Yau's conjecture for nonlocal minimal surfaces

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Abstract

We introduce nonlocal minimal surfaces on closed manifolds and establish a far-reaching Yau-type result: in every closed, n-dimensional Riemannian manifold we construct infinitely many nonlocal s-minimal surfaces. We prove that, when $s \in (0,1)$ is sufficiently close to 1, the constructed surfaces are smooth for $n = 3$ and $n = 4$, while for $n \geq 5$ they are smooth outside of a closed set of dimension $n - 5$.

Moreover, we prove surprisingly strong regularity and rigidity properties of finite Morse index s-minimal surfaces such as a "finite Morse index Bernstein-type result". These properties make nonlocal minimal surfaces ideal objects on which to apply min-max variational methods.

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1 Introduction

1.1 The "classical" Yau conjecture

The existence and regularity of minimal hypersurfaces in closed manifolds is one of the central questions in Riemannian geometry. Yau's conjecture (raised in 1982 by S.-T. Yau [101]) is a particularly famous and archetypal problem. It states that every closed three dimensional manifold must contain infinitely many smooth minimal surfaces. This problem exposes the enormous difficulties in applying variational methods to the area functional defined on the class of "surfaces".

Yau's conjecture was recently established by K. Irie, F. C. Marques, and A. Neves [71] (in the case of generic metrics) and by A. Song [96] (in full generality):

Theorem 1.1 ([71, 96])**.** *Let (M^n, g) be a closed Riemannian manifold of dimension $3 \leq n \leq 7$. Then, there exists an infinite number of smooth, closed, minimal hypersurfaces in M.*

To construct an infinite number of minimal surfaces, one must consider non-stable critical points of the area functional (since generic closed 3-manifolds contain only a finite number of stable minimal surfaces). These surfaces are naturally constructed using min-max (i.e., mountain-pass type) methods. The use of min-max methods for the area functional goes back to Almgren [3, 4] and afterwards Pitts [83] in the 1960s.

Several essential difficulties arise when trying to construct minimal surfaces employing a min-max scheme. The principal underlying issue is that, in the case of the area functional for surfaces, min-max sequences can be extremely noncompact, unless we work with very weak notions of convergence.

Indeed, suppose we are given a sequence of minimal surfaces with uniformly bounded Morse index. This is only a best case scenario, as the accumulation points of min-max sequences, if they exist, will be finite index minimal surfaces. A concrete instance of this best case scenario would be a sequence of shrinking catenoids that converge to a hyperplane (with multiplicity two). In this example the sequence does convergence in some weak sense (namely, in the sense of varifolds). However, the limitting object, a "double hyperplane", has arguably very few things in common with the catenoids that approximate it (e.g. their topologies or their total curvatures are completely different!). More generally, it is possible to guarantee that every sequence of minimal surfaces with uniformly bounded index and area will have converging subsequences, provided one chooses a weak enough notion of convergence. However, as in the example of the catenoid, weak convergence has undesired side effects: different sequences of minimal surfaces with "interesting topologies" may yield, in the limit, the same object with a loss of topology (with an integer multiplicity).

We emphasize that, as the example of the catenoid shows, there is no way to avoid the side effects: they must come with any convergence weak enough to guarantee compactness. Hence, a crucial difficulty in the proof of Theorem 1.1 is the extraordinary difficulty to ensure that multiple minimal surfaces, with bigger and bigger areas, constructed via a min-max method are distinct (and not the same one counted multiple times). Another very delicate question is the control on the topology of the minimal surfaces constructed via min-max (e.g. to construct minimal spheres in a given manifold). Such issues have only been solved, in particular cases, through a huge effort in several outstanding works, including: Almgren [2, 3, 4], Pitts [83], Schoen-Simon [92], Marques-Neves [76, 77, 78, 79], Irie-Marques-Neves [71] and Song [96]; or Simon-Smith [95], Haslhofer-Ketover [70] and Wang-Zhou [99], to cite a few.

1.2 Purpose of the paper

In this paper we introduce nonlocal minimal (hyper)surfaces —in the spirit of Caffarelli-Roquejoffre-Savin [25]— on closed Riemannian manifolds.

We must develop the theory of these new geometric objects from the beginning, starting by their most fundamental properties such as a monotonicity formula for stationary surfaces, the definition of surfaces with Morse index bounded by m and how it implies (almost!) stability in one out of $m+1$ disjoint domains, etc.

We obtain surprisingly strong estimates applying to finite Morse index nonlocal minimal surfaces, that do not hold for classical minimal surfaces. These estimates confer finite Morse index nonlocal minimal surfaces exceptional compactness and regularity properties, thanks to which we establish far-reaching existence and regularity results including a nonlocal analog of Theorem 1.1. Let us give a quick selection/highlights of our results here:

(i) Any closed manifold of dimension $n \geq 3$ must contain infinitely many nonlocal minimal (hyper)surfaces (that is, the nonlocal analog of Yau's conjecture). More precisely, given $s \in (0,1)$, for every $\mathfrak{p} \in \mathbb{N}$ there exist an s-minimal surface with Morse index $\leq \mathfrak{p}$ and energy comparable to $\mathfrak{p}^{s/n}$. These surfaces are smooth in low dimensions, and smooth away from a closed lower dimensional set in every dimension.

(ii) For $n \in \{3,4\}$ and $s \in (0,1)$ sufficiently close to 1 (the limit case $s=1$ formally corresponds to classical minimal surfaces), the following holds:

- Any smooth (embedded) s-minimal hypersurface of finite Morse index in $\mathbb{R}^n$ must be a hyperplane.
- In a closed n-dimensional manifold M^n, any sequence of smooth s-minimal (hyper)surfaces with uniformly bounded Morse index automatically satisfies uniform curvature and sheet separation estimates. As a consequence, any such sequence has a subsequence that converges smoothly and with multiplicity one to a (smooth) submanifold. In particular, if all the elements of the sequence are homeomorphic to the same topological space $\mathbb{X}$ then the limit is also homeomorphic to $\mathbb{X}$.

A main purpose of this paper is to demonstrate that nonlocal minimal surfaces are an ideal class on which to apply min-max methods, as they seem to prevent almost every pathology that arises for classical minimal surfaces (multiplicity, loss of topology). Thanks to their exceptional compactness and regularity properties, Morse theory for nonlocal minimal surfaces is in some sense as "flawless" as finite-dimensional Morse theory, at least from the functional analysis (i.e. compactness) perspective. It goes without saying that this is in striking contrast with the situation for classical minimal surfaces for the area functional.

For the reader interested in classical minimal surfaces, let us emphasize that nonlocal minimal surfaces approximate classical minimal surfaces (as the fractional parameter $s \in (0,1)$ converges to 1). So, in combination with the recent results in [36], which describe how stable nonlocal s-minimal surfaces converge towards classical minimal surfaces as $s \uparrow 1$, the results in the present paper give a powerful new method to construct classical minimal surfaces. This method resembles in many aspects the Allen-Cahn approximation in [67, 60, 61, 38], but presents several advantages (some of which are discussed in [36], and some of which will become evident this work).

Since we think that the paper can be of interest to readers who do not necessarily have any previous knowledge on nonlocal elliptic equations, we try give an accessible and mostly self-contained presentation. In particular we avoid non-essential cross references to the "nonlocal PDEs" literature. Moreover, we have spared no efforts in trying to make our proofs as efficient as possible.

1.3 Nonlocal minimal (hyper)surfaces on a closed Riemannian manifold

Nonlocal minimal (hyper)surfaces in $\mathbb{R}^n$ were first introduced and studied in [25]. In this section we define nonlocal minimal (hyper)surfaces on a closed Riemannian manifold, emphasizing the "canonical nature" of these new geometric objects.

Let (M^n, g) be an n-dimensional, closed Riemannian manifold, with $n \geq 2$. Let us start by giving a canonical definition of the fractional Sobolev seminorm $H^{s/2}(M)$. This can be done in at least three equivalent ways:

(i) Using the *heat kernel*[1] $H_M(t,p,q)$ of M, we can put

$$K_s(p,q) := \int_0^\infty H_M(p,q,t)\,\frac{dt}{t^{1+s/2}}. \tag{1}$$

We then define

$$[u]^2_{H^{s/2}(M)} := \iint_{M\times M} (u(p)-u(q))^2 K_s(p,q)\,dV_p\,dV_q. \tag{2}$$

The kernel $K_s(p,q)$ will be shown to be comparable to $\frac{1}{d(p,q)^{n+s}}$, and they coincide in the case $M = \mathbb{R}^n$ (up to a constant factor).

(ii) Following a *spectral approach*, we can set

$$[u]^2_{H^{s/2}(M)} = \sum_{k\geq 1} \lambda_k^{s/2} \langle u, \varphi_k\rangle^2_{L^2(M)} \tag{3}$$

where $\{\varphi_k\}_k$ is an orthonormal basis of eigenfunctions of the Laplace-Beltrami operator $(-\Delta_g)$ and $\{\lambda_k\}_k$ are the corresponding eigenvalues. For $s = 2$ this gives the usual $[u]^2_{H^1(M)}$ seminorm.

(iii) Considering a *Caffarelli-Silvestre type extension (cf. [28, 9])*, namely, a degenerate-harmonic extension problem in one extra dimension, we can set

$$[u]^2_{H^{s/2}(M)} = \inf\left\{ \int_{M\times\mathbb{R}_+} z^{1-s} |\widetilde{\nabla} U(p,z)|^2\,dV_p dz \quad \text{s.t.} \quad U(x,0) = u(x) \right\}.$$

Here $\widetilde{\nabla}$ denotes the Riemannian gradient of the manifold $\widetilde{M} = M \times \mathbb{R}_+$, with respect natural product metric $\widetilde{g} = g + dz \otimes dz$, and the infimum is taken over all U belonging to the weighted Hilbert space $\widetilde{H}^1(\widetilde{M})$ (see Definition 2.18 for the precise definition of this space, and we refer to Section 2 in general for all the basic properties of this extension characterization).

We will prove that (i)-(iii) define the same norm (not merely equivalent norms), up to explicit multiplicative constants. We emphasize that this gives a canonical definition of the $H^{s/2}(M)$ seminorm on a closed manifold.

The fractional perimeter on the whole closed manifold M is defined as follows.

[1] As customary, by heat kernel here we mean the fundamental solution of the heat equation $\partial_t u = \Delta u$ on M, where Δ denotes the Laplace-Beltrami operator on M.

Definition 1.2. Given $s \in (0,1)$ and a (measurable) set $E \subset M$, we define the *s-perimeter of E* as

$$\mathrm{Per}_s(E) := [\chi_E]^2_{H^{s/2}(M)} = \frac{1}{4}[\chi_E - \chi_{E^c}]^2_{H^{s/2}(M)} = 2\int_E \int_{E^c} K_s(p,q)dV_p dV_q\,, \tag{4}$$

where χ_E is the characteristic function of E, $E^c := M \setminus E$ and $[\cdot]^2_{H^{s/2}(M)}$ is defined by (2).

From the estimates that we will prove for K_s, one can see that for every set $E \subset M$ with smooth boundary, one has that $(1-s)\mathrm{Per}_s(E) \to \mathrm{Per}(E)$ as $s \uparrow 1$ (up to a multiplicative dimensional constant, see [14] and also [42, 31, 6] for further details on the computation in the case of $\mathbb{R}^n$).

Moreover, it is convenient to define localized or relative versions of the fractional perimeter, somewhat analogous to the classical relative perimeter.

Definition 1.3. Given a bounded open set $\Omega \subset M$ (with Lipchitz boundary), a *relative s-perimeter* in Ω is a functional denoted by $\mathrm{Per}_s(\,\cdot\,,\Omega)$ and satisfying the following two properties:

(I) $\mathrm{Per}_s(E,\Omega) - \mathrm{Per}_s(F,\Omega) = \mathrm{Per}_s(E) - \mathrm{Per}_s(F)$ for all (measurable) sets E and F that coincide outside Ω and $\mathrm{Per}_s(F) < \infty$.

(II) $\mathrm{Per}_s(E,\Omega) < \infty$ if ∂E is a smooth submanifold in a neighbourhood of the compact set $\overline{\Omega}$.

Throughout the paper we fix a relative s-perimeter defined similarly as in [25] (there for the case of the Euclidean space $\mathbb{R}^n$). We define the *relative s-perimeter of E in* Ω as

$$\mathrm{Per}_s(E,\Omega) := \iint_{(M\times M)\setminus(\Omega^c\times\Omega^c)} (\chi_E(p) - \chi_E(q))^2 K_s(p,q)dV_p dV_q\,,$$

where $\Omega^c := M \setminus \Omega$ is the complement of Ω. With this definition, one can easily check that properties (I) and (II) above holds. Moreover, it follows directly from its definition that the previous notion of relative s-perimeter satisfies the following properties.

- $\mathrm{Per}_s(E,\Omega) = \mathrm{Per}_s(E^c,\Omega)$ for every (measurable) $E \subset M$.
- If $E \subset \Omega$ or $E^c \subset \Omega$ then $\mathrm{Per}_s(E,\Omega) = \mathrm{Per}_s(E)$, where $\mathrm{Per}_s(E)$ is the s-perimeter on the whole manifold M defined in (4).
- Let $\Omega_1, \Omega_2 \subset M$ with $|\Omega_1 \cap \Omega_2| = 0$. Then $\mathrm{Per}_s(E, \Omega_1 \cup \Omega_2) \geq \mathrm{Per}_s(E,\Omega_1) + \mathrm{Per}_s(E,\Omega_2)$.
- Let $E_1, E_2 \subset M$ with $|E_1 \cap E_2| = 0$. Then $\mathrm{Per}_s(E_1 \cup E_2, \Omega) \leq \mathrm{Per}_s(E_1,\Omega) + \mathrm{Per}_s(E_2,\Omega)$.

Remark 1.4. Notice that there would be other possibilities to define a relative s-perimeter. For example, in view of the "spectral definition" (3) of the fractional perimeter, we could have defined a different relative perimeter by $\mathrm{Per}_s(E,\Omega) = \sum_{k\geq 1} \lambda_k^{s/2} \langle \chi_E, \varphi_k \rangle^2_{L^2(\Omega)}$. It is easy to check that this satisfes properties (I)-(II) above as well.

In this work we will denote by $\mathfrak{X}(\mathcal{U})$ the space of smooth vector fields in $\mathcal{U} \subseteq M$, by $\mathrm{spt}(X)$ the support of X and $\mathfrak{X}_c(\mathcal{U})$ the space of smooth vector fields with compact support in $\mathcal{U}$.

Definition 1.5. Let (M,g) be a closed Riemannian manifold. Given $s \in (0,1)$, the boundary ∂E of a set $E \subset M$ is said to be an *s-minimal surface* if $\mathrm{Per}_s(E) < \infty$ and, for every $X \in \mathfrak{X}(M)$, we have

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Per}_s(\psi_X^t(E)) = 0\,,$$

where $\psi_X^t : M \times \mathbb{R} \to M$ denotes the flow of X at time t.

The previous definition admits a natural local version.

Definition 1.6. Let (M,g) be a closed Riemannian manifold. Given $\mathcal{U} \subset M$ open, the boundary ∂E of a set $E \subset M$ is said to be an *s-minimal surface* in $\mathcal{U}$ if for every Lipchitz domain Ω with compact closure such that $\overline{\Omega} \subset \mathcal{U}$ we have $\mathrm{Per}_s(E,\Omega) < \infty$, and for every smooth and compactly supported vector field $X \in \mathfrak{X}_c(\mathcal{U})$ with $\mathrm{spt}(X) \subset \overline{\Omega}$ we have

$$\frac{d}{dt}\Big|_{t=0}\mathrm{Per}_s(\psi_X^t(E),\Omega) = 0\,.$$

Definition 1.7 (Morse index and stability)**.** Let (M,g) be a closed Riemannian manifold and ∂E be an s-minimal surface in $\mathcal{U} \subset M$ open (as in Definition 1.6). Then, ∂E is said to have *Morse index* at most m in $\mathcal{U}$ if for every Lipchitz domain Ω with compact closure such that $\overline{\Omega} \subset \mathcal{U}$, for every $(m+1)$ vector fields $X_0,\ldots,X_m \in \mathfrak{X}_c(\mathcal{U})$ with $\cup_{i=0}^m \mathrm{spt}(X_i) \subset \overline{\Omega} \subset \mathcal{U}$ we have $\mathrm{Per}_s(E,\Omega) < \infty$ and there exists some linear combination $X = a_0X_0 + \ldots + a_mX_m$ with $a_0^2 + a_1^2 + \ldots a_m^2 = 1$ such that

$$\frac{d^2}{dt^2}\Big|_{t=0}\mathrm{Per}_s(\psi_X^t(E),\Omega) \geq 0.$$

In the particular case $m = 0$, we say that ∂E is *stable* in $\mathcal{U}$.

Remark 1.8. As we prove[2] in Lemma 2.16 and Lemma 2.17, if $\mathrm{Per}_s(E,\Omega) < \infty$ and $X \in \mathfrak{X}_c(\mathcal{U})$ is such that $\mathrm{spt}(X) \subset \overline{\Omega}$ then the map $t \mapsto \mathrm{Per}_s(\psi_X^t(E),\Omega)$ is well-defined for all t and of class C^∞. Thus, the previous definitions are meaningful.

1.4 Main results

As explained before, one of the main goals of this paper is to establish the existence of infinitely many s-minimal surfaces on every closed manifold:

Theorem 1.9 (Fractional Yau-type result). *Let (M^n,g) be an n-dimensional, closed Riemannian manifold, with $n \geq 2$. Then, for every natural number $\mathfrak{p} \geq 1$, there exists an s-minimal surface $\Sigma^\mathfrak{p} = \partial E^\mathfrak{p}$ with Morse index at most $\mathfrak{p}$ —in the sense of Definition 1.7— and fractional perimeter*

$$C^{-1}\mathfrak{p}^{s/n} \leq (1-s)\mathrm{Per}_s(E^\mathfrak{p}) \leq C\mathfrak{p}^{s/n}, \tag{5}$$

for some $C = C(M) > 1$. In particular, M contains infinitely many s-minimal surfaces. Moreover, these surfaces are viscosity solutions to the NMS (i.e. Nonlocal Minimal Surface) equation (see Proposition 3.26), and satisfy the structural properties (17)-(19) in Proposition 1.30.

The regularity of the constructed surfaces depends on the classification of stable s-minimal cones (An open subset $E \subset \mathbb{R}^n$ is said to be a cone if E is an open set and $\lambda E = E$ for all $\lambda > 0$).

Definition 1.10. Given $s \in (0,1)$, we define the critical dimension n_s^* as the minimum dimension $n \geq 3$ such that there exists a smooth and stable s-minimal cone in $\mathbb{R}^n \setminus \{0\}$ which is not a hyperplane.

By [20] and [36], $n_s^* \geq 5$ for all $s \in (s_0,1]$, where $s_0 \in (0,1)$ is a universal constant. It is conjectured that in fact $n_s^* = 7$ for all s sufficiently close to 1. For $n = 8$ the Simons cone $E = \{x_1^2 + x_2^2 + x_3^2 + x_4^2 < x_5^2 + x_6^2 + x_7^2 + x_8^2\} \subset \mathbb{R}^8$, which is a minimizer in the classical case $s = 1$, is easily shown to be stable for all $s \in (s_0,1)$, for some $s_0 < 1$ sufficiently close to 1, so that $n_s^* \leq 8$ in this case[3].

We now state a regularity result for the constructed surfaces, which will be proved in Section 4.5.

Theorem 1.11 (Size of the singular set). *For $n \geq 3$, the surfaces $\{\Sigma^\mathfrak{p}\}_{\mathfrak{p}\in\mathbb{N}}$ of Theorem 1.9 are smooth submanifolds outside of a closed set $\mathrm{sing}(\Sigma^\mathfrak{p})$ of Hausdorff dimension at most $n - n_s^*$. In particular, $\mathrm{sing}(\Sigma^\mathfrak{p}) = \varnothing$ if $n < n_s^*$ (and this holds for $n = 3,4$ and s close to 1, since $n_s^* \geq 5$). Moreover, in the case $n = n_s^*$ the set $\mathrm{sing}(\Sigma^\mathfrak{p})$ is discrete.*

[2] Notice that for functions taking values in $\{\pm1\}$ the potential part of the energy vanishes and the Sobolev part of the energy gives the fractional perimeter.

[3] More generally, the natural generalization of the Simons cone to higher (even) n has been shown in [50] to be stable in dimension $n \geq 14$ for any $s \in (0,1)$. In particular, $n_s^* \leq 14$ for any $s \in (0,1)$.

The surfaces in Theorem 1.9 will be constructed as limits as $\varepsilon \to 0^+$ of solutions to the fractional Allen-Cahn equation on M. We emphasize that —in sharp contrast to the case of classical minimal surfaces— the Allen-Cahn approximation does not really play a crucial role in our construction. We just use it so that we are able to apply standard min-max existence results of critical points (like those in the book by Ghossoub [62]). What really makes our construction easier, in comparison with the classical case $s = 1$, are the very strong a priori estimates satisfied by finite Morse index s-minimal surfaces for $s < 1$, see Remark 1.18. Corresponding analog estimates are satisfied by Allen-Cahn solutions with bounded index, which allows us to send $\varepsilon \to 0$ without problems. In contrast, in the classical case this passage to the limit is really delicate: one is forced to use varifold convergence, and then multiplicity and neck pinching situations need to be ruled out. This requires generic metric assumptions and has only been done for $n = 3$ in [38].

Definition 1.12 (Fractional Allen-Cahn energy)**.** Let $s \in (0, 2)$ and $\varepsilon > 0$. Given $v : M \to \mathbb{R}$, we define the *fractional Allen-Cahn (abbr. A-C) energy* of v on the open set $\Omega \subseteq M$ as

$$\mathcal{E}_\Omega(v) := \mathcal{E}^{\mathrm{Sob}}_\Omega(v) + \mathcal{E}^{\mathrm{Pot}}_\Omega(v), \tag{6}$$

where

$$\mathcal{E}^{\mathrm{Sob}}_\Omega(v) := \frac{1}{4}\iint_{M\times M\setminus\Omega^c\times\Omega^c} (v(p) - v(q))^2 K_s(p, q) dV_p dV_q, \quad \mathcal{E}^{\mathrm{Pot}}_\Omega(v) := \varepsilon^{-s}\int_\Omega W(v)\, dx\,,$$

and $W(v) = \frac{1}{4}(1 - v^2)^2$ is the standard quartic double-well potential with wells at ± 1. We will sometimes denote $\mathcal{E}_\Omega$ by $\mathcal{E}^{\varepsilon,s}_\Omega$ or $\mathcal{E}^{\varepsilon}_\Omega$ if we want to stress the dependence of the energy from ε and/or s.

Note that, with this definition of the Allen-Cahn energy, we have

$$\mathcal{E}_\Omega(\chi_E - \chi_{E^c}) = \mathcal{E}^{\mathrm{Sob}}_\Omega(\chi_E - \chi_{E^c}) = \mathrm{Per}_s(E, \Omega)\,,$$

and

$$\mathcal{E}_{\Omega_1\cup\Omega_2}(v) \le \mathcal{E}_{\Omega_1}(v) + \mathcal{E}_{\Omega_2}(v)\,.$$

The double-well potential penalizes functions which are not identical to ± 1, and that is why one expects to find nonlocal s-minimal surfaces as the limits of critical points of this energy when $\varepsilon \to 0$.

A function $u : M \to \mathbb{R}$ is a critical point of $\mathcal{E}_\Omega$ if and only if it solves the fractional Allen-Cahn equation

$$(-\Delta)^{s/2}u + \varepsilon^{-s}W'(u) = 0 \quad \text{in } \Omega\,. \tag{7}$$

Here $(-\Delta)^{s/2}$ is the fractional Laplacian on (M, g), and it can be represented as (see Section 2 for details)

$$(-\Delta)^{s/2}u(p) = \int_M (u(p) - u(q))K_s(p, q)\, dV_q\,. \tag{8}$$

We also have a definition of Morse index, related to the second variation of the energy.

Proposition 1.13 (Second variation)**.** *Let $\Omega \subset M$ be an open set. Let $u \in H^{s/2}(M)$ be a critical point of $\mathcal{E}_\Omega$. Then, given $\xi \in C^1_c(\Omega)$, the second variation of $\mathcal{E}_\Omega$ at u is given by*

$$\mathcal{E}''_\Omega(u)[\xi, \xi] = \frac{1}{4}\iint_{(M\times M)\setminus(\Omega^c\times\Omega^c)} |\xi(p) - \xi(q)|^2 K_s(p, q)\, dV_p\, dV_q + \varepsilon^{-s}\int_\Omega W''(u)\xi^2\, dV\,. \tag{9}$$

Definition 1.14 (Morse index)**.** Let $\Omega \subset M$ an open set, and let $u \in H^{s/2}(M)$ be a critical point of $\mathcal{E}_\Omega$. The *Morse index* of u in Ω, denoted by $m_\Omega(u)$, is defined as the maximum dimension among all linear subspaces $\mathcal{L} \subset C^1_c(\Omega) \subset H^{s/2}(M)$ such that $\mathcal{E}''_\Omega(u)$ is negative definite on $\mathcal{L}$. Moreover, we say that u is *stable* in Ω if $m_\Omega(u) = 0$.

