+\lambdabar(a,\mathbf{u}),\quad \text{for every } a\in\pi_{d-1}X.$$

*Proof of Theorem A(I).* Pick a $\mathsf{u}$-whiskered representative $A_\mathsf{u}$ for $a\in\pi_{d-1}X$ as in Definition 3.6. Then by Lemma 3.9 we have

$$\mathsf{dax}_\mathsf{u}(a) := \mathsf{Dax}(F_{A_\mathsf{u}}) = \mathsf{Dax}_x(F_{A_\mathsf{u}})+\mathsf{Dax}_y(F_{A_\mathsf{u}})$$

for $F_{A_\mathsf{u}}\colon\mathbb{S}^{d-2}\to\mathrm{Imm}_\partial(\mathbb{D}^1,X)$ given by immersed arcs $F_{A_\mathsf{u}}(\vec{t}) := \mathsf{u}_{\leq\theta_L}\cdot A_\mathsf{u}(\vec{t})\cdot\mathsf{u}_{\geq\theta_R}$ whose double points are either self-intersections of $A_\mathsf{u}(\vec{t})$ (type $x$), or intersections of $A_\mathsf{u}(\vec{t})$ with $\mathsf{u}$ (type $y$).

On the other hand, recall from Remark 3.10 that a $\mathsf{u}_-$-whiskered representative is given by

$$F_{A_{\mathsf{u}_-}}(\vec{t}) := \mathsf{u}_{\leq\theta_L}\cdot A_\mathsf{u}(\vec{t})\cdot\mathsf{u}'_{\geq\theta_R}.$$

Therefore, $\mathsf{dax}(a) = \mathsf{Dax}(F_{A_{\mathsf{u}_-}}) = \mathsf{Dax}_x(F_{A_\mathsf{u}})$. Namely, there is no change in the set of $x$-points and their associated loops, but now there are no $y$-points since we assume the arc $\mathsf{u}_-$ is "short enough", i.e. $A_\mathsf{u}(\vec{t})\cap\mathsf{u}'_{\geq\theta_R}=\emptyset$. Thus, we need only show that $\mathsf{Dax}_y(F_{A_\mathsf{u}}) = \lambdabar(a,\mathsf{u})$.

We have $y_j = A(\vec{t_j})(\theta_j^-) = \mathsf{u}(\theta_j^+)$, for some $\vec{t_j}\in\mathbb{I}^{d-2}$, $\theta_j^-\in[\theta_L,\theta_R]$, $\theta_j^+\in[\theta_R,1]$, see Figure 6. By Lemma 3.9, the associated Dax double point loop is given by

$$g_{y_j} = \mathsf{u}_{\leq\theta_L}\cdot A(\vec{t_j})_{\leq\theta_j^-}\cdot\mathsf{u}^{-1}_{\leq\theta_j^+}.$$

This is precisely equal to $\lambda_{y_j}(A,\mathsf{u})$ from (21) since $\lambda_{y_j}(A)=\mathsf{u}_{\leq\theta_L}\cdot A(\vec{t}_j)_{\leq\theta_j^-}$ and $\lambda_{y_j}(\mathsf{u})=\mathsf{u}_{\leq\theta_j^+}$ (cf. Figures 1 and 8). Moreover, any $g_{y_j}\simeq 1$ is not counted towards $\mathsf{dax}_{\mathsf{u}}(a)$ by definition, see (19), so we will have $\mathsf{Dax}_y(F_{A_{\mathsf{u}}})=\lambda\!\!\!\lambda(a,\mathbf{u})$ once we check that the signs agree.

To this end, the Dax sign $\varepsilon_{y_j}$ compares the orientation of the vector space

$$dA|_{(\vec{t}_j,\theta_j^-)}(T\mathbb{I}^{d-2}\oplus T\mathbb{D}^1)\oplus d\mathsf{const}_{\mathsf{u}}|_{\theta_j^+}(T\mathbb{I}^{d-2}\oplus T\mathbb{D}^1)\quad\text{ to }\quad T_{\vec{t}_j}\mathbb{I}^{d-2}\oplus T_{y_j}X,$$

while the sign $\varepsilon_{y_j}(A,\mathsf{u})$ for $\lambda$ compares $dA|_{(\vec{t}_j,\theta_j^-)}(T\mathbb{S}^{d-1})\oplus d\mathsf{u}|_{\theta_j^+}(T\mathbb{D}^1)$ to $T_{y_j}X$, or equivalently

$$T_{\vec{t}_j}\mathbb{I}^{d-2}\oplus dA|_{(\vec{t}_j,\theta_j^-)}(T\mathbb{S}^{d-1})\oplus d\mathsf{u}|_{\theta_j^+}(T\mathbb{D}^1)\quad\text{ to }\quad T_{\vec{t}_j}\mathbb{I}^{d-2}\oplus T_{y_j}X.$$

Two out of four displayed oriented spaces are the same, while the other two differ by $(d-2)(d-1)$ transpositions, which is an even number, so $\varepsilon_{y_j}=\varepsilon_{y_j}(A,\mathsf{u})$. $\square$

We immediately obtain Corollary B(i): if $A$ is embedded then all arcs in the family $F_{A_{\mathsf{u}_-}}$ are embedded so $\mathsf{dax}(a)=\mathsf{Dax}_x(F_{A_{\mathsf{u}_-}})=0$. Moreover, Corollary B(ii) is also immediate since

$$\mathsf{dax}_{g\mathsf{u}}(a)-\mathsf{dax}_{\mathsf{u}}(a)=\mathsf{dax}(a)+\lambda\!\!\!\lambda(a,g\mathbf{u})-(\mathsf{dax}(a)+\lambda\!\!\!\lambda(a,\mathbf{u}))=\lambda\!\!\!\lambda(a,g\mathbf{u})-\lambda\!\!\!\lambda(a,\mathbf{u}).$$

Let us now prove part (II) of Theorem A: for any $a\in\pi_{d-1}X$ and $g\in\pi_1X$ we claim that

$$\mathsf{dax}(ga)\;=\;g\mathsf{dax}(a)\overline{g}\;-\;\lambda\!\!\!\lambda(ga,g)+\lambda\!\!\!\lambda(g,ga),$$

recalling from (22) that $\lambda\!\!\!\lambda(g,ga)=(-1)^{d-1}\overline{\lambda\!\!\!\lambda(ga,g)}$.

*Proof of Theorem A(II).* Pick a $\mathsf{u}_-$-whiskered representative $A_{\mathsf{u}_-}\colon\mathbb{I}^{d-2}\times[\theta_L,\theta_R]\to X$ as in Definition 3.6 so that $\mathsf{Dax}(F_{A_{\mathsf{u}_-}})=\mathsf{Dax}_x(F_{A_{\mathsf{u}_-}})$ is the sum over all self-intersections of $A_{\mathsf{u}_-}(\vec{t})$ (see the previous proof). We now describe a $\mathsf{u}_-$-whiskered representative of $ga$.

