TIGHT BOUNDS FOR DIVISIBLE SUBDIVISIONS

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Abstract. Alon and Krivelevich proved that for every n-vertex subcubic graph H and every integer $q \geq 2$ there exists a (smallest) integer $f = f(H, q)$ such that every K_f-minor contains a subdivision of H in which the length of every subdivision-path is divisible by q. Improving their superexponential bound, we show that $f(H, q) \leq \frac{21}{2}qn + 8n + 14q$, which is optimal up to a constant multiplicative factor.

1. Introduction

Enjoying a long tradition in graph theory, the typical extremal question asks which conditions we can impose on a graph to force its containment of a given subgraph. Famous examples of this class of problems are Turán's theorem [35], describing the smallest average degree that guarantees the existence of a complete subgraph of specified size, and Dirac's Theorem [16], which gives a sharp minimum degree threshold for the existence of a Hamiltonian cycle. In this paper we shall be concerned with the existence of subdivisions of a fixed graph, and well-known results in this direction are due to Bollobás and Thomason [6] and Komlós and Szemerédi [27], who showed that any graph of average degree $\Theta(t^2)$ contains a subdivision of K_t.

While results of this kind are of fundamental importance in extremal and structural graph theory, they share the shortcoming that they require the host graphs to be reasonably dense, and do not, for instance, yield anything for graphs with bounded maximum degree. These very sparse graphs arise naturally in several applications, and it is therefore of interest to study other structural conditions, different from the average or minimum degree, that guarantee the existence of desired subgraphs (necessarily themselves of small maximum degree). We follow this line of research by solving a problem introduced by Alon and Krivelevich [1] regarding the existence of so-called *divisible subdivisions* in graphs containing a large clique minor.

Before stating our main result, let us describe the necessary definitions and background. Throughout this paper, a *K_f-minor* is defined as a graph G whose vertex set is partitioned into f disjoint non-empty sets $X_1, \ldots, X_f$, such that for every $i \in [f]$, the induced subgraph $G[X_i]$ is connected, and for every $i \neq j \in [f]$, there exists at least one edge in G with endpoints in X_i and X_j. The sets $X_1, \ldots, X_f$ are referred to as *supernodes* or *branch sets* of the K_f-minor G.

Given a graph H, a *subdivision* of H is any graph H' obtained from H by replacing its edges with internally vertex-disjoint paths connecting the original endpoints of the edges in H. A vertex in H' corresponding to an original vertex of H is called a *branch vertex* of H', while all remaining vertices are called *subdivision vertices*. The paths in H' replacing the edges of H are called *subdivision paths*. A subdivision H' of H is called *q-divisible* if all its subdivision paths are of length divisible by q.

Subdivisions and minors of graphs with constraints on the lengths of paths have received significant attention in the literature. For example, Alon, Krivelevich and Sudakov [3] showed that every n-vertex graph of average degree εn, for fixed $\varepsilon > 0$, contains a subdivision of K_k in which every subdivision-path has length 2 (thus, a 2-divisible subdivision), where $k = \Omega(\sqrt{n})$. For more results of this nature, we refer the reader to [5, 19, 17, 18, 20, 21, 22, 23, 25, 26, 31, 34].

Another standout result is due to Thomassen [32], who gave general sufficient conditions for finding subdivisions with modular constraints on the lengths of the subdivision paths. He proved that, given any graph H and, for every edge $e \in E(H)$, an assignment of two natural numbers $d(e)$ and $k(e)$, there exists an integer c (depending only on H and the sequences $d(e), k(e)$)

Date: November 15, 2021.

The first author was supported by the Deutsche Forschungsgemeinschaft (DFG) project 415310276. The second author was supported in part by SNSF Grant 200021_196965. The third author was supported by an ETH Zürich Postdoctoral Fellowship.

such that every graph of chromatic number at least c contains a subdivision of H in which each edge $e \in E(H)$ is replaced by a subdivision path whose length is congruent to $d(e)$ modulo $k(e)$. Furthermore, if for no edge $e \in E(H)$ the number $d(e)$ is odd while $k(e)$ is even, then there exists an integer c' such that for every graph of minimum degree at least c' the same conclusion holds. The latter result in particular shows that for every graph H and every integer q, any graph of sufficiently large minimum degree (in terms of H and q) contains a q-divisible H-subdivision.

As alluded to earlier, the results mentioned above, and other results on parity-constrained subdivisions in the literature, only apply to dense graphs. Thus, if the average degree of the host graph is small (just slightly above 2, say), then no sufficient conditions for divisible subdivisions were known. Alon and Krivelevich [1] filled this gap by providing a much more general sufficient condition in the case that H is subcubic.

Theorem 1.1. *For every graph H with $\Delta(H) \leq 3$ and every integer $q \geq 2$ there exists a (smallest) integer $f = f(H, q) \geq 1$ such that every K_f-minor contains a q-divisible subdivision of H as a subgraph.*

The main advantage of the result of Alon and Krivelevich relies in the fact that it is possible to find (arbitrarily) large complete minors even in classes of graphs with bounded maximum degree. For example, it was shown by Kawarabayashi and Reed [24] that every graph without sublinear separators contains a large complete minor; in particular it is known (c.f. Krivelevich [28]) that this property holds for essentially all graphs (even those of bounded degree) with sufficiently good expansion properties. Another interesting class of graphs for which the result applies are graphs of minimum degree at least 3 and large girth (see the results in [33, 15, 29]). Hence, Theorem 1.1 guarantees the existence of q-divisible H-subdivisions in these sparse graph classes.

Theorem 1.1 is qualitatively optimal in the sense that for every $q \geq 2$ there exist complete K_f-minors for arbitrarily large f with maximum degree 3 and such that every path between a pair of vertices of degree 3 has length divisible by q. Such a minor would not contain a subdivision of a graph with maximum degree at least 4, nor would it contain a subdivision of a cubic graph with a subdivision path of non-zero length modulo q.

However, the result proved in [1] was quantitatively far from optimal: following the proof of Theorem 1.1 in [1], it gives at best an upper bound on $f(H, q)$ which is of magnitude $(q^2n)^{q^3n}$, where $n = v(H)$. In contrast, the best lower bound on $f(H, q)$ for subcubic graphs H on n vertices and m edges we are aware of is $m(q-1)+n$, obtained by considering the complete graph of order $m(q-1)+n-1$ (which is too small to host a q-divisible subdivision of H).

Consequently, Alon and Krivelevich posed the problem of improving the bound on $f(H, q)$. In this paper we determine the value of $f(H, q)$ for all possible choices of H and q up to a constant multiplicative error.