For $3 \le n < n^*_s$, we prove a strong regularity and separation result for s-minimal surfaces which are limits of Allen-Cahn solutions with bounded index (as in our case), and which will be proved in Section 4.4.

Definition 1.15 (Family of Allen-Cahn limits). A surface $\Sigma \subset M$ is said to belong to the class $\mathcal{A}_m(M)$ if $\Sigma = \partial E$ and there exists a sequence of functions $u_j : M \to (-1, 1)$ which are solutions to the Allen-Cahn equation (7) on M, with Morse index $m(u_j) \le m$ for all j, and parameters $\varepsilon_j \to 0$, such that $u_j \to u_0 := \chi_E - \chi_{E^c}$ in $L^1(M)$.

Theorem 1.16 (Uniform regularity and separation). *Let $s \in (0,1)$ and $3 \le n < n_s^*$. Let (M^n, g) be an n-dimensional, closed Riemannian manifold satisfying the flatness assumption $\mathrm{FA}_3(M, g, p, 1, \varphi)$ around p (see Definition 1.21). Assume that $\partial E \in \mathcal{A}_m(M)$ is an Allen-Cahn limit (see Definition 1.15).*
*Then ∂E is a $C^{1,\alpha}$ hypersurface for some $\alpha \in (0,1)$, with uniform regularity and separation estimates around p. That is, there exists a radius $R = R(n,s,m) > 0$ such that, after a rotation, $\varphi^{-1}(\partial E) \cap \left(\mathcal{B}_R^{n-1}(0) \times [-R, R]\right)$ is the graph of a **single** function $f : \mathcal{B}_R^{n-1}(0) \times \{0\} \to [-R, R]$ inside the chart, and*

$$\|f\|_{C^{1,\alpha}(B_R^{n-1} \times \{0\})} \le C(n,s,m)\,.$$

As an immediate application of Theorem (1.16) and Arzelà-Ascoli we obtain:

Corollary 1.17. *Let (M^n, g) be a closed Riemannian manifold, and let $s \in (0,1)$ and $3 \le n < n_s^*$. Then, every sequence $\Sigma_k = \partial E_k \in \mathcal{A}_m(M)$ admits a subsequence converging to some Σ_∞ in the strongest possible sense of convergence for submanifolds. In particular, if all elements Σ_k of the sequence are homeomorphic to the same topological space $\mathbb{X}$, then the limit Σ_∞ is also homeomorphic to $\mathbb{X}$.*

We now make an important remark.

Remark 1.18. Define the class $\mathcal{A}'_m(M)$ consisting of surfaces $\Sigma = \partial E \subset M$ such that there exists a sequence of s-minimal surfaces $\Sigma_j = \partial E_j$ of class C^2, with Morse index at most m for all j, such that $E_j \to E$ in $L^1(M)$. Then, the result of Theorem 1.16 would also hold for surfaces in $\mathcal{A}'_m(M)$ (and in particular for surfaces which are a priori known to be C^2), with a similar proof but with several technical modifications.

We conclude this section with a technical remark about dimension $n = 2$, which can be skipped on a first reading.

Remark 1.19. In dimension $n = 2$, the same regularity results would hold for limits of **stable** solutions of the Allen-Cahn equation (or for stable s-minimal surfaces which are of class C^2). Instead of arguing using the classification of stable cones in the proofs, one would argue that Lemma 4.16 is also true in the case when $n = 2$ and $E \subset \mathbb{R}^2$ is a cone which is the limit of **stable** Allen-Cahn solutions (for $n \ge 3$, the extra dimensions in the proof of Lemma 4.16 are used to reduce to an almost-stable case; for $n = 2$ if we assume stability the proof still goes through).

1.5 Other highlighted results and overview of the paper

1.5.1 Existence of min-max solutions to Allen-Cahn.

In Section 3.1 we exhibit in a simple manner the existence of critical points of the Allen-Cahn energy (6) on M, employing a min-max theorem as in [60]. Then, we prove lower and upper bounds for the energies of the constructed solutions. The complete statement of our result is the following.

Theorem 1.20 (Existence of min-max Allen-Cahn solutions). *Let (M^n, g) be an n-dimensional, closed Riemannian manifold, and fix $s_0 \in (0,1)$. Let $\mathfrak{p} \ge 1$ be a natural number (the number of min-max parameters) and $s \in (s_0, 1)$. Then, there exists $\varepsilon_{\mathfrak{p}} > 0$ (depending on M, s and $\mathfrak{p}$) such that for all $\varepsilon \in (0, \varepsilon_{\mathfrak{p}})$, there exists a solution $u_{\varepsilon,\mathfrak{p}}$ to the Allen-Cahn equation (7) on M with Morse index $m(u_{\varepsilon,\mathfrak{p}}) \le \mathfrak{p}$. Moreover, there exists $C > 1$ depending only on M and s_0 such that*

$$C^{-1}\mathfrak{p}^{s/n} \le (1-s)\,\mathcal{E}_M^{\varepsilon,s}(u_{\varepsilon,\mathfrak{p}}) \le C\mathfrak{p}^{s/n}\,. \tag{10}$$

After proving this result, our main goal will be to show that, for fixed $\mathfrak{p}$, as $\varepsilon \to 0$ a subsequence of the $u_{\varepsilon,\mathfrak{p}}$ converges in a strong sense to a fractional minimal surface $\Sigma^{\mathfrak{p}} = \partial E^{\mathfrak{p}} \subset M$, meaning in particular that

$$\mathrm{Per}_s(E^{\mathfrak{p}}) = \lim_{\varepsilon \to 0} \mathcal{E}_M^{\varepsilon,s}(u_{\varepsilon,\mathfrak{p}})\,.$$

Together with the bound given by (10), we get for every $\mathfrak{p} \in \mathbb{N}$ a fractional minimal surface $\Sigma^{\mathfrak{p}} = \partial E^{\mathfrak{p}}$ with fractional perimeter $\mathrm{Per}_s(E^{\mathfrak{p}}) \sim \mathfrak{p}^{s/n}$. This perimeter growth shows that the family of surfaces $\{\Sigma^{\mathfrak{p}}\}_{\mathfrak{p}\in\mathbb{N}}$ necessarily forms an infinite set, thus proving the fractional Yau's conjecture.

For this reason, a large portion of the article is devoted to studying the properties of solutions to the Allen-Cahn equation with a uniform upper bound on their Morse index. We now state and explain the main results of Sections 3.2 and 3.3.

1.5.2 Estimates for finite Morse index solutions to Allen-Cahn

In Section 3.2, we prove several estimates for finite Morse index solutions to the Allen-Cahn equation.

In order to precisely quantify the dependence of the constants in the estimates on the geometry of the ambient manifold, the notion of "local flatness assumption" will be very useful (this quantification will be important when we perform blow-up arguments). Let us introduce it below.

Here, as in the rest of the paper, $\mathcal{B}_R(0)$ denotes the Euclidean ball of radius R centered at 0 of $\mathbb{R}^n$, and $B_R(p)$ denotes the metric ball on M of radius R and center p.

Definition 1.21 (Local flatness assumption)**.** Let (M^n, g) be an n-dimensional Riemannian manifold and $p \in M$. For $R > 0$, we say that (M, g) satisfies the ℓ-th order *flatness assumption at scale R around the point p, with parametrization φ*, abbreviated as $\mathrm{FA}_\ell(M, g, R, p, \varphi)$, whenever there exists an open neighborhood V of p and a diffeomorphism

$$\varphi : \mathcal{B}_R(0) \to V, \quad \text{with } \varphi(0) = p\,,$$

such that, letting $g_{ij} = g\left(\varphi_*\left(\frac{\partial}{\partial x^i}\right), \varphi_*\left(\frac{\partial}{\partial x^j}\right)\right)$ be the representation of the metric g in the coordinates φ^{-1}, we have

$$(1 - \tfrac{1}{100})|v|^2 \le g_{ij}(x)v^i v^j \le (1 + \tfrac{1}{100})|v|^2 \quad \forall\, v \in \mathbb{R}^n \text{ and } \forall\, x \in \mathcal{B}_R(0)\,, \tag{11}$$

and

$$R^{|\alpha|}\left|\frac{\partial^{|\alpha|} g_{ij}(x)}{\partial x^\alpha}\right| \le \tfrac{1}{100} \quad \forall \alpha \text{ multi-index with } 1 \le |\alpha| \le \ell, \text{ and } \forall x \in \mathcal{B}_R(0). \tag{12}$$

Remark 1.22. Notice that for any smooth closed Riemannian manifold (M, g), given $\ell \ge 0$, there exists $R_0 > 0$ for which $\mathrm{FA}_\ell(M, g, R_0, p, \varphi_p)$ is satisfied for all $p \in M$, where φ_p can be chosen to be the restriction of the exponential map[4](of M) at p to the (normal) ball $\mathcal{B}_{R_0}(0) \subset T_pM \cong \mathbb{R}^n$.

Remark 1.23. The notion above of local flatness is used in our results to stress the fact that, once the local geometry of the manifold is controlled in the sense of Definition 1.21, then our estimates are independent of M. Interestingly, this makes our estimates of local nature even though the equation we deal with is nonlocal.

Remark 1.24. Throughout the paper the following scaling properties will be used several times.

(a) Given $M = (M, g)$ and $r > 0$, we can consider the "rescaled manifold" $\widehat{M} = (M, r^2 g)$. When performing this rescaling, the new heat kernel $H_{\widehat{M}}$ satisfies

$$H_{\widehat{M}}(p, q, t) = r^{-n} H_M(p, q, t/r^2)\,.$$

As a consequence, the "rescaled kernel" $\widehat{K}_s$ defining the s-perimeter on $\widehat{M}$ satisfies

$$\widehat{K}_s(p, q) = r^{-(n+s)} K_s(p, q).$$

(b) Concerning the flatness assumption, it is easy to show that $\mathrm{FA}_\ell(M, g, R, p, \varphi) \Rightarrow \mathrm{FA}_\ell(M, g, R', p, \varphi)$ for all $R' < R$ and $\mathrm{FA}_\ell(M, g, R, p, \varphi) \Leftrightarrow \mathrm{FA}_\ell(M, r^2 g, R/r, p, \varphi(r\,\cdot\,))$.

(c) Similarly, if $\mathrm{FA}_\ell(M, g, R, p, \varphi)$ holds, and $q \in \varphi(\mathcal{B}_R(0))$ is such that $\mathcal{B}_\varrho(\varphi^{-1}(q)) \subset \mathcal{B}_R(0)$, then $\mathrm{FA}_\ell(M, r^2 g, \varrho/r, q, \varphi_{\varphi^{-1}(q), r})$ holds, where $\varphi_{x,\rho} := \varphi(x + \rho\,\cdot\,)$.

[4]That is $\varphi_p = (\exp_p \circ\, i)|_{\mathcal{B}_{R_0}(0)}$ for any isometric identification of $i : \mathbb{R}^n \to TM_p$

One of the main results in the present work is the following estimate, to be proved in Section 3.2.3.

Theorem 1.25 (BV estimate). *Let M be a closed n-dimensional Riemannian manifold for which* $\mathrm{FA}_2(M, g, R, p, \varphi)$ *holds —see Definition 1.21. Let $s \in (0,1)$ and $u : B_R(p) \to (-1,1)$ be a solution of the Allen-Cahn equation (7) in $B_R(p) \subset M$ with parameter ε, and with Morse index $m_{B_R(p)}(u) \leq m$. Then*

$$\int_{B_{R/2}(p)} |\nabla u| dx \leq CR^{n-1},$$

for some $C = C(n, s, m)$.

Note that u the BV estimate above holds uniformly in the Allen-Cahn parameter ε. The nomenclature of "BV" is to be read as "*bounded variation*".

Remark 1.26. Our proof of Theorem 1.25 gives a control on the behaviour of the constant $C(n, s, m)$ as $s \uparrow 1$. More precisely, for fixed $s_\circ \in (0,1)$ we have $C(n, s, m) \leq C(n, s_\circ, m)/(1-s)$ for all $s \in (s_\circ, 1)$. In view of the results from [36], the sharp asymptotic for s close to 1 is expected to be $C(n, s, m) \leq C(n, s_\circ, m)/(1-s)^{1/2}$.

Another important result is a bound on the Sobolev and Potential parts of the energies, obtained in Section 3.2.5:

Theorem 1.27 (Energy estimate). *Let $u : M \to (-1,1)$ be a solution of (7) in $B_R(p) \subset M$ with parameter ε and Morse index $m_{B_R(p)}(u) \leq m$. Suppose that* $\mathrm{FA}_2(M, g, R, p, \varphi)$ *holds —see Definition 1.21. Then*

$$\mathcal{E}^{\mathrm{Sob}}_{B_{R/2}(p)}(u) \leq CR^{n-s}, \tag{13}$$

and there exists $\varepsilon_0 = \varepsilon_0(n, s, m)$ such that for $\varepsilon < \varepsilon_0$

$$\mathcal{E}^{\mathrm{Pot}}_{B_{R/2}(p)}(u) \leq C\left(\frac{\varepsilon}{R}\right)^{\beta} R^{n-s}, \tag{14}$$

where $C = C(n, s, m)$ and $\beta := \min\left(\frac{1-s}{2}, s\right) > 0$.

Section 3.2.4 will prove the following result, and which will give, among other things, that the level sets of Allen-Cahn solutions converge to the limit (hyper)surfaces in the Hausdorff distance of sets.

Proposition 1.28 (Density estimates). *Let $u : M \to (-1,1)$ be a solution of (7) in $B_R(p) \subset M$ with Morse index $m_{B_R(p)}(u) \leq m$, and suppose that* $\mathrm{FA}_2(M, g, R, p, \varphi)$ *holds —see Definition 1.21. Then, there exist positive constants ω_0, C_0 and ε_0, depending only on n, s, and m, such that the following holds: whenever $\varepsilon \leq \varepsilon_0$, $R \geq C_0\varepsilon$ and*

$$R^{-n} \int_{B_R(p)} |1 + u_\varepsilon| \, dx \leq \omega_0 \qquad \left(\text{respectively,} \quad R^{-n} \int_{B_R(p)} |1 - u_\varepsilon| \, dx \leq \omega_0\right), \tag{15}$$

then

$$\left\{u_\varepsilon \geq -\tfrac{9}{10}\right\} \cap B_{R/2}(p) = \varnothing \qquad \left(\text{respectively,} \quad \left\{u_\varepsilon \leq \tfrac{9}{10}\right\} \cap B_{R/2}(p) = \varnothing\right). \tag{16}$$

1.5.3 Convergence results

In Section 3.3, these estimates that we have just stated are used to show the convergence, as $\varepsilon \to 0$, of solutions of (7) to a limit interface.

Remark 1.29. We will use many times the fact that, for a set of finite perimeter, we can replace E by a representative (that is, equivalent up to a negligible set) $\widetilde{E}$ such that all the points in the topological boundary $\partial\widetilde{E}$ have density different from zero or one.

The complete statement of our convergence result is the following.

Theorem 1.30. ***(Convergence as*** $\varepsilon \to 0^+$***).*** *Fix* $s \in (0,1)$*. Let* u_{ε_j} *be a sequence of solutions of (7) on* M *with parameters* $\varepsilon_j \to 0$ *and Morse index* $m(u_{\varepsilon_j}) \le m$*. Then, there exist a subsequence, still denoted by* u_{ε_j}*, and a nonlocal s-minimal surface* $\Sigma = \partial E$ *with Morse index at most m in the weak sense, such that*

$$u_{\varepsilon_j} \xrightarrow{H^{s/2}} u_0 = \chi_E - \chi_{E^c}\,.$$

In particular $\mathcal{E}_M^{\mathrm{Sob}}(u_{\varepsilon_j}) \to \mathrm{Per}_s(E) = \mathcal{E}_M^{\mathrm{Sob}}(u_0)$*. Moreover,* $\mathcal{E}_M^{\mathrm{Pot}}(u_{\varepsilon_j}) \to 0 = \mathcal{E}_M^{\mathrm{Pot}}(u_0)$*.*

In addition, up to changing E in a set of measure zero, we have

$$\mathrm{int}(E) \supseteq \left\{p \in M \,:\, \liminf_{r\downarrow 0} \tfrac{|E\cap B_r(p)|}{|B_r(p)|} = 1\right\}, \tag{17}$$

$$M \setminus \overline{E} \supseteq \left\{p \in M \,:\, \limsup_{r\downarrow 0} \tfrac{|E\cap B_r(p)|}{|B_r(p)|} = 0\right\}, \tag{18}$$

$$\Sigma = \left\{p \in M \,:\, \tfrac{|E\cap B_r(p)|}{|B_r(p)|} \in [c, 1-c] \quad \forall r \in (0, r_\circ(p)), \ \ \text{for some } r_\circ(p) > 0\right\}, \tag{19}$$

where $\Sigma = \partial E$ *represents the topological boundary of E as usual. Also, for all given* $c \in (-1,1)$

$$d_{\mathrm{H}}(\{u_{\varepsilon_j} \ge c\}, E) \to 0\,, \ \ \text{as } j \to \infty\,, \tag{20}$$

where $d_{\mathrm{H}}(X,Y) = \inf\{\rho > 0 \,:\, X \subseteq \bigcup_{y\in Y} B_\rho(y) \text{ and } Y \subseteq \bigcup_{x\in X} B_\rho(x)\}$ *denotes the standard Hausdorff distance between subsets of M.*

As explained in Section 1.5.1, this result combined with Theorem 1.20 will give Theorem 1.9.

1.5.4 Regularity in low dimensions

Sections 4.1–4.5 are devoted to proving the uniform regularity and separation estimate in low dimensions of Theorem 1.16, as well as the result of Theorem 1.11 on the size of the singular set in higher dimensions.

First, Sections 4.1 and 4.2 define and describe the properties of blow-ups of s-minimal surfaces, in particular when they are the limits of Allen-Cahn solutions with bounded index.

Then, in Section 4.3 it is shown that such blow-ups converge to a single hyperplane in $\mathbb{R}^n$, under the assumption that stable s-minimal cones in $\mathbb{R}^n$ are flat; that is, when $n < n_s^*$ is less than the critical dimension of Definition 1.10. This classification result for blow-ups is used in Section 4.4 to prove Theorem 1.16. The proof is done by a blow-up and contradiction strategy to show that the surfaces are flat at some fixed scale, and an improvement of flatness theorem[5] which holds for all nonlocal minimal surfaces which are viscosity solutions of the zero nonlocal mean curvature equation, a criticality condition much weaker than minimality.

Finally, a dimension-reduction argument combined with the previous strategy allows to prove Theorem 1.11 for all n.

1.5.5 Bernstein and De Giorgi type results

Section 4.6 establishes the validity of the "finite Morse index versions" of the nonlocal De Giorgi and Bernstein conjectures, once again under the assumption of the classification of stable cones. This represents a remarkable departure from the behavior of classical minimal surfaces, and of solutions to the classical (local) Allen-Cahn equation, with bounded index.

The proof of both results uses the same strategy as the proof, in Section 4.3, of the fact that blow-up limits of s-minimal surfaces satisfying a certain list of properties, which were in particular satisfied by limits of Allen-Cahn, need to be half-spaces.

[5]This improvement of flatness theorem was proved on $\mathbb{R}^n$ in the seminal article [25] which first defined nonlocal minimal surfaces, and the version of it on manifolds has been recently proved in [81].

The Bernstein conjecture (today theorem) states that graphical complete minimal hypersurfaces must be hyperplanes in low dimensions. See [2, 11, 37, 48, 58, 84, 94] for related generalizations to the classes of minimizing and stable hypersurfaces.

In Section 4.6 we establish:

Theorem 1.31 (Finite index nonlocal Bernstein-type result)**.** *Let $s \in (0,1)$ and $3 \le n < n_s^*$, where n_s^* is the critical dimension (see Definition 1.10).*

Then, any finite Morse index s-minimal surface in $\mathbb{R}^n$ of class C^2 is a half-space.

Under the assumption of stability (Morse index zero) the previous theorem was established in [19] and in the case of minimizers if follows from [25].

The De Giorgi conjecture is a famous related statement about certain entire solutions to the Allen-Cahn equation being one-dimensional, or equivalently about their level sets being hyperplanes in low dimensions. See [1, 5, 52, 64, 87, 88, 89, 19] for related previous results in the minimizing and stable cases. In Section 4.6, we show:

Theorem 1.32 (Finite index nonlocal De Giorgi-type result)**.** *Let $s \in (0,1)$ and $3 \le n < n_s^*$, where n_s^* is the critical dimension (see Definition 1.10).*

Then, every finite Morse index solution u of $(-\Delta)^{s/2}u + W'(u) = 0$ in $\mathbb{R}^n$ is a 1D layer solution, namely, $u(x) = \phi(e \cdot x)$ for some $e \in \mathbb{S}^{n-1}$ and increasing function $\phi : \mathbb{R} \to (-1,1)$.

Under the assumption of stability (Morse index zero) the previous theorem was established in [19] and for minimizers it followed from [25, 47].

Remark 1.33. Recall that $n_s^* \ge 5$ for $s \in (s_0, 1)$ for some universal constant $s_0 \in (0,1)$. In particular, Theorems 1.31 and 1.32 hold for $n = 3, 4$ and $s \in (s_0, 1)$.

2 The fractional setting on manifolds

Throughout the paper (unless otherwise stated) (M, g) will be a closed (i.e. compact and without boundary) Riemannian manifold of dimension n.

2.1 The fractional Laplacian on (M, g)

Taking inspiration from the case of $\mathbb{R}^n$ in [25], in this section we give several equivalent definitions for the fractional Laplacian on a closed Riemannian manifold (M, g). It is natural here to define the fractional powers $(-\Delta)^{s/2}$ for any $s \in (0,2)$. On the other hand, in the rest of the paper we will always restrict to $s \in (0,1)$: the norm $H^{s/2}(M)$ used to define the fractional perimeter is only considered with $s \in (0,1)$ since the $H^{1/2}$ energy of a characteristic function is infinite.

2.1.1 Spectral and singular integral definitions

The fractional Laplacian $(-\Delta)^{s/2}$ can be define as the $s/2$-th power (in the sense of spectral theory) of the usual Laplace-Beltrami operator on a Riemannian manifold, through Bochner's subordination.

Given $\lambda > 0$ and $s \in (0,2)$, the following numerical formula holds:

$$\lambda^{s/2} = \frac{1}{\Gamma(-s/2)} \int_0^\infty (e^{-\lambda t} - 1) \frac{dt}{t^{1+s/2}} \,. \tag{21}$$

Formally applying the above relation to the operator $L = (-\Delta)$ in place of λ, one obtains the following definition for the fractional Laplacian.

Definition 2.1 (**Spectral definition**). Let $s \in (0,2)$. The fractional Laplacian $(-\Delta)^{s/2}$ is the operator that acts on regular functions u by

$$(-\Delta)^{s/2} u = \frac{1}{\Gamma(-s/2)} \int_0^\infty (e^{t\Delta} u - u) \frac{dt}{t^{1+s/2}} \,. \tag{22}$$

Here, the expression $e^{t\Delta} u$ is to be understood as the solution of the heat equation on M at time t and with initial datum u —we refer to [72] for details.

Remark 2.2. On a closed Riemannian manifold a closely related definition of the fractional Laplacian is available: if $\{\phi_k\}_{k=1}^\infty$ is an $L^2(M)$ orthonormal basis of eigenfunctions for $(-\Delta)$ with eigenvalues

$$0 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_k \xrightarrow{k\to\infty} +\infty$$

and $u \in L^2(M)$ then

$$(-\Delta)^{s/2} u = \sum_{k=1}^\infty \lambda_k^{s/2} \langle u, \phi_k \rangle_{L^2(M)} \phi_k \,.$$

Since the solution to the heat equation on M with initial datum an eigenfunction ϕ_k is given by $e^{t\Delta}\phi_k = e^{-\lambda_k t}\phi_k$, the above definition is easily shown to be identical to (22) by first observing that they coincide for eigenfunctions (thanks to (21)), and then extending the result by approximation. In [97] all the details of this equivalence are carried out in the case of certain positive second order operators with discrete spectrum on a domain $\Omega \subset \mathbb{R}^n$. In our case of $(-\Delta)$ on a closed Riemannian manifold, the proof is then completely analogous. Nevertheless, this characterization will not be used in what follows and is given only as complementary information.

The second definition for the fractional Laplacian, closely related to the spectral one, expresses it as a singular integral. It will be our working definition in a substantial portion of the article.

Definition 2.3 (**Singular integral definition**). The fractional Laplacian $(-\Delta)^{s/2}$ of order (of differentiation) $s \in (0,2)$ is the operator that acts on a regular function u by

$$(-\Delta)^{s/2} u(p) = \int_M (u(p) - u(q)) K_s(p,q) \, dV_q \,, \tag{23}$$

where $K_s(p,q) : M \times M \to \mathbb{R}$ is given by[6]

$$K_s(p,q) = \frac{s/2}{\Gamma(1-s/2)} \int_0^\infty H_M(p,q,t) \frac{dt}{t^{1+s/2}} \,, \tag{24}$$

and where $H_M : M \times M \times (0,\infty) \to \mathbb{R}$ denotes the usual heat kernel on M.