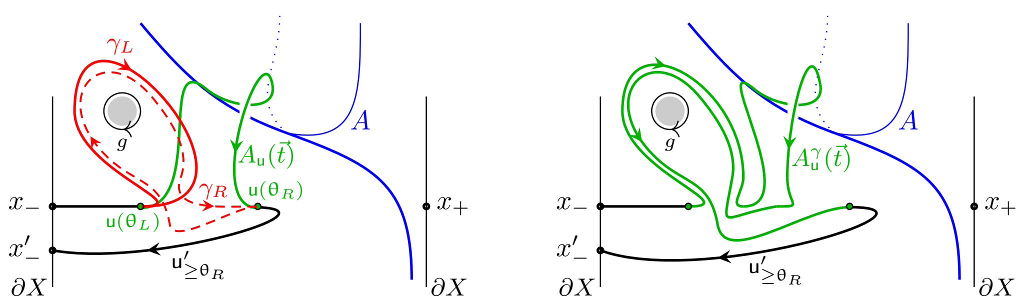


**Figure 9.** *Left:* A loop $\gamma_L$ based at $\mathsf{u}(\theta_L)$, and its pushoff $\gamma_R$ based at $\mathsf{u}(\theta_R)$. *Right:* A $\mathsf{u}_-$-whiskered representative $A^{\gamma}_{\mathsf{u}_-}(\vec{t})$ of $ga$ consist of the old arc $A_{\mathsf{u}_-}(\vec{t})$ and the "new whisker", made of $\gamma_L$ and $\gamma_R^{-1}$.

Firstly, we can represent $g\in\pi_1X$ by $\gamma=\mathsf{u}_{\leq\theta_L}\cdot\gamma_L\cdot\mathsf{u}^{-1}_{\leq\theta_L}$ for an embedded loop $\gamma_L$ based at $\mathsf{u}(\theta_L)$, such that $\gamma_L\times pr_2\colon\mathbb{D}^1\times\mathbb{I}^{d-2}\to X\times\mathbb{I}^{d-2}$ intersects the interior of $A_{\mathsf{u}_-}$ transversely and in a finite number of points $(z_j,\vec{t}_j)\in X\times\mathbb{I}^{d-2}$. Similarly, let $\gamma_R$ be a copy of $\gamma_L$ based at $\theta_R$ instead, i.e. $\gamma_R\simeq\mathsf{u}|^{-1}_{[\theta_L,\theta_R]}\cdot\gamma_L\cdot\mathsf{u}|_{[\theta_L,\theta_R]}$, see the left part of Figure 9. We define $A^{\gamma}_{\mathsf{u}_-}(\vec{t})$ as

$$\gamma_L\cdot A_{\mathsf{u}_-}(\vec{t})\cdot\gamma_R^{-1},$$

modified into an immersed arc in the obvious way, see the right part of Figure 9. We claim that $A^{\gamma}_{\mathsf{u}_-}$ is a $\mathsf{u}_-$-whiskered representative of $ga$. Namely, recall that the action of $g\in\pi_1X$ on $a\in\pi_{d-1}X$ agrees with the conjugation action of $\pi_0\Omega X$ on $\pi_{d-2}\Omega X$ and note that $A^{\gamma}_{\mathsf{u}_-}$ is precisely homotopic to such a pointwise conjugate.

Therefore, $\mathsf{dax}(ga)\coloneqq\mathsf{Dax}(F_{A^{\gamma}_{\mathsf{u}_-}})\in\mathbb{Z}[\pi_1X\setminus 1]$ is the sum of signed nontrivial double point loops associated to double points of the immersed arcs

$$F_{A^{\gamma}_{\mathsf{u}_-}}(\vec{t})\coloneqq\mathsf{u}_{\leq\theta_L}\cdot A^{\gamma}_{\mathsf{u}_-}(\vec{t})\cdot\mathsf{u}'_{\geq\theta_R},\quad\text{ for }\vec{t}\in\mathbb{I}^{d-2}.$$

Such points can be divided into two groups.

(old) Each double point $z_i$ of some $F_{A_{\mathsf{u}_-}}(\vec{t}_i)$ appears also in $F_{A^\gamma_{\mathsf{u}_-}}(\vec{t}_i)$.

(new) Each intersection point $z_j$ that contributes to $\lambda(a,g)$, gives a pair of intersection points: $z_j^L \in A_{\mathsf{u}_-} \cap \gamma_L$ and $z_j^R \in A_{\mathsf{u}_-} \cap \gamma_R^{-1}$, see Figure 10.

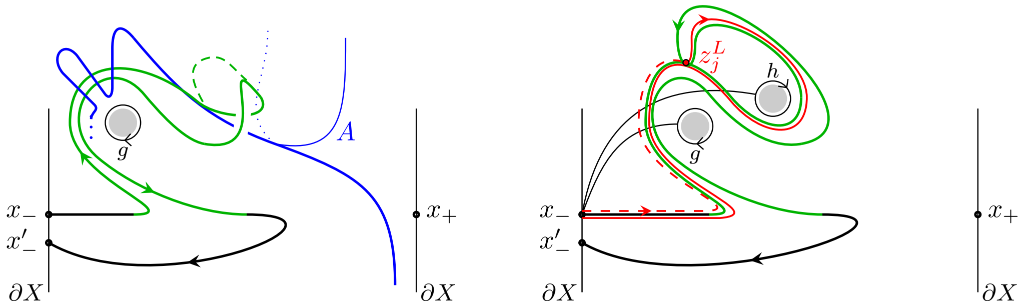


**Figure 10.** *Left:* For every intersection point of $a$ and $g$ there are two double points $z_j^L, z_j^R$ of $A^\gamma_{\mathsf{u}_-}$. Namely, the dashed arc (part of $A_{\mathsf{u}_-}(\vec{t})$) moves across the sphere $a$ and for some $\vec{t}$ it hits back into $\gamma_L$ and $\gamma_R$. *Right:* The associated loop of $z_j^L$ goes along $\gamma$ (dashed), then jumps to $A^\gamma_{\mathsf{u}_-}(\vec{t})$ (solid red). It is homotopic to $h$.

In the first case the associated loop $g_{z_i}$ for $\mathsf{dax}(ga)$ is the one which contributed to $\mathsf{dax}(a)$ but now conjugated by $g$, as the new whisker is used for both paths $F_{A^\gamma_{\mathsf{u}_-}}(\vec{t}_i)_{\leq\theta_i^-}$ and $F_{A^\gamma_{\mathsf{u}_-}}(\vec{t}_i)_{\leq\theta_i^+}$ that comprise $g_{z_i}$. This gives $g\mathsf{dax}(a)\overline{g}$, which is exactly the first term on the right hand side of (II).

In the second case, the associated Dax double point loops are respectively

$$\begin{aligned} g_{z_j^L} &= [\lambda_{z_j}(\gamma_L)\cdot\lambda_{z_j}(A^\gamma_{\mathsf{u}_-})^{-1}] = [\lambda_{z_j}(\gamma, A^\gamma_{\mathsf{u}_-})],\\ g_{z_j^R} &= [\lambda_{z_j}(A^\gamma_{\mathsf{u}_-})\cdot\lambda_{z_j}(\gamma_R)^{-1}] = [\lambda_{z_j}(A^\gamma_{\mathsf{u}_-},\gamma)], \end{aligned}$$

where we used the definition of $\lambda$ from (21). Thus, we will exactly obtain the two remaining terms of (II), namely $\lambda(g,ga) = (-1)^{d-1}\lambda(ga,g)$ and $-\lambda(ga,g)$ respectively, once we check the signs.

