Ramsey number of 1-subdivisions of transitive tournaments

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Abstract

The study of problems concerning subdivisions of graphs has a rich history in extremal combinatorics. Confirming a conjecture of Burr and Erdős, Alon proved in 1994 that subdivided graphs have linear Ramsey numbers. Later, Alon, Krivelevich and Sudakov showed that every n-vertex graph with at least εn^2 edges contains a 1-subdivision of the complete graph on $c_\varepsilon\sqrt{n}$ vertices, resolving another old conjecture of Erdős. In this paper we consider the directed analogue of these problems and show that there exists a constant C such that every tournament on at least Ck^2 vertices contains the 1-subdivision of a transitive tournament on k vertices. This is tight up to the constant multiplicative factor and confirms a conjecture of Girão, Popielarz and Snyder.

1 Introduction

Given a graph G, a *subdivision of* G is a graph obtained by replacing its edges with internally vertex-disjoint paths of arbitrary length. More specifically, the 1*-subdivision of* G is the subdivision in which the length of these paths is 2. Problems concerning subdivisions of graphs have been extensively studied in extremal combinatorics. For example, a celebrated conjecture of Mader [8] from 1967, says that there exists some constant C such that every graph of average degree at least Ck^2 contains a subdivision of K_k. This was confirmed only 30 years later, by Bollobás and Thomason [3] and independently by Komlós and Szemerédi [7].

One of the central topics in discrete mathematics is the study of Ramsey numbers. The *Ramsey number*, $r(G)$, of a graph G is the smallest number N such that every 2-coloring of K_N contains a monochromatic copy of G. A well known conjecture of Erdős and Burr [4] was that subdivisions of graphs in which each subdivision path is of length at least 2, have Ramsey number which is linear in the number of vertices. Alon [1] resolved this in 1994, showing that every graph on n vertices in which no two vertices of degree at least 3 are adjacent has Ramsey number at most $12n$. Later, Alon, Krivelevich and Sudakov [2] proved a stronger density-type result for cliques, showing that every n-vertex graph with at least εn^2 edges contains the 1-subdivision of a complete graph on $c_\varepsilon\sqrt{n}$ vertices. This proved an old conjecture of Erdős [5].

In this paper, we study analogues of these problems in the framework of directed graphs. Notice that in this context it is only sensible to consider embedding *acyclic* graphs in host digraphs, since in general the host digraph might not contain a cycle. Therefore, we will only consider subdivisions of the transitive tournament T_k on k vertices. Secondly, it is not possible to give a density-type statement as it was done in the result of Alon, Krivelevich and Sudakov [2]. Indeed, note that an orientation of the edges of the Turán graph $T(n,k)$ with k parts in which the direction of an edge between two parts conforms to a previously specified ordering of the parts, does not even contain a path of length k. Hence, only in very dense host directed graphs can we hope to embed an arbitrary subdivision of T_k, let alone the 1-subdivision (Scott [9], in fact, proved that one can find a non-specified subdivision of T_k inside of every n-vertex digraph with more edges than $T(n,k)$). This naturally leads to the following Ramsey-type question: How many vertices should a tournament have in order to contain the 1-subdivision of T_k?

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The *oriented Ramsey number*, $\overrightarrow{r}(H)$, of an oriented graph H is the smallest number N such that every tournament on N vertices contains a copy of H. The above question then asks for the oriented Ramsey number of the 1-subdivision of T_k. This problem was raised by Girão, Popielarz and Snyder [6], who gave an upper bound of $O(k^2 \log^3 k)$. They also conjectured that 1-subdivisions of T_k actually have linear oriented Ramsey number. In this paper we prove this conjecture.

Theorem 1.1. *Every tournament on 10^4k^2 vertices contains the 1-subdivision of T_k.*

Clearly, the above result is tight up to the constant multiplicative factor since the 1-subdivision of T_k has at least $k^2/2$ vertices. In the next section, we will give some preliminaries and then prove the result in Section 3. We finish with some brief concluding remarks.

