HIGHER HOMOTOPY GROUPS IN LOW DIMENSIONAL TOPOLOGY

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Talk based on: `https://arxiv.org/abs/2105.13032`

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Introduction

Spaces of embeddings

- Fix $1 \leq k \leq d$. Let M be a compact smooth d-dimensional manifold and $\sigma \colon S^{k-1} \hookrightarrow \partial M$ a smooth embedding. Recall that this means that σ is *injective*, and at any $x \in S^{k-1}$ the derivative $d\sigma_x$ is *injective*.

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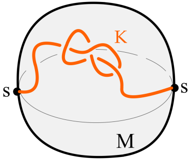

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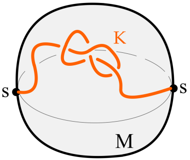

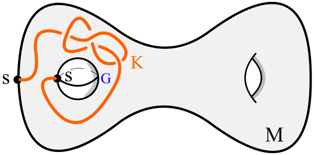

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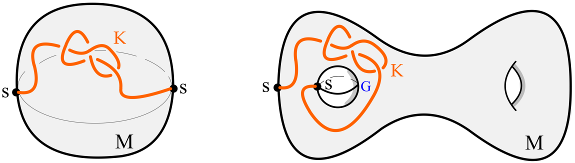

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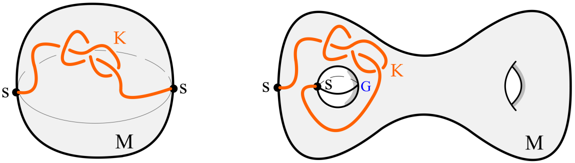

- Setting with a dual: If there exists $\mathbf{G}\colon \mathbb{S}^{d-k} \hookrightarrow \partial M$, such that $\mathbf{G}$ has trivial normal bundle and $\mathbf{G} \pitchfork \mathbf{s} = \{pt\}$. Like in the example on the right! We also assume $\mathrm{Emb}_\partial(\mathbb{D}^k; M)$ is nonempty, and fix a basepoint [a].

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- More recently, intensively studied is the set of 2-knots $_0\,\mathrm{Emb}_{@}(D^2; M)$ for a 4-manifold M. This can be huge – for example, "spinning" a classical knot gives a 2-knot in $_0\,\mathrm{Emb}_{@}(S^2; R^4) = {}_0\,\mathrm{Emb}_{@}(D^2; D^4)$.

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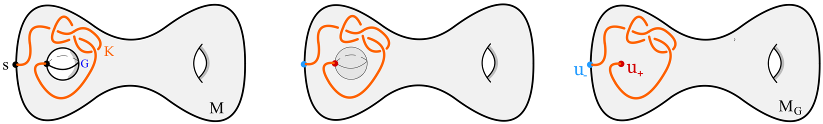
- A sample open problem: Is there a nontrivial 2-knot $S^2 \;,\!\; R^4$ which is trivial locally flatly?
- In this talk: we shall compute $_0\,\mathrm{Emb}_{@}(D^2; M)$ in the setting with a dual!
- Although usually only the sets of components are considered, we will see that higher homotopy groups of embedding spaces are also useful.

Space Level Light Bulb Theorem

Theorem [K-Teichner]

In the setting with a dual, if we denote $M_G := M \ [\ {}_G \, h^{d \quad k+1}$, then



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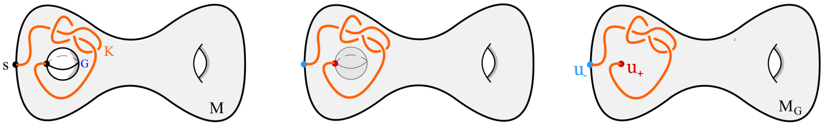

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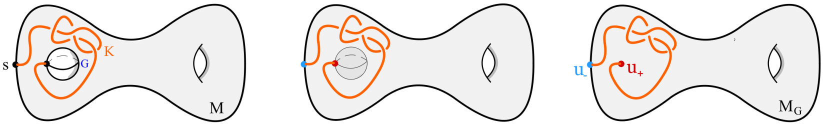
- Here $\mathrm{Emb}''_{@}(\mathrm{D}^{k \quad 1}; M_G)$ with the boundary condition given by $\mathsf{u}_0 := @\mathsf{u}_+$, and denotes the space of loops based at $\mathsf{u}_+ := \mathsf{s} \setminus \mathsf{h}^{\mathsf{d} \quad \mathsf{k}+1}$.