Remark 2.4. If the compact manifold M is replaced by the Euclidean space $\mathbb{R}^n$ then

$$K_s(x,y) = \frac{s/2}{\Gamma(1-s/2)} \int_0^\infty H_{\mathbb{R}^n}(x,y,t) \frac{dt}{t^{1+s/2}} = \frac{s/2}{\Gamma(1-s/2)} \int_0^\infty \left(\frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} \right) \frac{dt}{t^{1+s/2}} = \frac{\alpha_{n,s}}{|x-y|^{n+s}} \,,$$

where

$$\alpha_{n,s} = \frac{2^s \Gamma\left(\frac{n+s}{2}\right)}{\pi^{n/2} |\Gamma(-s/2)|} = \frac{s 2^{s-1} \Gamma\left(\frac{n+s}{2}\right)}{\pi^{n/2} \Gamma(1-s/2)}. \tag{25}$$

Hence we recover the usual form of the fractional Laplacian on $\mathbb{R}^n$.

[6] Note that $\frac{1}{|\Gamma(-s/2)|} = \frac{s/2}{\Gamma(1-s/2)}$.

The equivalence[7] between the two definitions (22) and (23) is immediately seen by expressing the solution $e^{t\Delta}u$ to the heat equation in terms of the initial datum u as

$$(e^{t\Delta}u)(p) = \int_M u(q) H_M(p,q,t)\, dV_q\,,$$

using $\int_M H_M(p,q,t)\, dV_q = 1$ and changing the order of integration. Note that, on a noncompact Riemannian manifold, the mass preservation property $\int_M H_M(p,q,t)\, dq = 1$ may fail leading to undesired phenomena such as the fractional Laplacian of a constant being different from zero. It is thus natural to assume that M is stochastically complete (although actually the local estimates of this paper do not need such an assumption).

2.1.2 Properties of the kernel

This section gives important estimates on the singular kernel $K_s(p,q)$.

Several times in the paper we will need to compare locally the heat kernel $H_M(p,q,t)$ or the singular kernel $K_s(p,q)$ with the standard ones on $\mathbb{R}^n$. A first result is the following:

Lemma 2.5. *Let g be a smooth metric on $\mathbb{R}^n$ such that $\frac{|v|^2}{4} \le g_{ij}(x)v^iv^j \le 4|v|^2$ and $|Dg_{ij}(x)| \le 1$ for all $x, v \in \mathbb{R}^n$. Denote $M := (\mathbb{R}^n, g)$ and let K_s be defined by (24). Then, there exists positive constants $c_i = c_i(n)$ for $1 \le i \le 6$ such that*

$$\frac{c_1}{t^{n/2}} e^{-\frac{|x-y|^2}{c_2 t}} \le H_M(x,y,t) \le \frac{c_3}{t^{n/2}} e^{-\frac{|x-y|^2}{c_4 t}}\,,$$

and

$$c_5 \frac{\alpha_{n,s}}{|x-y|^{n+s}} \le K_s(x,y) \le c_6 \frac{\alpha_{n,s}}{|x-y|^{n+s}}\,,$$

for all $(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times [0,\infty)$.

Proof. The two sided estimates for the heat kernel H_M follows directly from the classical parabolic estimates of Aronson [8]. The second inequality follows by integrating the first one, from the definition (24) of $K_s(x,y)$. □

In all the section we will use the (standard) multi-index notation for derivatives. A multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ will be an n-tuple of nonnegative integers (in other words $\alpha \in \mathbb{N}^n$). We define

$$|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

For a function $f : \mathbb{R}^n \to \mathbb{R}$ is of class C^ℓ we shall use the notation

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} f := \frac{\partial^{\alpha_1+\alpha_2+\cdots\alpha_n} f}{(\partial x^1)^{\alpha_1} (\partial x^2)^{\alpha_2} \cdots (\partial x^n)^{\alpha_n}}\,.$$

For $\alpha = 0$, we put $\frac{\partial^{|0|}}{\partial x^0} f := f$.

The first property concerns the behaviour of the kernel around a point satisfying flatness assumptions.

Proposition 2.6. *Let (M,g) be a Riemannian n-manifold, not necessarily closed, $s \in (0,2)$ and let $p \in M$. Assume $\mathrm{FA}_\ell(M,g,R,p,\varphi)$ holds and denote $K(x,y) := K_s(\varphi(x), \varphi(y))$.*

Given $x \in \mathcal{B}_R(0)$, let $A(x)$ denote the positive symmetric square root of the matrix $(g_{ij}(x))$ —g_{ij} being the metric in coordinates φ^{-1}— and, for $x, z \in \mathcal{B}_{R/2}(0)$, define

$$k(x,z) := K(x, x+z) \quad \text{and} \quad \widehat{k}(x,z) := k(x,z) - \alpha_{n,s} \frac{\det\big(A(x)\big)}{|A(x)z|^{n+s}}.$$

[7] With our choice of constants, this is an equality and not only an equivalence.

Then

$$\left|\widehat{k}(x,z)\right| \le R^{-1}\frac{C(n,s)}{|z|^{n+s-1}} \quad \textit{for all } x,z\in\mathcal{B}_{R/4}(0)\,, \tag{26}$$

and, for every multi-indices α,β with $|\alpha|+|\beta|\le\ell$, we have

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial z^\beta}k(x,z)\right| \le \frac{C(n,s,\ell)}{|z|^{n+s+|\beta|}} \quad \textit{for all } x,z\in\mathcal{B}_{R/4}(0). \tag{27}$$

Moreover, for all $x\in\mathcal{B}_{R/4}(0)$ and for all $q\in M\setminus\varphi(\mathcal{B}_R(0))$ we have

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}K_s(\varphi(x),q)\right| \le \frac{C(n,\ell)}{R^{n+s}}\,, \tag{28}$$

and

$$\int_{M\setminus\varphi(\mathcal{B}_R(0))}\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}K_s(\varphi(x),q)\right|dV_q \le \frac{C(n,\ell)}{R^s}, \tag{29}$$

for every multi-index α with $|\alpha|\le\ell$.

To prove Proposition 2.6 we will need several preliminary lemmas with properties of the heat kernel of M. Their proofs are rather standard and are included in the appendix for the convenience of the reader.

Lemma 2.7 (Appendix A)**.** *Let (M,g) be a Riemannian n-manifold, $p\in M$, and assume $\mathrm{FA}_0(M,g,1,p,\varphi)$ holds. Then*

$$1-C\exp(-c/t) \le \int_{\varphi(\mathcal{B}_{1/2}(0))}H_M(p,q,t)dV_q \le 1, \quad \textit{for all } t>0,$$

with $C,c>0$ depending only on n.

Lemma 2.8 (Appendix A)**.** *Under the same assumptions as in Proposition 2.6, for all $q\in M\setminus\varphi(\mathcal{B}_1(0))$ we have*

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}H_M(\varphi(x),q,t)\right| \le C\exp(-c/t), \quad \textit{for } (x,t)\in\mathcal{B}_{1/2}(0)\times[0,\infty) \tag{30}$$

and for every multi-index α with $|\alpha|\le\ell$, with $C,c>0$ depending only on n and ℓ.

Lemma 2.9 (Appendix A)**.** *Under the same assumptions as in Proposition 2.6. We have*

$$\int_{M\setminus\varphi(\mathcal{B}_1(0))}\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}H_M(\varphi(x),q,t)\right|dV_q \le C\exp(-c/t), \quad \textit{for } (x,t)\in\mathcal{B}_{1/2}(0)\times[0,\infty) \tag{31}$$

and for every multi-index α with $|\alpha|\le\ell$, with $C,c>0$ depending only on n and ℓ.

Lemma 2.10 (Appendix A)**.** *Let (M,g) and (M',g') be two Riemannian n-manifolds. Assume that both M and M' satisfy the flatness assumptions $\mathrm{FA}_\ell(M,g,1,p,\varphi)$ and $\mathrm{FA}_\ell(M',g,1,p',\varphi')$ respectively, and suppose that $g_{ij}\equiv g'_{ij}$ in $\mathcal{B}_1(0)$ in the coordinates induced by φ^{-1} and $(\varphi')^{-1}$.*

Then, letting $H(x,y,t):=H_M(\varphi(x),\varphi(y),t)$ and $H'(x,y,t):=H_{M'}(\varphi'(x),\varphi'(y),t)$, we have that the difference $(H-H')(x,y,t)$ is of class C^ℓ in $\mathcal{B}_{1/2}(0)\times\mathcal{B}_{1/2}(0)\times[0,\infty)$ and

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial y^\beta}(H-H')(x,y,t)\right| \le C\exp(-c/t) \quad \textit{for } (x,y,t)\in\mathcal{B}_{1/2}(0)\times\mathcal{B}_{1/2}(0)\times[0,\infty),$$

whenever α and β are multi-indexes satisfying $|\alpha|+|\beta|\le\ell$, with $C,c>0$ depending only on n and ℓ.

As a first consequence of Lemma 2.10 we have that the following "local version" of Lemma 2.5 above also holds.

Lemma 2.11. *Let $s_0 \in (0,2)$ and $s \in (s_0, 2)$. Let (M,g) be a Riemannian n-manifold and $p \in M$. Assume that* $\mathrm{FA}_1(M,g,p,1,\varphi)$ *holds. Then*

$$c_7 \frac{\alpha_{n,s}}{|x-y|^{n+s}} \le K_s(\varphi(x),\varphi(y)) \le c_8 \frac{\alpha_{n,s}}{|x-y|^{n+s}},$$

for all $x, y \in \mathcal{B}_{1/2}(0)$, where $c_7, c_8 > 0$ depends on n and s_0.

Proof. Take $\eta \in C_c^\infty(\mathcal{B}_1(0))$ with $\chi_{\mathcal{B}_{1/2}(0)} \le \eta \le \chi_{\mathcal{B}_1(0)}$ and let $g'_{ij} := g_{ij}\eta + (1-\eta)\delta_{ij}$. This is a metric on $\mathbb{R}^n$ with $g'_{ij} = g_{ij}$ in $\mathcal{B}_{1/2}(0)$. Denote by K_s, K'_s and H, H' the singular kernels and heat kernels of (M,g) and $(\mathbb{R}^n, g')$ respectively. Then, by Lemma 2.10 applied to the manifolds (M,g) and $(\mathbb{R}^n, g')$ we have, for $x, y \in \mathcal{B}_{1/4}(0)$:

$$\begin{aligned} \left|K_s(\varphi(x),\varphi(y)) - K'_s(x,y)\right| &\le \frac{s/2}{\Gamma(1-s/2)} \int_0^\infty \left|H(\varphi(x),\varphi(y),t) - H'(x,y,t)\right| \frac{dt}{t^{1+s/2}} \\ &\le \frac{Cs}{\Gamma(1-s/2)} \int_0^\infty e^{-c/t} \frac{dt}{t^{1+s/2}} \le C(2-s), \end{aligned}$$

for some dimensional $C = C(n)$. Then, the result follows directly by Lemma 2.5 (and the explicit formula (25) for $\alpha_{n,s}$) for $x, y \in \mathcal{B}_{1/4}(0)$, and the conclusion also holds for $x, y \in \mathcal{B}_{1/2}(0)$ by a standard covering argument. □

Proposition 2.12 (Appendix A)**.** *Assume that $M = (\mathbb{R}^n, g)$ with $g = (g_{ij}(x))$ satisfying*

$$\tfrac{1}{2}\mathrm{id} \le (g_{ij}) \le 2\mathrm{id} \qquad \text{and} \qquad \left|\frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{ij}\right| \le 1 \quad \text{for all } |\alpha| \le \ell, \tag{32}$$

for some $\ell \ge 1$. Let $H(x,y,t)$ be heat kernel of M.

For $x \in \mathbb{R}^n$ let $A(x)$ denote the (unique) positive definite symmetric square root of the matrix $g(x) = (g_{ij}(x))$, and define $h(z,x,t)$ by the identity

$$H(x,y,t) = \frac{1}{t^{n/2}} h\left(\frac{A(x)(y-x)}{\sqrt{t}}, x, t\right).$$

Define also:

$$h_\circ(x,z,t) = h_\circ(z) := \frac{1}{(4\pi)^{n/2}} e^{-|z|^2/4} \qquad \text{and} \qquad \widehat{h} := h - h_\circ.$$

Then, there are positive dimensional C and c such that

$$|\widehat{h}| \le C \min(1, \sqrt{t}) e^{-c|z|^2} \quad \text{for all } (x,z,t) \in \mathbb{R}^n \times \mathbb{R}^n \times (0,\infty).$$

Morever, we have

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial z^\beta} h\right| \le Ce^{-c|z|^2} \quad \text{for all } (x,z,t) \in \mathbb{R}^n \times \mathbb{R}^n \times (0,1) \text{ and } \alpha, \beta \text{ with } |\alpha| + |\beta| \le \ell,$$

for positive constants C and c depending only on n and ℓ.

Now, we have all the ingredients to give the proof of Proposition 2.6.

Proof of Proposition 2.6. Note that the statement is scaling invariant. Hence, with no loss of generality, we assume that $R = 1$. Moreover, it suffices to consider the case $M = (\mathbb{R}^n, g)$, $p = 0$, $\varphi = \mathrm{id}$, and g_{ij} satisfying the assumptions of Proposition 2.12:

Indeed, similarly to the proof of Corollary 2.11, in the general case we can fix a radially nonincreasing cutoff function $\eta \in C_c^\infty(\mathcal{B}_1)$ such that $\eta \equiv 1$ in $\mathcal{B}_{7/8}$ and consider the "extended" metric $g'_{ij} := g_{ij}\eta +$

$\delta_{ij}(1-\eta)$. Observe that (M,g) and $(\mathbb{R}^n,g')$ the assumptions of Lemma 2.10 with $M'=\mathbb{R}^n$ and $\varphi'=\mathrm{id}$. Let $H(x,y,t)$ and $H'(x,y,t)$ be defined as in Lemma 2.10.

Recall that, by definition, for all $x,y\in\mathcal{B}_1(0)$

$$K(x,y)=K_s(\varphi(x),\varphi(y))=c_s\int_0^\infty H_M(\varphi(x),\varphi(y),t)\frac{dt}{t^{1+s/2}}=c_s\int_0^\infty H(x,y,t)\frac{dt}{t^{1+s/2}},\tag{33}$$

where $c_s=\frac{s/2}{\Gamma(1-s/2)}$. Let likewise

$$K'(x,y)=c_s\int_0^\infty H'(x,y,t)\frac{dt}{t^{1+s/2}}.$$

Now, thanks to Lemma 2.10 we obtain, for all $x,y\in\mathcal{B}_{1/2}$:

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial y^\beta}(K-K')(x,y,t)\right|\le c_s\int_0^\infty\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial y^\beta}(H-H')(x,y,t)\right|\frac{dt}{t^{1+s/2}}\le Cs\int_0^\infty e^{-c/t}\frac{dt}{t^{1+s/2}}\le C.$$

So, as claimed, we are left to proving the estimate for the $M=(\mathbb{R}^n,g)$, $p=0$, $\varphi=\mathrm{id}$, and g_{ij} satisfying the assumptions of Proposition 2.12.

Recalling (33), notice that

$$k(x,z)=K(x,x+z)=c_s\int_0^\infty H(x,x+z,t)\frac{dt}{t^{1+s/2}}=c_s\int_0^\infty h\left(\frac{A(x)z}{\sqrt{t}},x,t\right)\frac{dt}{t^{n/2+1+s/2}}.\tag{34}$$

Also, recalling that $h_\circ(z):=(4\pi)^{-n/2}e^{-|z|^2/4}$, we have

$$\begin{aligned}\widehat{k}(x,z)=k(x,z)-\alpha_{n,s}\frac{\det(A(x))}{|A(x)z|^{n+s}}&=c_s\int_0^\infty\left(h\left(\frac{A(x)z}{\sqrt{t}},x,t\right)-h_\circ\left(\frac{A(x)z}{\sqrt{t}}\right)\right)\frac{dt}{t^{n/2+1+s/2}}\\&=c_s\int_0^\infty\widehat{h}\left(\frac{A(x)z}{\sqrt{t}},x,t\right)\frac{dt}{t^{n/2+1+s/2}},\end{aligned}$$

Therefore using the heat kernel estimates from Proposition 2.12 (and noticing $|A(x)z|\ge\frac{1}{\sqrt{2}}|z|$ for all x,z by assumption) we obtain

$$|\widehat{k}(x,z)|\le c_s\int_0\left|\widehat{h}\left(\frac{A(x)z}{\sqrt{t}},x,t\right)\right|\frac{dt}{t^{n/2+1+s/2}}\le Cs\int_0^\infty\sqrt{t}\exp(-c|z|/\sqrt{t})\frac{dt}{t^{n/2+1+s/2}}=C|z|^{1-n-s}.$$

This proves (26). Similarly, the estimates (27) follow diferentating (34) and using the corresponding estimates for derivatives of the heat kernel from 2.12.

Finally, (28) and (29) follow analogously integrating the heat kernel estimates in Lemmas 2.8 and 2.9, respectively. □

The next property concerns the behaviour of the kernel when the two points p and q are separated from each other.

Proposition 2.13. *Let (M,g) be a Riemannian n-manifold and $s\in(0,2)$. Assume that for some $p,q\in M$ both $\mathrm{FA}_\ell(M,g,1,p,\varphi_p)$ and $\mathrm{FA}_\ell(M,g,1,q,\varphi_q)$ hold, and suppose that $\varphi_p(\mathcal{B}_1(0))\cap\varphi_q(\mathcal{B}_1(0))=\varnothing$. Put $K_{pq}(x,y):=K_s(\varphi_p(x),\varphi_q(y))$. Then*

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial y^\beta}K_{pq}(x,y)\right|\le C(n,\ell)\quad\text{for all } |x|<\tfrac12 \text{ and } |y|<\tfrac12,$$

whenever $|\alpha|+|\beta|\le\ell$.

Proof of Propositon 2.13. Let $H_*(x,y,t) := H_M(\varphi_p(x),\varphi_q(y),t)$. It follows from Lemma 2.8 that

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}H_*(x,y,t)\right| \le C\exp(-c/t)$$

for all $|x| < \frac{3}{4}$ and $|y| < \frac{3}{4}$, where C and c depend only on n, and $|\alpha|$.

We now use that (by symmetry of the heat kernel in p and q), for each $x \in \mathcal{B}_{1/2}$ fixed, the function $u(y,t) := \frac{\partial^{|\alpha|}}{\partial x^\alpha}H_*(x,y,t)$ is solution of the heat equation $u_t = Lu$, in the ball $|y| < 1$, where L denotes the Laplace-Beltrami (with respect to y, in local coordinates). Since $|u| \le C\exp(-c/t)$ in $\mathcal{B}_{3/4}\times(0,\infty)$, reasoning exactly as in the proof of Lemma 2.8 (only that now the spatial variables are y instead of x) we obtain

$$\left|\frac{\partial^{|\beta|}}{\partial y^\beta}u(y,t)\right| \le C\exp(-c/t),$$

for some new positive constants C and c depending only on n, and $|\beta|$. This shows:

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial y^\beta}H_*(x,y,t)\right| \le C\exp(-c/t)$$

Then the proposition follows immediately after noticing that, by definition,

$$K_{pq}(x,y) = \frac{s/2}{\Gamma(1-s/2)}\int_0^\infty H_*(x,y,t)\frac{dt}{t^{1+s/2}},$$

and hence

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial y^\beta}K_{pq}(x,y)\right| = \left|\frac{s/2}{\Gamma(1-s/2)}\int_0^\infty \frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial y^\beta}H_*(x,y,t)\frac{dt}{t^{1+s/2}}\right| \le Cs\int_0^\infty \exp(-c/t)\frac{dt}{t^{1+s/2}} \le C\,,$$

for some constant $C > 0$ that depends only on n and ℓ, and this concludes the proof. ☐

We next study how the Allen-Cahn energy behaves under inner variations. For this, we need to first study how the singular kernel K_s behaves when translating its arguments under the flow of a vector field.

Proposition 2.14. *Let (M,g) be a closed n-dimensional Riemannian manifold and $s \in (0,2)$. Consider any smooth vector field $X \in \mathfrak{X}(M)$, and fix points $p,q \in M$. Writing ψ^t for the flow of X at time t, then the kernel satisfies*

$$\left|\frac{d^\ell}{dt^\ell}\bigg|_{t=0}K_s(\psi^t(p),\psi^t(q))\right| \le C(1+K_s(p,q)). \tag{35}$$

for some constant $C = C(M,s,\ell,\max_{0\le k\le\ell}\|\nabla^k X\|_{L^\infty(M)})$.

Proof. This follows from the estimates of Proposition 2.6, in particular by (27) and (28). We prove the result just for $\ell = 1$, as the general case just follows by induction by the very same arguments. Let $R = R(M) > 0$ be such that the flatness assumption $\mathrm{FA}_\ell(M,g,16R,p,\varphi_p)$ holds for every $p \in M$; such an R exists by Remark 1.22. We split in two cases.

Case 1: $q \in \varphi_p(\mathcal{B}_{4R}(0))$.

In this case, denoting $K(x,y) := K_s(\varphi_p(x),\varphi_p(y))$ and $k(x,z) := K(x,x+z)$ as in Proposition 2.6, we have that

$$K_s(\psi^t\circ\varphi_p(x),\psi^t\circ\varphi_p(y)) = K(\psi_p^t(x),\psi_p^t(y)) = k(\psi_p^t(x),\psi_p^t(y)-\psi_p^t(x))\,,$$

where ψ_p^t is the flow of $\xi = (\varphi_p)^*X$, i.e. the vector field $\xi = \xi_p \in \mathfrak{X}(\mathcal{B}_{16R}(0))$ such that $X\circ\varphi_p = (\varphi_p)_*\xi$. Then, for all $x,y \in \mathcal{B}_{2R}(0)$ we have:

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0}K(\psi_p^t(x),\psi_p^t(y)) &= \frac{d}{dt}\bigg|_{t=0}k(\psi_p^t(x),\psi_p^t(y)-\psi_p^t(x))\\ &= \frac{\partial k}{\partial x^\alpha}(x,y-x)\xi^\alpha(x) + \frac{\partial k}{\partial z^\alpha}(x,y-x)(\xi^\alpha(x)-\xi^\alpha(y))\,,\end{aligned}$$

where sum over repeated indices is to be understood. Hence, by (27) of Proposition 2.6 we get

$$\left|\frac{d}{dt}\bigg|_{t=0} K(\psi_p^t(x),\psi_p^t(y))\right| \le \frac{C}{|y-x|^{n+s}}\|\xi\|_{L^\infty} + \frac{C}{|y-x|^{n+s+1}}\|D\xi\|_{L^\infty}|y-x|$$
$$\le \frac{C}{|y-x|^{n+s}} \le CK(x,y)$$

for some $C = C(n,s,\|\xi\|_{C^{0,1}})$, where in the last line we have also used Lemma 2.11. Finally, evaluating this inequality at $x=0$ and $y=\varphi_p^{-1}(q)$ we obtain

$$\left|\frac{d}{dt}\bigg|_{t=0} K(\psi^t(p),\psi^t(q))\right| = \left|\frac{d}{dt}\bigg|_{t=0} K(\psi_p^t(0),\psi_p^t(y))\right| \le CK(0,y) = CK_s(p,q),$$

as wanted.

Case 2: $q \notin \varphi_p(\mathcal{B}_{4R}(0))$. Then $\mathrm{FA}_\ell(M,g,R,q,\varphi_q)$ holds and the sets $\varphi_p(\mathcal{B}_R(0))$ and $\varphi_q(\mathcal{B}_R(0))$ are disjoint. Hence, by Proposition 2.13 the kernel $K_{pq}(x,y) := K_s(\varphi_p(x),\varphi_q(y))$ is smooth (with uniform estimates on all derivatives) in the domain $\mathcal{B}_{R/2}(0)\times\mathcal{B}_{R/2}(0)$. Hence

$$\frac{d}{dt}\bigg|_{t=0} K_s(\psi^t\circ\varphi_p(x),\psi^t\circ\varphi_q(x)) = \frac{d}{dt}\bigg|_{t=0} K_{pq}(\psi_p^t(x),\psi_p^t(y))$$
$$= \frac{\partial K_{pq}}{\partial x^\alpha}(x,y)\xi_p^\alpha(x) + \frac{\partial K_{pq}}{\partial y^\alpha}(x,y)\xi_q^\alpha(y).$$

Using Proposition 2.13 to bound the derivatives of K_{pq}, and then evaluating at $(x,y)=(0,0)$ gives

$$\left|\frac{d}{dt}\bigg|_{t=0} K_s(\psi^t(p),\psi^t(q))\right| \le \frac{C}{R^{n+s}},$$

for some $C = C(n,s,\|\xi_p\|_{L^\infty},\|\xi_q\|_{L^\infty})$.