2 Preliminaries

We will refer to the 1-subdivision of the transitive tournament on k vertices as just a subdivision (on k vertices or of size k). The *base set* of such a subdivision will denote the set of those k vertices. The *subdivision vertices* of the subdivision are the other vertices. Given such a base set B, we can naturally order its vertices as such - for vertices $u, v \in B$, we write $u <_B v$ if there is a subdivision vertex w so that $u \to w \to v$. Two subdivisions are said to be *disjoint* if they are vertex-disjoint.

Definition 2.1. Let T be a tournament. For two vertices u and v we say that u *s-dominates* v if there is a set S of size at least s such that $u \to S \to v$. Otherwise we say that u is *s-close* to v, and (u, v) is an s-close ordered pair. Similarly, for sets B_1, B_2, we say that B_1 s-dominates B_2 if every vertex in B_1 s-dominates every vertex in B_2.

2.1 Appending two subdivisions

We first make a trivial observation regarding when two disjoint subdivisions can be appended to each other in order to form a larger one.

Observation 2.2. *Let B_1 and B_2 be base sets of two disjoint subdivisions such that B_2 is k^2-dominating B_1, where $k := |B_1| + |B_2|$. Then $B_1 \cup B_2$ is the base set of a subdivision.*

We now give a more refined version of the above.

Lemma 2.3. *Let B_1 and B_2 be base sets of two disjoint subdivisions and suppose that B_2 is $4|B_1|k$-dominating B_1, where $k := |B_1| + |B_2|$. Then either $B_1 \cup B_2$ is the base set of a subdivision or there exist at least $2|B_1|k$ ordered pairs (u, v) in B_2 with $u <_{B_2} v$ which are k^2-close.*

Proof. Let N be the number of ordered pairs (u, v) in B_2 with $u <_{B_2} v$ which are k^2-close, and assume then that $N < 2|B_1|k$. Let us construct a new subdivision whose base set is $B_1 \cup B_2$. The ordering of the vertices in this subdivision will conform to the orderings of the given subdivisions using B_1 and B_2, with all vertices in B_2 coming before all vertices in B_1. For each of the N ordered pairs in B_2 which are k^2-close, let the subdivision vertex which connects them be the same one as used in the given subdivision. Similarly, for every ordered pair of vertices in B_1, again use the subdivision vertex coming from the subdivision given before. Now, since B_2 is $4|B_1|k$-dominating B_1, for every pair $(u, v) \in B_2 \times B_1$ there are at least $4|B_1|k - \binom{|B_1|^2}{2} - N - k > |B_1||B_2|$ available vertices which have not been used for previous subdivision vertices (or base vertices), and so we can greedily define subdivision vertices for those pairs. Finally, since every pair (u, v) in B_2 for which we have not yet found a subdivision vertex is not k^2-close, there are enough vertices for each pair to greedily complete our subdivision. $\square$

3 Proof of Theorem 1.1

We prove the statement by induction on k. The statement trivially holds for $k = 1$, so let us assume that $k > 1$ and that the claim holds for all smaller values. Let T be a tournament on $n = 10000k^2$ vertices, and suppose for the sake of contradiction that it does not contain a subdivision of size k. We first prove a lemma which shows that, under the given assumption, T in a certain sense must look random-like.