Locally around $z_j^R$ the first strand (the one with $\theta_-$) moves across $A^\gamma_{\mathsf{u}_-}$, while the second (the one with $\theta_+$) is constantly $\gamma^{-1}$, which is exactly the sign of $\varepsilon_{z_j^R}(A^\gamma_{\mathsf{u}_-},\gamma^{-1})$ so the opposite of the sign for $\lambda(ga,g)$. On the other hand, for $z_j^L$ the first strand is constantly $\gamma$ and the second moves across $A_{\mathsf{u}_-}$, so indeed $\varepsilon^{\mathsf{Dax}}_{z_j^L} = \varepsilon_{z_j^L}(\gamma, A^\gamma_{\mathsf{u}_-}) = (-1)^{d-1}\varepsilon_{z_j^L}(A^\gamma_{\mathsf{u}_-},\gamma)$. This completes the proof of (II). □

*Proof of Corollary B.* For the parts (i) and (ii) see the paragraph after the proof of Theorem A(I) above. The part (iii) follows from

$$\begin{aligned} \mathsf{dax}_{g\mathsf{u}}(ga) - g\mathsf{dax}_{\mathsf{u}}(a)\overline{g} &= \mathsf{dax}(ga) + \lambda\!\!\!\lambda(ga, g\mathbf{u}) - g\big(\mathsf{dax}(a) + \lambda\!\!\!\lambda(a,\mathbf{u})\big)\overline{g}\\ &= \big(\mathsf{dax}(ga) - g\mathsf{dax}(a)\overline{g}\big) + \lambda\!\!\!\lambda(ga,g\mathbf{u}) - g\lambda\!\!\!\lambda(a,\mathbf{u})\overline{g}\\ &= \lambda\!\!\!\lambda(g,ga) - \lambda\!\!\!\lambda(ga,g) + \big(\lambda\!\!\!\lambda(ga,g) + g\lambda\!\!\!\lambda(a,\mathbf{u})\overline{g}\big) - g\lambda\!\!\!\lambda(a,\mathbf{u})\overline{g}\\ &= \lambda\!\!\!\lambda(g,ga), \end{aligned}$$

where we applied (I) twice, reordered the terms, then used (II) and Lemma 3.12, and finally cancelled some terms. We prove (iv) by applying (ii) and then (iii):

$$\begin{aligned} \mathsf{dax}_{\mathsf{u}}(ga) &= \mathsf{dax}_{g\mathsf{u}}(ga) - \lambda\!\!\!\lambda(ga,g\mathbf{u}) + \lambda\!\!\!\lambda(ga,\mathbf{u})\\ &= g\mathsf{dax}_{\mathsf{u}}(a)\overline{g} + \lambda\!\!\!\lambda(g,ga) - \lambda\!\!\!\lambda(ga,g\mathbf{u}) + \lambda\!\!\!\lambda(ga,\mathbf{u}). \end{aligned}$$

Finally, (v) follows from (iv), (i) and Lemma 3.12. □

## 4. Knotted circles

In this section we study the space $\mathrm{Emb}(\mathbb{S}^1, N)$ of smooth embeddings of a circle into a compact manifold $N$ of dimension $d \geq 3$ (possibly with $\partial N \neq \emptyset$). The inclusion into the space of immersions $\mathfrak{i}_N\colon \mathrm{Emb}(\mathbb{S}^1, N) \hookrightarrow \mathrm{Imm}(\mathbb{S}^1, N)$ induces for any basepoint $\mathsf{s} \in \mathrm{Emb}(\mathbb{S}^1, N)$ isomorphisms

$$\pi_n\mathfrak{i}_N\colon \pi_n(\mathrm{Emb}(\mathbb{S}^1, N), \mathsf{s}) \xrightarrow{\cong} \pi_n(\mathrm{Imm}(\mathbb{S}^1, N), \mathsf{s}), \quad 0 \leq n \leq d-4, \tag{23}$$

similarly as in (12) for arcs. Thus, the lowest homotopy group potentially distinguishing embedded from immersed circles is again $\pi_{d-3}$, and $\ker(\pi_{d-3}\mathfrak{i}_N)$ is isomorphic to the cokernel of the connecting map $\delta_{\mathfrak{i}_N}\colon \pi_{d-2}(\mathrm{Imm}(\mathbb{S}^1, N), \mathsf{s}) \to \pi_{d-2}(\mathrm{Imm}(\mathbb{S}^1, N), \mathrm{Emb}(\mathbb{S}^1, N), \mathsf{s})$.

The first part of Theorem C is a computation of $\pi_n(\mathrm{Imm}(\mathbb{S}^1, N), \mathsf{s})$ for $n \leq d-3$, which we carry out in Section 4.1. In Section 4.2 we relate the space $\mathrm{Emb}(\mathbb{S}^1, N)$ to $\mathrm{Emb}_\partial(\mathbb{D}^1, N\setminus D^d)$, and use this to prove Theorem C(II) which computes $\ker(\pi_{d-3}\mathfrak{i}_N)$ (restated as Theorem 4.6), see Section 4.3.

4.1. **Immersed circles.** Let $\mathbb{S}(N)$ denote the unit sphere subbundle of the tangent bundle of $N$, and $\Lambda\mathbb{S}(N) := \mathrm{Map}(\mathbb{S}^1, \mathbb{S}(N))$ its free loop space, i.e. the space of all (nonbased) loops in $\mathbb{S}(N)$. Fix a neighbourhood $[e, e+\varepsilon] \subseteq \mathbb{S}^1$ of a point $e \in \mathbb{S}^1$ and en embedding $\overline{e}\colon [e, e+\varepsilon] \hookrightarrow N$.

**Proposition 4.1.** *Taking unit derivatives gives a weak homotopy equivalence of fibration sequences*

$$\begin{tikzcd}
r^{-1}(\overline{e}) \arrow[r, hook] \arrow[d, "\mathsf{d}", "\simeq"'] & \mathrm{Imm}(\mathbb{S}^1, N) \arrow[r, "r"] \arrow[d, "\mathsf{d}", "\simeq"'] & \mathrm{Imm}([e, e+\varepsilon], N) \arrow[d, "\mathsf{d}_e", "\simeq"'] \\
\Omega\mathbb{S}(N) \arrow[r, hook] & \Lambda\mathbb{S}(N) \arrow[r, "\mathsf{ev}_e"] & \mathbb{S}(N)
\end{tikzcd}$$

*where $r$ restricts an immersion to $[e, e+\varepsilon] \subseteq \mathbb{S}^1$ and $\mathsf{ev}_e$ evaluates a map at $e$. Moreover, in the bottom sequence the connecting map sends $\gamma \in \Omega\mathbb{S}(N)$ to the concatenated loop $\gamma \cdot \mathsf{ds} \cdot \gamma^{-1} \in \Omega\mathbb{S}(N)$.*

*Proof.* The map $\mathsf{d}_e\colon \mathrm{Imm}([e, e+\varepsilon], N) \to \mathbb{S}(N)$ taking the unit derivative at $e \in \mathbb{S}^1$ is a weak equivalence, since the arc can be arbitrarily shorten (see [Sma58, Lem. 6.2]). The restriction map $r$ is a Serre fibration by Smale [Sma58], so the fibre $r^{-1}(\overline{e})$ over any basepoint is weakly equivalent to the fibre $\mathrm{Imm}_*(\mathbb{S}^1, N) := (\mathsf{d}_e \circ r)^{-1}(\overline{e})$, the space of *based immersions*. Now, Smale also shows that $\mathsf{d}\colon \mathrm{Imm}_*(\mathbb{S}^1, N) \to \Omega\mathbb{S}(N)$ is a weak equivalence. Thus, in the above diagram the leftmost vertical map $\mathsf{d}$ is also a weak equivalence, implying the middle map $\mathsf{d}$ is as well.