Theorem 1.2. *Let H be an n-vertex graph with $e(H) = m$ and $\Delta(H) \leq 3$. Then, for every integer $q \geq 2$, it holds that*

$$m(q-1)+n \leq f(H, q) \leq 7mq + 8n + 14q,$$

and hence $f(H, q) = \Theta(mq + n)$.

In the special case where H is a cycle, Alon and Krivelevich [1] proved that for some constant C, every K_f-minor with $f \geq Cq \log q$ contains a cycle of length divisible by q. The correct order of magnitude in this case was determined independently by Mészáros and the third author [30] and by Arsovski (personal communication), who showed that $f \geq Cq$ is sufficient. Theorem 1.2 thus completes the picture by extending these sharp bounds to all subcubic graphs.

In fact, we obtain Theorem 1.2 as a special case of a more general result, which naturally generalises the setting of divisible subdivisions to that of Abelian groups, as also mentioned by Alon and Krivelevich. Given an abelian group $(A, +)$, we call a graph *A-weighted* if each edge of the graph is equipped with a weight $a \in A$. This setting is studied in a subfield of algebraic Ramsey theory known as *zero-sum theory*, and we refer the reader to the survey of Caro [12] for an overview of the area. One important graph-theoretic parameter studied in the area is the *zero-sum Ramsey number*, which measures the smallest size of a weighted complete graph in which one is guaranteed to find a desired subgraph of total weight zero. We refer to [2, 4, 7, 8, 9, 10, 11, 13, 14] for examples of results on this parameter.

Here, instead of a q-divisible subdivision, we aim for finding a subdivision of a fixed graph H such that the sum of the weights along any subdivision path equals $0 \in A$, and call such a subdivision an *A-divisible H-subdivision.*

The bound we will obtain for this problem depends on the parameter of an abelian group defined by

$$\sigma(A) = \max_{B \le A} \left\{ \frac{|\{a \in A : 2a \in B\}|}{|B|} \right\}.$$

For example, observe that for every integer $q \ge 2$, we have $\sigma(\mathbb{Z}_q) = 1$ if q is odd and $\sigma(\mathbb{Z}_q) = 2$ if q is even. With this notation in place, we can now state our result.

Theorem 1.3. *For every subcubic graph H with n vertices and m edges and for every finite abelian group $(A, +)$, it holds that every A-weighted K_f-minor with*

$$f \ge 7m|A| + 4n\sigma(A) + 14|A|$$

contains an A-divisible H-subdivision.

We can deduce Theorem 1.2 directly from Theorem 1.3 as follows: given a K_f-minor G with $f \ge 7mq + 8n + 14q$, let $A = (\mathbb{Z}_q, +)$ and assign weight $1 \in A$ to each edge of G. We can now apply Theorem 1.3, observing that $f \ge 7m|A| + 4n\sigma(A) + 14|A|$. In this labelling of G, a $\mathbb{Z}_q$-divisible H-subdivision is precisely a q-divisible H-subdivision, and hence we recover the conclusion of Theorem 1.2.

Notation and Organisation. We use standard notation throughout, but highlight some key terminology here to avoid any confusion.

Given a graph G and a finite abelian group $(A, +)$, an *A-weighting* of G is defined to be an assignment $w : E(G) \to A$ of elements from A to the edges of G. Given an A-weighting w of a graph G and a subgraph H of G, we denote by $w(H) = \sum_{e \in E(H)} w(e)$ the total weight of H in G, where the summation is in the abelian group. An A-weighted K_f-minor for some $f \ge 1$ is simply a K_f-minor equipped with an A-weighting (which we, if not defined otherwise, always denote by w).

If G is an A-weighted graph for some abelian group $(A, +)$, we say that a subdivision of H contained as a subgraph in G is an *A-divisible subdivision* if the sum of all edge weights along any subdivision path in the H-subdivision equals 0.

Given an abelian group $(A, +)$ and subsets $A_1, A_2, \ldots, A_k \subseteq A$, we denote by $A_1 + \cdots + A_k = \{a_1 + \cdots + a_k : a_i \in A_i, i \in [k]\} \subseteq A$ the sumset of $A_1, \ldots, A_k$. The sum of an empty list of subsets of A is defined to be equal to $\{0\}$. Given a subset $S \subseteq A$, we denote by $\langle S \rangle$ the set of elements in the subgroup generated by the elements in S.

The remainder of this paper is laid out as follows. In the following section, we present some preliminary definitions and results that will be used in our proof. In Section 3, we prove our main result, Theorem 1.3. Finally, in Section 4, we provide some concluding remarks and open questions.

2. Preliminaries

In this section we shall introduce the notion of connectors, which play a pivotal role in our proof. We shall define them and prove some initial results, leading up to Proposition 2.10, which concerns the existence of connectors in A-weighted K_f minors (in what follows, A is an arbitrary finite abelian group and $f \ge 4$ is an integer).

2.1. Reduced minors and connectors.

Since our main result is about finding certain subgraphs in A-weighted complete minors, it will be helpful for us to restrict our attention to minors which are in some sense minimal. This property is captured in the following definition.

Definition 2.1. Let G be an A-weighted K_f-minor. We say that G is a *reduced minor* if the following hold:

- Each supernode induces a tree in G;
- Every leaf in each tree induced by a supernode is adjacent to a vertex in another supernode;

- There is exactly one edge between any two supernodes;
- $\delta(G) \geq 3$.

Remark 2.2. *Let G be an A-weighted K_f-minor, where $f \geq 4$. We can get a reduced minor G' from G in the following natural way. We start by, for each supernode N, replacing $G[N]$ by one of its spanning trees. Next, between every pair of supernodes, we remove excess edges until exactly one connecting edge remains. We continue by, for every supernode N, successively removing from the tree $G[N]$ leaves that do not have any neighbours outside N. Notice that after each of those operations, our graph is a K_f-minor with minimum degree at least 2.*

Finally, for every vertex v with only two neighbours u_1 and u_2, we delete v and introduce the edge $\{u_1, u_2\}$ with weight $w(\{u_1, u_2\}) = w(\{u_1, v\}) + w(\{v, u_2\})$. Observe that u_1 and u_2 cannot previously have been adjacent, since if they were, then v, u_1 and u_2 would form a triangle. However, since $f \geq 4$, v is of too low degree to be its own supernode, and therefore must be in the same supernode as at least one of u_1 and u_2. Then, if u_1 and u_2 are in the same supernode N, we would have a cycle in $G[N]$, while otherwise there would be two edges between their supernodes.