Putting together the two cases above we get

$$\left|\frac{d}{dt}\bigg|_{t=0} K_s(\psi^t(p),\psi^t(q))\right| \le C(1+K_s(p,q)),$$

for some $C = C(M,n,s,\|X\|_{C^1(M)})$ and conclude the proof.

□

We also record a version of Proposition 2.14 which depends only on local quantities:

Proposition 2.15. *Let (M,g) be a closed n-dimensional Riemannian manifold and $s\in(0,2)$. Assume that the flatness assumption $\mathrm{FA}_\ell(M,g,R,p,\varphi)$ holds, and let $X\in\mathfrak{X}(M)$ be a smooth vector field supported on $\varphi(\mathcal{B}_{R/4})$. Writing ψ^t for the flow of X at time t, then for every $x,y\in\mathcal{B}_{R/4}(0)$ we have*

$$\left|\frac{d^\ell}{dt^\ell}\bigg|_{t=0} K_s(\psi^t(\varphi(x)),\psi^t(\varphi(y)))\right| \le CK_s(\varphi(x),\varphi(y)) \le C\frac{\alpha_{n,s}}{|x-y|^{n+s}}, \tag{36}$$

for some constant $C = C(n,s,\|X\|_{C^\ell(\varphi(\mathcal{B}_R))})$. Moreover, given $T>0$ we have that, for all $0\le t\le T$,

$$\left|\frac{d^\ell}{dt^\ell} K_s(\psi^t(\varphi(x)),\psi^t(\varphi(y)))\right| \le C_T K_s(\varphi(x),\varphi(y)) \le C_T\frac{\alpha_{n,s}}{|x-y|^{n+s}}, \tag{37}$$

where $C_T = C_T(n,s,T,\|X\|_{C^\ell(\varphi(\mathcal{B}_{R/4}))})$.

Proof. By scaling we can assume $R = 1$. The second inequality in both (36) and (37) then follows from Lemma 2.11. As for the first inequality of (36), it follows from the proof of Case 1 in Proposition 2.14, since it only depends on local estimates for X. Finally, (37) can be deduced from (36). Indeed, note that for all $1 \le k \le \ell$ and $0 \le t \le T$,

$$\left| \frac{d^k}{dt^k} K_s(\psi^t(\varphi(x)), \psi^t(\varphi(y))) \right| = \left| \frac{d^k}{dr^k}\bigg|_{r=0} K_s(\psi^{t+r}(\varphi(x)), \psi^{t+r}(\varphi(y))) \right| \le C_0 K_s(\psi^t(\varphi(x)), \psi^t(\varphi(y))) \,, \tag{38}$$

with $C_0 = C_0(n, s, \|X\|_{C^\ell(\varphi(\mathcal{B}_1))})$. Thus, we are only left with proving that

$$K_s(\psi^t(\varphi(x)), \psi^t(\varphi(y))) \le C_T K_s(\varphi(x), \varphi(y))$$

for some $C_T = C_T(n, s, T, \|X\|_{C^\ell(\varphi(\mathcal{B}_1))})$. But this follows itself from (38), with $k = 1$, since we can write the inequality as

$$\frac{d}{dt}\left[e^{-C_0 t} K_s(\psi^t(\varphi(x)), \psi^t(\varphi(y)))\right] \le 0 \,,$$

and integrating we find that

$$K_s(\psi^t(\varphi(x)), \psi^t(\varphi(y))) \le e^{C_0 T} K_s(\varphi(x), \varphi(y))$$

for every $0 \le t \le T$. □

Proposition 2.14 is used to bound time derivatives of the energy of "flown objects" by their energy at time zero.

Lemma 2.16. *Let $s \in (0, 2)$ and $v \in H^{s/2}(M)$ be a function with $|v| \le 1$. Let $X \in \mathfrak{X}(M)$ be a smooth vector field and $v_t := v \circ \psi^{-t}$, where ψ^t is the flow of X at time t. Then, for all $T > 0$ there holds*

$$\sup_{0<t<T} \left| \frac{d^\ell}{dt^\ell} \mathcal{E}_M(v_t) \right| \le C\big(1 + \mathcal{E}_M(v)\big) \,,$$

for some constant $C = C(M, s, \ell, T, \max_{0 \le k \le \ell} \|\nabla^k X\|_{L^\infty(M)})$.

Proof. Let C denote a constant that depends only on M, s, ℓ, T and $\max_{0 \le k \le \ell} \|\nabla^k X\|_{L^\infty(M)}$.

The idea of the proof is to change variables using the flow ψ^t in the corresponding integrals defining the Allen-Cahn energy, and after that to exchange integration and differentiation.

Let us start with the Sobolev part of the energy. We have:

$$\begin{aligned} \frac{d^\ell}{dt^\ell} \mathcal{E}^{\mathrm{Sob}}_M(v_t) &= \frac{d^\ell}{dt^\ell} \iint |v(\psi^{-t}(p)) - v(\psi^{-t}(q))|^2 K_s(p, q)\, dV_p\, dV_q \\ &= \frac{d^\ell}{dt^\ell} \iint |v(p) - v(q)|^2 K_s(\psi^t(p), \psi^t(q))\, J_t(p) J_t(q)\, dV_p\, dV_q \\ &= \iint |v(p) - v(q)|^2 \frac{d^\ell}{dt^\ell}\Big[K_s(\psi^t(p), \psi^t(q))\, J_t(p) J_t(q)\Big] dV_p\, dV_q. \end{aligned} \tag{39}$$

Since $0 < t < T$, the derivatives in time of the Jacobians J_t can of course be bounded by a constant C with the right dependencies. What remains in order to bound (39) by $C(1 + \mathcal{E}^{\mathrm{Sob}}_M(v))$ is to control the first k-th derivatives in time of $K_s(\psi^t(p), \psi^t(q))$ by $C(1 + K_s(p, q))$, for all $0 < t < T$. The main bound is given by Proposition 2.14, which gives for all $1 \le k \le \ell$:

$$\left| \frac{d^k}{dt^k} K_s(\psi^t(p), \psi^t(q)) \right| = \left| \frac{d^k}{dr^k}\bigg|_{r=0} K_s(\psi^{t+r}(p), \psi^{t+r}(q)) \right| \le C(1 + K_s(\psi^t(p), \psi^t(q)) \,. \tag{40}$$

Now, integrating this inequality for $k = 1$ similarly to how we proceeded in the proof of Lemma 2.15, we conclude that

$$K_s(\psi^t(p), \psi^t(q)) \le C(1 + K_s(p, q)), \quad \text{for all } 0 < t < T \,. \tag{41}$$

We can now go back to (39) and apply the bounds that we just derived. We get that

$$
\begin{aligned}
\left|\frac{d^\ell}{dt^\ell}\mathcal{E}_M^{\mathrm{Sob}}(v_t)\right| &\le \iint |v(p)-v(q)|^2 \frac{d^\ell}{dt^\ell}\Big[K_s(\psi^t(p),\psi^t(q))\, J_t(p)J_t(q)\Big]\, dV_p\, dV_q \\
&\le C\iint |v(p)-v(q)|^2(1+K_s(p,q))\, dV_p dV_q \\
&= C(1+\mathcal{E}_M^{\mathrm{Sob}}(v))
\end{aligned}
$$

for all $0<t<T$, where C has the right dependencies.

The potential part of the energy is simpler to deal with. Indeed, we have

$$
\frac{d^\ell}{dt^\ell}\mathcal{E}_M^{\mathrm{Pot}}(v_t) = \frac{d^\ell}{dt^\ell}\int \varepsilon^{-s}W(v(\psi^{-t}(p))dV_p = \int \varepsilon^{-s}W(v(p))\frac{d^\ell}{dt^\ell}J_t(p)dV_p, \tag{42}
$$

from which we directly conclude that

$$
\left|\frac{d^\ell}{dt^\ell}\mathcal{E}_M^{\mathrm{Pot}}(v_t)\right| \le C\mathcal{E}_M^{\mathrm{Pot}}(v),
$$

finishing the proof. ☐

Lemma 2.16 has a local version, which comes from applying local estimates for the kernel instead.

Lemma 2.17. *Let M satisfy the flatness assumptions $\mathrm{FA}_\ell(N,g,p,R,\varphi)$. Let $s\in(0,2)$ and $v\in H^{s/2}(M)$ be a function with $|v|\le 1$. Let $X\in\mathfrak{X}(M)$ be a smooth vector field supported on $\varphi(\mathcal{B}_{R/2})$, and put $v_t := v\circ\psi_X^{-t}$, where ψ_X^t is the flow of X at time t. Then, for all $T>0$ there holds*

$$
\sup_{0<t<T}\left|\frac{d^\ell}{dt^\ell}\mathcal{E}_{\varphi(\mathcal{B}_{R/2})}(v_t)\right| \le C(1+\mathcal{E}_{\varphi(\mathcal{B}_{R/2})}(v)),
$$

for some constant $C=C(s,\ell,T,\max_{0\le k\le\ell}\|\nabla^k X\|_{L^\infty(\varphi(\mathcal{B}_{R/2}))})$.

Proof. We modify the proof of Lemma 2.16 accordingly. First, by scaling, it suffices to prove the Lemma in the case $R=1$. Since X is supported on $\varphi(\mathcal{B}_{1/2})$, the integrand in (39) is supported then on

$$
\begin{aligned}
(N\times N)\setminus(\varphi(\mathcal{B}_{1/2})^c\times\varphi(\mathcal{B}_{1/2})^c) = &\Big[(\varphi(\mathcal{B}_{2/3})\times\varphi(\mathcal{B}_{2/3}))\setminus(\varphi(\mathcal{B}_{1/2})^c\times\varphi(\mathcal{B}_{1/2})^c)\Big]\cup \\
&\cup\Big[(\varphi(\mathcal{B}_{1/2})\times(N\setminus\varphi(\mathcal{B}_{2/3})))\cup((N\setminus\varphi(\mathcal{B}_{2/3}))\times\varphi(\mathcal{B}_{1/2}))\Big],
\end{aligned}
$$

so that

$$
\begin{aligned}
&\left|\frac{d^k}{dt^k}\mathcal{E}_M^{\mathrm{Sob}}(v_t)\right| = \left|\frac{d^k}{dt^k}\mathcal{E}_{\varphi(\mathcal{B}_{1/2})}^{\mathrm{Sob}}(v_t)\right| \\
&= \Bigg|\iint_{(\varphi(\mathcal{B}_{2/3})\times\varphi(\mathcal{B}_{2/3}))\setminus(\varphi(\mathcal{B}_{1/2})^c\times\varphi(\mathcal{B}_{1/2})^c)} |v(p)-v(q)|^2\frac{d^k}{dt^k}\Big[K(\psi_Y^t(p),\psi_Y^t(q))J_t(p)J_t(q)\Big]\, dV_p\, dV_q \\
&\qquad + 2\iint_{\varphi(\mathcal{B}_{1/2})\times(N\setminus\varphi(\mathcal{B}_{2/3}))} |v(p)-v(q)|^2\frac{d^k}{dt^k}\Big[K(\psi_Y^t(p),q)J_t(p)\Big]\, dV_p\, dV_q\Bigg| \\
&\le C\iint_{(\mathcal{B}_{2/3}\times\mathcal{B}_{2/3})\setminus(\mathcal{B}_{1/2}^c\times\mathcal{B}_{1/2}^c)} |v(\varphi(x))-v(\varphi(y))|^2\left|\frac{d^k}{dt^k}\Big[K(\varphi(\psi_X^t(x)),\varphi(\psi_X^t(y)))J_t(\varphi(x))J_t(\varphi(y))\Big]\right| dx\, dy \\
&\qquad + C\iint_{\mathcal{B}_{1/2}\times(N\setminus\varphi(\mathcal{B}_{2/3}))} |v(\varphi(x))-v(q)|^2\left|\frac{d^k}{dt^k}\Big[K(\varphi(\psi_X^t(x)),q)J_t(\varphi(x))\Big]\right| dx\, dV_q\,.
\end{aligned}
$$

Bounding the derivatives in time of the Jacobians by a constant with the right dependencies, using (36) to bound the kernel in the first double integral, and using (29) to bound the integral in q in the second double integral by a constant, we conclude that

$$\left|\frac{d^k}{dt^k}\mathcal{E}^{\mathrm{Sob}}_{\varphi(\mathcal{B}_{1/2})}(v_t)\right| \leq C(1+\mathcal{E}^{\mathrm{Sob}}_{\varphi(\mathcal{B}_{1/2})}(v)) .$$

Regarding the potential part of the energy, from the computation in (42) we readily find that

$$\left|\frac{d^\ell}{dt^\ell}\mathcal{E}^{\mathrm{Pot}}_{\varphi(\mathcal{B}_{1/2})}(v_t)\right| \leq C\mathcal{E}^{\mathrm{Pot}}_{\varphi(\mathcal{B}_{1/2})}(v) \tag{43}$$

where C has the right dependencies, which completes the proof.

□

2.1.3 Caffarelli-Silvestre type extension

Definition 2.18. We define the weighted Sobolev space

$$\widetilde{H}^1(\mathbb{R}^n \times (0,\infty)) = H^1(\mathbb{R}^n \times (0,\infty), z^{1-s}dxdz)$$

as the completion of $C_c^\infty(\mathbb{R}^n \times [0,\infty))$ with the norm

$$\|U\|^2_{\widetilde{H}^1} := \|U\|^2_{L^2(\mathbb{R}^n\times(0,\infty),z^{1-s}dxdz)} + \|\widetilde{D}U\|^2_{L^2(\mathbb{R}^n\times(0,\infty),z^{1-s}dxdz)},$$

where $\widetilde{D}U = (\frac{\partial U}{\partial x^1},\ldots,\frac{\partial U}{\partial x^n},\frac{\partial U}{\partial z})$ denotes the Euclidean gradient in $\mathbb{R}^{n+1}$. This is a Hilbert space with the natural inner product that induces the norm above. It is a known fact that any $U \in \widetilde{H}^1(\mathbb{R}^n \times (0,\infty))$ has a well defined trace in $L^2(\mathbb{R}^n)$ that we denote by $U(x,\cdot)$.

The following essential result by Caffarelli and Silvestre shows that fractional powers of the Laplacian on $\mathbb{R}^n$ can be realized as a Dirichlet-to-Neumann map via an extension problem.

Theorem 2.19 ([28])**.** *Let $s \in (0,2)$ and $u \in H^{s/2}(\mathbb{R}^n)$. Then, there is a unique solution $U = U(x,z)$: $\mathbb{R}^n \times [0,+\infty) \to \mathbb{R}$ among functions in $\widetilde{H}^1(\mathbb{R}^n \times (0,\infty))$ to the problem*

$$\begin{cases} \Delta_x U + \frac{\partial^2 U}{\partial z^2} + \frac{1-s}{z}\frac{\partial U}{\partial z} = 0, & \text{on } \mathbb{R}^n \times (0,\infty) \\ U(x,0) = u(x) & \text{for } x \in \mathbb{R}^n, \end{cases} \tag{44}$$

and it verifies that

$$\lim_{z\to 0^+} z^{1-s}\frac{\partial U}{\partial z}(x,z) = -\beta_s^{-1}\,(-\Delta)^{s/2}u(x), \tag{45}$$

where Δ_x denotes the standard Laplacian on $\mathbb{R}^n$ and β_s is a positive constant that depends only on s.

In [28] three different proofs of this fact are presented, but each of these proofs relies on some additive structure of the base space. To prove that the extension procedure produces the fractional power of the Laplacian also on a Riemannian manifold, which is the setting we are interested in, one has to rely on different ideas. It was proved by P.R. Stinga in [97] that the unique solution to (44) verifying (45) admits the explicit representation

$$U(x,z) = \frac{z^s}{2^s\Gamma(s/2)}\int_0^\infty e^{t\Delta}u(x)\,e^{-\frac{z^2}{4t}}\,\frac{dt}{t^{1+s/2}}, \tag{46}$$

expressing U in terms of the solution to the heat equation $e^{t\Delta}u$. The proof of this fact does not rely on the additive structure of $\mathbb{R}^n$ and can be carried out in the same way on closed Riemannian manifolds.

Theorem 2.20. *Let (M^n, g) be a closed Riemannian manifold, let $s \in (0,2)$ and $u \in H^{s/2}(M)$. Consider the product manifold $\widetilde{M} = M \times (0, +\infty)$ endowed with the natural product metric*[8]*. Then, there is a unique solution $U : M \times (0, \infty) \to \mathbb{R}$ among functions in $\widetilde{H}^1(\widetilde{M})$ to*

$$\begin{cases} \widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla} U) = 0 & \text{in } \widetilde{M}, \\ U(p,0) = u(p) & \text{for } p \in \partial\widetilde{M} = M, \end{cases} \tag{47}$$

and it verifies that

$$\lim_{z\to 0^+} z^{1-s}\frac{\partial U}{\partial z}(p,z) = -\beta_s^{-1}(-\Delta)^{s/2}u(p)\,, \tag{48}$$

where

$$\beta_s = \frac{\Gamma(1-s/2)}{2^{s-1}\Gamma(s/2)} \tag{49}$$

and the fractional Laplacian on the right-hand side is defined by either (22) or (23).

Proof. The fact that a solution among functions in $\widetilde{H}^1(\widetilde{M})$ exists and is unique follows by direct minimization of the associated energy $v \mapsto \int_{\widetilde{M}} |\widetilde{\nabla} v|^2 z^{1-s} dV dz$ over $\widetilde{H}^1(\widetilde{M})$. Moreover, as briefly mentioned above, the explicit representation (46) of the unique such solution $U \in \widetilde{H}^1(\widetilde{M})$ in terms of the solution to the heat equation holds also on any closed Riemannian manifold with the same proof, see [32] for the details. Then, (48) can be shown with this formula exactly as in [97]. □

2.1.4 Modifications of definitions for the Euclidean space and other noncompact manifolds

We observe that (4) can also be used to define the fractional perimeter for any (possibly noncompact) complete manifold N for which a fractional Hilbert norm $H^{s/2}(N)$ is defined.[9]

Definition (i) of the $H^{s/2}$ norm generalizes well to noncompact M. We refer to the survey [65] for the construction and properties of the heat kernel on complete, non-compact Riemannian manifolds. In the case of the Euclidean space $\mathbb{R}^n$, this viewpoint is consistent (i.e. coincides) with the original definition by Caffarelli, Roquejoffre, and Savin in [25] (see also Remark 2.4).

Notice that Definition (ii) of the $H^{s/2}$ norm does not generalize well to the case of non-compact M. (Although in very particular cases of highly symmetric manifolds such as the Euclidean space $\mathbb{R}^n$ o the hyperbolic space $\mathbb{H}^n$, one could possibly replace the Fourier series by Fourier transform.) However, definition (iii), via the extension property, also generalizes well to the case of non-compact manifolds. Still, some extra assumptions are needed in order to establish the equivalence between (i) and (iii) —see [9] for a related discussion concerning the definition of the fractional Laplacian on noncompact manifolds.

In any case, it is worth emphasizing that the equivalence between (i) and (iii) can be verified for many noncompact manifolds —including Euclidean space, the hyperbolic space, etc.— with (essentially) the same proof as the one given here for compact manifolds. We also notice that Definitions 1.6 and 1.7 make complete sense even when M is replaced by any of these non-compact manifolds.

It will be clear from our proofs that, in the case of noncompact manifolds for which the equivalence of (i) and (iii) can be established, s-minimal surfaces will enjoy the same (local) properties as the ones established here in the case of compact manifolds (e.g. the monotonicity formula), with almost identical proofs.

2.2 The fractional Sobolev energy

We define the fractional Sobolev seminorm $[u]_{H^{s/2}(M)}$ for $s \in (0,2)$ as

$$[u]^2_{H^{s/2}(M)} = 2\int_M u(-\Delta)^{s/2}u\, dV, \tag{50}$$

[8] That is, the metric defined by $\tilde{g}((\xi_1, z_1), (\xi_2, z_2)) = g(\xi_1, \xi_2) + z_1 z_2$, and where $\widetilde{\operatorname{div}}$ and $\widetilde{\nabla}$ denote the divergence and Riemannian gradient with respect to this product metric respectively.

[9] At least when E is bounded with smooth boundary.

where $(-\Delta)^{s/2}u$ is defined by (23). Then, the associated functional space $H^{s/2}(M)$ is

$$H^{s/2}(M) = \{u \in L^2(M) : [u]^2_{H^{s/2}(M)} < \infty\}\,, \tag{51}$$

and is called the *fractional Sobolev space of order* $s/2$. This is a Hilbert space with norm given by

$$\|u\|^2_{H^{s/2}(M)} = \|u\|^2_{L^2(M)} + [u]^2_{H^{s/2}(M)}\,.$$

The equivalent formulas for the fractional Laplacian given in Section 2.1 show that the fractional Sobolev seminorm can be expressed also in the following forms.

Proposition 2.21. *The fractional Sobolev seminorm* (50) *is equal to both*

$$[u]^2_{H^{s/2}(M)} = \iint_{M\times M} (u(p)-u(q))^2 K_s(p,q)\, dV_p dV_q \tag{52}$$

and

$$[u]^2_{H^{s/2}(M)} = \beta_s \inf_{v\in \widetilde{H}^1(\widetilde{M})} \left\{ \int_{\widetilde{M}} |\widetilde{\nabla} v|^2 z^{1-s}\, dV dz \,:\, v(q,0) = u(q) \right\}. \tag{53}$$

Moreover, this infimum is attained by the unique $U \in \widetilde{H}^1(\widetilde{M})$ given by Theorem 2.20, and hence also

$$[u]^2_{H^{s/2}(M)} = \beta_s \int_{\widetilde{M}} |\widetilde{\nabla} U|^2 z^{1-s}\, dV dz\,, \tag{54}$$

where β_s is defined in (49)*.*

Proof. Formula (52) is easily seen to be equal to (50) just by expanding the square and using that the total mass of the heat kernel H_M is equal to one for all times. Then, (53) and (54) follow by Theorem 2.20 and its proof. □

We end this section by recalling the following simple interpolation result. After a finite covering argument, it implies in particular that the characteristic function χ_E of any set of finite perimeter $E \subset M$ is in $H^{s/2}(M)$, as long as $s \in (0,1)$.

Proposition 2.22. *Let $s \in (0,1)$, and let $u : \mathcal{B}_1 \subset \mathbb{R}^n \to [-1,1]$ be a function of bounded variation. Then,*

$$\iint_{\mathcal{B}_1\times\mathcal{B}_1} \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\, dx\, dy \le \frac{C(n)}{(1-s)s} [u]^s_{BV(\mathcal{B}_1)} \|u\|^{1-s}_{L^1(\mathcal{B}_1)}\,.$$

Proof. See for instance Proposition 4.2 in [15]. □

2.3 Monotonicity formula for stationary points of semilinear elliptic functionals and s-minimal surfaces

The monotonicity formula for s-minimal surfaces has been obtained, to our knowledge, only for minimizers[10] and on $\mathbb{R}^n$ in the article [25]. Here we prove the analogous (local) monotonicity formula on a Riemannian manifold, with a proof that holds simultaneously for any s-minimal surface, that is for any stationary point of the fractional perimeter regardless of second variation or regularity, and also for any stationary point of a semilinear elliptic functional with a nonnegative potential term, hence including the fractional Allen-Cahn energy. For $r > 0$ and $p \in M$ denote

$$\begin{aligned} B_r(p) &= \{q \in M \,:\, d_g(q,p) < r\}\,,\\ \widetilde{B}^+_r(p,0) &= \{(q,z) \in \widetilde{M} \,:\, d_{\widetilde{g}}((q,z),(p,0)) < r\}\,,\\ \partial \widetilde{B}^+_r(p,0) &= \partial\left(\widetilde{B}^+_r(p,0)\right)\\ \partial^+ \widetilde{B}^+_r(p,0) &= \partial \widetilde{B}^+_r(p,0) \cap \{z>0\}\,. \end{aligned} \tag{55}$$

[10]It is pointed out in [19] that the proof in [25] also implies the monotonicity formula for smooth stable s-minimal surfaces on $\mathbb{R}^n$.

In all this section, since there will be no possible ambiguity, we will use ∇ instead of $\widetilde{\nabla}$ to denote the gradient in $\widetilde{M}$ with respect to the product metric.