For the last claim, first note that the connecting map goes from the space $\Omega\mathbb{S}(N)$ based at $\mathsf{const}_*$ to $\Omega\mathbb{S}(N)$ based at $\mathsf{ds}$. By definition, it takes a loop $\gamma$ in $\mathbb{S}(N)$, lifts it (using the homotopy lifting property of $\mathsf{ev}_e$) to a path in $\Lambda\mathbb{S}(N)$ which at time $t=1$ equals $\mathsf{ds}$, and then evaluates at the other endpoint[1] $t=0$. We simply observe that the path $t \mapsto \gamma|_{[t,1]} \cdot \mathsf{ds} \cdot \gamma|_{[t,1]}^{-1}$ is one such lift, since it equals $\mathsf{ds}$ for $t=1$, and for each $t$ its value at $e$ is $\gamma(t)$. ∎

Corresponding to a basepoint $\mathsf{s} \in \mathrm{Imm}(\mathbb{S}^1, N)$ is the point $\mathsf{ds} \in \Lambda\mathbb{S}(N)$. Note that it is important to keep track of basepoints, since the components of $\Lambda\mathbb{S}(N)$ are in general not even homotopy equivalent (so neither are components of $\mathrm{Imm}(\mathbb{S}^1, N)$). However, the components of $\Omega\mathbb{S}(N)$ are: the postconcatenation $- \cdot (\mathsf{ds})^{-1}\colon \Omega\mathbb{S}(N) \to \Omega\mathbb{S}(N)$, $\gamma \mapsto \gamma \cdot (\mathsf{ds})^{-1}$ is a homotopy equivalence, with an obvious inverse $\gamma \mapsto \gamma \cdot (\mathsf{ds})$ (note that $(\mathsf{ds})^{-1} \neq \mathsf{d}(\mathsf{s}^{-1})$).

Thus, writing $(- \cdot (\mathsf{ds})^{-1}) \circ \mathsf{d} = (\mathsf{d}-) \cdot (\mathsf{ds})^{-1}$ we have a diagram of fibration sequences

$$\begin{tikzcd}
r^{-1}(\overline{e}) \arrow[r, hook] \arrow[d, "(\mathsf{d}-)\cdot(\mathsf{ds})^{-1}"', "\simeq"] & \mathrm{Imm}(\mathbb{S}^1, N) \arrow[r, "r"] \arrow[d, "\mathsf{d}"', "\simeq"] & \mathrm{Imm}([e, e+\varepsilon], N) \arrow[d, "\mathsf{d}_e"', "\simeq"] \\
\Omega\mathbb{S}(N) \arrow[r, "\cdot(\mathsf{ds})"] & \Lambda\mathbb{S}(N) \arrow[r, "\mathsf{ev}_e"] & \mathbb{S}(N)
\end{tikzcd} \tag{24}$$

[1] We use this convention as we prefer to define the group commutator by $[a, b] = aba^{-1}b^{-1}$. If lifts would start at $t = 0$, we would get $\gamma^{-1} \cdot \mathsf{ds} \cdot \gamma$, so we would have to use $a^{-1}b^{-1}ab$ instead.

so that now the basepoint of $\Omega\mathbb{S}(N)$ is $\mathsf{const}_*$, and the bottom connecting map is $\gamma \mapsto \gamma{\cdot}\mathsf{ds}{\cdot}\gamma^{-1}{\cdot}\mathsf{ds}^{-1}$.

Using the canonical isomorphisms $\pi_n(\Omega\mathbb{S}(N), \mathsf{const}_*) \cong \pi_{n+1}(\mathbb{S}(N), e)$ and the correspondence of Samelson and Whitehead products (see, for example, [Kos20, App.B]), we view this connecting map as the self-map of $\pi_{n+1}\mathbb{S}(N)$ given by

$$a \mapsto [a, \mathbf{ds}]_W = \begin{cases} a\,(\mathbf{ds}\,a)^{-1}, & n = 0, \\ a - \mathbf{ds}\,a, & n > 0. \end{cases}$$

This is the Whitehead product with the homotopy class $\mathbf{ds} \in \pi_1\mathbb{S}(N)$ of $\mathsf{ds}$. As before, $\mathbf{ds}\,a$ denotes the usual action of $\mathbf{ds} \in \pi_1\mathbb{S}(N)$ on $a \in \pi_n\mathbb{S}(N)$ ("the change of whisker"). For $n = 0$ this is the conjugation action, so the formula $a\,(\mathbf{ds}\,a)^{-1}$ is exactly the commutator $[a, \mathbf{ds}] \in \pi_1\mathbb{S}(N)$.

**Corollary 4.2.** *There is a bijection $\pi_0 \operatorname{Imm}(\mathbb{S}^1, N) \cong {}^{\pi_1\mathbb{S}(N)}\!/\!_{[a,x]\,\sim\,1}$ and group extensions*

$$\pi_{n+1}\mathbb{S}(N)\big/\langle a - \mathbf{ds}\,a\rangle \rightarrowtail \pi_n(\operatorname{Imm}(\mathbb{S}^1, N), \mathsf{s}) \twoheadrightarrow \big\{b \in \pi_n\mathbb{S}(N) \mid b = \mathbf{ds}\,b\big\}.$$

*for all $n \geq 1$. Moreover, for $n \leq d-3$ the space $\mathbb{S}(N)$ can be replaced by $N$ and $\mathbf{ds}$ by $\mathbf{s} \in \pi_1 N$.*

The first part follows from the long exact sequence in homotopy groups and (24), noting that the set $\pi_0\Lambda\mathbb{S}(N)$ is in bijection with the set of orbits of the action of $\pi_1\mathbb{S}(N)$ on itself via the connecting map $a \mapsto [a, x]$ for varying $x = \mathbf{ds}$. The last sentence follows since the projection $\mathbb{S}(N) \to N$ is $(d-1)$-connected (as the fibre is $\mathbb{S}^{d-1}$).

**Remark 4.3.** If $\mathbf{ds} = 1 \in \pi_1\mathbb{S}(N)$ (equivalently $\mathbf{s} = 1 \in \pi_1 N$), we have split extensions

$$\pi_{n+1}\mathbb{S}(N) \rightarrowtail \pi_n(\operatorname{Imm}(\mathbb{S}^1, N); 1) \rightleftarrows \pi_n\mathbb{S}(N).$$

The splitting comes from the fact that $\mathsf{ev}_e$ has a section, sending $x \in \mathbb{S}(N)$ to the constant loop $\mathsf{const}_x$. Note that this section is basepoint preserving if and only if we choose a constant loop for a basepoint of $\Lambda\mathbb{S}(N)$ (which we can do in this case since $\mathsf{s}$ is nullhomotopic). △

4.2. **Reducing circles to arcs.** We next give a fibration for embedded circles analogous to (24).

**Proposition 4.4.** *Taking the unit derivative at $e \in \mathbb{S}^1$ gives a fibration sequence*

$$\operatorname{Emb}_\partial(\mathbb{D}^1, N \setminus D^d) \xrightarrow{-\cdot\,\overline{e}} \operatorname{Emb}(\mathbb{S}^1, N) \xrightarrow{\mathsf{d}_e} \mathbb{S}(N),$$

*where the first map glues together along the boundary the neat arc $\overline{e}\colon [e, e+\varepsilon] \hookrightarrow D^d \subseteq N$ with the given neat arc $K\colon \mathbb{D}^1 = \mathbb{S}^1 \setminus [e, e+\varepsilon] \hookrightarrow N \setminus D^d$.*

The boundary condition for $K \in \operatorname{Emb}_\partial(\mathbb{D}^1, N \setminus D^d)$ is given by the derivative of $\overline{e}$ at the boundary, so that gluing them gives a smooth embedding $K \cdot \overline{e}\colon \mathbb{S}^1 \hookrightarrow N$.