Hence, after completing these steps, we obtain an A-weighted K_f-minor G', which we say is a reduced graph *of G. Crucially, note that every path in G' is obtained as a contraction of a path in G with the same endpoints and the same weight.*

The reduction of A-weighted clique minors described above naturally gives rise to a containment relation as follows: Given two numbers $f_1, f_2 \in \mathbb{N}$ such that $f_1 \leq f_2$, an A-weighted K_{f_1}-minor G_1 and an A-weighted K_{f_2}-minor G_2, we say that G_1 is a reduced sub-minor of G_2, in symbols, $G_1 \preceq_A G_2$, if G_1 is a reduced minor obtained from G_2 by first deleting all vertices in a subset of its supernodes, and then applying reduction operations as described above to the remaining complete minor.

Pause to note that $\preceq_A$ forms a transitive relation on the set of A-weighted clique minors, and, just as above, if $G_1 \preceq_A G_2$ for two A-weighted clique-minors, then every path in G_1 corresponds to a contraction of a path in G_2 with the same endpoints and the same weight. In particular, if G_1 contains an A-divisible subdivision of a graph H, then G_2 contains such a subdivision as well, which even uses the same set of branch vertices.

When working with a reduced K_f-minor, it is often convenient to view the graph at the level of its supernodes. However, when we then try to embed a subdivision H' of a cubic graph H, we need to identify individual vertices within the supernodes to act as the branch vertices of H'. The following proposition allows us to do so.

Proposition 2.3. *Let T be a tree, and let v_1, v_2 and v_3 be (not necessarily distinct) vertices in T. Then there exists a vertex $\bar{v}$ that is connected by internally vertex-disjoint paths (possibly of length zero) to v_1, v_2 and v_3. We call $\bar{v}$ the central vertex in T with respect to v_1, v_2 and v_3.*

Proof. Let P be the unique path between v_1 and v_2. If v_3 is on P, then we set $\bar{v} = v_3$. If not, consider the path from v_3 to v_2, and let $\bar{v}$ be the first vertex on this path that lies on P. ∎

We conclude this subsection by introducing connectors, which are central to our proof. Roughly speaking, these are subgraphs of weighted graphs that contain many paths of different weights between a specified pair of vertices.

Definition 2.4. Given an A-weighted graph G, a *connector* is a subgraph of G consisting of the union of disjoint cycles $C_1, \dots, C_\ell$ together with paths $P_0, P_1, \dots, P_\ell$ such that all paths are mutually disjoint and internally vertex disjoint from the cycles, and such that the last vertex of P_i is in C_{i+1} for all $0 \leq i \leq \ell - 1$, and the first vertex of P_i is in C_i for all $i \in [\ell]$. For all $i \in [\ell]$, let x_i and y_i be the weights of the two paths contained in C_i that connect P_{i-1} to P_i.

For a subset $S \subseteq A$ we say that a connector is an S-*connector* if $S \subseteq \sum_{i \in [\ell]} \{0, x_i - y_i\}$. We will refer to the first vertex of P_0 and the last vertex of P_ℓ as the first and last vertex of the connector, respectively (see Figure 1), and we call them the *endpoints* of the connector.

The *base path* of a connector is the path P between the endpoints such that, for each i, $P \cap C_i$ is the path of weight y_i. By *switching* the path used in C_i to the one which is of weight x_i, observe that the weight of P changes by $x_i - y_i$. Hence, by definition of a S-connector, by doing the appropriate switches, for every $s \in S$ we can get a path Q of weight $w(Q) = w(P) + s$ between the endpoints of the connector.

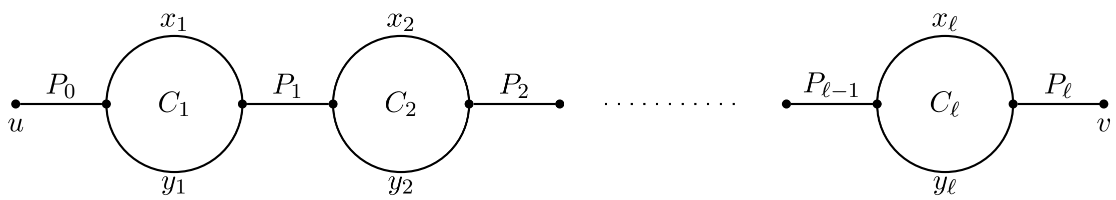

Figure 1. Illustration of a connector with endpoints u and v. For each $i \in [\ell]$, the labels x_i and y_i represent the weights of the two paths in C_i which connect P_{i-1} to P_i.

2.2. **Permissible cycles and B-restricted minors.** The main goal of this section is to prove Proposition 2.10, which concerns the existence of connectors and plays an important role in the proof of Theorem 1.3. In order to lay the groundwork for this result, we next define special types of paths and cycles in complete minors.

Definition 2.5. Let G be a reduced K_f-minor.

- We say that a path P in G is *permissible* if for every supernode N we have that $G[N] \cap P$ is either empty or a path.
- We say that a cycle C in G is *permissible* if for every supernode N we have that $G[N] \cap C$ is either empty or a path, except for at most one supernode N_1 for which $G[N_1] \cap C$ can be the union of two vertex-disjoint paths.

Remark 2.6. *Notice that if a permissible cycle C has a supernode N_1 for which $G[N_1] \cap C$ is the union of two disjoint paths P_1 and P_2, then C is incident to at least five supernodes. Indeed, observe that the four edges in C which are incident to the endpoints of P_1 and P_2, and which are not in N_1, have their other endpoint in mutually distinct supernodes. This is because G is a reduced minor, so there is exactly one edge between two supernodes. Hence, C meets at least four other supernodes besides N_1.*

We will also need the following definition before stating the results of this subsection.

Definition 2.7. Let G be an A-weighted K_f-minor and let $B \leq A$ be a subgroup. Then we say that G is *B-restricted* if the following hold:

- Every permissible cycle C in G has weight $w(C) \in B$.
- Every edge e has weight $w(e)$ such that $2w(e) \in B$.

Note that every A-weighted K_f-minor is trivially A-restricted, and so this definition is only of interest when B is a proper subgroup of A. The following lemma states that in order to show a weighted minor is B-restricted, it suffices to check permissible cycles incident to few supernodes.

Lemma 2.8. *Let A be an abelian group, let $f \geq 5$ and let $B \leq A$. Let G be a reduced A-weighted K_f-minor such that every permissible cycle C whose vertices are contained in at most five supernodes satisfies $w(C) \in B$. Then G is B-restricted.*

Proof. Suppose $B \neq A$ as otherwise the claim holds trivially. Let us first show that $2w(e) \in B$ for every edge e in G. Denote by u, v the endpoints of an edge e, and let N_1 and N_2 be the supernodes such that $u \in N_1$ and $v \in N_2$.

