HOMOTOPY GROUPS OF SOME EMBEDDING SPACES

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Based on the joint work with Peter Teichner (MPIM Bonn)
https://arxiv.org/abs/2105.13032

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Motivation

- Consider compact smooth manifolds V and X with nonempty boundary, with $k := \dim V$, and $d := \dim X$ such that $1 \quad k \quad d$.
- **General goal.** Study the homotopy type of the space

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- Recall that a smooth map K is an embedding if it is *injective* and at any $v \in V$ the derivative $dK|_v$ is *injective*, and K is neat if it is transverse to the boundary and $K(V) \cap \partial X = K(\partial V)$.

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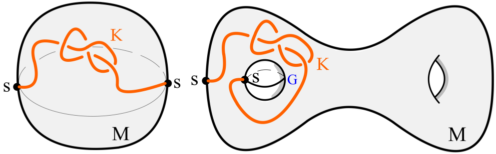

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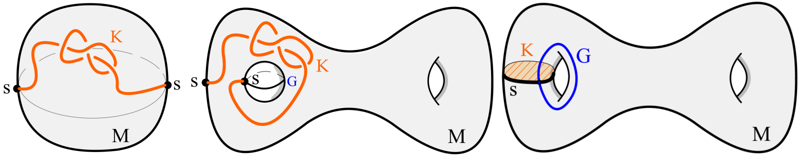

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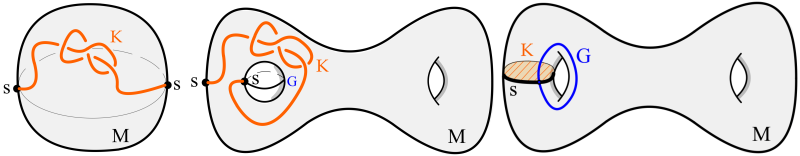

- For $V = \mathbb{D}^k$, the setting with a dual: if there exists $G\colon \mathbb{S}^{d-k} \hookrightarrow \partial X$, such that G has trivial normal bundle and $G \pitchfork \mathbf{s} = \{pt\}$. Like pictures 2 and 3!

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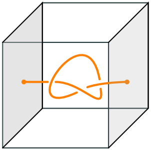

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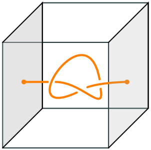

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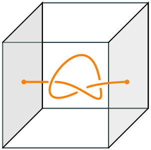

- Recently, intensively studied is the set of (long) 2-knots in a 4-manifold M:

$$\pi_0\,\mathrm{Emb}_\partial(\mathbb{D}^2; M)$$

This can be huge – for example, "spinning" a classical knot gives a 2-knot in $\pi_0\,\mathrm{Emb}_\partial(\mathbb{S}^2; \mathbb{R}^4) = \pi_0\,\mathrm{Emb}_\partial(\mathbb{D}^2; \mathbb{D}^4)$.

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where $X := M \cup_G h^{d-k+1}$.

Recall that setting with a dual means: we have a d-manifold M and embedding $s = \partial U: \mathbb{S}^{k-1} \hookrightarrow \partial M$, such that there exists $G: \mathbb{S}^{d-k} \hookrightarrow \partial M$ with trivial normal bundle and such that $G \pitchfork s = \{pt\}$.

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The main result today, and applications

How to compute homotopy groups?

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where Dax is defined on the image of the realisation map ∂r and is its explicit inverse, and $rel_{1,d} := \emptyset$ and $rel_{\ell,d} := \langle g - (-1)^d \ell g : g \in \pi_1 X\rangle$ if $\ell \geq 2$

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- We make this more explicit, and compute many classes of examples in K' 21.

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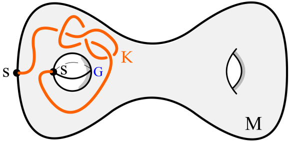

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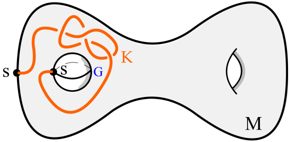

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$\Longrightarrow$ We recover (and generalise) LBT for spheres of Gabai ’20 and Schneiderman–Teichner ’21.

- Moreover, we get an (unexpected) group structure on $\pi_0 \mathrm{Emb}_\partial(\mathrm{D}^2; M)$!