Theorem 2.23 (Monotonicity formula). *Let (M^n, g) be an n-dimensional, closed Riemannian manifold. Let $s \in (0,2)$ and*

$$\mathcal{E}(v) = [v]^2_{H^{s/2}(M)} + \int_M F(v)\, dV,$$

where F is any smooth nonnegative function. Let $u : M \to \mathbb{R}$ be stationary for $\mathcal{E}$ under inner variations, meaning that $\mathcal{E}(u) < \infty$ and for any smooth vector field X on M there holds $\frac{d}{dt}\big|_{t=0}\mathcal{E}(u \circ \psi^t_X) = 0$, where ψ^t_X is the flow of X at time t. For $(p_\circ, 0) \in \widetilde{M}$ define

$$\Phi(R) := \frac{1}{R^{n-s}}\left(\beta_s \int_{\widetilde{\mathcal{B}}^+_R(p_\circ,0)} z^{1-s}|\nabla U(p,z)|^2\, dV_p dz + \int_{B_R(p_\circ)} F(u)\, dV\right),$$

where U is the unique solution given by Theorem 2.20. Then, there exist constants $C = C(n)$ and $R_{\max} = R_{\max}(M, p_\circ) > 0$ with the following property: whenever $R_\circ \le R_{\max}$ and K is an upper bound for all the sectional curvatures of M in $B_{R_\circ}(p_\circ)$, then

$$R \mapsto \Phi(R)e^{C\sqrt{K}R} \quad \text{is nondecreasing for} \quad R < R_\circ\,,$$

and the inequality

$$\Phi'(R) \ge -C\sqrt{K}\Phi(R) + \frac{s}{R^{n-s+1}}\int_{B_R(p_\circ)} F(u)\, dV + \frac{2\beta_s}{R^{n-s}}\int_{\partial^+\widetilde{\mathcal{B}}^+_R(p_\circ,0)} z^{1-s}\langle \nabla U, \nabla d\rangle^2\, d\widetilde{\sigma}$$

holds for all $R < R_0$, with $d(\cdot) = d_{\widetilde{g}}((p_\circ, 0), \cdot\,)$ the distance function on $\widetilde{M}$ from the point $(p_\circ, 0)$.

Moreover, in the particular case where $M = \mathbb{R}^n$, $F \equiv 0$, $s \in (0,1)$, and $u = \chi_E - \chi_{E^c}$ is a stationary set for the fractional s-perimeter, there holds

$$\Phi'(R) = \frac{2\beta_s}{R^{n-s}}\int_{\partial^+\widetilde{\mathcal{B}}^+_R(p_\circ,0)} z^{1-s}\langle \nabla U, \nabla d\rangle^2\, dx dz \ge 0\,,$$

which shows that Φ is nondecreasing and that it is constant if and only if E is a cone.

Remark 2.24. It will follow from the proof that the radius $R_{\max}$ in Theorem 2.23 can be taken to be $R_{\max} = \mathrm{inj}_M(p_\circ)/4$. Moreover, since M is compact $R_{\max}$ is uniformly bounded below as $R_{\max}(M, p_\circ) \ge \mathrm{inj}_M/4$, for all $p_\circ \in M$.

Proof of Theorem 2.23. Since during the entire proof the point $p_\circ \in M$ will be fixed, we will not specify the center of the balls in what follows, as this will always be $(p_\circ, 0)$ for balls inside $\widetilde{M}$ and $p_\circ$ for balls on M. We divide the proof into two steps.

Step 1. First, we show that if u is stationary for the energy $\mathcal{E}(v) = [v]^2_{H^{s/2}(M)} + \int_M F(v)$ under inner variations, then its Caffarelli-Silvestre extension U is stationary for the energy

$$U \mapsto \beta_s \int_{\widetilde{M}} z^{1-s}|\nabla U|^2\, dV dz + \int_M F(U|_M)\, dV\,,$$

under inner variations on $\widetilde{M}$ given by vector fields Y on $\widetilde{M}$ such that $Y|_M$ is tangent to M.

Recall that the Caffarelli-Silvestre extension of u given by (47).

Let Y be a vector field on $\widetilde{M}$ such that $Y|_M$ is tangent to M, and let ψ^t_Y denote its flow at time t. Let also V_t be the Caffarelli-Silvestre extension of $u \circ \psi^t_{Y|_M}$, for any $t \in \mathbb{R}$. By the minimality of the extension in the energy space we have

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\beta_s \int_{\widetilde{M}} z^{1-s}|\nabla(U\circ\psi_Y^{-t})|^2 &= \lim_{t\to 0}\frac{1}{t}\left(\beta_s\int_{\widetilde{M}} z^{1-s}|\nabla U|^2 - \beta_s\int_{\widetilde{M}} z^{1-s}|\nabla(U\circ\psi_Y^t)|^2\right)\\
&\le \lim_{t\to 0}\frac{1}{t}\left(\beta_s\int_{\widetilde{M}} z^{1-s}|\nabla U|^2 - \beta_s\int_{\widetilde{M}} z^{1-s}|\nabla V_t|^2\right)\\
&= \lim_{t\to 0}\frac{[u]^2_{H^{s/2}(M)} - [u\circ\psi_Y^t]^2_{H^{s/2}(M)}}{t}\\
&= \frac{d}{dt}\Big|_{t=0}[u\circ\psi_Y^{-t}]^2_{H^{s/2}(M)},
\end{aligned}
$$

and likewise

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\beta_s \int_{\widetilde{M}} z^{1-s}|\nabla(U\circ\psi_Y^{-t})|^2 &= \lim_{t\to 0}\frac{1}{t}\left(\beta_s\int_{\widetilde{M}} z^{1-s}|\nabla(U\circ\psi_Y^{-t})|^2 - \beta_s\int_{\widetilde{M}} z^{1-s}|\nabla U|^2\right)\\
&\ge \lim_{t\to 0}\frac{1}{t}\left(\beta_s\int_{\widetilde{M}} z^{1-s}|\nabla V_{-t}|^2 - \beta_s\int_{\widetilde{M}} z^{1-s}|\nabla U|^2\right)\\
&= \lim_{t\to 0}\frac{[u\circ\psi_Y^{-t}]^2_{H^{s/2}(M)} - [u]^2_{H^{s/2}(M)}}{t}\\
&= \frac{d}{dt}\Big|_{t=0}[u\circ\psi_Y^{-t}]^2_{H^{s/2}(M)}.
\end{aligned}
$$

Hence

$$
\frac{d}{dt}\Big|_{t=0}\beta_s \int_{\widetilde{M}} z^{1-s}|\nabla(U\circ\psi_Y^{-t})|^2 = \frac{d}{dt}\Big|_{t=0}[u\circ\psi_Y^{-t}]^2_{H^{s/2}}.
$$

Since u is stationary for the energy $\mathcal{E}(v) = [v]^2_{H^{s/2}(M)} + \int_M F(v)\,dV$ under inner variations, this shows that U is stationary for the energy $U \mapsto \beta_s \int_{\widetilde{M}} z^{1-s}|\nabla U|^2\,dVdz + \int_M F(U|_M)\,dV$ under inner variations on $\widetilde{M}$, with vector fields Y as above, and this concludes the first step.

Step 2. We now compute such an inner variation for a suitably chosen Y. First, the variation of the potential part of the energy is

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\int_M F(u\circ\psi_Y^{-t})\,dV &= \frac{d}{dt}\Big|_{t=0}\int_M F(u)J_t(p)\,dV_p\\
&= \int_M F(u)\operatorname{div}_g(Y|_M)\,dV.
\end{aligned}
\tag{56}
$$

The quantity $\operatorname{div}_g(Y|_M)$ will be estimated later. We now focus on computing the variation for the Sobolev part of the energy. Once again, we change variables in the integral using the flow ψ_Y^t, obtaining

$$
\int_{\widetilde{M}} z^{1-s}|\nabla(U\circ\psi_Y^{-t})|^2\,dVdz = \int_{\widetilde{M}}(z\circ\psi_Y^t)^{1-s}|\nabla(U\circ\psi_Y^{-t})|^2\circ\psi_Y^t\,J_t(p,z)\,dV_pdz\,. \tag{57}
$$

Now we choose the vector field Y. We take $Y = \eta(d)d\nabla d$, where $d = d_{\widetilde{g}}((p_\circ,0),\cdot)$ is the distance on $\widetilde{M}$ from the point $(p_\circ,0)$ and $\eta = \eta_\delta$ is a single variable smooth function with $\eta \equiv 1$ on $[0,R]$, decreasing to zero on $[R,R+\delta]$, and $\eta \equiv 0$ on $[R+\delta,+\infty)$. Since the distance $d_{\widetilde{g}}((p_\circ,0),\cdot)$ restricts to the distance $d_g(p_\circ,\cdot)$ on M when computed on points on $\widetilde{M}$ with $z=0$, clearly $Y|_M$ is tangent to M. We want to exchange the order of derivation and integration in (57), hence we compute separately the three terms that will appear in doing so. For the first term, using $d^2_{\widetilde{g}}((p,z),(p_\circ,0)) = d^2_g(p,p_\circ) + z^2$ and the definition of Y we see that

$$
\frac{d}{dt}\Big|_{t=0}(z\circ\psi_Y^t)^{1-s} = (1-s)z^{-s}\eta(d)z = (1-s)z^{1-s}\eta(d)\,.
$$

As for the second term that will appear, a simple computation —see for example the lines after Lemma 3.1 in [59]— shows that

$$\frac{d}{dt}\Big|_{t=0}|\nabla(U\circ\psi_Y^{-t})|^2(\psi_Y^t(x)) = -2\langle\nabla_{\nabla U}Y,\nabla U\rangle\,.$$

Moreover, we also have

$$\begin{aligned}\langle\nabla_{\nabla U}Y,\nabla U\rangle &= \langle\nabla_{\nabla U}(\eta(d)d\nabla d),\nabla U\rangle\\ &= \langle\nabla U,\nabla\eta(d)\rangle\langle d\nabla d,\nabla U\rangle + \eta(d)\langle\nabla_{\nabla U}(d\nabla d),\nabla U\rangle\\ &= \eta'(d)\langle\nabla U,\nabla d\rangle\langle d\nabla d,\nabla U\rangle + \eta(d)\langle\nabla_{\nabla U}(d\nabla d),\nabla U\rangle\\ &= d\eta'(d)|\langle\nabla U,\nabla d\rangle|^2 + \eta(d)\langle\nabla_{\nabla U}(d\nabla d),\nabla U\rangle\,.\end{aligned}$$

Notice that K is also an upper bound for all the sectional curvatures on $\widetilde{M}$ in $\widetilde{B}_{R_\circ}(p_\circ,0)$, and that $\mathrm{inj}_M(p_\circ)=\mathrm{inj}_{\widetilde{M}}(p_\circ,0)$. Thus, by Lemma C.1 applied to $V=\nabla U$,

$$\langle\nabla_{\nabla U}Y,\nabla U\rangle = d\eta'(d)|\langle\nabla U,\nabla d\rangle|^2 + \eta(d)(1+O(\sqrt{K}R))|\nabla U|^2$$

for all $R<\min\left\{R_\circ,\frac{1}{\sqrt{K}}\right\}$. Lastly, for the remaining factor in the integral, Lemma (C.2) gives that

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}J_t = \widetilde{\mathrm{div}}(Y) &= \eta'(d)d|\nabla d|^2 + \eta(d)\widetilde{\mathrm{div}}(d\nabla d)\\ &= d\eta'(d) + \eta(d)(n+1)(1+O(\sqrt{K}R))\,,\end{aligned}$$

in $B_R(p_\circ)$, for $R<\min\left\{R_\circ,\frac{1}{\sqrt{K}}\right\}$.

Now, analogously applying Lemma (C.2) on M instead of $\widetilde{M}$ to (56), we already find an estimation for the potential energy:

$$\frac{d}{dt}\Big|_{t=0}\int_M F(u\circ\psi_Y^{-t})\,dV = \int_M F(u)\big(d\eta'(d)+\eta(d)n(1+O(\sqrt{K}R))\big)\,dV\,.$$

Moreover, it follows from (the local version of) Bonnet-Myers' theorem that $R_\circ < R_{\max} := \mathrm{inj}_M(p_\circ)/4 < \min\left\{\mathrm{inj}_M(p_\circ),\frac{1}{\sqrt{K}}\right\}$, and this will be our final choice of $R_{\max}$ for the statement. From now on, we always consider $R<R_\circ\le R_{\max}=\mathrm{inj}_M(p_\circ)/4$.

Regarding the Sobolev part of the energy, exchanging differentiation and integration and substituting the estimates we have obtained so far gives:

$$\begin{aligned}&\frac{d}{dt}\Big|_{t=0}\int_{\widetilde{M}} z^{1-s}|\nabla(U\circ\psi_Y^{-t})|^2\\ &\quad= \int_{\widetilde{M}}(1-s)z^{1-s}\eta(d)|\nabla U|^2 + z^{1-s}\big(-2d\eta'(d)|\langle\nabla U,\nabla d\rangle|^2 - 2\eta(d)(1+O(\sqrt{K}R))|\nabla U|^2\big)\\ &\qquad+\int_{\widetilde{M}} z^{1-s}|\nabla U|^2\big(d\eta'(d)+\eta(d)(n+1)(1+O(\sqrt{K}R))\big)\,dV dz\\ &\quad= (n-s)(1+O(\sqrt{K}R))\int_{\widetilde{B}^+_{R+\delta}} z^{1-s}|\nabla U|^2\eta(d) + \int_{\widetilde{B}^+_{R+\delta}\setminus\widetilde{B}^+_R} z^{1-s}d\eta'(d)\big(|\nabla U|^2-2|\langle\nabla U,\nabla d\rangle|^2\big).\end{aligned}$$

Adding the expressions for the potential and Sobolev parts of the energy, we get

$$\begin{aligned}&\frac{d}{dt}\Big|_{t=0}\left(\beta_s\int_{\widetilde{M}} z^{1-s}|\nabla(U\circ\psi_Y^{-t})|^2 + \int_M F(u\circ\psi_Y^{-t})\right)\\ &\quad= (n-s)(1+O(\sqrt{K}R))\beta_s\int_{\widetilde{B}^+_{R+\delta}}\eta(d)z^{1-s}|\nabla U|^2\\ &\qquad+\beta_s\int_{\widetilde{B}^+_{R+\delta}\setminus\widetilde{B}^+_R} d\eta'(d)z^{1-s}(|\nabla U|^2-2\langle\nabla U,\nabla d\rangle^2)\\ &\qquad+n(1+O(\sqrt{K}R))\int_{B_{R+\delta}}\eta(d)F(u) + \int_{B_{R+\delta}\setminus B_R} d\eta'(d)F(u)\,.\end{aligned}$$

By stationarity of u and Step 1 we know that the left-hand side is equal to 0 for every Y, thus the right-hand side vanishes for all $\eta = \eta_\delta$ defined as above. Since this holds for all $\delta > 0$, we now let $\delta \searrow 0$ so that η_δ converges to the characteristic function of $[0, R]$. This gives (for a.e. $R \in (0, R_\circ)$)

$$\begin{aligned} 0 =& (n-s)(1+O(\sqrt{K}R))\beta_s \int_{\widetilde{B}_R^+} z^{1-s}|\nabla U|^2 - R\beta_s \int_{\partial \widetilde{B}_R^+} z^{1-s}|\nabla U|^2 + 2R\beta_s \int_{\partial^+ \widetilde{B}_R^+} (\partial_\nu U)^2 \\ &+ n(1+O(\sqrt{K}R)) \int_{B_R} F(u) - R\int_{\partial B_R} F(u). \end{aligned}$$

Rearranging the terms and multiplying by R^{-n+s-1}, we deduce that

$$\begin{aligned} -\frac{(n-s)}{R^{n-s+1}} &\left(\beta_s \int_{\widetilde{B}_R^+} z^{1-s}|\nabla U|^2 + \int_{B_R} F(u)\right) + \frac{1}{R^{n-s}}\left(\beta_s \int_{\partial \widetilde{B}_R^+} z^{1-s}|\nabla U|^2 + \int_{\partial B_R} F(u)\right) \\ &\geq -\frac{C\sqrt{K}}{R^{n-s}}\left(\beta_s \int_{\widetilde{B}_R^+} z^{1-s}|\nabla U|^2 + \int_{B_R} F(u)\right) + \frac{2\beta_s}{R^{n-s}} \int_{\partial^+ \widetilde{B}_R^+} z^{1-s}\langle \nabla U, \nabla d\rangle^2 + \frac{s}{R^{n-s+1}}\int_{B_R} F(u)\,, \end{aligned}$$

for some absolute constant $C > 0$. In other words,

$$\Phi'(R) \geq -C\sqrt{K}\Phi(R) + \frac{2\beta_s}{R^{n-s}} \int_{\partial^+ \widetilde{B}_R^+} z^{1-s}\langle \nabla U, \nabla d\rangle^2 + \frac{s}{R^{n-s+1}}\int_{B_R} F(u)\, dV\,,$$

and this implies in particular that

$$\frac{d}{dR}\left(e^{C\sqrt{K}R}\Phi(R)\right) \geq 0 \quad \text{for all} \quad R < R_\circ.$$

Lastly, in the case where $M = \mathbb{R}^n$, $F \equiv 0$, $s \in (0,1)$ and $u = \chi_E - \chi_{E^c}$ is a stationary set for the fractional s-perimeter, instead of the two bounds used above

$$\begin{aligned} \langle \nabla_{\nabla U}(d\nabla d), \nabla U\rangle &= (1+O(\sqrt{K}R))|\nabla U|^2\,, \\ \widetilde{\operatorname{div}}(d\nabla d) &= (n+1)(1+O(\sqrt{K}R))\,, \end{aligned}$$

given respectively by Lemma (C.1) and (C.2), one has the equalities

$$\begin{aligned} \langle \nabla_{\nabla U}(d\nabla d), \nabla U\rangle_{\mathbb{R}^{n+1}} &= |\nabla U|^2\,, \\ \operatorname{div}_{\mathbb{R}^{n+1}}(d\nabla d) &= n+1\,, \end{aligned}$$

where U is the extension of $u = \chi_E - \chi_{E^c}$. Thus, following the proof one finds the exact expression

$$\Phi'(R) = \frac{2\beta_s}{R^{n-s}} \int_{\partial^+ \widetilde{B}_R^+(p_\circ, 0)} z^{1-s}\langle \nabla U, \nabla d\rangle^2\, dxdz \geq 0\,.$$

In particular, Φ is constant if and only if $\langle \nabla U, \nabla d\rangle = 0$, that is, if and only if E is dilation-invariant for dilations with center at $p_\circ \in \mathbb{R}^n$. With this we conclude the proof. ☐

3 Existence of min-max solutions to Allen-Cahn and convergence to a limit nonlocal minimal surface

3.1 Existence results – Proof of Theorem 1.20

In what follows, (M^n, g) will be an arbitrary closed, n-dimensional Riemannian manifold, and $s_0 \in (0,1)$ will be fixed. Moreover, we will use the notation $\mathcal{E}^\varepsilon_M(v)$ for the Allen-Cahn energy from (6) to make the parameter ε explicit in the notation.

3.1.1 Min-max procedure

The solutions in Theorem 1.20 are obtained using an equivariant min-max procedure, based on the construction in [60] and the min-max theorems of [62], [63] and [73]. Since the topology of $H^{s/2}(M)$ is trivial, this is done by exploiting the $\mathbb{Z}_2$-symmetry of the functional $\mathcal{E}_M^\varepsilon$. Indeed, we consider the family $\mathcal{F}_\mathfrak{p}$ of all sets $A \subset H^{s/2}(M)\backslash\{0\}$ which are continuous odd images of $\mathfrak{p}$-spheres:

$$\mathcal{F}_\mathfrak{p} := \left\{A = f(\mathbb{S}^\mathfrak{p}) \;:\; f \in C^0(\mathbb{S}^p; H^{s/2}(M)\backslash\{0\}) \;\text{ and }\; f(-x) = -f(x)\;\forall\, x \in \mathbb{S}^\mathfrak{p}\right\}.$$

Remark 3.1. This min-max family has been chosen for simplicity, but other min-max families can be considered; see the seminal article [73] by Lazer-Solimini, as well as the discussion in Remark 3.7 of [60]. In particular, one can obtain solutions in Theorem 1.20 which also satisfy lower bounds for their (extended) Morse indices. We nevertheless remark that a growth for the (proper!) index of the solutions is already implied in our case, by combining the lower energy bound in Theorem 1.20 with the upper energy bounds in Theorem 1.27 (which will be proved later).

For fixed ε, the min-max value of the family $\mathcal{F}_\mathfrak{p}$ is defined as

$$c_{\varepsilon,\mathfrak{p}} := \inf_{A\in\mathcal{F}_\mathfrak{p}} \sup_{u\in A} \mathcal{E}_M^\varepsilon(u). \tag{58}$$

Note that, defining $T(u) := \max\{-1, \min\{u, +1\}\}$ the truncation of u between the values ±1, we have that $|T(u)|(x) \le 1$ for all $x \in M$ and $\mathcal{E}_M^\varepsilon(T(u)) \le \mathcal{E}_M^\varepsilon(u)$. Hence

$$c_{\varepsilon,\mathfrak{p}} = \inf_{A\in\mathcal{F}_\mathfrak{p}} \sup_{u\in A} \mathcal{E}_M^\varepsilon(u) = \inf_{A\in\widetilde{\mathcal{F}}_\mathfrak{p}} \sup_{u\in A} \mathcal{E}_M^\varepsilon(u)\,,$$

where

$$\widetilde{\mathcal{F}}_\mathfrak{p} = \{A \in \mathcal{F}_\mathfrak{p} : |u| \le 1 \;\text{ for all }\; u \in A\}.$$

This shows that we can consider, in the arguments that follows, that the functions in the min-max sets have absolute values pointwise bounded by one. The proof of Theorem 1.20 relies on the existence result given by the min-max scheme and the following bound on the min-max values.

Theorem 3.2. *Let (M^n, g) be a compact Riemannian manifold, $s_0 \in (0,1)$ and $s \in (s_0, 1)$. Then, for every $\mathfrak{p} \in \mathbb{N}$ there exists $\varepsilon_\mathfrak{p} > 0$, depending on M, s and $\mathfrak{p}$, such that the min-max values (58) satisfy*

$$\frac{C^{-1}}{1-s}\mathfrak{p}^{s/n} \le c_{\varepsilon,\mathfrak{p}} \le \frac{C}{1-s}\mathfrak{p}^{s/n}\,, \quad \text{for all } \varepsilon \in (0, \varepsilon_\mathfrak{p})\,, \tag{59}$$

for some constant $C = C(M, s_0)$.

Proof. The proof of this is contained in subsections 3.1.2 and 3.1.3 below, which deal with the lower bound and upper bound respectively. □

To apply the existence result we need the energy $\mathcal{E}_M^\varepsilon$ to satisfy the Palais-Smale condition along appropriately bounded sequences, and this is addressed by the next lemma.

Lemma 3.3. *Let $\varepsilon > 0$ and $s \in (0,1)$. Suppose that $(u_k)_k \subset H^{s/2}(M)$ is a sequence of functions satisfying $|u_k| \le 1$, $|\mathcal{E}_M^\varepsilon(u_k)| \le C$, and $d\mathcal{E}_M^\varepsilon(u_k) \to 0$ strongly in $H^{s/2}(M)$. Then, there is a subsequence of $(u_k)_k$ converging strongly in $H^{s/2}(M)$.*

Proof. The proof is an adaptation of Proposition 2.25 in [16]. We just prove the statement for $\varepsilon = 1$, as exactly the same proof works for every fixed $\varepsilon > 0$. The boundedness of the energies $\mathcal{E}_M(u_k)$ gives the convergence

$$u_k \xrightarrow{L^2} u \;\text{ and }\; u_k \xrightarrow{H^{s/2}} u$$

of a subsequence, that we do not relabel, to some $u \in H^{s/2}(M)$. To upgrade the convergence to be in the strong sense, we use the particular form of the functional. First, note that given $v \in H^{s/2}(M)$ we have

$$\begin{aligned} d\mathcal{E}^{\varepsilon}_M(u)[v] &= \frac{1}{4}\iint (u(p)-u(q))(v(p)-v(q))\, K_s(p,q)\, dV_p dV_q + \int W'(u) v\, dV \\ &= \lim_{k\to\infty} \frac{1}{4}\iint (u_k(p)-u_k(q))(v(p)-v(q))\, K_s(p,q)\, dV_p dV_q + \int W'(u_k) v\, dV \\ &= \lim_{k\to\infty} d\mathcal{E}^{\varepsilon}_M(u_k)[v] = 0\,, \end{aligned}$$

where we used $|u_k| \le 1$ to pass to the limit in the term $\int W'(u_k)v$. In other words, u is a critical point of $\mathcal{E}^{\varepsilon}_M$. From this we deduce that

$$\begin{aligned} 0 &= \lim_{k\to\infty} \left(d\mathcal{E}^{\varepsilon}_M(u_k)[u_k-u] - d\mathcal{E}^{\varepsilon}_M(u)[u_k-u] \right) \\ &= \mathcal{E}^{\mathrm{Sob}}_M(u_k-u) + \int (W'(u_k)-W'(u))(u_k-u)\, dV, \end{aligned}$$

and since the second term tends to zero, the first term must do so as well. This proves that $u_k \to u$ strongly in $H^{s/2}(M)$ and concludes the proof. □

Theorem 3.4. *For every $\mathfrak{p} \in \mathbb{N}$, there exists $\varepsilon_{\mathfrak{p}} > 0$ such that: for all $\varepsilon \in (0, \varepsilon_{\mathfrak{p}})$ there exists $u_{\varepsilon,\mathfrak{p}} \in H^{s/2}(M)$ which is a critical point of $\mathcal{E}^{\varepsilon}_M$ with $\mathcal{E}^{\varepsilon}_M(u_{\varepsilon,\mathfrak{p}}) = c_{\varepsilon,\mathfrak{p}}$ and Morse index $m(u_{\varepsilon,\mathfrak{p}}) \le \mathfrak{p}$ (see Definition 1.14).*

Proof. Since $\mathcal{E}^{\varepsilon}_M$ satisfies the Palais-Smale condition along appropriately bounded sequences (see Lemma 3.3 above) and since $d^2\mathcal{E}^{\varepsilon}_M$ is a Fredholm operator at critical points, the min-max theorems in [62] imply that there exists a critical point $u_{\varepsilon,\mathfrak{p}}$ for $\mathcal{E}^{\varepsilon}_M$ at energy level $c_{\varepsilon,\mathfrak{p}}$ and with Morse index $m(u_{\varepsilon,\mathfrak{p}}) \le \mathfrak{p}$. To be precise, there is one detail that we have to address. The min-max theorem in [62] needs to be applied on a complete, connected Banach manifold X on which $\mathbb{Z}_2$ acts freely, and apparently, we are applying it to the punctured space $X = H^{s/2}(M) \setminus \{0\}$, which is not complete. Nevertheless, for every $\varepsilon > 0$ the min-max value $c_{\varepsilon,\mathfrak{p}}$ is defined and, as we will prove in Subsection 3.1.3, the upper bound (59) for $c_{\varepsilon,\mathfrak{p}}$ holds independently of ε. Since the energy of the zero function tends to infinity as $\varepsilon \searrow 0$, this shows that there is $\varepsilon_{\mathfrak{p}} > 0$ such that, for all $\varepsilon \in (0, \varepsilon_{\mathfrak{p}})$, every min-max sequence is uniformly separated from $0 \in H^{s/2}(M)$, hence the min-max theorem can be applied and this concludes the proof of existence. □

Hence, to obtain Theorem 1.20 we are only left with proving the lower and upper bounds in (59). The analog bounds in the case of min-max families of hypersurfaces and their areas were first proved by Gromov in [66]. In [68], Guth gave an elegant new proof of the result by Gromov on which the proof of our bounds is based. See also [76] for an adaptation of the proof by Guth to the setting of closed manifolds, as well as [60] for the adaptation to the classical Allen-Cahn case. Nevertheless, our proof of the bounds is closer to (and also simpler than) those in [68] or [76], thanks to the fact that sets of finite perimeter embed naturally[11] in the same function space $H^{s/2}(M)$ as their fractional Allen-Cahn approximations, as long as $s \in (0,1)$.