- Supscript ″ means each embedded disk is equipped with a “push-off”(...).



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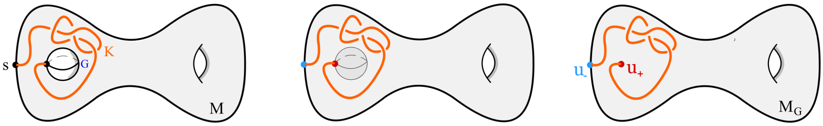

Example: $k = 1; d = 3$

This recovers the classical LBT: isotopy classes of arcs in a 3-manifold M with ends on two components of $@M$, one of which is S^2, are in bijection with $\;_1(M \; [_G \, h^3).$ $\;=)\;$ a knot in the chord for a light bulb can be unknotted!

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$k = 3:$ $\pi_0 \mathrm{Emb}_\partial(D^3; S^1 \times D^3) = \pi_1 \mathrm{Emb}_\partial(D^2; D^4)$, cf. Budney–Gabai.

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$k=3$: $\pi_0\mathrm{Emb}_\partial(\mathrm{D}^3;\mathrm{S}^1 \times \mathrm{D}^3) = \pi_1\mathrm{Emb}_\partial(\mathrm{D}^2;\mathrm{D}^4)$, cf. Budney–Gabai.

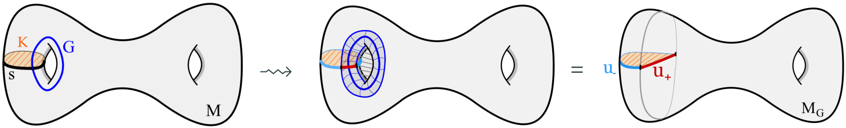
$k=d$: Recovers a theorem (and proof) of Cerf '68:

$$\mathrm{Diff}^+_\partial(\mathrm{D}^d) = \mathrm{Emb}_\partial(\mathrm{D}^d;\mathrm{D}^d) \simeq \Omega\mathrm{Emb}_\partial(\mathrm{D}^{d-1};\mathrm{D}^d).$$

In particular, $\pi_0\mathrm{Diff}^+_\partial(\mathrm{D}^4) = \pi_1(\mathrm{Emb}_\partial(\mathrm{D}^3;\mathrm{D}^4);\iota)$. Open: is this nontrivial?

Cerf’s trick

Cerf's trick: Proof of Space Level LBT



$K\colon D^k \hookrightarrow M$, with $\partial K = s$ $\qquad$ $J\colon D^k \hookrightarrow M_G := M \cup_G h^{d-k+1}$, with $\partial J = u_- \cup u_+$

□

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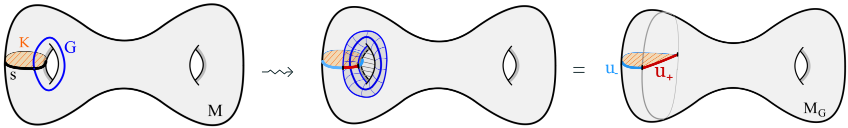

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Can go back by removing a tubular neighbourhood of u_+ in M_G, and can show

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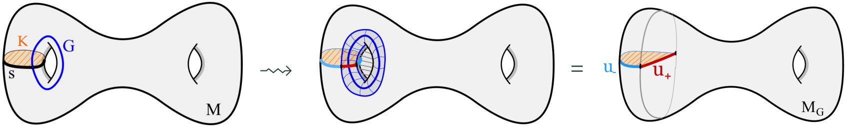

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$$\mathrm{Emb}_\partial^{\varepsilon}(D^k; M) \simeq \mathrm{Emb}_\partial^{\varepsilon}(ᗡ^k; M_G).$$

Now consider the fibration sequence (due to Cerf):

$$\mathrm{Emb}_\partial^{\varepsilon}(ᗡ^k; M_G) \hookrightarrow \mathrm{Emb}_{D}^{\varepsilon}(ᗡ^k; M_G) \xrightarrow{K \mapsto K|_{D^{\varepsilon}_+}} \mathrm{Emb}_\partial^{\varepsilon}(D^{k-1}; M_G)$$