First consider the case that $N_1 = N_2$. Since u and v are of degree at least three (since G is a reduced minor), let x_1, x_2 be two of the other neighbours of u, and x_3, x_4 two other neighbours of v. For each $i \in [4]$ do the following. If x_i is already in another supernode $X_i \neq N_1$, let P_i be the path consisting of one vertex x_i, and let $s_i = x_i$. Otherwise, let t_i be a leaf in the maximal subtree of $G[N_1]$ which contains x_i, but which does not contain u and v, and let s_i be a neighbor of t_i in another supernode X_i (by the definition of a reduced minor, every leaf has such a neighbour). Now, let P_i be the path obtained by concatenating the unique path between x_i and t_i in $G[N_1]$ with the edge $\{t_i, s_i\}$. Next, let Q_1 be the unique path between s_1 and s_3 in the tree $G[X_1 \cup X_3]$, and similarly Q_2 the path between s_2 and s_4 in $G[X_2 \cup X_4]$ (see Figure 2). Note that by construction we have that the cycle C_1 formed by the paths $P_1 - Q_1 - P_3 - x_3 - v - u - x_1$ is a permissible cycle incident with three supernodes, as is the cycle C_2 consisting of $P_2 - Q_2 - P_4 - x_4 - v - u - x_2$, so we have that $w(C_1), w(C_2) \in B$. Also note that the cycle C with edge-set $(E(C_1) \cup E(C_2)) \setminus \{e\}$ is permissible and incident to five supernodes, hence we conclude $2w(e) = w(C_1) + w(C_2) - w(C) \in B$.

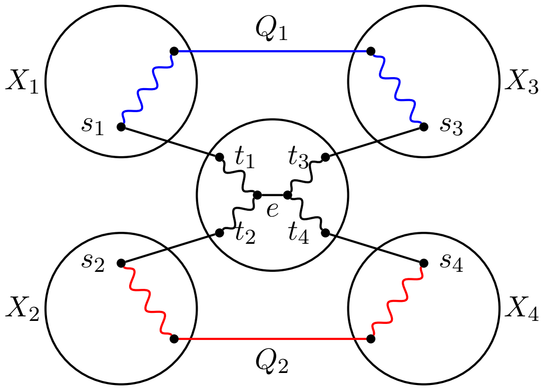

Figure 2. The union of the blue curves depicts the path Q_1, and the union of the red curves depicts Q_2.

Next, suppose $N_1 \neq N_2$. Let M be a supernode different from N_1, N_2 and let $\{n_1, m_1\}$ be the unique edge between N_1 and M, and let $\{n_2, m_2\}$ be the unique edge between N_2 and M, where $m_i \in M$ and $n_i \in N_i$. Let Q_1 be the path between m_1 and m_2 in M. Let P_1 be the path in N_1 from u to n_1, and let P_2 be a path (internally vertex-disjoint from P_1) in N_1 which starts at u, goes to a leaf in $G[N_1]$, and finishes at a vertex s_1 in another supernode M_1 different from N_1, N_2, M, such that $V(P_1) \cap M_1 = \{s_1\}$. Analogously, we find a path P_3 from v to n_2 in N_2, and a path P_4 from v to a vertex $s_2 \in M_2$, for a supernode M_2 different from N_1, N_2, M, and such that $V(P_4) \cap M_2 = \{s_2\}$. Finally, let Q_2 be the path in the tree $G[M_1 \cup M_2]$ which connects s_1 to s_2. Again we have constructed two cycles C_1 and C_2, consisting of paths $\{v, u\} - P_1 - \{n_1, m_1\} - Q_1 - \{m_2, n_2\} - P_3$ and $\{v, u\} - P_2 - Q_2 - P_4$, and as in the case when $N_1 = N_2$, we get a permissible cycle C (with edge-set $(E(C_1) \cup E(C_2)) \setminus \{e\}$) contained in the union of at most five supernodes. We conclude again that $2w(e) = w(C_1) + w(C_2) - w(C) \in B$.

Now let us show that all permissible cycles C have $w(C) \in B$, and for the sake of contradiction assume the contrary. Let C be a permissible cycle with $w(C) \notin B$ that minimises the number of supernodes it intersects. By assumption, C is incident to at least six supernodes. Look at the cyclic ordering of those supernodes based on their appearance on C (where at most one supernode appears two times, as shown in Figure 3).

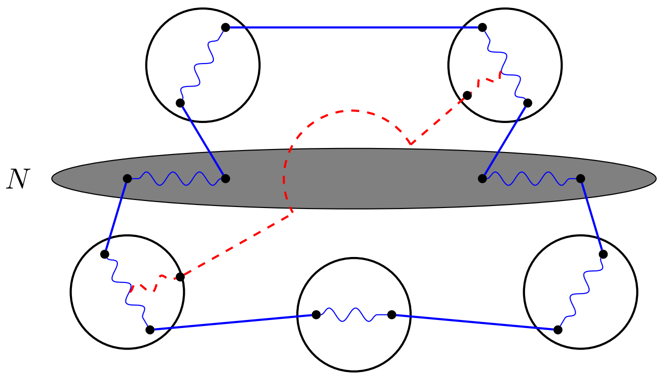

Figure 3. The circles represent the supernodes that contain a single path in the cycle C for which we assumed that $w(C) \notin B$, while the ellipse represents the supernode N that intersects C in two paths. The curved blue lines are paths in C, while straight blue lines are edges in C. The dashed red curve represents the path P between two supernodes which splits C into two permissble cycles, each of which is incident to fewer supernodes than C.

Let N_1 and N_2 be two supernodes that intersect C in exactly one path each, P_1 and P_2 respectively, such that N_1 and N_2 are at least three apart in the cyclic ordering of supernodes.

Now, let P be the shortest path in $G[N_1 \cup N_2]$ between P_1 and P_2. The union $C \cup P$ then splits into two cycles C_1 and C_2 whose intersection is the path P. But since C_1 and C_2 are also permissible, and since they are incident to fewer supernodes than C, by our minimality assumption on C we have that $w(C_1), w(C_2) \in B$. But this means that

$$w(C) = \underbrace{w(C_1)}_{\in B} + \underbrace{w(C_2)}_{\in B} - \underbrace{2w(P)}_{\in B} \in B,$$

since $2w(e) \in B$ for each edge e in the path P, as shown in the first part of the proof. This gives the desired contradiction and finishes the proof. ■

We continue with a simple lemma which, roughly speaking, will be used to reveal some information about the cycle weights in a graph in which a particular connector cannot be extended by another cycle. It is one ingredient for the proposition which follows after the lemma.