3.1.2 Lower bound

In the proofs of both the upper and lower bounds, we will make use of the following simple fact.

Lemma 3.5. *Let (M^n, g) be a closed, n-dimensional Riemannian manifold. Then, there exist positive constants C_0, C_1, C_2 depending only on M such that: for every $\mathfrak{p} \in \mathbb{N}$ there exist N disjoint balls $B_r(q_1), \dots B_r(q_N)$ with*

$$\begin{aligned} C_1\, \mathfrak{p} \le N \le C_2\, \mathfrak{p}\,, \\ r = C_0\, \mathfrak{p}^{-1/n}\,, \end{aligned}$$

and

$$\bigcup_{i\le N} B_{3r}(q_i) = M\,.$$

[11] Via functions of the form $\chi_E - \chi_{E^c}$.

Proof. Since M is compact, by a comparison argument there exists a constant $c = c(M) > 1$ such that

$$c^{-1}r^n \le \mathcal{H}^n(B_r(q)) \le c\, r^n\,, \quad \text{for all } q \in M\,,\ r < \mathrm{inj}(M)\,. \tag{60}$$

We claim that, for $C_0 = \mathrm{inj}(M)/3$, the statement holds. Indeed, with this choice $r = C_0\, \mathfrak{p}^{-1/n} \le \mathrm{inj}(M)/3$ for all $\mathfrak{p} \in \mathbb{N}$. Consider the cover $\bigcup_{q\in M} B_r(q)$ of M, and let $B_r(q_1), \dots, B_r(q_N)$ be a maximal disjoint collection. Then, by maximality $\bigcup_{i\le N} B_{3r}(q_i) = M$. Moreover, comparing volumes

$$c^{-1}r^nN \le \sum_{i\le N} \mathcal{H}^n(B_r(q_i)) \le \mathrm{Vol}(M) \le \sum_{i\le N} \mathcal{H}^n(B_{3r}(q_i)) \le c3^nr^nN\,,$$

which by the choice of r implies

$$C_1\,\mathfrak{p} := \frac{\mathrm{Vol}(M)}{c3^nC_0^n}\,\mathfrak{p} \le N \le \frac{c\mathrm{Vol}(M)}{C_0^n}\,\mathfrak{p} =: C_2\,\mathfrak{p}\,.$$

□

The proof of the lower bound depends on the next two lemmas.

Lemma 3.6. *Let $\{B_r(q_i)\}_{i=1}^{\mathfrak{p}}$ be a family of $\mathfrak{p}$ balls on M. Then, given any $A \in \mathcal{F}_{\mathfrak{p}}$, there exists some $u \in A$ such that*

$$\int_{B_r(q_i)} u = 0 \quad \text{for all} \quad i = 1, \dots, \mathfrak{p}.$$

Proof. This is simply a consequence of the Borsuk-Ulam theorem. Indeed, let A be the continuous odd image of $f : \mathbb{S}^{\mathfrak{p}} \to H^{s/2}(M) \setminus \{0\}$. Define the (odd) function

$$g : H^{s/2}(M) \to \mathbb{R}^{\mathfrak{p}} \quad \text{by} \quad g(u) := \left(\int_{B_r(q_1)} u, \dots, \int_{B_r(q_{\mathfrak{p}})} u \right).$$

Then $g \circ f : \mathbb{S}^{\mathfrak{p}} \to \mathbb{R}^{\mathfrak{p}}$ is an odd, continuous map, and by the Borsuk-Ulam theorem there exists $a \in \mathbb{S}^{\mathfrak{p}}$ with $g \circ f(a) = 0$. Hence, taking $u = f(a)$ finishes the proof. □

For the next lemma it will be convenient to define the local part of the Sobolev energy as follows. Recall

$$\begin{aligned}\mathcal{E}^{\mathrm{Sob}}_{\Omega}(v) &= \frac{1}{4}\iint_{M\times M\setminus \Omega^c\times\Omega^c} (v(p)-v(q))^2K_s(p,q)dV_pdV_q \\ &= \frac{1}{4}\iint_{\Omega\times\Omega} (v(p)-v(q))^2K_s(p,q)dV_pdV_q + \frac{1}{2}\iint_{\Omega\times\Omega^c} (v(p)-v(q))^2K_s(p,q)dV_pdV_q\,.\end{aligned}$$

Then, we set

$$\mathcal{E}^{\mathrm{Sob}}|_{\Omega}(v) := \frac{1}{4}\iint_{\Omega\times\Omega} (v(p)-v(q))^2K_s(p,q)dV_pdV_q\,.$$

Moreover, we also denote by

$$\mathrm{Per}_s|_{\Omega}(E) := \mathcal{E}^{\mathrm{Sob}}|_{\Omega}(\chi_E - \chi_{E^c}) \tag{61}$$

the local part of the nonlocal perimeter.

We stress that $\mathrm{Per}_s|_{\Omega}(E) \le \mathrm{Per}_s(E,\Omega)$, with equality iff $E \cap \Omega = \emptyset$ or $E^c \cap \Omega = \emptyset$.

With this notation, let us recall the fractional isoperimetric inequality (which is implied, actually equivalent, to the fractional Sobolev inequality for example in [56]).

Let $s \in (0,1)$. Then is $c_{iso} = c_{iso}(n,s) > 0$ such that for every $E \subset \mathcal{B}_1$

$$\mathrm{Per}_s|_{\mathcal{B}_1}(E) \ge c_{iso} \min\left\{ \frac{|E|}{|\mathcal{B}_1|}, \frac{|\mathcal{B}_1 \setminus E|}{|\mathcal{B}_1|} \right\}^{\frac{n-s}{n}}, \tag{62}$$

and actually if $s \in (s_0,1)$ then $c_{iso} \ge \frac{c'}{1-s}$ for some $c' = c'(n,s_0)$.

Lemma 3.7. *Let $s_0 \in (0,1)$. Then, there exists a constant $c_0 = c_0(n,s_0) > 0$ such that the following holds: Let $q \in M$, and assume that the flatness hypothesis $\mathrm{FA}_1(M,g,q,2R_0,\varphi)$ holds. Given $s \in (s_0,1)$, there exists $\varepsilon_0 = \varepsilon_0(n,s) > 0$ such that: for every $r \le R_0$ and $\varepsilon \le \varepsilon_0 r$, given any $u \in H^{s/2}(M)$ with $|u| \le 1$ and $\int_{B_r(q)} u = 0$ there holds*

$$\mathcal{E}^{\varepsilon}|_{B_r(q)}(u) \ge \frac{c_0}{1-s} r^{n-s}\,.$$

Proof. Let $\psi : B_{R_0(q)} \to \mathbb{R}^n$ be normal coordinates at q, and let g_{ij} denote the metric in these coordinates. It is not difficult to show that $\mathrm{FA}_1(M,g,q,2R_0,\varphi)$ implies

$$\frac{1}{C}\mathrm{id} \le (g_{ij})(x) \le C\mathrm{id} \quad \text{in } \mathcal{B}_{R_0}$$

for some $C > 1$ depending only on n.

Let $v = u \circ \psi^{-1}$. By assumption we have $\int_{\mathcal{B}_r} v\sqrt{|g|}\,dx = 0$, and $|v| \le 1$. This implies

$$-1 + 1/C \le \fint_{\mathcal{B}_r} v \le +1 - 1/C\,.$$

By Lemma 2.11

$$\mathcal{E}^{\mathrm{Sob}}|_{B_r(q)}(u) + \varepsilon^{-s}\int_{B_r(q)} W(u)\,dV \ge \frac{1}{C}\left(\frac{1}{4}\iint_{\mathcal{B}_r\times\mathcal{B}_r} \frac{|v(x)-v(y)|^2}{|x-y|^{n+s}}\,dxdy + r^{-s}(\varepsilon/r)^{-s}\int_{\mathcal{B}_r} W(v)\,dx\right). \tag{63}$$

We claim that the right-hand side of (63) is bounded below by $\frac{c_{iso}}{4C^2} r^{n-s}$, provided $\varepsilon/r \le \varepsilon_0(n,s)$, where c_{iso} is the constant in the fractional isoperimetric inequality (62).

To prove this lower bound, by scaling invariance we may assume without loss of generality $r = 1$. We argue by contradiction. Suppose there exist sequences $\varepsilon_k \downarrow 0$ and $v_k \in H^{s/2}(\mathcal{B}_1)$ with $|v_k| \le 1$, $\fint_{\mathcal{B}_1} v_k \in [-1+1/C, 1-1/C]$, but such that

$$\frac{1}{4}\iint_{\mathcal{B}_1\times\mathcal{B}_1} \frac{|v_k(x)-v_k(y)|^2}{|x-y|^{n+s}}\,dxdy + \varepsilon_k^{-s}\int_{\mathcal{B}_1} W(v_k)\,dx < \frac{c_{iso}}{4C}\,. \tag{64}$$

In particular, $\|v_k\|^2_{H^{s/2}(\mathcal{B}_1)}$ is uniformly bounded in k and $\int_{\mathcal{B}_1} W(v_k) \to 0$. Hence, up to subsequences (that we do not relabel) we have that $v_k \rightharpoonup v_\infty$ weakly in $H^{s/2}(\mathcal{B}_1)$ and $v_k \to v$ almost everywhere. Then, by Fatou's Lemma, $\int_{\mathcal{B}_1} W(v_\infty) = 0$ and therefore $|v_\infty| = 1$ almost everywhere. Moreover

$$\fint_{\mathcal{B}_1} v_\infty \in [-1+1/C, 1-1/C]\,.$$

This implies that $v_\infty = \chi_E - \chi_{E^c}$ for some set $E \subset \mathcal{B}_1$ with $\frac{1}{|\mathcal{B}_1|}\min\{|E|, |\mathcal{B}_1 \setminus E|\} \ge \frac{1}{2C}$. By the lower-semicontinuity of the Sobolev seminorm, the fractional isoperimetric inequality (62) and (64) we get

$$\frac{c_{iso}}{2C} \le \mathrm{Per}_s|_{\mathcal{B}_1}(E) \le \frac{1}{4}\liminf_{k\to\infty}\iint_{\mathcal{B}_1\times\mathcal{B}_1} \frac{|v_k(x)-v_k(y)|^2}{|x-y|^{n+s}}\,dxdy \le \frac{c_{iso}}{4C}\,,$$

a contradiction.

Going back to (63), we have therefore proved that there exists $\varepsilon_0 = \varepsilon_0(n,s) > 0$ such that, for every $\varepsilon \le \varepsilon_0 r$ and every u as in the statement

$$\mathcal{E}^{\varepsilon}|_{B_r(q)}(u) \ge \frac{c_{iso}}{4C^2} r^{n-s}\,.$$

Since the constant $c_{iso} = c_{iso}(n,s)$ for the fractional isoperimetric inequality satisfies that $c_{iso} \ge \frac{c'(n,s_0)}{1-s}$ for some $c' = c'(n,s_0) > 0$, we conclude the desired result. □

We can now give the proof of the lower bound.

Proof of Theorem 3.2 (part 1). The lower bound in (59) follows, in a simple manner, from the lemmas above: Given $\mathfrak{p} \in \mathbb{N}$, by Lemma 3.5 find $N \geq C_1\mathfrak{p}$ disjoint balls $\{B_r(q_i)\}_{i=1}^N$ in M with radius $r = C_0\,\mathfrak{p}^{-1/n}$. Moreover, up to taking C_1 bigger and C_0 smaller, we can also assume that $r < R_0$, where R_0 is such that $\mathrm{FA}_1(M, g, q, 2R_0, \varphi)$ holds for all $q \in M$ (see Remark 1.22). Given any $A \in \mathcal{F}_\mathfrak{p}$, by Lemma 3.6 there exists $u \in A$ such that

$$\int_{B_r(q_i)} u = 0 \quad \text{for all} \quad i = 1, \dots, N.$$

Then, by Lemma 3.7, for $\varepsilon \leq \varepsilon_0 r$ we have

$$\mathcal{E}^\varepsilon|_{B_r(q_i)}(u) \geq \frac{c_0}{1-s} r^{n-s} \quad \text{for all} \quad i = 1, \dots N,$$

which by the choice of r implies

$$\mathcal{E}^\varepsilon_M(u) \geq \sum_{i=1}^N \mathcal{E}^\varepsilon|_{B_r(q_i)}(u) \geq N \frac{c_0}{1-s} r^{n-s} \geq \frac{C^{-1}}{1-s}\mathfrak{p}^{s/n},$$

for some constant C that depends only on M and s_0. Since we have found such a $u \in A$ given any $A \in \mathcal{F}_p$, we deduce the desired lower bound. □

3.1.3 Upper bound

While the lower bound required finding a function with high energy inside every admissible set $A \in \mathcal{F}_\mathfrak{p}$, the upper bound requires finding a single admissible set A such that all its elements have "low" energy. We will explicitly construct a continuous odd map $f : \mathbb{S}^\mathfrak{p} \to H^{s/2}(M) \setminus \{0\}$ so that all the elements in $A = f(\mathbb{S}^\mathfrak{p})$ have controlled energy. These functions will be of the form $\chi_E - \chi_{E^c}$, for some set $E \subset M$, and our task is then to bound the fractional perimeter of these sets.

The proof of the upper bound in (59) goes as follows.

Proof of Theorem 3.2 (part 2). By Lemma 3.5 (with $\mathfrak{p}$ replaced by k), for every $k \geq 1$ there exist $N \leq C_2 k$ disjoint balls $B_r(q_1), \dots, B_r(q_N)$ of radius $r = C_0 k^{-1/n}$ and with $\bigcup_{i \leq N} B_{3r}(q_i) = M$. Moreover, recalling the proof of Lemma 3.5, the bounds (60) hold for such an r.

Now, given $(a_0, a_1, \dots, a_\mathfrak{p}) \in \mathbb{S}^\mathfrak{p}$ consider the polynomial $P_a(z) = a_0 + a_1 z + \ldots + a_p z^\mathfrak{p}$ and name $\{\alpha_1, \dots, \alpha_\ell\}$ its real roots in increasing order, so that $\ell \leq \mathfrak{p}$. In $\mathbb{R}^n$ consider the set

$$E := \bigcup_{i=1}^N \mathcal{B}_{3r}(3r(2i+1), 0, \dots, 0)\,;$$

these are N aligned balls, tangent to each other, with centers on the x_1-axis. Now we split the set E into two disjoint subsets $E = E^+ \cup E^-$. Given the real roots $\{\alpha_1, \dots, \alpha_\ell\}$, assign the set $E \cap \{x_1 \leq \alpha_1\}$ to E^+ if $P_a(z) \geq 0$ for all $z \in (-\infty, \alpha_1]$, and else assign it to E^- if $P_a(z) \leq 0$ for all $z \in (-\infty, \alpha_1]$. Then, analogously assign $E \cap \{\alpha_1 < x_1 \leq \alpha_2\}$ to E^+ if $P_a(z) \geq 0$ for all $z \in (\alpha_1, \alpha_2]$, and assign it to E^- if $P_a(z) < 0$ for $z \in (\alpha_1, \alpha_2]$. Repeat this procedure until E is divided into the two subsets E^+ and E^-. Note that there are at most $\mathfrak{p}$ transitions[12] between E^+ and E^-, and thus E^+ has perimeter at most $|\partial E^+| \leq N|\partial \widetilde{B}_{3r}| + (6r)^{n-1}\mathfrak{p}$. Now, basically we want to do the same on the balls $\{B_{3r}(q_i)\}_{i=1}^N$ on M identifying $B_{3r}(q_i)$ with $\mathcal{B}_{3r}(3r(2i+1), 0, \dots, 0)$ via the exponential map, that we consider as a map

$$\exp_{q_i} : \mathcal{B}_{3r}(3r(2i+1), 0, \dots, 0) \to B_{3r}(q_i)\,.$$

In order to do so, we first have to make the covering $\{B_{3r}(q_i)\}_{i=1}^N$ of M disjoint. For this purpose, for all $1 \leq i \leq N$ we consider

$$Q_i := B_{3r}(q_i) \setminus \bigcup_{j \leq i-1} B_{3r}(q_j)\,,$$

[12] This corresponds to the case when $P_a(z)$ has $\mathfrak{p}$ distinct real roots in the interval $(0, 6rN)$.

and note that $\{Q_i\}_{i=1}^N$ is a disjoint partition of M with $Q_i \subset B_{3r}(q_i)$. Let $u_a : M \to \{+1, -1\}$ be defined as

$$u_a(q) := \begin{cases} +1 & \text{if } \ q \in Q_i \text{ and } (\exp_{q_i})^{-1}(q) \in E^+, \\ -1 & \text{if } \ q \in Q_i \text{ and } (\exp_{q_i})^{-1}(q) \in E^-. \end{cases}$$

Set $\Sigma_a := \partial\{u_a = 1\}$. By interpolation (use for example Proposition 2.22 applied to a covering of M with small enough balls) there exists $C = C(M, s_0)$ so that

$$\begin{aligned} \mathcal{E}_M^{\mathrm{Sob}}(u_a) &= \frac{1}{4}[u_a]^2_{H^{s/2}(M)} = \mathrm{Per}_s(\{u_a = 1\}) \\ &\leq \frac{C}{1-s}\mathcal{H}^{n-1}(\Sigma_a)^s \ \mathrm{Vol}(\{u_a = 1\}) \leq \frac{C}{1-s}\mathcal{H}^{n-1}(\Sigma_a)^s. \end{aligned} \tag{65}$$

Moreover, by (60) we have

$$\begin{aligned} \mathcal{H}^{n-1}(\Sigma_a) &\leq C(N|\partial\mathcal{B}_{3r}| + r^{n-1}\mathfrak{p}) \\ &\leq C(kr^{n-1} + \mathfrak{p}r^{n-1}) = C(k^{1/n} + \mathfrak{p}k^{-1+1/n}), \end{aligned}$$

for all $k \geq 1$, with C that depends only on M. Clearly, the optimal value of the right hand side is attained for $k = \mathfrak{p}$ and gives $\mathcal{H}^{n-1}(\Sigma_a) \leq C\mathfrak{p}^{1/n}$. This, together with (65), implies

$$\mathcal{E}_M^{\mathrm{Sob}}(u_a) \leq \frac{C}{1-s}\mathfrak{p}^{s/n}, \tag{66}$$

for all $a \in \mathbb{S}^{\mathfrak{p}}$, with C depending only on M and s_0.
Now, from the definition of $E^{\pm}$ it is clear that $u_{-a}(x) = -u_a(x)$. The map $\mathbb{S}^{\mathfrak{p}} \ni a \mapsto u_a \in H^{s/2}(M) \setminus \{0\}$ is then a continuous, odd map, and thus defines an element of $\mathcal{F}_{\mathfrak{p}}$. Together with the bound (66), which holds for all $a \in \mathbb{S}^{\mathfrak{p}}$, this concludes the proof. □

One should compare the simplicity of this construction, with the one in [60] for the classical Allen-Cahn equation and the classical perimeter, In that case, to define the $\mathfrak{p}$-sweepouts with the correct interface one has to consider functions which are compositions of:

(i) The solution to the 1-dimensional Allen-Cahn equation with parameter $\varepsilon > 0$.

(ii) A "modified" distance function, measuring the distance to hyperplanes $\{x_1 \leq c\}$ (which play a similar role to those in our construction) but also accounting for the complex parts of the roots of the polynomials in order to smooth out the cancellations of the leaves.

Using this composition of functions is necessary in the classical Allen-Cahn case in order to regularize the construction, as characteristic functions of sets of finite perimeter do not belong to $H^1(M)$, while they do belong to $H^{s/2}(M)$ for $s < 1$.

Remark 3.8. Notice that, for every fixed $\mathfrak{p}$, both the proofs of the lower bound and the upper bound in (59) rely on the fact that there is the same "critical scale" $r = C\mathfrak{p}^{-1/n}$ in the construction. This is given by dividing M in $N \sim \mathfrak{p}$ disjoint patches of volume of order $\sim 1/\mathfrak{p}$. The lower bound shows that, given any $A \in \mathcal{F}_{\mathfrak{p}}$, there is one element of A that has zero average - see Lemma 3.6 - in each of these patches, and in particular this element has energy uniformly bounded from below of order $\mathfrak{p}^{s/n}$. On the other hand, the upper bound shows that this configuration, i.e. making the transitions take place in $N \sim \mathfrak{p}$ disjoint patches that cover M, is also (of the order of) the best that one can achieve.

As a consequence of the results above, we deduce our complete existence result.

Proof of Theorem 1.20. The statement follows from combining the existence result of Theorem 3.4 and the bounds on the min-max values given by Theorem 3.2. □

3.2 Estimates for Allen-Cahn solutions with bounded Morse index

By Theorem 2.20, notice that u is a solution to the Allen-Cahn equation

$$(-\Delta)^{s/2}u + \varepsilon^{-s}W'(u) = 0 \quad \text{on } M\,,$$

if and only if the Caffarelli-Silvestre extension U, i.e. the unique solution to (47), solves

$$\begin{cases} \widetilde{\operatorname{div}}(z^{1-s}\nabla U) = 0\,, & \text{in } \widetilde{M} \qquad (67)\\ \lim_{z\to 0^+} z^{1-s}U_z(\,\cdot\,,z) = \beta_s^{-1}\varepsilon^{-s}W'(u(\cdot))\,, & \text{on } M\,. \qquad (68)\end{cases}$$

Recall the definition of finite Morse index solutions of Definition 1.14.

3.2.1 Finite Morse index and almost stability

For critical points of local functionals, it is well known that having Morse index bounded by m implies stability in one out of every $m+1$ disjoint open sets. In the nonlocal case this is not the case anymore, but in one of the sets we will be able to obtain a weaker, quantitative lower bound on the second derivative which we will refer to as *almost stability*.