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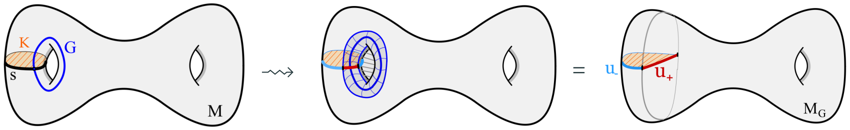

$K\colon D^k\ ,!\ \ M$, with $@K = s$ $\qquad J\colon ɑ^k\ ,!\ \ M_G := M\ [\ _G\, h^d\ \ ^{k+1}$, with $@J = u\ \ [\ u_+$

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The total space is contractible (shrink the half-disk to its u'' -collar), so:

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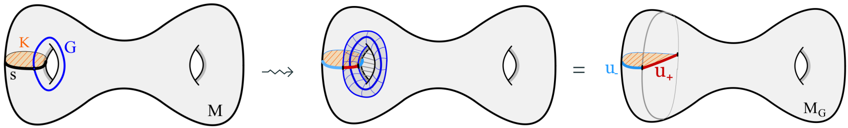

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The total space is contractible (shrink the half-disk to its u_-^ε-collar), so:

$$\mathrm{Emb}^\varepsilon_{\partial}(\mathrm{D}^{k-1}; M_G) \underset{\mathrm{fol}^\varepsilon_a}{\overset{\mathrm{amb}_a}{\rightleftarrows}} \mathrm{Emb}_{\partial^\varepsilon}(\mathbb{D}^k; M_G)$$

where: amb_a is the connecting map (use the family ambient isotopy theorem to extend loops), $\mathrm{fol}^\varepsilon_a(K)$ is the loop of ε-augmented $(k-1)$-disks foliating the sphere $a \cup K$. □

LBT for 2-disks in 4-manifolds

The 4D setting with a dual

Let M be an oriented compact smooth 4-manifold together with

- a knot $\mathsf{s}\colon \mathbb{S}^1 \hookrightarrow \partial M$,
- an embedded sphere $G\colon \mathbb{S}^2 \hookrightarrow \partial M$,

so that s and G intersect transversely and positively in a single point. Recall that we study the set of isotopy classes $\mathrm{Emb}_\partial[\mathbb{D}^2; M] := {}_0\,\mathrm{Emb}_\partial(\mathbb{D}^2; M)$ of neat smooth embeddings $K\colon \mathbb{D}^2 \hookrightarrow M$ which on $\partial\mathbb{D}^2$ agree with s.

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By Space Level LBT we have $\mathrm{Emb}_\partial[\mathbb{D}^2; M] := \pi_1\,\mathrm{Emb}^{\varepsilon}_\partial(\mathbb{D}^1; M \cup_G h^3)$ and we can compute the latter group! Moreover, we can interpret the resulting group structure on the original set, as follows.

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- Let $m = \mathsf{s}(\ i) \in M$ be the basepoint and denote $= {}_1(M; m\ )$,
- Let $\mathbb{Z}[\ ]$ be the group ring, and $\mathbb{Z}[\ r 1]$ the subgroup of $\mathbb{Z}[\ r 1] := \{r = \sum {}_i g_i : g_i \neq 1\}$ of those $\sum {}_i g_i$ that are equal to $\sum {}_i g_i^{\ 1}$,

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- a knot $\mathsf{s}\colon \mathbb{S}^1 \hookrightarrow \partial M$,
- an embedded sphere $G\colon \mathbb{S}^2 \hookrightarrow \partial M$,

so that s and G intersect transversely and positively in a single point. Recall that we study the set of isotopy classes $\mathbf{Emb}_\partial[\mathbb{D}^2; M] := {}_0\,\mathbf{Emb}_\partial(\mathbb{D}^2; M)$ of neat smooth embeddings $K\colon \mathbb{D}^2 \hookrightarrow M$ which on $\partial\mathbb{D}^2$ agree with s.

By Space Level LBT we have $\mathbf{Emb}_\partial[\mathbb{D}^2; M] := {}_1\,\mathbf{Emb}^{''}_\partial(\mathbb{D}^1; M \cup_G h^3)$ and we can compute the latter group! Moreover, we can interpret the resulting group structure on the original set, as follows.