$k = d-1: \ \mathrm{Emb}_\partial(\mathrm{D}^{d-1}; \mathrm{S}^1 \times \mathrm{D}^{d-1}) \simeq \mathrm{Emb}_\partial(\mathrm{D}^{d-2}; \mathrm{D}^d)$

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Theorem (Cerf ’68)

There is a homotopy equivalence $\mathrm{Diff}^+_\partial(\mathrm{D}^d) \simeq \mathrm{Emb}_\partial(\mathrm{D}^{d-1}; \mathrm{D}^d)$. In particular,

$$\pi_0 \mathrm{Diff}^+_\partial(\mathrm{D}^4) = \pi_1(\mathrm{Emb}_\partial(\mathrm{D}^3; \mathrm{D}^4); \mathrm{a}).$$

$k = 2:\ \mathrm{Emb}_{\partial}(\mathrm{D}^2; M) \simeq \mathrm{Emb}^{\varepsilon}_{\partial}(\mathrm{D}^1; X).$

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Open problem

Is $\pi_0\,\mathrm{Diff}^{+}_{\partial}(\mathrm{D}^4)$ trivial? Compute it.

See Budney-Gabai, Gay, Watanabe for some candidate diffeomorphisms.

Metastable homotopy groups

Stable, metastable, meta^{2}stable...(?)

A generic smooth immersion $V^\ell \looparrowright X^d$ has transverse self-intersections only of multiplicity $n \leq \frac{d}{d-\ell}$.

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- *Goodwillie–Klein–Weiss* embedding calculus.
  - Construct a tower of spaces $\mathrm{P}_n(V;X)$, $n \geq 1$, with:
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  - $\mathrm{Emb}(V;X) \to \mathrm{P}_n(V;X)$ is $(nd-(n+1)\ell-(2n-1))$-connected (hard!).
  - Use homotopy theoretic tools to study $\mathrm{P}_n(V;X)$.

About the lowest degree in the metastable range

- Therefore, part 1) in the Main Theorem, which said

  $$p_u\colon \pi_n(\mathrm{Emb}_\partial(\mathrm{D}^\ell; X); u) \cong \pi_n(\mathrm{Imm}_\partial(\mathrm{D}^\ell; X); u) \cong \pi_{n+\ell} X, \qquad \text{for } 0 \leq n \leq d-2\ell-2.$$

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    It turns out this is given as the image of a certain homomorphism $\mathrm{dax}\colon \pi_{d-\ell}X \to \mathbb{Z}[\pi_1X\setminus 1]$.
  - The desired kernel is the cokernel of ∂_{Imm}.

About the lowest degree in the metastable range

Theorem [Dax '72]

There is an isomorphism $_{d \; 2` \; 1}(\mathrm{Imm}(V;X);\mathrm{Emb}(V;X);u) \; = \; {}_0(C_u; \; {}_u)$, the degree 0 normal bordism group of a certain space C_u with a stable normal bundle ${}_u$ over it.

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Theorem [K–Teichner '22]

There is an isomorphism $\mathrm{Dax}\colon \pi_{d-2\ell-1}(\mathrm{Imm}(V,X),\mathrm{Emb}(V,X);u) \to \mathbb{Z}[\pi_1 X]_{rel_{\ell,d}}$ given as follows: represent a relative class by a "perfect" map

$$F\colon (\mathbb{I}^{d-2\ell-1}, \mathbb{I}^{d-2\ell-2}\times\{0\}, \mathbb{I}^{d-2\ell-2}\times\{1\}\cup\partial\mathbb{I}^{d-2\ell-2}\times\mathbb{I}) \to (\mathrm{Imm},\mathrm{Emb},u)$$

i.e. F is smooth and its track

$$\vec{F}\colon \mathbb{I}^{d-2\ell-1}\times V \to \mathbb{I}^{d-2\ell-1}\times X,\quad (\vec{t},v)\mapsto(\vec{t},F(\vec{t},v)),$$

has no triple points and double points $(\vec{t}_i,x_i)\in\mathbb{I}^{d-2\ell-1}\times V$ for $i=1,\dots,r$ are isolated and transverse.

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has no triple points and double points $(\vec{t}_i;x_i)\in\mathbb{I}^{d-2\ell-1}\times V$ for $i=1;\dots;r$ are isolated and transverse. Then $\mathrm{Dax}([F])=\sum_{i=1}^{r}\varepsilon_{(\vec{t}_i;x_i)}g_{(\vec{t}_i;x_i)}$ is the sum of signed double point loops of $\widetilde{F}$.

The realisation map and the Dax invariant

Moreover, the inverse of Dax can be made explicit: for $g \in \pi_1 X \setminus 1$ the relative homotopy class $\partial \mathrm{r}(g)$ is given by

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We can compute this in many classes of examples! See [K ’21].

Thank you!