*Proof.* The restriction map $r\colon \operatorname{Emb}(\mathbb{S}^1, N) \to \operatorname{Emb}([e, e+\varepsilon], N)$ is a fibration by the Cerf–Palais theorem [Cer61; Pal60]. For the base space of this fibration we have weak homotopy equivalences $\mathsf{d}_e\colon \operatorname{Emb}([e, e+\varepsilon], N) \simeq \operatorname{Imm}([e, e+\varepsilon], N) \simeq \mathbb{S}(N)$, since the shrinking from the argument in the proof of Proposition 4.1 can be done through embeddings.

Finally, for the fibre $r^{-1}(\overline{e})$ over the basepoint $\overline{e}\colon [e, e+\varepsilon] \hookrightarrow N$, we claim that there is an equivalence $r^{-1}(\overline{e}) \simeq \operatorname{Emb}_\partial(\mathbb{D}^1, N \setminus D^d)$. We claim that there is a retraction of $r^{-1}(\overline{e})$ onto its subspace consisting of those $K\colon \mathbb{S}^1 \hookrightarrow N$ which intersect an open tubular neighbourhood $\nu_{\overline{e}} \coloneqq \operatorname{im}\overline{e} \times \mathbb{D}^{d-1}_\epsilon \cong \mathbb{D}^d \subseteq N$ *precisely* in $K([e, e+\varepsilon]) = \operatorname{im}\overline{e}$, since then we will have

$$r^{-1}(\overline{e}) \simeq \operatorname{Emb}_\partial(\mathbb{S}^1 \setminus [e, e+\varepsilon], N \setminus \nu_{\overline{e}}) \cong \operatorname{Emb}_\partial(\mathbb{D}^1, N \setminus D^d).$$

The retraction can be constructed by integrating a vector field defined on each punctured normal disk $\overline{e}(p) \times (\mathbb{D}^{d-1}_\epsilon \setminus \{0\})$ for $p \in [e, e+\varepsilon]$ by pointing radially outwards. This gives a homotopy from the identity map on $\nu_{\overline{e}} \setminus \operatorname{im}\overline{e}$ to a smooth self-map $\varphi$ with the image $\varphi(\nu_{\overline{e}} \setminus \operatorname{im}\overline{e}) \subseteq \operatorname{im}\overline{e} \times (\mathbb{D}^{d-1}_\epsilon \setminus \mathbb{D}^{d-1}_{\epsilon/3})$.

Since $K\colon \mathbb{S}^1 \hookrightarrow N$ belongs to $r^{-1}(\overline{e})$ if it agrees with $\overline{e}$ on $[e, e+\varepsilon]$, does not intersect $\overline{e}$, the composite $\varphi \circ K$ is well defined and belongs to the desired subspace. $\square$

Let $\mathsf{s}\colon \mathbb{S}^1 \hookrightarrow N$ be the basepoint in $\mathrm{Emb}(\mathbb{S}^1, N)$, so that $\mathsf{u} := \mathsf{s}|_{\mathbb{S}^1\setminus[e,e+\varepsilon]}$ is the basepoint in the fibre $\mathrm{Emb}_\partial(\mathbb{D}^1, N\setminus D^d)$, and $\mathsf{u}\cdot\overline{e} = \mathsf{s}$.

Using the inclusion $\mathsf{i}_N\colon \mathrm{Emb}(\mathbb{S}^1, N) \hookrightarrow \mathrm{Imm}(\mathbb{S}^1, N)$ we combine the last fibration sequence with those from (24), to obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Emb}_\partial(\mathbb{D}^1, N\setminus D^d) \arrow[r, hook, "-\cdot\overline{e}"] \arrow[d, "\mathsf{j}_N\mathsf{d}_\mathsf{u}"'] & \mathrm{Emb}(\mathbb{S}^1, N) \arrow[r, "\mathsf{d}_e"] \arrow[d, "\mathsf{d}\mathsf{i}_N"'] & \mathbb{S}(N) \arrow[d, equal] \\
\Omega\mathbb{S}(N) \arrow[r, "-\cdot(\mathsf{ds})"] & \Lambda\mathbb{S}(N) \arrow[r, "\mathsf{ev}_e"] & \mathbb{S}(N)
\end{tikzcd} \tag{25}$$

Let us explain the label of the left vertical map. It is obtained by precomposing $(\mathsf{d}-)\cdot(\mathsf{ds})^{-1}$ from (24) with $-\cdot\overline{e}$. Since $\overline{e} = \mathsf{u}^{-1}\cdot\mathsf{s}$, this takes $K \in \mathrm{Emb}_\partial(\mathbb{D}^1, N\setminus D^d)$ to

$$\begin{aligned}
\left((\mathsf{d}-)\cdot(\mathsf{ds})^{-1}\right)(K\cdot\overline{e}) &= \mathsf{d}(K\cdot\overline{e})\cdot(\mathsf{ds})^{-1}\\
&= (\mathsf{d}K\cdot(\mathsf{du})^{-1}\cdot\mathsf{ds})\cdot(\mathsf{ds})^{-1}\\
&= \mathsf{d}K\cdot(\mathsf{du})^{-1}.
\end{aligned}$$

Therefore, this is the composite of the map $\mathsf{d}_\mathsf{u}\colon \mathrm{Imm}_\partial(\mathbb{D}^1, N\setminus D^d) \to \Omega\mathbb{S}(N\setminus D^d)$, defined by $\mathsf{d}_\mathsf{u}(K) = (\mathsf{d}K)\cdot(\mathsf{du})^{-1}$, and the map $\mathsf{j}_N\colon \Omega\mathbb{S}(N\setminus D^d) \hookrightarrow \Omega\mathbb{S}(N)$ induced by the inclusion $N\setminus D^d \subseteq N$.

From the vertical and horizontal long exact sequences of homotopy groups, we see that $-\cdot\overline{e}$ induces a weak equivalence $\mathrm{hofib}_{\mathsf{const}_*}(\mathsf{j}_N\mathsf{d}_\mathsf{u}) \simeq \mathrm{hofib}_{\mathsf{ds}}(\mathsf{d}\mathsf{i}_N)$, for the basepoint $* = \overline{e}(e) \in N$.

Moreover, by Corollary 4.2 the following commutative diagram has exact rows and columns.

$$\begin{tikzcd}
\pi_{d-1}(\mathbb{S}N) \arrow[r, "a\mapsto a-\mathsf{ds}\,a"] \arrow[d] & \pi_{d-1}(\mathbb{S}N) \arrow[r] \arrow[d, "\delta_{\mathsf{j}_N\mathsf{d}_\mathsf{u}}"] & \pi_{d-2}\Lambda\mathbb{S}(N) \arrow[r, two heads] \arrow[d, "\delta_{\mathsf{di}_N}"] & \left\{b\in\pi_{d-2}N \mid b = \mathsf{s}\,b\right\} \arrow[d] \\
0 \arrow[r] \arrow[d] & \pi_{d-3}\,\mathrm{hofib}_*(\mathsf{j}_N\mathsf{d}_\mathsf{u}) \arrow[r, tail, two heads, "\cong"] \arrow[d] & \pi_{d-3}\,\mathrm{hofib}_{\mathsf{ds}}(\mathsf{di}_N) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\pi_{d-2}N \arrow[r, "\delta_{\mathsf{d}_e}"] \arrow[d, equal] & \pi_{d-3}(\mathrm{Emb}_\partial(\mathbb{D}^1, N\setminus D^d), \mathsf{u}) \arrow[r, "\pi_{d-3}(-\cdot\overline{e})"] \arrow[d, "\pi_{d-3}(\mathsf{j}_N\mathsf{d}_\mathsf{u},\mathsf{u})"] & \pi_{d-3}(\mathrm{Emb}(\mathbb{S}^1, N), \mathsf{s}) \arrow[r, "\pi_{d-3}\mathsf{d}_e"] \arrow[d, "\pi_{d-3}(\mathsf{di}_N,\mathsf{s})"] & \pi_{d-3}N \arrow[d, equal] \\
\pi_{d-2}N \arrow[r, "b\mapsto b-\mathsf{s}\,b"] & \pi_{d-2}N \arrow[r, "\pi_{d-3}(-\cdot\mathsf{ds})"] & \pi_{d-3}(\Lambda\mathbb{S}(N), \mathsf{ds}) \arrow[r] & \pi_{d-3}N.
\end{tikzcd} \tag{26}$$

The desired group $\ker\pi_{d-3}(\mathsf{di}_N, \mathsf{s})$ is the cokernel of the connecting map $\delta_{\mathsf{di}_N}$. We apply the snake (aka kernel–cokernel sequence) lemma to the top two rows with the leftmost top term omitted.