Lemma 2.9. *Let S and T be subsets of elements of a finite abelian group A, where $0 \in S$. Suppose that $S + \{t\} \subseteq S$ for all $t \in T$. Then $\langle \{t \mid t \in T\} \rangle \subseteq S$.*

Proof. We want to show that for every integer k and for every sequence $t_1, \dots, t_k$ of (not necessarily distinct) elements of T, we have $\sum_{i=1}^{k} t_i \in S$. We show this by induction on k.

For $k = 1$, note that S contains 0, so by assumption $0 + t_1 \in S$. For $k \geq 2$, by the induction hypothesis we may assume $\sum_{i=1}^{k-1} t_i \in S$. Then we have $\sum_{i=1}^{k} t_i = \left(\sum_{i=1}^{k-1} t_i\right) + t_k \in S + \{t_k\} \subseteq S$, as desired. ■

The following result is one of the main building blocks of our proof. It states that in a B-restricted minor we can either find a B-connector of small size, or we can delete a small number of supernodes and be left with a B'-restricted minor for a proper subgroup $B' < B$.

Proposition 2.10. *Let A be a finite abelian group, $B \leq A$ a subgroup, and let G be a reduced B-restricted A-weighted K_f-minor. Then at least one of the following two claims holds:*

- *G contains a B-connector F which intersects at most $7|B|$ supernodes, such that for each endpoint v of F and the supernode N containing v we have $V(F) \cap N = \{v\}$. Furthermore, the base path of F is permissible.*
- *There is a proper subgroup $B' < B$ and a reduced subminor $G' \preceq_A G$ such that G' is a B'-restricted A-weighted reduced $K_{f'}$-minor, where $f' \geq f - 7|B|$.*

Proof. Note that we may assume $f > 7|B|$, as otherwise the second claim is trivially true by letting G' be a trivial subgraph of G. We will attempt to construct the connector by finding a sequence of subsets $S_0 \subsetneq S_1 \subsetneq \ldots \subsetneq S_t = B$ for some $t \leq |B|$, and for each $i \in [t]$ an S_i-connector intersecting at most $7i$ supernodes. For every $i \in [t-1]$, the S_{i+1}-connector will extend the previously constructed S_i-connector with vertices from at most seven new supernodes.

Let $S_0 = \emptyset$, and for technical reasons, with slight abuse of notation, let the empty subgraph be our first connector. Suppose for some $0 \leq i \leq t-1$ we have found an S_i-connector where $S_i \subsetneq B$, and let us find an S_{i+1}-connector. Consider the graph obtained by removing from G all supernodes that have a vertex in the S_i-connector, and denote by G_i a reduced minor of that graph. Note that G_i is a $K_{f'}$-minor for $f' \geq f - 7i > 7|B| - 7i > 7$ and that $G_i \preceq_A G$.

Let C be a permissible cycle in G_i that is incident to at most five supernodes. Note that C consists of vertices that are in at least three different supernodes (as any pair of supernodes induces a tree in a reduced minor), and choose T_1, T_2 and T_3 to be three distinct supernodes that intersect C in precisely one non-empty path (see Remark 2.6). Now, let N_1, N_2 and N_3 be three supernodes in G_i disjoint from C.

In what follows, to simplify our notation, we will use the same names for corresponding supernodes in G and in the reduced subminor G_i, anticipating that a supernode might lose some vertices and that some edges can be contracted when passing from G to G_i. We also use the same names for subgraphs H of G_i, which correspond to subdivisions of H in G. Observe that by Remark 2.2, the weight of the subdivision paths in G is the weight of the corresponding edge of H in G_i.

For $j \in [3]$, let $u_j \in V(G_i)$ be the vertex in N_j that has a neighbour in T_j, and let Q_j be the shortest path from u_j to $C \cap T_j$ in $G_i[N_j \cup T_j]$, observing that the interval vertices of Q_j all lie in T_j (see Figure 4). The endpoints of these paths split C into three paths in G_i, whose weights we denote by x_1, x_2 and x_3, in such a way that for every $j \in [3]$ the segment of C between the endpoints of Q_{j-1} and Q_{j+1} is of total weight x_j[1]. Let $\delta_1(C) = x_1 + x_2 - x_3$, $\delta_2(C) = x_2 + x_3 - x_1$ and $\delta_3(C) = x_3 + x_1 - x_2$.

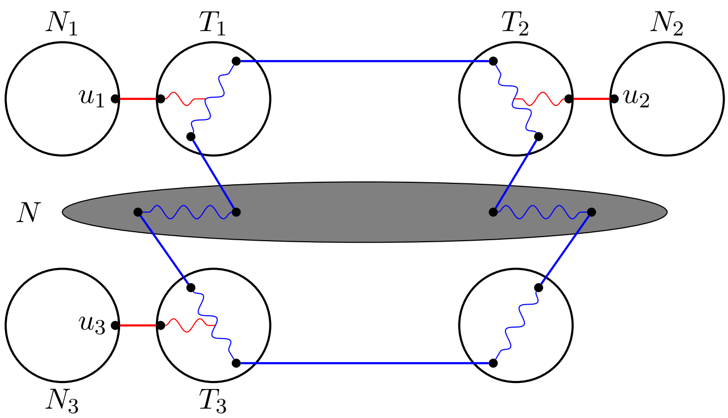

Figure 4. The circles represent the supernodes that contain a single path in the cycle C, while the ellipse represents the supernode N that intersects C in two paths. The red curves from u_j to the blue paths represent the paths Q_j.

Suppose first that there is some $j \in [3]$ such that $S_i + \{0, \delta_j(C)\} \neq S_i$; by relabelling, we may assume that $j = 1$. Then the endpoints of Q_1 and Q_2 split C into two paths P_1 and P_2 of weights x_3 and $x_1 + x_2$. Now we complete the construction of the S_{i+1}-connector, where we set $S_{i+1} = S_i + \{0, \delta_1(C)\}$. If $i = 0$, then let the S_{i+1}-connector be the concatenation of Q_1, C and Q_2. Otherwise, let z be the last vertex of the S_i-connector that we want to extend, and let Z be the supernode in G containing it. Prolong the last path of this connector by attaching to z the shortest path from z to Q_1 in $G[Z \cup N_1]$, and then attach to this path the remaining part of Q_1 whose endpoint is on the cycle C. Now, attach to the connector the cycle C. To complete the S_{i+1}-connector, we wish to attach the path Q_2. Note that the last vertex u_2 of Q_2 was the only vertex in Q_2 from N_2 in the graph G_i, but when we lift from G_i to G, the path Q_2 may have more vertices from N_2. Hence we remove all vertices from $Q_2 \cap N_2$ except the one adjacent to T_2, and attach the obtained path to C. In this fashion, we maintain the desired property of the last vertex in the connector, as the vertex in Q_2 adjacent to T_2 is now the only vertex from N_2 in the connector. Also note that that we added vertices from at most seven new supernodes to the S_i-connector to make this S_{i+1}-connector. Finally, P_1 is a permissible path,[2] and by construction it is easy to see that the base path of the S_{i+1}-connector is also permissible.