- Let $m = \mathsf{s}(\ i) \in M$ be the basepoint and denote $= {}_1(M; m)$,
- Let $\mathbb{Z}[\ ]$ be the group ring, and $\mathbb{Z}[\ \setminus 1]$ the subgroup of $\mathbb{Z}[\ \setminus 1] := \{r = \sum {}_i g_i : g_i \neq 1\}$ of those $\sum {}_i g_i$ that are equal to $\sum {}_i g_i^{\ 1}$,
- Let $\mathrm{dax}\colon {}_3 M \to \mathbb{Z}[\ \setminus 1]$ be the homomorphism defined in terms of the Dax invariant Dax of the classes of loops of arcs in M_G (...).

Theorem [K-Teichner] There is an exact sequence of sets

$$\frac{\mathbb{Z}[\ \ r1]}{\mathrm{dax}(\ \ _3M)} \underset{\mathrm{Dax}}{\overset{+\ \mathrm{fm}(\ \ )^G}{\rightleftarrows}} \mathrm{Emb}_{@}[\mathrm{D}^2; M] \xrightarrow{j} \mathrm{Map}_{@}[\mathrm{D}^2; M] \xrightarrow{2} \frac{\mathbb{Z}[\ \ r1]}{hr\ \ \ \bar{r}i}$$

In detail:

Theorem [K-Teichner] There is an exact sequence of sets

$$\mathbb{Z}[\ \ r\,1] \quad \text{dax}(\ \ {}_3M) \underset{\text{Dax}}{\overset{+\ \text{fm}(\ )^G}{\rightleftarrows}} \text{Emb}_@[\mathrm{D}^2;M] \xrightarrow{\ j\ } \text{Map}_@[\mathrm{D}^2;M] \xrightarrow{\ \ 2\ } \mathbb{Z}[\ \ r\,1]\ hr \quad \bar{r}i$$

In detail:

- Wall's self-intersection invariant $\ _2$ is surjective;
- a map $f\colon \mathrm{D}^2\ \ !\ \ M$, $@f = \mathsf{s}$, is homotopic to an embedding iff $\ _2(f) = 0$;

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LBT for 2-disks

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$$\frac{\mathbb{Z}[\ \ 1]}{\mathrm{dax}(\ {}_3 M)} \underset{\mathrm{Dax}}{\overset{+\,\mathrm{fm}(\ )^G}{\rightleftarrows}} \mathrm{Emb}_@[\mathbb{D}^2; M] \xrightarrow{j} \mathrm{Map}_@[\mathbb{D}^2; M] \xrightarrow{\ 2\ } \mathbb{Z}[\ \ 1]\ hr \quad \bar{r}i$$

In detail:

- Wall's self-intersection invariant ${}_2$ is surjective;
- a map $f\colon \mathbb{D}^2 \ ! \ M$, $@f = \mathsf{s}$, is homotopic to an embedding iff ${}_2(f) = 0$;
- embeddings homotopic to $K\colon \mathbb{D}^2\ ,!\ M$ are obtained from K by the action $+\,\mathrm{fm}(r)^G$: do finger moves along r, and then Norman tricks;
- $\mathrm{Dax}(\ ; K)\colon j\ {}^1[K]\ !\ \frac{\mathbb{Z}[\ \ 1]}{\mathrm{dax}(\ {}_3 M)}$ is the inverse of this action

( ) the relative Dax invariant, given by a clever count of double point loops in a homotopy to K, detects the action:

$$\mathrm{Dax}(K + \mathrm{fm}(r)^G; K) = [r]:$$

Picture of LBT for 2-disks

Theorem [K-Teichner] There is an exact sequence of groups

$$\frac{\mathbb{Z}[\quad r\,1]}{\mathrm{dax}(\quad {}_3 M)} \underset{\mathrm{Dax}}{\overset{+\ \mathrm{fm}(\ )^G}{\rightleftarrows}} \mathrm{Emb}_{@}[\mathbb{D}^2; M] \xrightarrow{j} \mathrm{Map}_{@}[\mathbb{D}^2; M] \xrightarrow{2} \frac{\mathbb{Z}[\quad r\,1]}{hr \quad \bar{r} i}$$

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- A similar construction by Gabai ('21).

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- A similar construction by Gabai ('21).

- We recover LBT for spheres of Gabai ('20) and Schneiderman–Teichner ('21).

Thank you!