SELECTED HOMEWORK SOLUTIONS

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HW6.1

Consider the ring of Laurent polynomials $R := \mathbb{Z}[t^{\pm 1}]$, equipped with the involution $\overline{p(t)} = p(t^{-1})$. Let

$$A = \begin{pmatrix} a(t) & \overline{c(t)} \\ c(t) & b(t) \end{pmatrix}$$

be a Hermitian matrix over R.

a) Construct a compact, oriented 4-manifold $M \simeq \mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^2$, whose intersection form λ_M is in some basis given precisely by A.

b) Show that up to units $t^{\pm n}$, $\det(A)$ can be read off from ∂M and that $\pi_1(\partial M) \twoheadrightarrow \mathbb{Z}$.

Solution 6.1. Recall that A Hermitian means that $A = \overline{A^T}$, so we have $a(t) = a(t^{-1})$ and $b(t) = b(t^{-1})$. Therefore, we can write

$$a(t) = \sum_{n \in \mathbb{Z}} k_n t^n = k_0 + \sum_{n \geq 1} k_n (t^n + t^{-n}) \tag{1}$$

for some coefficients $k_n \in \mathbb{Z}$.

a) The goal is to construct the 4-manifold M_L analogously to what we did in the class for a two-component link L (see Class 9), but now with a three-component link instead. We can start with the *unlink* $L = (L_1, L_2, L_3) : \sqcup^3 \mathbb{S}^1 \hookrightarrow \mathbb{S}^3$ bounding disjoint undisks $(f_1, f_2, f_3) : \sqcup^3 \mathbb{D}^2 \hookrightarrow \mathbb{D}^4$ and *modify* L_1 and L_2 in the complement of L_3 using certain finger moves that we determine later. Then we take out a tubular neighbourhood of f_3 and attach two 2-handles to L_1 and L_2 (with framings which we also have to choose[1]), so that we get a 4-manifold:

$$M_L := (\mathbb{D}^4 \setminus \nu f_3) \cup_{L_1^f r} \mathbb{D}^2 \times \mathbb{D}^2 \cup_{L_2^f r} \mathbb{D}^2 \times \mathbb{D}^2$$

In order to determine what modifications to make, let us first see how they will relate with the intersection form of M_L.

[1] See Class 16 for an introduction to Kirby calculus.

Since both 2-handles are attached to loops that are null-homotopic in the complement of the undisk, M_L is homotopy equivalent to the wedge $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^2$ (independently of what modifications we choose). We know that π_2 of this wedge is a free R-module on two generators. Therefore, we can (either by understanding this homotopy equivalence or by knowing a bit of Kirby calculus) calculate:

$$\pi_2(M_L) = \pi_2(\widetilde{M}_L) = H_2(\widetilde{M}_L) \cong R\langle \widetilde{S}_1, \widetilde{S}_2 \rangle$$

where $\pi_1(M_L) \cong \mathbb{Z}$ is generated by the meridian t to the undisk f_3. We can represent the generator S_1 by an immersed sphere built from the immersed disk for L_1 (obtained from the undisk f_1 by the modifications) glued along L_1 to the core of the corresponding 2-handle; for S_2 we do the same using the modified undisk f_2 and capping it off by the core of the other 2-handle. Those two spheres together with some whiskers to the basepoint are denoted $\widetilde{S}_1$ and $\widetilde{S}_2$.
The intersection matrix is in this basis given by $\lambda_{ij} := \lambda_M(\widetilde{S}_i, \widetilde{S}_j)$, for $1 \leq i, j \leq 2$. Thus, it only remains to choose modifications of the unlink, so that starting from (λ_{ij}) we get to the matrix A.
Recall the formula to calculate intersections of a sphere with its parallel push-off (see Class 6):

$$\lambda_M(\widetilde{S}_i, \widetilde{S}_i) = \mu(\widetilde{S}_i) + \overline{\mu(\widetilde{S}_i)} + e(\nu \widetilde{S}_i)$$