If we are able to iterate this process, after at most $|B|$ steps we would obtain the desired B-connector (meeting at most $7|B|$ supernodes). On the other hand, if an iteration fails, it must mean that for some $0 \leq i \leq |B| - 1$ and for all $j \in [3]$, we have $S_i + \{0, \delta_j(C)\} = S_i$ for every choice of a permissible cycle C incident to at most five supernodes. Let $\mathcal{C}$ be the collection of those cycles in G_i. We conclude, by Lemma 2.9, that the group $B' = \langle\{\delta_j(C) \mid j \in [3], C \in \mathcal{C}\}\rangle$, is contained in S_i. Since we do not yet have a B-connector, we must have $B' < B$. Furthermore, observe that $w(C) = \delta_1(C) + \delta_2(C) + \delta_3(C) \in B'$ for all $C \in \mathcal{C}$. Appealing to Lemma 2.8, we deduce that G_i is a B'-restricted reduced $K_{f'}$-minor, where, since we have only lost the supernodes in the S_i-connector, $f' \geq f - 7i \geq f - 7|B|$. ■

[1] summation of indices with modular arithmetic

[2] Either all supernodes incident to C intersect C in at most one path, and then it is clear that the same holds for P_1. Otherwise, by Remark 2.6 we know that exactly 5 supernodes are incident to C, and the cyclic ordering of the supernodes as they appear on C is $T, -, -, T, -, -$ (where T is the repeated supernode, and each '-' sign is uniquely assigned to a distinct supernode). Now it is clear that for every choice of P_1 as in the proof, P_1 intersects $G[T]$ in at most one path.

3. The proof

We are now ready to prove our main result, which we restate here for the convenience of the reader.

Theorem 1.3. *For every subcubic graph H with n vertices and m edges and for every finite abelian group $(A, +)$, it holds that every A-weighted K_f-minor with*

$$f \geq 7m|A| + 4n\sigma(A) + 14|A|$$

contains an A-divisible H-subdivision.

Proof. Let G be an A-weighted K_f-minor, which we may assume is reduced (see Remark 2.2). The idea of the proof is to find $m = e(H)$ disjoint connectors in G, each of which is contained in the union of at most $7|A|$ supernodes. We will route the subdivision paths through these connectors, so that we can then apply switches within the connectors to ensure each path is of weight $0 \in A$.

We begin by constructing A-connectors for as long as we are able. If at some point no further such connector can be found, this will reveal some structural information about the edge weights in the remaining graph, as given by Proposition 2.10. This information will allow us to pass to some subgroup $A' < A$, and we shall then construct A'-connectors instead. We repeat this process until we have the desired number of connectors, keeping track of their various types. We shall then use these in conjunction with the remaining vertices to build an A-divisible subdivision of H.

We initially set $f_0 = f$, $B_0 = A$ and $G_0 = G$. For each iteration of our process, suppose we have $i \geq 0$ and f_i, B_i and G_i such that $G_i \preceq_A G$ is a B_i-restricted A-weighted reduced K_{f_i}-minor; we shall then find f_{i+1}, B_{i+1} and G_{i+1} as follows.

Applying Proposition 2.10 to G_i, we either find a B_i-connector F_i using at most $7|B_i|$ supernodes, or we obtain a proper subgroup $B' < B_i$ and a reduced subminor $G' \preceq_A G_i \preceq_A G$ that is a B'-restricted reduced $K_{f'}$-minor, where $f' \geq f - 7|B_i|$.

(1) In the former case, we increment our iteration counter, and let G_{i+1} be the graph obtained by reducing G_i after removing the supernodes used in F_i. We set $B_{i+1} = B_i$ and we let f_{i+1} be the number of supernodes in G_{i+1}, noting that $f_{i+1} \geq f_i - 7|B_i|$.
(2) In the latter case, we remain in the current iteration, but update $G_i = G'$, $B_i = B'$ and we set $f_i = f'$. We repeat the process, applying Proposition 2.10 to G_i again, until we encounter the first case.

Every time we enter case (2) in this inductive process, the size of the group B_i we are working with decreases by a factor of at least two. Thus, we encounter case (2) at most $\lceil \log_2 |A| \rceil$ times. Moreover, after $j \geq 0$ instances of case (2), the size of the group B_i is at most $2^{-j}|A|$. Since we lose at most $7|B_i|$ supernodes every time we fall into case (2), we can bound the total number of supernodes lost in case (2) during the process by $14|A| = \sum_{j \geq 0} 7 \cdot 2^{-j}|A|$.

As the number of occurrences of the second case is bounded, we must eventually meet the first case m times, after which we will have the desired m connectors. Each connector costs us at most $7|B_i| \leq 7|A|$ supernodes, and so by the time we reach the final graph G_m, we still have at least $f - m \cdot 7|A| - 14|A| \geq 4n\sigma(A)$ supernodes, and for each $0 \leq i < m$ we have a B_i-connector $F_i \subseteq G_i$.

Now pick $4n\sigma(A)$ of the supernodes in G_m and group them into clusters of four. For each such cluster of supernodes N_1, N_2, N_3 and N_4, let n_2, n_3 and n_4 be the vertices in N_2, N_3 and N_4 respectively that have a neighbour in N_1. Mark the central vertex in $G_m[N_1]$ for the neighbours of n_2, n_3 and n_4 in N_1. Let $\{x_i \mid i \in [n\sigma(A)]\}$ be the collection of the marked central vertices, and denote by X_i the supernode of x_i for each i, and we call X_i the *central* supernode of its cluster.

Next we show that we can pick n of those central vertices

$$\{y_i \mid i \in [n]\} \subseteq \{x_i \mid i \in [n\sigma(A)]\},$$

where we denote the supernodes of y_i by Y_i, such that, for every pair $i, j \in [n]$, the weight of the unique path between y_i and y_j in the tree $G_m[Y_i \cup Y_j]$ is in B_m.