Definition 3.9 (Almost stability). Let $\Omega \subset M$ open and $u : M \to (-1,1)$ be a critical point of $\mathcal{E}_\Omega$. Given $\Lambda \in \mathbb{R}$, we say that u is Λ-*almost stable in* Ω if

$$\mathcal{E}''_\Omega(u)[\xi,\xi] \geq -\Lambda\|\xi\|^2_{L^1(\Omega)} \quad \forall\,\xi \in C^1_c(\Omega)\,.$$

Lemma 3.10 (Finite Morse index and almost stability)**.** *Let $u : M \to (-1,1)$ be a solution of the Allen-Cahn equation $(-\Delta)^{s/2}u + \varepsilon^{-s}W'(u) = 0$ on M with Morse index at most m (see Definition 1.14, with $\Omega = M$). Consider a collection $\mathcal{U}_1,\ldots,\mathcal{U}_{n+1}$ of $(n+1)$ disjoint open subsets of M, and set*

$$\Lambda := m \max_{i\neq j} \sup_{\mathcal{U}_i\times\mathcal{U}_j} K(p,q)\,.$$

Then, there is (at least) one set $\mathcal{U}_k$ among $\mathcal{U}_1,\ldots,\mathcal{U}_{n+1}$ such that u is Λ-almost stable in $\mathcal{U}_k$, that is

$$\mathcal{E}''(u)[\xi,\xi] \geq -\Lambda\|\xi\|^2_{L^1(\mathcal{U}_k)} \quad \forall\,\xi \in C^1_c(\mathcal{U}_k)\,,$$

Proof. We prove the Lemma just for $m = 1$ for the sake of clarity, the proof goes on exactly the same for general m. Let $\xi_1 \in C^\infty_c(\mathcal{U}_1)$ and $\xi_2 \in C^\infty_c(\mathcal{U}_2)$. Testing the second variation of the Allen-Cahn energy, explicited in (9), with linear combinations of ξ_1 and ξ_2 gives

$$\mathcal{E}''(u)[a\xi_1 + b\xi_2] = a^2\mathcal{E}''(u)[\xi_1,\xi_1] + b^2\mathcal{E}''(u)[\xi_2,\xi_2] - 2ab\iint_{\mathcal{U}_1\times\mathcal{U}_2}\xi_1(p)\xi_2(q)K(p,q)\,.$$

Since $K(p,q) \leq \Lambda$ for all $(p,q) \in \mathcal{U}_1\times\mathcal{U}_2$, the interaction term can be bounded as

$$\begin{aligned} -2ab\iint_{\mathcal{U}_1\times\mathcal{U}_2}\xi_1(p)\xi_2(q)K(p,q) &\leq 2ab\Lambda\|\xi_1\|_{L^1(\mathcal{U}_1)}\|\xi_2\|_{L^1(\mathcal{U}_2)} \\ &\leq a^2\Lambda\|\xi_1\|^2_{L^1(\mathcal{U}_1)} + b^2\Lambda\|\xi_2\|^2_{L^1(\mathcal{U}_2)}. \end{aligned}$$

Hence

$$\mathcal{E}''(u)[a\xi_1 + b\xi_2] \leq a^2\Big(\underbrace{\mathcal{E}''(u)[\xi_1,\xi_1] + \Lambda\|\xi_1\|^2_{L^1(\mathcal{U}_1)}}_{=:F_1(\xi_1)}\Big) + b^2\Big(\underbrace{\mathcal{E}''(u)[\xi_2,\xi_2] + \Lambda\|\xi_2\|^2_{L^1(\mathcal{U}_2)}}_{=:F_2(\xi_2)}\Big). \qquad (69)$$

We want to show that either $F_1(\xi_1) \geq 0$ for all $\xi_1 \in C^\infty_c(\mathcal{U}_1)$ or $F_2(\xi_2) \geq 0$ for all $\xi_2 \in C^\infty_c(\mathcal{U}_2)$. Suppose neither of these two held, then there would exist $\xi_1 \in C^\infty_c(\mathcal{U}_1)$, $\xi_2 \in C^\infty_c(\mathcal{U}_2)$ such that $F_1(\xi_1) < 0$ and $F_2(\xi_2) < 0$. This would imply, however, that (69) is negative for all $(a,b) \neq (0,0)$, thus contradicting that the Morse index of u is at most one. □

3.2.2 Control of $\mathcal{E}^{\mathrm{Pot}}$ by $\mathcal{E}^{\mathrm{Sob}}$

For the next results, recall the notation for balls in the extended manifold (55).

Lemma 3.11. *Let $R \in (0,1]$, and assume M satisfies flatness assumption $\mathrm{FA}_2(M, g, 2R, p, \varphi)$. Let $\varepsilon > 0$, $s \in (0,1)$ and $u : M \to (-1,1)$ be a solution of the Allen-Cahn equation*

$$(-\Delta)^{s/2}u + \varepsilon^{-s}W'(u) = 0 \tag{70}$$

in $B_R(p)$, that is Λ-almost stable in $B_R(p)$, in the sense of Definition 3.9, for $\Lambda \in (0, \Lambda_0/R^{n+s}]$. Then, there exists a positive constant $C = C(n, \Lambda_0)$ such that, for all $a \in (0,1]$

$$R^{s-n}\mathcal{E}^{\mathrm{Pot}}_{B_{R/2}(p)}(u) \le C\left(\frac{\beta_s}{a}R^{s-n}\int_{\widetilde{B}^+_R(p,0)} z^{1-s}|\widetilde{\nabla}U|^2 + a + (\varepsilon/R)^s R^{s-n}\mathcal{E}^{\mathrm{Pot}}_{B_R(p)}(u)\right). \tag{71}$$

In particular, since $|u| \le 1$ for $a = 1$ we have

$$R^{s-n}\mathcal{E}^{\mathrm{Pot}}_{B_{R/2}(p)}(u) \le C\left(\beta_s R^{s-n}\int_{\widetilde{B}^+_R(p,0)} z^{1-s}|\widetilde{\nabla}U|^2 + 1\right). \tag{72}$$

Proof. Notice that, as $\varepsilon > 0$ is arbitrary, the statement is scaling invariant. Indeed, if $u : B_R(p) \to (-1,1)$ is a Λ-almost stable solution of (70) with parameter ε and Λ, then, on the rescaled manifold $(M, R^{-2}g)$, u is an $(R^{n+s}\Lambda)$-almost stable solution of (70) with parameter ε replaced by ε/R, and Λ replaced by $R^{n+s}\Lambda \le \Lambda_0$ since $R \le 1$ by hypothesis. Hence, we can assume $R = 1$.

In what follows, C denotes a general constant that depends only on n. To compare the potential energies on M with the Sobolev energies in the extended manifold, we need a well chosen cutoff function $\widetilde{\eta}$ defined on the extended manifold $\widetilde{M}$. To this aim, let $\widetilde{\eta}$ solve

$$\begin{cases} \widetilde{\mathrm{div}}(z^{1-s}\widetilde{\nabla}\widetilde{\eta}) = 0 & \text{in } \widetilde{B}^+_1(p,0)\,, \\ \widetilde{\eta} = 0 & \text{on } \partial^+\widetilde{B}_1(p,0)\,, \\ \widetilde{\eta} = \eta_0 & \text{on } B_1(p)\times\{0\}\,, \end{cases} \tag{73}$$

where $\eta_0 = \varphi \circ \eta \in C^2_c(B_1(p))$ and η is a fixed cutoff function with $\eta = 1$ in $\mathcal{B}_{2/3}(0)$ and $\eta = 0$ outside $\mathcal{B}_{3/4}(0)$. First, since $\mathrm{FA}_2(M, g, 2, p, \varphi)$ holds, by the estimates of Lemma B.4 we have for all $q \in B_1(p)$

$$\beta_s\left|\big(-z^{1-s}\partial_z\widetilde{\eta}\big)(q, 0^+)\right| \le C \quad \text{and} \quad \beta_s\int_{\widetilde{B}^+_1(p,0)} z^{1-s}|\widetilde{\nabla}\widetilde{\eta}|^2 \le C\,,$$

for some dimensional constant $C = C(n)$. Note also that $|\widetilde{\eta}| \le 1$. Then

$$\begin{aligned}\mathcal{E}^{\mathrm{Pot}}_{B_{1/2}(p)} &= \frac{\varepsilon^{-s}}{4}\int_{B_{1/2}(p)}(1-u^2)^2 \le \frac{\varepsilon^{-s}}{4}\int_{B_1(p)}(1-u^2)^2\eta_0^2 \\ &= \frac{1}{4}\left(\varepsilon^{-s}\int_{B_1(p)}u^2(1-u^2)^2\eta_0^2 + \varepsilon^{-s}\int_{B_1(p)}(1-u^2)^3\eta_0^2\right) =: \frac{1}{4}(I+J)\,.\end{aligned}$$

On the one hand by (70) and the divergence theorem

$$
\begin{aligned}
I &= \int_{B_1(p)} \varepsilon^{-s} u^2(1-u^2)\eta_0^2 = \int_{B_1(p)} u\eta_0^2 (-\Delta)^{s/2} u \\
&= \beta_s \int_{B_1(p)} u\eta_0^2 \left(-z^{1-s} U_z\right)(\,\cdot\,, 0^+) \\
&= \beta_s \int_{\widetilde{B}_1^+(p,0)} \widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla} U\, U \widetilde{\eta}^2) \\
&= \beta_s \int_{\widetilde{B}_1^+(p,0)} z^{1-s}(|\widetilde{\nabla} U|^2 \widetilde{\eta}^2 + 2\widetilde{\eta} U \widetilde{\nabla}\widetilde{\eta}\cdot \widetilde{\nabla} U) \\
&\le \beta_s \int_{\widetilde{B}_1^+(p,0)} z^{1-s}\Big((1+\tfrac{1}{a})|\widetilde{\nabla} U|^2 \widetilde{\eta}^2 + aU^2 |\widetilde{\nabla}\widetilde{\eta}|^2 \Big) \\
&\le \frac{C\beta_s}{a} \int_{\widetilde{B}_1^+(p,0)} z^{1-s}|\widetilde{\nabla} U|^2 + Ca\,,
\end{aligned}
$$

where β_s is the constant defined in (49) (see also Proposition 2.21) and we have used (67), (68) and Young's inequality in the second to last line.

On the other hand, since $W''(u) = 1 - 3u^2$ and u is Λ-almost stable

$$
\begin{aligned}
J &= \int_{B_1(p)} \varepsilon^{-s}(1-3u^2+2u^2)\big((1-u^2)\eta_0\big)^2 \\
&\le \mathcal{E}^{\mathrm{Sob}}_{B_1(p)}((1-u^2)\eta_0) + \Lambda\bigg(\int_{B_1(p)} \big|(1-u^2)\eta_0\big| \bigg)^2 + 2I \\
&\le \underbrace{\frac{\beta_s}{4}\int_{\widetilde{B}_1^+(p,0)} z^{1-s}\big|\widetilde{\nabla}\big((1-U^2)\widetilde{\eta}\big)\big|^2}_{=:J_1} + C\underbrace{\int_{B_1(p)} (1-u^2)^2\eta_0^2}_{=:J_2} + 2I.
\end{aligned}
$$

Here we have bounded $\mathcal{E}^{\mathrm{Sob}}((1-u^2)\eta_0)$ by J_1 since the former is the infimum of $\frac{\beta_s}{4}\int z^{1-s}|\widetilde{\nabla} V|^2$ over all the extensions V of $(1-u^2)\eta$, and $(1-U^2)\eta_0$ is one such extension. Now, since $\widetilde{\eta} \equiv 0$ on $\partial^+\widetilde{B}_1(p,0)$ and $\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla}\widetilde{\eta}) = 0$ we have

$$
\begin{aligned}
J_1 &= \frac{\beta_s}{4}\int_{\widetilde{B}_1^+(p,0)} z^{1-s}\Big(4U^2|\widetilde{\nabla} U|^2\widetilde{\eta}^2 + \tfrac{1}{2}\widetilde{\nabla}\big((1-U^2)^2\big)\cdot\widetilde{\nabla}(\widetilde{\eta}^2) + (1-U^2)^2|\widetilde{\nabla}\widetilde{\eta}|^2 \Big) \\
&= \frac{\beta_s}{4}\bigg(4\int_{\widetilde{B}_1^+(p,0)} z^{1-s}U^2|\widetilde{\nabla} U|^2\widetilde{\eta}^2 - \int_{B_1(p)} z^{1-s}(1-u^2)^2\eta_0\partial_z\widetilde{\eta} + \\
&\qquad - \int_{\widetilde{B}_1^+(p,0)} (1-U^2)^2 \widetilde{\operatorname{div}}(z^{1-s}\widetilde{\eta}\widetilde{\nabla}\widetilde{\eta}) + \int_{\widetilde{B}_1^+(p,0)} (1-U^2)^2|\widetilde{\nabla}\widetilde{\eta}|^2 \bigg) \\
&= C\beta_s \int_{\widetilde{B}_1^+(p,0)} z^{1-s}U^2|\widetilde{\nabla} U|^2\widetilde{\eta}^2 + \int_{B_1(p)} (1-u^2)^2\eta_0\left(-\beta_s z^{1-s}\partial_z\widetilde{\eta}\right)(\,\cdot\,,0^+) \\
&\le C\beta_s \int_{\widetilde{B}_1^+(p,0)} z^{1-s}|\widetilde{\nabla} U|^2 + C\varepsilon^s \mathcal{E}^{\mathrm{Pot}}_{B_1(p)}\,,
\end{aligned}
$$

and also

$$
J_2 = \int_{B_1(p)} (1-u^2)^2\eta^2 \le C\varepsilon^s\mathcal{E}^{\mathrm{Pot}}_{B_1(p)}.
$$

Thus

$$
J = J_1 + CJ_2 + 2I \le C\left(\frac{\beta_s}{a}\int_{\widetilde{B}_1^+(p,0)} z^{1-s}|\widetilde{\nabla} U|^2 + a + \varepsilon^s\mathcal{E}^{\mathrm{Pot}}_{B_1(p)} \right),
$$

for some $C = C(n, \Lambda_0)$. Putting together the estimates above gives the result. □

3.2.3 BV estimate – Proof of Theorem 1.25

The aim of this subsection is to prove Theorem 1.25.

Lemma 3.12. *Assume M satisfies the local flatness assumption $\mathrm{FA}_2(M,g,1,p,\varphi)$. Let X be a vector field with $\operatorname{spt} X \subset B_{3/4}(p)$, and denote by $\xi : \mathcal{B}_1(0) \to \mathbb{R}^n$ the pull back $\xi := \varphi^* X$ of X via the chart φ^{-1}. Let $\varepsilon > 0$, $s \in (0,1)$ and $u : M \to (-1,1)$ be a solution of the Allen-Cahn equation $(-\Delta)^{s/2} u + \varepsilon^{-s} W'(u) = 0$ in $B_1(p)$. Then*

$$\mathcal{E}''(u)[\nabla_X u, \nabla_X u] \le C\left(\beta_s \int_{\widetilde{B}^+_{3/4}(p,0)} |\widetilde{\nabla} U|^2 z^{1-s} dV dz + \int_{B_{3/4}(p)} \varepsilon^{-s} W(u) dV\right), \tag{74}$$

where $C = C(n, \|\xi\|_{C^2(\mathcal{B}_1)})$ and U is the extension of u.

Proof. Denote by ψ_X^t the flow of X at time t and $u_t := u \circ \psi_X^{-t}$, then by definition

$$\mathcal{E}''(u)[\nabla_X u, \nabla_X u] = \frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{E}(u_t) = \lim_{t\to 0} \frac{\mathcal{E}(u_t) + \mathcal{E}(u_{-t}) - 2\mathcal{E}(u)}{t^2}. \tag{75}$$

Let $\widetilde{X}$ be any smooth extension of X in $\widetilde{B}^+_{3/4}(p,0)$ with support compactly contained in $\widetilde{B}^+_{3/4}(p,0)$, in other words $\widetilde{X}$ vanishes in a neighborhood of $\partial^+ \widetilde{B}^+_{3/4}(p,0)$, and such that $\widetilde{X}^{n+1} \equiv 0$. This last condition implies, if $\widetilde{\psi}^t$ is the flow of $\widetilde{X}$, that $\widetilde{\psi}^t$ leaves invariant the z-component in the extended manifold $\widetilde{M}$.

To bound the above we split the energy $\mathcal{E}$ in its Sobolev part and potential part. For the Sobolev part, by the minimality of the extension in the energy space

$$\mathcal{E}^{\mathrm{Sob}}(u_t) = \frac{\beta_s}{4} \int_{\widetilde{M}} z^{1-s} |\widetilde{\nabla} \widetilde{u}_t|^2 \, dV dz \le \frac{\beta_s}{4} \int_{\widetilde{M}} z^{1-s} |\widetilde{\nabla} U_t|^2 \, dV dz,$$

where β_s is the constant in Theorem 2.20. Here $\widetilde{u}_t$ is the extension of u_t and $U_t := U \circ \widetilde{\psi}^{-t}$. We emphasize that, with our current notation, U_t is not the extension of u_t, but instead the translation of U via $\widetilde{\psi}^t$ in the extended manifold $\widetilde{M}$. Denote

$$I(t) := \int_{\widetilde{M}} |\widetilde{\nabla} U_t|^2 z^{1-s} dV dz.$$

We then have

$$\begin{aligned}
\lim_{t\to 0} \frac{\mathcal{E}^{\mathrm{Sob}}(u_t) + \mathcal{E}^{\mathrm{Sob}}(u_{-t}) - 2\mathcal{E}^{\mathrm{Sob}}(u)}{t^2} &\le \frac{\beta_s}{4}\left(\lim_{t\to 0} \frac{I(t) + I(-t) - 2I(0)}{t^2}\right) \\
&= \frac{\beta_s}{4} \frac{d^2}{dt^2}\bigg|_{t=0} I(t) \\
&= \frac{\beta_s}{4} \frac{d^2}{dt^2}\bigg|_{t=0} \int_{\widetilde{B}^+} |\widetilde{\nabla} U_t|^2 z^{1-s} dV dz,
\end{aligned}$$

Now, since M satisfies local flatness assumption $\mathrm{FA}_2(M,g,1,p,\varphi)$, setting $\widetilde{\varphi}(x,z) = (\varphi(x), z)$, $\widetilde{\Omega} := \widetilde{\varphi}^{-1}(\widetilde{B}^+_{3/4}(p,0))$, $\phi_t := \widetilde{\varphi}^{-1} \circ \widetilde{\psi}^t \circ \widetilde{\varphi}$, and $\widetilde{U} := U \circ \widetilde{\varphi}$, we have

$$\begin{aligned}
I(t) &= \int_{\widetilde{B}^+_{3/4}(p,0)} |\widetilde{\nabla}(U \circ \widetilde{\psi}^{-t})|^2 z^{1-s} dV dz = \int_{\widetilde{\Omega}} \widetilde{g}^{ij} \partial_i (\widetilde{U} \circ \phi_{-t}) \partial_j (\widetilde{U} \circ \phi_{-t}) z^{1-s} \sqrt{|g|}\, dx dz \\
&= \int_{\widetilde{\Omega}} \widetilde{g}^{ij} (\partial_k \widetilde{U}) \circ \phi_{-t}) \partial_i \phi^k_{-t} \, ((\partial_l \widetilde{U}) \circ \phi_{-t}) \partial_j \phi^l_{-t} z^{1-s} \sqrt{|g|}\, dx dz \\
&= \int_{\phi_{-t}(\widetilde{\Omega})} (\partial_k U)(\partial_l U) \left(\widetilde{g}^{ij} \partial_i \phi^k_{-t} \partial_j \phi^l_{-t} \sqrt{|g|}\right) \circ \phi_t (\phi_t^{n+1})^{1-s} \, d\phi_t^1 \wedge \cdots \wedge d\phi_t^{n+1} \\
&= \int_{\widetilde{\Omega}} (\partial_k U)(\partial_l U) z^{1-s} \left(\widetilde{g}^{ij} \, (\partial_i \phi^k_{-t} \partial_j \phi^l_{-t} \sqrt{|g|}\right) \circ \phi_t \, |D\phi_t| \, dx dz
\end{aligned} \tag{76}$$

Hence

$$I''(0) = \int_{\widetilde{\Omega}} (\partial_k U)(\partial_l U) z^{1-s} \frac{\partial^2 F^{kl}}{\partial t^2}(0,x,z)\, dxdz\,,$$

where

$$F^{kl}(t,\cdot,\cdot) := \left(\widetilde{g}^{ij}\partial_i \phi^k_{-t}\partial_j \phi^l_{-t}\sqrt{|\widetilde{g}|}\right)\circ \phi_t\, |D\phi_t|.$$

Since $\phi : [0,\infty)\times\widetilde{\Omega} \to \mathbb{R}^{n+1}$ is the flow of $(\xi, 0)$, together with the flatness assumption, a direct computation shows that

$$\left\| \frac{\partial^2 F^{kl}}{\partial t^2}(0,\cdot) \right\|_{L^\infty(\widetilde{\Omega})} \le C\big(n, \|\xi\|_{C^2(\mathcal{B}_1)}\big).$$

Thus

$$I''(0) \le C \int_{\widetilde{\Omega}} \sum_{k=1}^{n+1} |\partial_k U|^2 z^{1-s}\, dxdz \le C \int_{\widetilde{B}^+_{3/4}(p,0)} |\widetilde{\nabla} U|^2 z^{1-s}\, dVdz\,,$$

where $C = C\big(n, \|\xi\|_{C^2(\mathcal{B}_1)}\big)$ and we have used the flatness assumption to compare the Euclidean metric on $\mathbb{R}^{n+1}$ to the one on $\widetilde{M}$.

Similarly, for the potential part of the energy

$$\lim_{t\to 0} \frac{\mathcal{E}^{\mathrm{Pot}}(u_t) + \mathcal{E}^{\mathrm{Pot}}(u_{-t}) - 2\mathcal{E}^{\mathrm{Pot}}(u)}{t^2} = \frac{d^2}{dt^2}\mathcal{E}^{\mathrm{Pot}}(u_t)\,.$$

Arguing as in the last part of the proof of Lemma 2.16 (the one regarding the potential part of the energy, with $\ell = 2$) we have

$$\frac{d^2}{dt^2}\mathcal{E}^{\mathrm{Pot}}(u_t) \le C\mathcal{E}^{\mathrm{Pot}}_{B_{3/4}(p)}(u) = C\int_{B_{3/4}(p)} \varepsilon^{-s} W(u)\, dV\,,$$

where $C > 0$ depends only on $\|\xi\|_{C^2(\mathcal{B}_1)}$ since by direct computation for the Jacobian

$$\left\| \frac{\partial^2}{\partial t^2}\left(\sqrt{|g|}|D\psi_t|\right)(0,\cdot) \right\|_{L^\infty} \le C\big(\|\xi\|_{C^2(\mathcal{B}_1)}\big)\,.$$

This, together with (75) and the bound for $I''(0)$, concludes the proof. □

Proposition 3.13 (Almost stability ⇒ BV). *Let $p \in M$, $s_0 \in (0,1)$, $s \in (s_0, 1)$ and assume that M satisfies the flatness assumption $\mathrm{FA}_2(M, g, 1, p, \varphi)$. Let $u : M \to (-1,1)$ be a solution of $(-\Delta)^{s/2}u + \varepsilon^{-s}W'(u) = 0$ which is Λ-almost stable in $B_1(p) \subset M$ (see Definition 3.9).*

Then, there exist constants Λ_0 and C, depending only on n and s_0, such that: if $\Lambda \le \Lambda_0$ then

$$\int_{B_{1/4}(p)} |\nabla u|\, dV \le \frac{C}{1-s}\,.$$

Remark 3.14. We emphasize that, in Theorem 3.13 above, the constant C on the right hand side is completely independent of the potential $\varepsilon^{-s}W$. In particular, C does not depend on ε.

Remark 3.15. The blow up rate $(1-s)^{-1}$ as $s \nearrow 1$ is not expected to be sharp, but $(1-s)^{-1/2}$ is; see Remark 2.3 in [19].

To prove Proposition 3.13 we will need the following lemma.

Lemma 3.16. *Let $n \ge 2$, $p \in M$, $s_0 \in (0,1)$, $s \in (s_0, 1)$ and assume that M satisfies the flatness assumption $\mathrm{FA}_2(M, g, R, p, \varphi)$. Then, there exist Λ_0 and C_0, depending only on n and s_0, such that the following property holds. Let $u : M \to (-1,1)$ be a solution of $(-\Delta)^{s/2}u + \varepsilon^{-s}W'(u) = 0$ which is Λ-almost stable in $B_R(p) \subset M$ for $\Lambda \le \Lambda_0/R^{n+s}$ (see Definition 3.9). Then, for every $\delta > 0$*

$$R^{1-n}\int_{B_{R/2}(p)} |\nabla u| dV \le \frac{C_0}{(1-s)\delta} + \delta R^{1-n}\int_{B_R(p)} |\nabla u| dV\,.$$

Proof. Since the statement is scaling invariant, as the constant C does not depend on ε, we can assume $R = 1$. See the beginning of the proof of Lemma 3.11 for details on the scaling.

We show that there exists a constant $C_0 = C_0(n, s_0) > 0$ such that, for any given $\delta > 0$, there holds

$$\|\nabla u\|_{L^1(B_{1/2}(p))} \leq \frac{C}{(1-s)\delta} + \delta \|\nabla u\|_{L^1(B_1(p))} .$$

In particular, this C does not depend on ε.