**Theorem 4.5.** *There is an exact sequence of groups*

$$\ker(\delta_{\mathsf{di}_N}) \longrightarrow \left\{b\in\pi_{d-2}N \mid b = \mathsf{s}\,b\right\} \xrightarrow{\delta^{whisk}_\mathsf{s}} \ker\pi_{d-3}(\mathsf{j}_N\mathsf{d}_\mathsf{u}, \mathsf{u}) \twoheadrightarrow \ker\pi_{d-3}(\mathsf{di}_N, \mathsf{s}).$$

*where $\delta^{whisk}_\mathsf{s}$ is a "parametrized change of the whisker": for $d = 3$ it sends $b = \mathsf{s}b\mathsf{s}^{-1} \in \pi_1 N$ to an embedded concatenation $b\cdot\mathsf{s}\cdot b^{-1}$, while for $d \geq 4$ it is a family version of this.*

*Moreover, if $\mathsf{s}$ is nullhomotopic then $\delta^{whisk}_\mathsf{s}$ is trivial.*

*Proof.* It remains to describe the connecting map $\delta^{whisk}_\mathsf{s}$ in the kernel–cokernel sequence, by definition given as a restriction of $\delta_{\mathsf{d}_e}$. This in turn has a description similar to the connecting map for $\mathsf{ev}_e$ given in the proof of Proposition 4.1. Namely, for $\gamma \in \Omega\mathbb{S}(N)$ we let $\delta_{\mathsf{d}_e}(\gamma) := B_0|_{\mathbb{S}^1\setminus[e,e+\varepsilon]}$ where we lift the loop $\gamma$ to a path $B_t \in \mathrm{Emb}(\mathbb{S}^1, N)$, i.e. an isotopy of "non-based" embedded circles, such that $B_1 = \mathsf{s}$ and for every $t \in [0,1]$ the unit derivative at $e$ is equal to $\mathsf{d}_e(B_t) = \beta_t$. Note that by the commutativity of (25) each $B_t \in \mathrm{Emb}(\mathbb{S}^1, N)$ is homotopic to the loop $\gamma|_{[t,1]} \cdot \mathsf{ds} \cdot \gamma|^{-1}_{[t,1]} \in \Lambda\mathbb{S}(N)$, which was used in the definition of $\delta_{\mathsf{ev}_e}$.

Thus, to obtain $\delta^{whisk}_{\mathsf{s}}(b)$ we represent $b \in \pi_{d-2}N$ by a map $\beta\colon \mathbb{I}^{d-3} \times \mathbb{I} \to \mathbb{S}(N)$, lift it to an isotopy $B\colon \mathbb{I}^{d-3} \times \mathbb{I} \to \mathrm{Emb}(\mathbb{S}^1, N)$ (i.e. $\mathsf{d}_e \circ B = \beta$) with $B(\vec{t}, t) = \mathsf{s}$ for $\vec{t} \in \partial\mathbb{I}^{d-3}$ or $t = 1$, and let

$$\delta^{whisk}_{\mathsf{s}}(b) \coloneqq [B(-,0)|_{\mathbb{S}^1 \setminus [e,e+\varepsilon]} \colon (\mathbb{I}^{d-3}, \partial\mathbb{I}^{d-3}) \to (\mathrm{Emb}_\partial(\mathbb{D}^1, N \setminus D^d), \mathsf{u})].$$

Then each $B(\vec{t}, t)$ is homotopic to $\beta_{\vec{t},\geq t} \cdot \mathsf{s} \cdot \beta^{-1}_{\vec{t},\geq t}$, so the family $B(-,0)$ is an embedded version of the map $\beta_{-,\geq 0} \cdot \mathsf{s} \cdot \beta^{-1}_{-,\geq 0} \colon \mathbb{S}^{d-3} \to \Omega\mathbb{S}(N)$, and can thus be viewed as a "parametrized whisker change". Moreover, since we assume $b - \mathbf{s}\, b = 0$ it follows that $B(-,0)$ is based homotopic to $\mathsf{const}_{\mathsf{s}}$, so that $\delta^{whisk}_{\mathsf{s}}(b)$ is indeed in the kernel of $\pi_{d-3}(\mathsf{j}_N \mathsf{d}_{\mathsf{u}}, \mathsf{u})$.

In particular, for $d = 3$ this means that we simply isotope $\mathsf{s}$ by dragging its part $\overline{e} = \mathsf{s}|_{[e,e+\varepsilon]}$ around a loop $\beta \in \Omega N$ representing $b \in \pi_1 N$, until this part comes back to its initial position (we will have to slightly push away the rest of $\mathsf{s}$). The resulting embedded circle $B(0)$ is clearly freely isotopic to $\mathsf{s}$, and based homotopic to $\beta \cdot \mathsf{s} \cdot \beta^{-1}$ which is based homotopic to $\mathsf{s}$. Thus, the restriction of $B(0)$ to $\mathbb{S}^1 \setminus [e, e+\varepsilon]$ is an arc homotopic to $\mathsf{u}$ rel. endpoints, but possibly not isotopic to it. Note that for $\mathbf{s} = 1$ every $\delta^{whisk}_{\mathsf{s}}(b)$ is clearly isotopic to $\mathsf{u}$.

In fact, for nullhomotopic $\mathsf{s}$ and any $d \geq 3$ the map $\pi_n \mathsf{d}_e$ has a section for any $n \geq 0$, implying that $\delta_{\mathsf{d}_e} = 0$, so $\delta^{whisk}_{\mathsf{s}}$ is trivial in this case. Namely, any family $\mathbb{S}^n \to \mathbb{S}(N)$ lifts by the parametrised ambient isotopy extension theorem to a family $\mathbb{I}^n \to \mathrm{Emb}(\mathbb{S}^1, N)$, which is constantly $\mathsf{s}$ on the boundary except possibly on $\{1\} \times \mathbb{I}^{n-1}$. However, we can choose the ambient isotopy so that it encompasses a ball containing $\mathsf{s}$. Thus, the rest of $\mathsf{s}$ is also being dragged along (not only $\mathsf{s}([e, e+\varepsilon])$) in the isotopy, giving a desired lift $\mathbb{S}^k \to \mathrm{Emb}(\mathbb{S}^1, N)$. □

### 4.3. Computing the kernel.

It remains to prove the following statement from Theorem C.