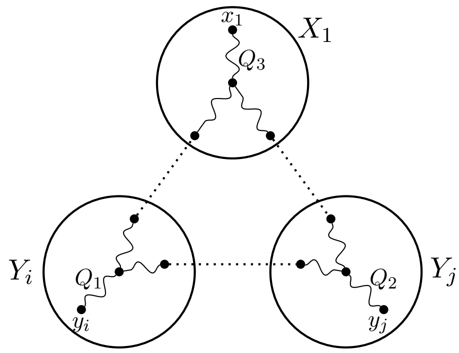

Figure 5. The dotted lines represent edges between two supernodes, while the solid curves represent paths in G_m

Indeed, let $B' := \{a \in A \mid 2a \in B_m\}$. Since G_m is B_m-restricted, we have that every edge e in G_m satisfies $w(e) \in B'$. Noting that B_m is, trivially, a subgroup of B', we can define the quotient group $B^* = B'/B_m$ and, by definition of σ, we have $|B^*| = \frac{|B'|}{|B_m|} \leq \sigma(A)$.

We now colour the vertices x_i with the elements of B^*. We start by giving $x_1 \in X_1$ the colour $0 + B_m \in B^*$. Then, for each $i \geq 2$, let P_{x_i} be the unique path between x_1 and x_i in $G[X_1 \cup X_i]$, and colour the vertex x_i with $w(P_{x_i}) + B_m \in B^*$.

We choose the vertices y_i arbitrarily from the largest colour class, which is of size at least $\frac{n\sigma(A)}{|B^*|} \geq \frac{n\sigma(A)}{\sigma(A)} = n$. Let us show that the path P_{ij} between y_i and y_j in $G_m[Y_i \cup Y_j]$ is of weight in B_m for all pairs $i, j \in [n]$. Let $P_i = P_{y_i}, P_j = P_{y_j}$, $Q_1 = P_i \cap P_{ij}$, $Q_2 = P_j \cap P_{ij}$, and $Q_3 = P_i \cap P_j$, and let the respective weights of the latter three paths[3] be q_1, q_2 and q_3 respectively (see Figure 5). Let C be the permissible cycle formed by the edges in $E[P_i \cup P_j \cup P_{ij}] \setminus E[Q_1 \cup Q_2 \cup Q_3]$. Observe that $w(P_{ij}) = w(C) + 2(q_1 + q_2 + q_3) - (w(P_i) + w(P_j))$. Since G_m is B_m-restricted and since we chose y_i and y_j to be of the same colour, we have $w(P_i) + w(P_j) \in 2w(P_i) + B_m = B_m$. Furthermore, since C is permissible in G_m, we have $w(C) \in B_m$. Finally $2(q_1 + q_2 + q_3) \in B_m$, again from G_m being B_m-restricted. This yields $w(P_{ij}) \in B_m$, as we wanted.

We now set about building an A-divisible subdivision of H. If we denote by $\{v_i \mid i \in [n]\}$ the vertices of H, y_i will be the branch vertex corresponding to v_i. Let $\mathcal{V} = \{y_i \mid i \in [n]\}$. For a fixed arbitrary ordering $e_1, \ldots, e_m$ of the edges of H, we now show how to construct the subdivision paths between the corresponding branch vertices in $\mathcal{V}$.

For each $k \in [m]$, the subdivision path of weight $0 \in A$ corresponding to the edge e_k will be constructed within the graph G_k. Since passing to a reduced subminor does not change the weights of paths, the corresponding path in G will also be of weight 0. After describing these paths, we will explain why they are internally vertex-disjoint.

Importantly, as an invariant during the construction process we will require that for every edge e_k with $k \in [m]$ and with endpoints v_i and v_j, the subdivision path representing e_k is vertex-disjoint from all supernodes contained in clusters corresponding to vertices y_t with $t \in [n] \setminus \{i, j\}$, and that it intersects the clusters of y_i and y_j only in their central supernodes as well as at most one other supernode from the cluster.

Now suppose that for some $k \in [m]$ we have already constructed subdivision paths for $e_1, \ldots, e_{k-1}$, and let $e_k = \{v_i, v_j\}$. We thus need to build a subdivision path connecting y_i and y_j. Recall that y_i came from the supernode Y_i, which was part of a four-supernode cluster. Let Y_i^1, Y_i^2 and Y_i^3 be the other three supernodes from that cluster, and let y_i^1, y_i^2 and y_i^3 be the vertices in these supernodes that are adjacent to Y_i (for which y_i was the central vertex). We define $y_j^1 \in Y_j^1, y_j^2 \in Y_j^2$ and $y_j^3 \in Y_j^3$ similarly with respect to y_j.

By our invariant on how subdivision-paths interact with the clusters, at most two of the three non-central supernodes Y_i^1, Y_i^2, Y_i^3 respectively Y_j^1, Y_j^2, Y_j^3 are intersected by subdivision-paths corresponding to $e_1, \ldots, e_{k-1}$. Hence, without loss of generality we may assume that Y_i^1 and Y_j^1 have not been used previously in the construction of subdivision paths. Let P be the path in

[3]Note that the intersections Q_1, Q_2, Q_3 indeed form paths (possibly of length zero), since $G_m[Y_i], G_m[Y_j]$ and $G_m[X_1]$ are trees.

$G_m[Y_i \cup Y_j]$ between y_i and y_j, Q_1^k the path in $G_m[Y_i \cup Y_i^1]$ between y_i and y_i^1, and Q_2^k the path in $G_m[Y_j \cup Y_j^1]$ between y_j and y_j^1. We then lift these paths to the corresponding paths in G_k; that is, we reverse any contractions that may have occurred when passing from G_k to G_m.

First, recall that our choice of y_i and y_j guarantees $w_{G_m}(P) \in B_m$, and so we have $w_{G_k}(P) \in B_m \leq B_k$. Next, note that the paths Q_1^k and Q_2^k may gain some additional vertices in G_k. We define $p_1 \in Y_i^1$ to be the vertex on Q_1^k that is adjacent to Y_i, and let $\bar{Q}_1^k \subseteq Q_1^k$ be the path in $G_k[Y_i \cup Y_i^1]$ from y_i to p_1. We define p_2 and $\bar{Q}_2^k$ analogously.