Lemma 3.1. *For all $R \subseteq V(T)$ there are least $0.99n$ vertices v so that $\frac{|R|}{2} - 150k^2 \leq |N_R^+(v)| \leq \frac{|R|}{2} + 150k^2$.*

Proof. Suppose otherwise and note that we can trivially assume that $|R| \geq 200k^2 = 0.02n$. Let us order the vertices in T as $v_1, \dots, v_n$ so that $|N_R^+(v_1)| \leq \ldots \leq |N_R^+(v_n)|$. Notice that it cannot be the case that for some $0.005n \leq i < j \leq i + 0.01n \leq 0.995n$, we have $|N_R^+(v_j)| \geq |N_R^+(v_i)| + k^2$. Indeed, let B_1 be a maximal base set of a subdivision contained in $\{v_1, \dots, v_i\}$ and B_2 a maximal base set of a subdivision in $\{v_{j+1}, \dots, v_n\}$. Then, B_2 is k^2-dominating B_1 since for each $u \in B_1$ and $w \in B_2$ we have

$$|N^+(w) \setminus N^+(u)| \geq |N_R^+(w)| - |N_R^+(u)| \geq |N_R^+(v_j)| - |N_R^+(v_i)| \geq k^2.$$

Hence, by Observation 2.2 and if $|B_1| + |B_2| \geq k$, we can greedily append subsets $B_1' \subseteq B_1$ and $B_2' \subseteq B_2$ to form a subdivision of size $|B_1'| + |B_2'| = k$, so we would achieve a contradiction. But note that $|B_1| + |B_2| \geq k$ holds indeed, since by induction

$$(|B_1| + |B_2|)^2 \geq \left(\sqrt{\frac{i}{10000}} + \sqrt{\frac{n-j}{10000}}\right)^2 \geq \frac{0.99n + 2\sqrt{i(n-j)}}{10000} \geq k^2,$$

where we used that $i - j \leq 0.01n$, $i \geq 0.005n$ and $j \leq 0.995n$.

Now, it must be the case that $|N_R^+(v_{0.005n})|$ and $|N_R^+(v_{0.995n})|$ differ by at most $100k^2$ since otherwise we could find i, j as in the above paragraph. Therefore, there exists some m such that at least $0.99n$ vertices v of T have $m - 50k^2 \leq |N_R^+(v)| \leq m + 50k^2$. To finish, at least $|R| - 0.01n = |R| - 100k^2$ of these vertices also belong to R and so, there are two such vertices u, v with $|N_R^+(v)| \geq 0.5(|R| - 100k^2) \geq |N_R^+(u)| - 100k^2$. Hence, $m + 50k^2 \geq 0.5(|R| - 100k^2) \geq m - 150k^2$ and so, $0.5|R| + 100k^2 \geq m \geq 0.5|R| - 100k^2$, which then concludes the proof. □

Now, let B be a set of vertices in T of maximal size b such that the following are satisfied:

- B is the base set of a subdivision.
- There is some $R \subseteq V(T)$ with $r := |R| \geq 0.2n$, such that either every vertex $v \in B$ has $|N_R^+(v)| \leq 0.2r + 8bk$ or every vertex has $|N_R^+(v)| \geq 0.8r - 8bk$.

Let S be the set of subdivision vertices of the given subdivision with base set B. Let us assume w.l.o.g. that every vertex $v \in B$ has $|N_R^+(v)| \leq 0.2r + 8bk$ (in the other case, the proof proceeds in an analogous manner). We first collect several simple claims implied by the above definition.

Claim 3.2. *There are at most $10000(b + 1)^2$ vertices u such that $|N_R^+(u)| \leq 0.2r + 8bk$.*

Proof. Assume that $b < k - 1$, since otherwise the claim is trivial given that T has $n = 10000k^2$ vertices. Now if there were more than $10000(b + 1)^2$ such vertices u, by induction we would have a base set of size larger than b formed by them. This would contradict the maximal choice of B. □

The next claim shows that B cannot be too large since then we could append it to a subdivision found among those vertices given by Lemma 3.1.

Claim 3.3. $b < 0.01k$.