So to obtain $\lambda_{11} = a(t)$ of the form as in (1), we will do for each $n \geq 1$ precisely k_n self-finger moves on L_1 in the complement of f_3, following a guiding arc which describes n full twists around f_3. This will contribute the term $k_n t^n$ to μ and $k_n t^{-n}$ to $\overline{\mu}$. Finally, to get the element k_0 we just change the Euler number $e(\nu \widetilde{S}_a)$ - this is easy since we can choose the framing for the attachment of the handle and this is equal to the Euler number of the normal bundle of the sphere S_1.
Everything is absolutely analogous for $\lambda_{22} = b(t)$, by modifying the component L_2. Now for $\lambda_{12} = \overline{\lambda_{21}}$ we do the finger moves between components L_1 and L_2, again using guiding arcs which go necessary number of times around the meridian t of f_3.

b) For the surjection $\pi_1(\partial M) \to \pi_1(M) \cong \mathbb{Z}\langle t \rangle$, observe that the meridian t to f_3 actually lives in the boundary of M.
Recall that the (equivariant) intersection form λ_M can be defined as the intersection form on the universal cover $\widetilde{M}$, or equivalently as a form on $H_2(\widetilde{M}) \cong H_2(M; R)$, the homology of M with local coefficients in R. The homology long exact sequence for $(M, \partial M)$ with R coefficients gives:

$$\begin{array}{ccccccc} H_2(M;R) & \xrightarrow{\iota_*} & H_2(M,\partial M;R) & \xrightarrow{\delta} & H_1(\partial M;R) & \longrightarrow & H_1(M;R) \\ \| & & \downarrow{\scriptstyle j\circ PD}\ \cong & & \| & & \downarrow \cong \\ H_2(M;R) & \xrightarrow{\Phi} & \mathrm{Hom}(H_2(M;R),R) & \xrightarrow{\delta\circ(j\circ PD)^{-1}} & H_1(\partial M;R) & \longrightarrow & 0 \end{array} \tag{2}$$

Here we used the Poincaré duality isomorphism $PD : H_2(M, \partial M; R) \to H^2(M; R)$ and a variant of the universal coefficients theorem[2] for cohomology:

$$0 \to \mathrm{Ext}^1_R(H_1(M;R), R) \to H^2(M;R) \xrightarrow{j} \mathrm{Hom}(H_2(M;R), R) \to 0$$

[2]Note that when R is not a PID the usual universal coefficient theorem for cohomology does not apply. However, there is a universal coefficent *spectral sequence* $E_2^{p,q} = Ext^q_R(H_p(M;R), R) \implies H^*(M;R)$. For $* = 2$ we actually do get a short exact sequence as stated.

where the first term is zero since $H_1(M;R) \cong H_1(\widetilde{M}) \cong 0$, so j is an isomorphism. Note that

$$H_2(M;R) = H_2(\widetilde{M}) \cong R\langle \widetilde{S}_1, \widetilde{S}_2 \rangle$$
$$\mathrm{Hom}(H_2(M;R),R) \cong R\langle \alpha_1, \alpha_2 \rangle$$

where we define α_i as the dual basis: $\alpha_i(\widetilde{S}_j) = \delta_{ij}$.

We now claim that the homomorphism $\Phi = j \circ PD \circ \iota_*$ in the lower row of (2) is given precisely by the intersection matrix $\lambda_M = A$. This will imply[3] that $H_1(\partial M;R)$ is the cokernel of Φ, hence it determines the determinant of A.

Write $\Phi(\widetilde{S}_i) = d_{i1}\alpha_1 + d_{i2}\alpha_2$ for some coefficients $d_{ij} \in R$ and calculate:

$$d_{ij} = \Phi(\widetilde{S}_i)(\widetilde{S}_j) = j \circ PD(\iota_*\widetilde{S}_i)(\widetilde{S}_j) = (\iota_*\widetilde{S}_i) \cdot \widetilde{S}_j = \lambda_{ij}$$

proving the claim (for the last equality, recall the correspondence of intersection and cup products: $PD(\iota_*\widetilde{S}_i) \frown \widetilde{S}_j = (\iota_*\widetilde{S}_i) \cdot \widetilde{S}_j$).

□

[3] Actually, we can say more: the generators of $H_1(\partial M;R)$ are represented by $\delta \circ (j \circ PD)^{-1}(\alpha_i) = \delta(C_i)$ where C_i is the cocore of the handle attached to L_i.