Now let t_1 and t_2 be the endpoints of the k^{th} connector F_k, which is a B_k-connector in G_k, and let T_1 and T_2 be their corresponding supernodes. Let $\bar{Q}_3^k$ be the path in $G_k[Y_i^1 \cup T_1]$ connecting p_1 to t_1, and let $\bar{Q}_4^k$ be the path in $G_k[Y_j^1 \cup T_2]$ between p_2 and t_2 (see Figure 6). Finally, let $\bar{Q}_5^k$ be the base path between t_1 and t_2 in the connector F_k. Observe that the concatenation of $P, \bar{Q}_1^k, \bar{Q}_3^k, \bar{Q}_5^k, \bar{Q}_4^k$ and $\bar{Q}_2^k$ gives a permissible cycle in G_k, whose weight is hence in B_k. Indeed, the base path $\bar{Q}_5^k$ is permissible by construction, while the other paths are contained within two supernodes, and hence must be permissible. As the paths do not share any supernodes beyond those of their common endpoints, it is then easy to verify that their union is permissible (with each supernode intersecting the cycle in at most one path).

Therefore, by removing the edges of P from the cycle (or, equivalently, by concatenating $\bar{Q}_1^k, \bar{Q}_3^k, \bar{Q}_5^k, \bar{Q}_4^k$ and $\bar{Q}_2^k$), we obtain a path $\bar{Q}^k$ in G_k between y_i and y_j whose weight is in B_k. We can thus perform the appropriate switches in the B_k-connector F_k (thereby modifying the path $\bar{Q}_5^k$) to find a path in G_k between y_i and y_j of weight $0 \in A$. Finally, this lifts to a path Q_{ij} between y_i and y_j in the original graph G of weight $0 \in A$, which is what we sought.

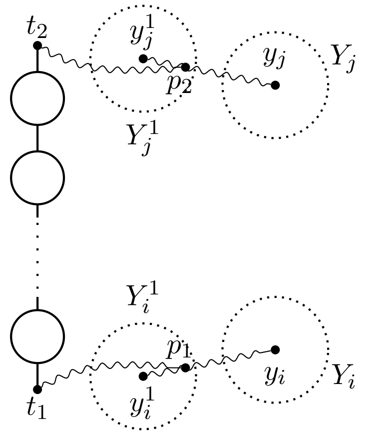

Figure 6. This figure illustrates how the path connecting y_i and y_j is constructed. Dotted circles represent supernodes, while squiggly lines represent paths.

To finish, we show that the subdivision paths we have constructed are internally vertex-disjoint. Indeed, with the exception of the supernodes $\{Y_i \mid i \in [n]\}$ housing the branch vertices $\{y_i \mid i \in [n]\}$, the subdivision paths pass through disjoint sets of supernodes. Within the supernode Y_i, since y_i was the central vertex for y_i^1, y_i^2 and y_i^3, it follows that the subdivision paths are disjoint apart from y_i, as required. Thus, the subdivision paths are internally vertex-disjoint, and we have constructed a genuine A-divisible subdivision of H. ■

4. Concluding remarks

In this paper, we have addressed a problem of Alon and Krivelevich [1] on divisible subdivisions, showing that if H is a subcubic graph on n vertices with m edges, A is a finite abelian group and $f \geq 7m|A|+4n\sigma(A)+14|A|$, then every A-weighted K_f-minor contains an A-divisible H-subdivision. In particular, by taking $A = (\mathbb{Z}_q, +)$, it follows that having $f \geq 7mq + 8n + 14q$ suffices to ensure the existence of an H-subdivision whose subdivision paths are all of length divisible by q.

This bound is tight up to a multiplicative constant. Indeed, since each subdivision path must have length at least q, a q-divisible H-subdivision requires at least $m(q-1)+n$ vertices. Thus, for $f = m(q-1)+n-1$, K_f itself is a K_f-minor without any q-divisible H-subdivisions. Having failed to find any constructions that yield a better lower bound, we suspect that this trivial bound may in fact be the true answer. Part of the difficulty in proving this is that K_f-minors can have a wide range of structures, and so it is easier to restrict the class of host graphs under consideration. A natural first step would be to only look at subdivisions of K_f.

Problem 4.1. *Given $q \in \mathbb{N}$ and a subcubic graph H with n vertices and m edges, is it true for $f = m(q-1)+n$ that every subdivision of K_f contains a q-divisible H-subdivision?*

Through some case analysis, we could answer this in the affirmative when $q = 2$ and $H = K_4$, providing some scant evidence in support of a positive answer. It would be interesting to see a general argument that applies to all q and H.

Recall that we seek q-divisible subdivisions because we cannot be guaranteed to find anything else — there are K_f-minors where every path between vertices of degree at least three has length divisible by q. Our proof shows that this is essentially the only obstruction, since the $\mathbb{Z}_q$-connectors allow us to obtain paths of any parity we wish. Thus, given $q \geq 2$, a subcubic graph H and, for every edge $e \in E(H)$, a residue $r_e \in \mathbb{Z}_q$, then in any K_f-minor G we can either find an H-subdivision such that, for each edge e, the subdivision path corresponding to e has length r_e modulo q, or we find a proper subgroup $W < \mathbb{Z}_q$ and a subgraph $G' \subseteq G$ that is a $K_{f'}$-minor for some $f' \geq f - 7qm$, such that every path P in G' between vertices of degree at least three has length satisfying $2\ell(P) \in W$. In particular, if q is prime, then it divides $\ell(P)$.

The other restriction we imposed is that the graph H should be subcubic. This is again necessary, as there are K_f-minors with maximum degree three. However, we can circumvent this obstruction by including a large minimum degree requirement. Thomassen [32] proved that for any graph H, all graphs of sufficiently large minimum degree contain q-divisible H-subdivisions (in fact, one can impose a much wider range of modular restrictions on the path lengths). In light of our results, it is natural to ask how much the bound on the minimum degree can be reduced when the host graph is a K_f-minor.

Problem 4.2. *Given $f, q \geq 2$ and a graph H, what is the smallest $d = d(H, q, f)$ such that every K_f-minor G with $\delta(G) \geq d$ contains a q-divisible H-subdivision?*

Finally, returning to the setting of group-weighted graphs, we observe that while our bound gives the correct order of magnitude in the case $A = \mathbb{Z}_q$, there is scope for improvement for other abelian groups. Indeed, when H is a cycle, Scheucher, Sidorenko and the third author (personal communication) show that when $A = \mathbb{Z}_2^d$, f need only grow logarithmically with the size of the group. It is then natural to presume that such savings can also be made for other subcubic graphs.

Problem 4.3. *Given $d \geq 2$ and a subcubic graph H on n vertices with $m = \Omega(n)$ edges, is there some $f = O(dm)$ such that every $\mathbb{Z}_2^d$-weighted K_f-minor contains a $\mathbb{Z}_2^d$-divisible H-subdivision?*

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