Proof. Clearly, $b < k$ by assumption. By Lemma 3.1, there are at least $0.99n - b^2 \geq 0.99n - k^2$ vertices with out-degrees in R between $0.5r - 150k^2$ and $0.5r + 150k^2$, and which do not belong to S. Therefore, by induction there exists a subdivision of size at least $0.99k$ consisting of these vertices - let its base set be B' so that also $|B'| < k$. Notice that B' is $4k^2$-dominating B since for each $u \in B$ and $v \in B'$ we have

$$|N^+(v) \setminus N^+(u)| \geq |N^+_R(v)| - |N^+_R(u)| \geq 0.5r - 150k^2 - 0.2r - 8bk \geq 4k^2.$$

Hence, by Observation 2.2, we can greedily append B and B' to form a subdivision of size $b + |B'|$, implying by assumption that $b < k - |B'| \leq 0.01k$. □

Finally, the last claim shows that due to the maximal choice of B, we cannot have many vertices with out-degrees in R slightly larger than $0.2r + 8bk$.

Claim 3.4. *There are at most* $10000bk$ *vertices* u *with* $0.2r + 8bk \leq |N^+_R(u)| \leq 0.2r + 12bk$.

Proof. Suppose otherwise. Then, by induction, there is a subdivision of size at least $\sqrt{kb}$ formed by these vertices. By Claim 3.3, we have $\sqrt{kb} \geq 10b > \frac{3}{2}b$ and so, this contradicts the maximal choice of the base set B. □

Given these claims, take now a base set B' of a subdivision which is disjoint to $B \cup S$, so that the following conditions are satisfied:

- Every $u \in B'$ has $N^+_R(u) \geq 0.2r + 12bk$.
- $b + |B'| = k$.

Indeed, we can find such a B' since by Claims 3.2 and 3.4, the number of vertices with out-degrees in R at least $0.2r + 12bk$ which do not belong to S is at least

$$n - 10000(b+1)^2 - 10000kb - b^2 \geq 10000(k-b)^2,$$

where we are also using that $b < 0.01k$ by Claim 3.3. Therefore, by induction the desired base set B' with size $k - |B|$ exists.

Now, we claim that this gives a contradiction, since we can append the base sets B, B' to form a subdivision of size k. Indeed, because of the first condition of B', we have that B' is $4bk$-dominating B. So, we can apply Lemma 2.3 to either get the desired subdivision, or to guarantee at least $2kb$ many k^2-close ordered pairs in B'. We show that the latter is not possible. Note first that there are at most b many vertices in B' with out-degrees at most $0.2n$. Indeed, otherwise these would form a larger base set than B and would then contradict the maximality of B (setting a new set $R := V(T)$). This implies that there are at most kb ordered pairs in B' which are k^2-close and which have a vertex of out-degree at most $0.2n$; hence, at least kb ordered pairs in B' are k^2-close and only contain vertices with out-degrees at least $0.2n$. By averaging, it follows that there must exist such a vertex v which is k^2-close to more than b vertices in B' - this contradicts the maximality of B since these vertices then form the base set of a subdivision and have out-degrees in $N^+(v)$ of size at least $|N^+(v)| - k^2 > 0.8|N^+(v)|$, where $|N^+(v)| \geq 0.2n$. □

4 Concluding remarks

In this paper we confirmed the conjecture of Girão, Snyder and Popielarz by showing the existence of a constant $C = 10^4$ such that the oriented Ramsey number of the 1-subdivision of the transitive tournament on k vertices is at most Ck^2. Conversely, a trivial lower bound of $\binom{k}{2} + k$ can be obtained by noting that this is precisely the number vertices in the 1-subdivision. Although we made no attempt to optimize the value of C obtained in our proof, it is clear that our arguments do not give a tight result. In fact, it is natural to ask whether a 'spanning' behaviour for this problem is true at least in an asymptotic form, i.e., if the oriented Ramsey number is $k^2/2 + o(k^2)$.

Note. After this paper was written, we learnt that simultaneously with our work, Kim, Lee and Seo (preprint *arXiv:2110.05002*) proved an $O(k^2 \log\log k)$ upper bound for the oriented Ramsey number of the 1-subdivision of T_k.

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