**Theorem 4.6.** *For $d \geq 4$ the Dax invariant for arcs induces an isomorphism from $\ker \pi_{d-3}(\mathsf{i}_N, \mathsf{s})$ to the abelian group with generators $g \in \pi_1 N \setminus 1$ and relations: $\mathsf{Dax}(\delta^{whisk}_{\mathsf{s}}(c)) = 0$ for all $c \in \pi_{d-2}N$ such that $c = \mathbf{s}\, c$, and $\mathsf{dax}_{\mathsf{u}}(a) = 0$ for all $a \in \pi_{d-1}(N \setminus D^d)$.*

We will use that $\ker(\pi_{d-3}\mathsf{i}_N, \mathsf{s}) \cong \ker(\pi_{d-3}\mathsf{d}\mathsf{i}_N, \mathsf{s})$ (as $\mathsf{d}$ is a homotopy equivalence by Proposition 4.1) and Theorem 4.5: we will first compute $\ker(\pi_{d-3}\mathsf{j}_N \mathsf{d}_{\mathsf{u}}, \mathsf{u})$ and then the image of $\delta^{whisk}_{\mathsf{s}}$.

**Lemma 4.7.** *The subgroups $\ker(\pi_{d-3}\mathsf{j}_N \mathsf{d}_{\mathsf{u}})$ and $\ker(\pi_{d-3} p_{\mathsf{u}})$ of $\pi_{d-3}\,\mathrm{Emb}_\partial(\mathbb{D}^1, N \setminus D^d)$ agree.*

*Proof.* First recall that $\mathsf{d}_{\mathsf{u}} \colon \mathrm{Emb}_\partial(\mathbb{D}^1, N \setminus D^d) \to \Omega\mathbb{S}(N \setminus D^d)$, is given by $K \mapsto \mathsf{d}K \cdot (\mathsf{du})^{-1}$, while $p_{\mathsf{u}} \colon \mathrm{Emb}_\partial(\mathbb{D}^1, N \setminus D^d) \to \Omega(N \setminus D^d)$ is given by $K \mapsto K \cdot \mathsf{u}^{-1}$. Thus, these maps agree on $\pi_{d-3}$, since $\pi_{d-2}\mathbb{S}(N \setminus D^d) = \pi_{d-2}(N \setminus D^d)$. The result then follows from $\pi_{d-3}(\mathsf{j}_N \mathsf{d}_{\mathsf{u}}) = \pi_{d-3}(\mathsf{j}_N) \circ \pi_{d-3}(\mathsf{d}_{\mathsf{u}})$, and the fact that the map $\pi_{d-3}(\mathsf{j}_N) \colon \pi_{d-2}(N \setminus D^d) \to \pi_{d-2}(N)$ is an isomorphism by the first part of the following standard result (see for example [Hat02, Lem.4.38]).

**Lemma 4.8.** *The natural inclusion induces isomorphisms $\pi_n(N \setminus D^d) \to \pi_n N$ for all $n \leq d-2$. Moreover, $\pi_d(N, N \setminus D^d) \cong \mathbb{Z}[\pi_1 N]\{\mathbf{\Phi}\}$, generated by the homotopy class of the attaching map $\Phi \colon (D^d, \mathbb{S}^{d-1}) \to (N, N \setminus D^d)$, so there is an exact sequence*

$$\pi_d(N \setminus D^d) \longrightarrow \pi_d N \longrightarrow \mathbb{Z}[\pi_1 N]\{\mathbf{\Phi}\} \longrightarrow \pi_{d-1}(N \setminus D^d) \twoheadrightarrow \pi_{d-1} N. \qquad \square$$

Note that Lemma 4.7 immediately implies Corollaries (i) and (ii) from the introduction.

As a side remark, in most cases $\mathbb{Z}[\pi_1 N]\{\mathbf{\Phi}\} \to \pi_{d-1}(N \setminus D^d)$ is injective, thanks to the following.

**Lemma 4.9.** *The map $\pi_d(N \setminus D^d) \to \pi_d N$ is not surjective if and only if the universal cover $\widetilde{N}$ is a rational homology sphere. Moreover, if for some $g \in \pi_1 N$ the class $g\mathbf{\Phi}$ is trivial in $\pi_{d-1}(N \setminus D^d)$, then $N$ is simply connected.*

*Proof.* We first claim that the image of $\pi_d(N\setminus D^d)\to\pi_d N$ precisely consists of degree 0 maps. Namely, the classes in $\pi_d N$ that come from $\pi_d(N\setminus D^d)$ are represented by nonsurjective maps, which have degree 0. Conversely, any degree 0 map $f\colon\mathbb{S}^d\to N$ is homotopic to a map missing a point in $N$, so after a homotopy we can assume it misses a neighbourhood $D^d$ of the point $\mathsf{s}(e)$, and hence lifts to a map $\mathbb{S}^d\to N\setminus D^d$.

Next, we show that there exists a class in $\pi_d N$ of nontrivial degree if and only if the universal cover $\widetilde{N}$ is a rational homology sphere. Namely, for $f\colon\mathbb{S}^d\to N$ and $\alpha\in H^k(N;\mathbb{Z})$ with $k\neq 0,d$ we have $f^*\alpha=0$, so $0=f_*([\mathbb{S}^d]\cap f^*\alpha)=n([N]\cap\alpha)$, where $n=\deg(f)\in\mathbb{Z}$ and $[N]\cap -$ is the Poincaré isomorphism. Thus, $n\neq 0$ implies that $N$ is a rational homology sphere, and so is $\widetilde{N}$, by the same argument applied to a lift $\widetilde{f}\colon\mathbb{S}^d\to\widetilde{N}$, which also must have nontrivial degree (equal to $n/|\pi_1N|$, so $\pi_1N$ is finite and actually trivial for even $d$, as seen using the Euler characteristic). Conversely, if $\widetilde{N}$ is a rational homology sphere then $\deg\colon\pi_n\widetilde{N}\otimes\mathbb{Q}\to H_d(\widetilde{N};\mathbb{Q})\cong\mathbb{Q}$ is an isomorphism by the rational Hurewicz theorem, so there exists $\widetilde{f}\colon\mathbb{S}^d\to\widetilde{N}$ whose degree $n$ is a rational generator. Then $f\colon\mathbb{S}^d\to\widetilde{N}\to N$ has degree $n|\pi_1X|\neq 0$.

The relative Hurewicz map takes $g\boldsymbol{\Phi}$ to $1\in\mathbb{Z}\cong H_d(N,N\setminus D^d)$, so if $f\in\pi_dN$ maps to $g\boldsymbol{\Phi}$, then $\deg(f)=1$. But since $|\pi_1N|$ has to divide $\deg(f)$, we get $|\pi_1N|=1$, proving the last claim. □

Combining Theorem 1.1, which computes $\ker(\pi_{d-3}p_{\mathsf{u}},\mathsf{u})\subseteq\pi_{d-3}\operatorname{Emb}_\partial(\mathbb{D}^1,N\setminus D^d)$ using the Dax invariant, with our kernel-cokernel sequence and Lemma 4.7, for $d\geq 4$ we obtain:

$$\begin{tikzcd}
\{b\in\pi_{d-2}N\mid b=\mathsf{s}\,b\} \arrow[r, "\delta_{\mathsf{s}}^{whisk}"] & \ker(\pi_{d-3}p_{\mathsf{u}},\mathsf{u}) \arrow[r, two heads] \arrow[d, two heads, shift right, "\mathsf{Dax}"'] \arrow[d, phantom, "\cong"] & \ker(\pi_{d-3}\mathsf{di}_N,\mathsf{s}) \\
& \mathbb{Z}[\pi_1(N\setminus D^d)\setminus 1]\big/\mathsf{dax}_{\mathsf{u}}(\pi_{d-1}(N\setminus D^d)) \arrow[u, hook, shift right, "\partial\mathfrak{r}"'] &
\end{tikzcd}$$

Thus, $\ker(\pi_{d-3}\mathsf{di}_N,\mathsf{s})$ is isomorphic to the quotient of $\ker(\pi_{d-3}p_{\mathsf{u}},\mathsf{u})$ by the image of $\delta_{\mathsf{s}}^{whisk}$, which is equal to the quotient of $\mathbb{Z}[\pi_1(N\setminus D^d)\setminus 1]$ by the subgroup of relations

$$rel_{\mathsf{s}}:=\mathsf{dax}_{\mathsf{u}}(\pi_{d-1}(N\setminus D^d))\oplus\{\mathsf{Dax}(\delta_{\mathsf{s}}^{whisk}(b))\mid b=\mathsf{s}\,b\in\pi_{d-2}N\}.$$

This finishes the proof of Theorem 4.6. □

4.4. **Circles in dimension three.** The discussion in this section so far gives us also the diagram

$$\begin{tikzcd}
\{b\in\pi_1N\mid b=\mathsf{s}\,b\} \arrow[r, "\delta_{\mathsf{s}}^{whisk}"] & \mathbb{K}(N\setminus\mathbb{D}^3,\mathbf{u}) \arrow[r, two heads] \arrow[d, two heads, "\mathsf{Dax}_{\mathsf{u}}"] & \mathbb{K}(N,\mathbf{s}) \\
& \mathbb{Z}[\pi_1(N\setminus D^d)\setminus 1]\big/\mathsf{dax}_{\mathsf{u}}(\pi_2(N\setminus D^d)) &
\end{tikzcd} \tag{27}$$

where $\mathbb{K}(N\setminus D^3,\mathbf{u})$ and $\mathbb{K}(N,\mathbf{s})$ are just sets, of respectively arcs homotopic to $\mathsf{u}$ and circles homotopic to $\mathsf{s}$, and the other two terms are groups.

Since in this dimension $\mathsf{s}\,b:=\mathsf{s}\cdot b\cdot\mathsf{s}^{-1}$, the leftmost term is the centralizer $\zeta(\mathbf{s})$ of $\mathbf{s}\in\pi_1N$. It acts on $\mathbb{K}(N\setminus\mathbb{D}^3,\mathbf{u})$ by sending $K\colon\mathbb{D}^1\hookrightarrow N\setminus\mathbb{D}^3$ to the arc $\delta_K^{whisk}(b)$ which is an embedded change of $K$ by the whisker $b$. Similarly as above, $\mathsf{Dax}_{\mathsf{u}}(\delta_K^{whisk}(b))$ counts double point loops in a homotopy from this arc to $\mathsf{u}$, which exists since $b\mathsf{s}b^{-1}$ is homotopic to $\mathsf{s}$.

Moreover, recall from Corollary 2.13 that we computed $\mathsf{dax}_{\mathsf{u}}(\pi_2(N\setminus D^d))$ in terms of the set $\mathcal{S}(N\setminus D^d)$ of generators of the $\mathbb{Z}[\pi_1N]$-module $\pi_2(N\setminus D^d)$, and that in Corollary 1.4 we saw that the sphere $\partial\mathbb{D}^3$ gives $\overline{g}-g\overline{\mathsf{s}}$. We also observe that $\lambda\!\!\!^{-}(ga,\mathbf{u})=\lambda\!\!\!^{-}(ga,\mathbf{s})$.

Recall that Theorem D asserts that there is a well-defined invariant $\mathsf{Dax}_{\mathsf{s}}$ from the set $\mathbb{K}(N,\mathbf{s})$ to the set of equivalence classes of the set-theoretic action of $b \in \zeta(\mathbf{s})$ via $b \star r = b \cdot r \cdot b^{-1} + \mathsf{Dax}(\delta_{\mathsf{s}}^{whisk}(b))$ on the group

$$\mathbb{Z}[\pi_1 N \setminus 1] \Big/ \left\langle \overline{g} - g\overline{\mathbf{s}},\ \lambda(ga,\mathbf{s}) - \lambda(ga,g) + \overline{\lambda(ga,g)} \mid g \in \pi_1 N,\ a \in \mathcal{S}(N \setminus \mathbb{D}^3) \right\rangle.$$

*Proof of Theorem D.* It remains to check that two different lifts $K, K' \in \mathbb{K}(N \setminus \mathbb{D}^3, \mathbf{s})$ of a knot $L \in \mathbb{K}(N,\mathbf{s})$ have the same Dax invariant modulo the given action. By the exactness of the sequence (27) we have $K' = \delta_K^{whisk}(b)$ for some $b = \mathbf{s}b \in \pi_1 N$. A homotopy from $\delta_K^{whisk}(b)$ to $\mathsf{u}$ can be written as a homotopy $h_1$ from $\delta_K^{whisk}(b)$ to $\delta_{\mathsf{u}}^{whisk}(b)$, where we use any homotopy from $K$ to $\mathsf{u}$, then followed by a homotopy $h_2$ from $\delta_{\mathsf{u}}^{whisk}(b)$ to $\mathsf{u}$. Therefore, using the additivity of the Dax invariant (see the proof of Lemma 3.5) we have

$$\mathsf{Dax}_{\mathsf{u}}(K') = \mathsf{Dax}_{\mathsf{u}}(\delta_K^{whisk}(b)) = \mathsf{Dax}(h_1) + \mathsf{Dax}(h_2) = b \cdot \mathsf{Dax}_{\mathsf{u}}(K) \cdot b^{-1} + \mathsf{Dax}_{\mathsf{u}}(\delta_{\mathsf{s}}^{whisk}(b)).$$

Therefore, $\mathsf{Dax}_{\mathsf{u}}(K)$ and $\mathsf{Dax}_{\mathsf{u}}(K')$ are in the same orbit of the action of $\zeta(\mathbf{s})$, as desired. □

**Remark 4.10.** In Theorem 2.14 we showed that $\mathsf{Dax}_{\mathsf{u}}$ is the universal type $\leq 1$ Vassiliev invariant of $\mathbb{K}(X,\mathbf{u})$, by writing any $v\colon \mathbb{K}(X,\mathbf{u}) \to A$ as $v(K) - v(\mathsf{u}) = w_v \circ \mathsf{Dax}(K)$. We had that $w_v$ is well defined since it vanished on the image of $\mathsf{dax}_{\mathsf{u}}(\pi_2 X)$.

The proof that for any $v\colon \mathbb{K}(N,\mathbf{s}) \to A$ we have $v(K) - v(\mathsf{s}) = w_v \circ \mathsf{Dax}_{\mathsf{s}}(K)$ follows the same steps. Since $w_v$ vanishes on $\mathsf{dax}_{\mathsf{u}}(\pi_2(N \setminus \mathbb{D}^d))$ by the same argument as for arcs, it remains to show $w_v$ is well defined modulo the action of $\zeta(\mathbf{s})$. Observe that for any $r$ we have $r = \mathsf{Dax}_{\mathsf{u}}(K)$ for some $K \in \mathbb{K}(N \setminus D^d, \mathbf{u})$, and we saw that $brb^{-1} + \mathsf{Dax}_{\mathsf{u}}(\delta_{\mathsf{s}}^{whisk}(b)$ is then equal to $\mathsf{Dax}_{\mathsf{u}}(K')$, for some arc $K'$ that closes up to the same knotted circle as $K$. Since $v$ is an invariant of circles, we conclude that $w_v(r) = w_v(\mathsf{Dax}_{\mathsf{s}}(K)) = w_v(\mathsf{Dax}_{\mathsf{s}}(K')) = w_v(brb^{-1} + \mathsf{Dax}_{\mathsf{u}}(\delta_{\mathsf{s}}^{whisk}(b)))$. △

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ETH Zürich, Department of Mathematics, Rämistrasse 101, 8092 Zürich, Switzerland

*Email address*: danica.kosanovic@math.ethz.ch