Let X be a vector field compactly supported in $B_{3/4}(p)$ to be chosen later, and let us denote $B := B_1(p)$ during this proof. Testing the almost stability inequality with $\xi = |\nabla_X u|$ gives

$$0 \leq \mathcal{E}''_B(u)[|\nabla_X u|, |\nabla_X u|] + \Lambda \|\nabla_X u\|^2_{L^1(B)} .$$

On the other hand,

$$\mathcal{E}''_B(u)[|\nabla_X u|, |\nabla_X u|] = \mathcal{E}''_B(u)[\nabla_X u, \nabla_X u] - 4 \int_B \int_B (\nabla_X u)_+(p)(\nabla_X u)_-(q) K_s(p,q)\, dV_p dV_q ,$$

thus we find that

$$4 \int_B \int_B (\nabla_X u)_+(p)(\nabla_X u)_-(q) K_s(p,q)\, dV_p dV_q \leq \mathcal{E}''_B(u)[\nabla_X u, \nabla_X u] + \Lambda \|\nabla_X u\|^2_{L^1(B)} . \tag{77}$$

Moreover, by Lemma 3.12 and Lemma 3.11 respectively, we have

$$\begin{aligned}\mathcal{E}''_B(u)[\nabla_X u, \nabla_X u] &\leq C \left(\beta_s \int_{\widetilde{B}^+_{3/4}(p,0)} |\widetilde{\nabla} U|^2 z^{1-s} dV dz + \mathcal{E}^{\mathrm{Pot}}_{B_{3/4}(p)}(u) \right) \\ &\leq C \left(\beta_s \int_{\widetilde{B}^+_1(p,0)} |\widetilde{\nabla} U|^2 z^{1-s} dV dz + 1 \right),\end{aligned}$$

for some $C = C(n, \|\xi\|_{C^2(\mathcal{B}_1(0))}, \Lambda_0)$, where $\xi^i = X^i \circ \varphi$ and Λ_0 will be chosen shortly depending only on n and s_0. Hence

$$4 \int_B \int_B (\nabla_X u)_+(p)(\nabla_X u)_-(q) K_s(p,q)\, dV_p dV_q \leq C \left(\beta_s \int_{\widetilde{B}^+} |\widetilde{\nabla} U|^2 z^{1-s} dV dz + 1 \right) + \Lambda \|\nabla_X u\|^2_{L^1(B)} .$$

Now, since by Lemma 2.11 there holds $K_s(p,q) \geq c_0 > 0$ for all $(p,q) \in B \times B$, for some constant $c_0 = c_0(n, s_0) > 0$, we have

$$\begin{aligned}\|(\nabla_X u)_+\|_{L^1(B)} \|(\nabla_X u)_-\|_{L^1(B)} &= \int_B \int_B (\nabla_X u)_+(p)(\nabla_X u)_-(q)\, dV_p dV_q \\ &\leq \frac{1}{c_0} \int_B \int_B (\nabla_X u)_+(p)(\nabla_X u)_-(q) K(p,q)\, dV_p dV_q .\end{aligned}$$

Also

$$\begin{aligned}\|(\nabla_X u)_+\|_{L^1(B)} - \|(\nabla_X u)_-\|_{L^1(B)} &= \int_B \nabla_X u \, dV = \int_B \langle \nabla u, X \rangle \, dV \\ &= \int_B \mathrm{div}(uX) - u\,\mathrm{div}(X) \, dV = \int_{\partial B} u \langle X, N \rangle \, d\sigma - \int_B u\,\mathrm{div}(X) \, dV ,\end{aligned}$$

where N is the outer unit normal vector field to ∂B. Then, since $|u| \leq 1$

$$\Big| \|(\nabla_X u)_+\|_{L^1(B)} - \|(\nabla_X u)_-\|_{L^1(B)} \Big| \leq \|X\|_{L^\infty(B)} + \|\mathrm{div}(X)\|_{L^\infty(B)} \leq C \left(\|\xi\|_{C_1(\mathcal{B}_1(0))} \right) \tag{78}$$

Hence, we get

$$\begin{aligned}\|\nabla_X u\|^2_{L^1(B)} &= \Big(\|(\nabla_X u)_+\|_{L^1(B)} + \|(\nabla_X u)_-\|_{L^1(B)}\Big)^2 \\ &= \Big(\|(\nabla_X u)_+\|_{L^1(B)} - \|(\nabla_X u)_-\|_{L^1(B)}\Big)^2 + 4\|(\nabla_X u)_+\|_{L^1(B)}\|(\nabla_X u)_-\|_{L^1(B)} \\ &\le C\beta_s \int_{\widetilde{B}^+} |\widetilde{\nabla} U|^2 z^{1-s} dV dz + C + \frac{\Lambda}{c_0}\|\nabla_X u\|^2_{L^1(B)}\,.\end{aligned}$$

Thus, for $\Lambda \le \frac{1}{2c_0} =: \Lambda_0$ we obtain

$$\|\nabla_X u\|^2_{L^1(B)} \le C\beta_s \int_{\widetilde{B}^+} |\widetilde{\nabla} U|^2 z^{1-s} dV dz + C\,.$$

We can now conclude Step 1 in the proof of Proposition 3.13. By Lemma B.3 with $R = 1$, $k = 0$ we have

$$\beta_s \int_{\widetilde{B}^+} |\widetilde{\nabla} U|^2 z^{1-s} dV dz \le \frac{C}{1-s}(1 + \|\nabla u\|_{L^1(B)})\,.$$

Thus, for every $\delta > 0$ by Young's inequality

$$\begin{aligned}\|\nabla_X u\|_{L^1(B)} &\le C + C\sqrt{\frac{1}{1-s}(1 + \|\nabla u\|_{L^1(B)})} \\ &\le \frac{C}{(1-s)\delta} + \delta\|\nabla u\|_{L^1(B)}\,.\end{aligned}$$

Now, let η be a smooth cutoff compactly supported in $B_{3/4}(p)$ and with $\eta \equiv 1$ on $B_{1/2}(p)$. Making the particular choice $X = \eta\frac{\partial}{\partial x^i}$ above and summing up from $i = 1$ to $i = n$, together with (11), gives

$$\|\nabla u\|_{L^1(B_{1/2}(p))} \le \frac{C}{(1-s)\delta} + \delta\|\nabla u\|_{L^1(B_1(p))}\,, \tag{79}$$

for some $C = C(n, s_0)$, and this concludes the proof.

□

Before giving the proof of Proposition 3.13, we recall a useful covering lemma by L. Simon [93].

Lemma 3.17 ([93]). *Let $\beta \in \mathbb{R}$, $M_0 > 0$, $\rho > 0$ and $\mathcal{S} : \mathfrak{B} \to [0, +\infty)$ be a nonnegative function defined on the family $\mathfrak{B}$ of open balls contained in the Euclidean ball $\mathcal{B}_\rho(0) \subset \mathbb{R}^n$ that is subadditive for finite unions, meaning that whenever $B \subset \bigcup_i B_i$ a finite union then $\mathcal{S}(B) \le \sum_i \mathcal{S}(B_i)$. Then, there exists a constant $\delta = \delta(n, \theta, \beta) > 0$ such that, if*

$$r^\beta \mathcal{S}(\mathcal{B}_{\theta r}(x_0)) \le \delta r^\beta \mathcal{S}(\mathcal{B}_r(x_0)) + M_0 \quad \text{whenever} \quad B_r(x_0) \subset \mathcal{B}_\rho(0),$$

then

$$\mathcal{S}(\mathcal{B}_{\rho/2}(0)) \le CM_0\,,$$

for some constant $C = C(n, \beta, \rho) > 0$.

We can now give the proof of Proposition 3.13.

Proof of Proposition 3.13 . Let Λ_0 and C_0 be the constants given by Lemma 3.16. Fix any Euclidean ball $\mathcal{B}_r(x) \subset \mathcal{B}_{3/4}(0)$. Consider the subadditive function (defined on the family of Euclidean balls contained in $\mathcal{B}_{3/4}(0)$)

$$\mathcal{S}(\mathcal{B}_r(x)) := \int_{\varphi(\mathcal{B}_r(x))} |\nabla u|\, dV\,.$$

Notice that $\mathrm{FA}_2(M, g, 1, p, \varphi)$ implies $B_{4r/5}(\varphi(x)) \subset \varphi(\mathcal{B}_r(x))$ and $\varphi(\mathcal{B}_{1/8}(x)) \subset B_{r/5}(\varphi(x))$. Hence, by Lemma 3.16 applied with $R = 4r/5$, for every $\delta > 0$ and $\mathcal{B}_r(x) \subset \mathcal{B}_{3/4}(0)$ we have

$$r^{1-n}\mathcal{S}(\mathcal{B}_{r/8}(x)) \le \delta r^{1-n}\mathcal{S}(\mathcal{B}_r(x)) + \frac{C}{(1-s)\delta}\,,$$

for some $C = C(n, s_0) > 0$. Using Lemma 3.17, taking δ the one given by the lemma with $\beta = 1 - n$, $\rho = 3/4$, $\theta = 1/8$, we find that

$$S(\mathcal{B}_{3/8}(0)) = \int_{\varphi(\mathcal{B}_{3/8}(0))} |\nabla u| \, dV \leq \frac{C}{1-s},$$

where C depends only on n and s_0. In particular, since $B_{1/4}(p) \subset \varphi(\mathcal{B}_{3/8}(0))$ this concludes the proof. □

Now we will prove the full BV estimate, by iteratively reducing to the almost-stable case thanks to a covering lemma which is inspired by the proof of Proposition 2.6 in [54].

In the following lemma we denote by $\mathcal{Q}_r(x) \subset \mathbb{R}^n$ the (hyper)cube of center x and side r.

Lemma 3.18. *Let $n \geq 1$, $m \geq 0$, $\theta \in (0,1)$, $D_0 > 0$ and $\beta > 0$. Let $\mathcal{S} : \mathfrak{B} \to [0, +\infty)$ be a subadditive[13] function defined on the family $\mathfrak{B}$ of the (hyper)cubes contained in $\mathcal{Q}_1(0) \subset \mathbb{R}^n$, such that*

(i) $\displaystyle\sup_{\{x : \mathcal{Q}_r(x) \in \mathfrak{B}\}} \mathcal{S}(\mathcal{Q}_r(x)) \to 0$ *as* $r \to 0$.

(ii) Whenever $\mathcal{Q}_r(x_0), \mathcal{Q}_r(x_1), \dots, \mathcal{Q}_r(x_m) \subset \mathcal{Q}_1(0)$ are $(m+1)$ disjoint cubes of the same side at pairwise distance at least $D_0 r$, then

$$\exists\, i \in \{0, 1, \dots, m\} \ \text{ such that } \ \mathcal{S}(\mathcal{Q}_{\theta r}(x_i)) \leq r^\beta M_0 .$$

Then

$$\mathcal{S}(\mathcal{Q}_{1/2}(0)) \leq C M_0 ,$$

for some $C = C(n, \theta, m, \beta, D_0) > 0$.

Proof. Let $\rho = 2^{-k}$, for a fixed integer $k > 1$, and consider the regular partition of $\mathcal{Q}_1(0)$ into 2^{kn} cubes of sidelength ϱ. Let us call $\mathfrak{F}_1 := \{\mathcal{Q}_i^1\}$ the family of cubes in this partition. In this way, clearly $\#\mathfrak{F}_1 \leq \rho^{-n}$. Let x_i^1 denote the center of the cube $\mathcal{Q}_i^1$ and, for every $\lambda > 0$ and cube $\mathcal{Q}$ of side r, let $\lambda \mathcal{Q}$ be the cube with the same center and side λr.

Now, we split the family $\mathfrak{F}_1$ as $\mathfrak{F}_1 = \mathfrak{G}_1 \cup \mathfrak{B}_1$ into the families of *good* and *bad* cubes as follows. Start by considering $\mathcal{Q}_1^1$, if there holds

$$\mathcal{S}(\theta \mathcal{Q}_1^1) \leq M_0 \rho^\beta \tag{80}$$

then it is considered a good cube, we assign it to $\mathfrak{G}_1$ and we remove it from $\mathfrak{F}_1$. On the other hand, if $\mathcal{Q}_1^1$ does not satisfy (80), then we assign it to the bad cubes $\mathfrak{B}_1$ and remove it from $\mathfrak{F}_1$. Moreover, if this happens, also all the cubes at distance less than $D_0 \rho$ from $\mathcal{Q}_1^1$ are considered bad as well, so they are assigned to $\mathfrak{B}_1$ and removed from $\mathfrak{F}_1$. Clearly, there are at most $(2D_0 + 3\sqrt{n})^n$ such cubes. We continue this procedure of splitting $\mathfrak{F}_1$ in good cubes and bad cubes until there are no cubes left.

By property (ii), we may have assigned cubes to the bad set $\mathfrak{B}_1$ at most at m steps. Since at each of these steps we removed at most $(2D_0 + 3\sqrt{n})^n$ cubes, this means that $\#\mathfrak{B}_1 \leq m(2D_0 + 3\sqrt{n})^n =: N_0$.

Regarding the good set $\mathfrak{G}_1$, we know it contains at most $\#\mathfrak{F}_1 \leq \rho^{-n}$ cubes, since this is just the total number of cubes in the cover. Moreover, by construction in every $\mathcal{Q} \in \mathfrak{G}_1$ we have

$$\mathcal{S}(\theta \mathcal{Q}) \leq M_0 \rho^\beta .$$

Hence

$$\mathcal{S}(\mathcal{Q}_\theta(0)) \leq \sum_{\mathcal{Q} \in \mathfrak{G}_1} \mathcal{S}(\theta \mathcal{Q}) + \sum_{\mathcal{Q} \in \mathfrak{B}_1} \mathcal{S}(\theta \mathcal{Q}) \leq M_0 \rho^{\beta - n} + \sum_{\mathcal{Q} \in \mathfrak{B}_1} \mathcal{S}(\theta \mathcal{Q}) .$$

The argument continues iteratively under the same scheme, on the union of the at most N_0 bad cubes that are in $\mathfrak{B}_1$. Consider the partition $\mathfrak{F}_2 := \{\mathcal{Q}_i^2\}$ of the cubes in $\mathfrak{B}_1$ obtained splitting each cube into 2^{kn}

[13] Meaning subadditive for finite unions of (hyper)cubes.

smaller cubes of side ρ^2. Notice that and $\#\mathfrak{F}_2 \le N_0\rho^{-n}$. Now assign cubes in $\mathfrak{F}_2$ to the good cubes $\mathfrak{G}_2$ or bad cubes $\mathfrak{B}_2$ as before; starting from $\mathcal{Q}_1^2$ assign it to $\mathfrak{G}_2$ if

$$\mathcal{S}(\theta\mathcal{Q}_1^2) \le M_0\rho^{2\beta}\,, \tag{81}$$

and remove it from $\mathfrak{F}_2$ if this happens. Else, if this is not the case we assign $\mathcal{Q}_1^2$ to the bad cubes $\mathfrak{B}_2$ and remove it, together with all the cubes in $\mathfrak{F}_2$ at distance less than $D_0\rho^2$ from it. Continue the procedure until there are no cubes left in $\mathfrak{F}_2$. By property (ii) again, exactly the same argument as in the first part shows that $\mathfrak{F}_2$ contains at most $N_0 = m(2D_0+3\sqrt{n})^n$ cubes assigned to the bad set, that is $\#\mathfrak{B}_2 \le N_0$. This produces a partition $\mathfrak{F}_2 = \mathfrak{G}_2 \cup \mathfrak{B}_2$, and we get

$$\sum_{\mathcal{Q}\in\mathfrak{B}_1} \mathcal{S}(\theta\mathcal{Q}) \le \sum_{\mathcal{Q}\in\mathfrak{G}_2} \mathcal{S}(\theta\mathcal{Q}) + \sum_{\mathcal{Q}\in\mathfrak{B}_2} \mathcal{S}(\theta\mathcal{Q}) \le N_0M_0\rho^{2\beta-n} + \sum_{\mathcal{Q}\in\mathfrak{B}_2} \mathcal{S}(\theta\mathcal{Q})\,.$$

Iterating this argument, after k steps we have always $\#\mathfrak{B}_k \le N_0$, and in particular by (i) and subadditivity

$$\mathcal{S}(\mathfrak{B}_k) \le \sum_{\mathcal{Q}\in\mathfrak{B}_k} \mathcal{S}(\mathcal{Q}) \to 0\,,$$

since each $\mathcal{Q} \in \mathfrak{B}_k$ has side $\rho^k \to 0$. Thus, the set of the points belonging to infinitely many bad families is $\mathcal{S}$-negligible. Hence[14]

$$\begin{aligned}\mathcal{S}(\mathcal{Q}_\theta(0)) &\le M_0\rho^{\beta-n} + N_0M_0\rho^{2\beta-n} + N_0M_0\rho^{3\beta-n} + \dots\\ &\le N_0M_0\rho^{\beta-n}\sum_{j\ge0}\rho^{j\beta} = \frac{N_0}{\rho^n(\rho^{-\beta}-1)}M_0\,.\end{aligned} \tag{82}$$

Now notice that $\mathcal{Q}_{1/2}(0)$ can be covered, for some $\xi = \xi_n$ dimensional constant, by $\xi_n\theta^{-n}$ many cubes of side $\theta/10$ such that the cube with the same center and side $1/10$ still is contained in $\mathcal{Q}_1(0)$. Since property (ii) is translation invariant, covering $\mathcal{Q}_{1/2}(0)$ in such a way gives

$$\mathcal{S}(\mathcal{Q}_{1/2}(0)) \le \frac{\xi_n\theta^{-n}N_0}{\rho^n(\rho^{-\beta}-1)}M_0 = \frac{\xi_n\theta^{-n}m(2D_0+3\sqrt{n})^n}{\rho^n(\rho^{-\beta}-1)}M_0\,,$$

and as this holds for every $\rho = 2^{-k}$, just choosing any fixed k gives the desired estimate.

□

Theorem 3.19. *Suppose that M satisfies the flatness assumption $\mathrm{FA}_2(M,g,1,p,\varphi)$, in the sense of Definition 1.21. Let $s_0 \in (0,1)$, $s \in (s_0,1)$ and $u : M \to (-1,1)$ be a solution of the Allen-Cahn equation (7) in $B_1(p) \subset M$ with parameter ε, and with Morse index $m_{B_1(p)}(u) \le m$. Then*

$$\int_{B_{1/2}(p)} |\nabla u|dx \le \frac{C}{1-s},$$

for some constant $C = C(n,s_0,m)$.

Proof. For a set $E \subset \mathbb{R}^n$ denote by $\lambda E := \{\lambda y \,:\, y \in E\}$. Consider the subadditive function[15]

$$\mathcal{S}(\mathcal{Q}) := \int_{\varphi\left(\frac{1}{2\sqrt{n}}\mathcal{Q}\right)} |\nabla u|\, dV\,,$$

defined on the cubes $\mathcal{Q} \subset \mathcal{Q}_1(0)$.

Claim. $\mathcal{S}$ satisfies properties (i) and (ii) of Lemma 3.18 with $M_0 = C/(1-s)$, $\beta = n-1$, $\theta = 1/8$, and D_0 depending only on n, s_0 and m.

[14] Note that we could also have stopped the exhaustion process when the error in the tail of (82) is less than the constant on the right-hand side, and we would have obtained the estimate with two times this constant.

[15] The factor $\frac{1}{2\sqrt{n}}$ inside $\varphi(\frac{1}{2\sqrt{n}}\mathcal{Q})$ is needed to have $\frac{1}{2\sqrt{n}}\mathcal{Q} \subset \mathcal{B}_{1/2}(0)$ for $\mathcal{Q} \subset \mathcal{Q}_1(0)$ in order to apply Lemma 2.11.

Proof of the claim. The first property is clear from the definition of $\mathcal{S}$. The second property is a consequence of the Morse index of u being at most m.

Indeed, let $\mathcal{Q}_r(x_0), \mathcal{Q}_r(x_1), \dots, \mathcal{Q}_r(x_m) \subset \mathcal{Q}_1(0)$ be $(m+1)$ disjoint cubes of the same side at pairwise distance at least $D_0 r$, and let $q_i := \varphi(x_i)$. Then, since $\mathcal{B}_{r/2}(x_i) \subset \mathcal{Q}_r(x_i)$ by Lemma 3.10 and Lemma 2.11 for at least one $\ell \in \{1, \dots m\}$, the inequality

$$\mathcal{E}''(u)[\xi, \xi] \geq -\frac{C_1 m}{(D_0 r/2)^{n+s}} \|\xi\|^2_{L^1(B_{r/2}(q_\ell))} \tag{83}$$

holds for all $\xi \in C_c^\infty(B_{r/2}(q_\ell))$, for some $C_1 = C_1(n, s_0)$. That is, u is a Λ-almost stable solution (in the sense of Definition 3.9) in $B_{r/2}(q_\ell)$ with $\Lambda = \frac{C_1 m}{(D_0 r/2)^{n+s}}$. Note that, in this case, on the re-scaled manifold $\widehat{M} := (M, (2/r)^2 g)$ we have that u is a $\Lambda(r/2)^{n+s}$-almost stable solution of $(-\Delta)^{s/2} u + (2\varepsilon/r)^{-s} W'(u) = 0$ in $\widehat{B}_1(q_\ell)$, and the flatness assumption $\mathrm{FA}_2(M, (2/r)^2 g, 1, q_\ell, \varphi_{x_\ell, r/2})$ holds.

Let Λ_0 be the constant given by Proposition 3.13. Then, there exists $D_0 = D_0(n, s_0, m) > 0$ so that u is a Λ-almost stable solution of the Allen-Cahn equation in $\widehat{B}_1(q_\ell)$ with $\Lambda = \frac{C_1 m}{(D_0 r/2)^{n+s}} \leq \Lambda_0$, for D_0 sufficiently large. Hence, by Proposition 3.13 we get

$$\int_{\widehat{B}_{1/4}(q_\ell)} |\nabla u|_{\widehat{g}} \, d\widehat{V} \leq \frac{C}{1-s},$$

and, since $\varphi\big(\mathcal{Q}_{\frac{r}{16\sqrt{n}}}(x_\ell)\big) \subset B_{r/8}(q_\ell)$, scaling back this inequality on M gives

$$\mathcal{S}(\mathcal{Q}_{r/8}(x_\ell)) \leq \int_{B_{r/8}(q_\ell)} |\nabla u| \, dV \leq \frac{C}{1-s} r^{n-1},$$

for some $C = C(n, s_0, m)$, and this concludes the proof of the claim.

Hence, by Lemma 3.18

$$\mathcal{S}(\mathcal{Q}_{1/2}(0)) = \int_{\varphi\left(\mathcal{Q}_{\frac{1}{4\sqrt{n}}}(0)\right)} |\nabla u| \, dV \leq \frac{C}{1-s}.$$

Now, the fact that the BV estimate holds in $B_{1/2}(p)$ follows by $\mathrm{FA}_2(M, g, 1, p, \varphi)$ and a standard covering argument, and this concludes the proof. □

As a corollary, simply by scaling we immediately get Theorem 1.25.

Proof of Theorem 1.25. Since $\mathrm{FA}_2(M, g, R, p, \varphi)$ holds, then the rescaled manifold $\widehat{M} := (M, R^{-2} g)$ satisfies $\mathrm{FA}_2(M, R^{-2} g, 1, p, \varphi_{0,R})$. Hence, Theorem 3.19 gives

$$\int_{B_{1/2}(p)} |\nabla u|_{\widehat{g}} \, d\widehat{V} \leq \frac{C}{1-s},$$

for some $C = C(n, s_0, m)$. Scaling back this inequality on M gives the result.

□

3.2.4 Density estimates

Proof of Proposition 1.28. Since the statement is scaling invariant, we prove the result just for $R = 1$. We argue by contradiction, suppose that $\int_{B_1(p)} |1 + u_\varepsilon| \, dV \leq \omega_0$ and that $\{u_\varepsilon \geq -\frac{9}{10}\} \cap B_{1/2}(p) \neq \emptyset$, for some $1 \geq C_0 \varepsilon$; the constant C_0 that will be chosen during the proof, depending only on n, s, and m. First, by continuity of u_ε and by taking $\omega_0 < |B_{1/2}(p)|$, there will be a point $q \in B_{1/2}(p)$ for which $|u_\varepsilon(q)| \leq \frac{9}{10}$.

Now, we claim that there exists a small constant $\alpha = \alpha(n, s)$ such that $[u_\varepsilon]_{C^\alpha(B_{\varepsilon/3}(q))} \leq C\varepsilon^{-\alpha}$ for all $\varepsilon \leq 1/10$. Indeed, let $\eta \in C_c^\infty(\mathcal{B}_2(0))$ be a cutoff function with $\chi_{\mathcal{B}_{3/2}(0)} \leq \eta \leq \chi_{\mathcal{B}_2(0)}$. Then, the function

$\widetilde{u}(x) := u_\varepsilon(\varphi(\varphi^{-1}(q) + \varepsilon x))\eta(x)$ is well defined on the whole $\mathbb{R}^n$, since $\widetilde{u}$ depends only on the values of u_ε in $\varphi(\mathcal{B}_{2\varepsilon}(\varphi^{-1}(q))) \subset \varphi(B_1)$. Now, by the flatness assumption $\mathrm{FA}_2(M, g, 1, p, \varphi)$ we have that $\widetilde{u}$ satisfies $|L\widetilde{u}| \le C$ in $\mathcal{B}_1(0)$, for some $C = C(n,s)$, where the kernel of L satisfies (137) by Proposition 2.6 – see in particular inequality (26). Hence, by Lemma B.5 we have that $[\widetilde{u}]_{C^\alpha(\mathcal{B}_{1/2}(0))} \le C$, for some $C = C(n,s) > 0$, thus $[u_\varepsilon]_{C^\alpha(B_{\varepsilon/3}(q))} \le C\varepsilon^{-\alpha}$ as desired.

Then, there is $\varepsilon_1 = \varepsilon_1(n,s) \le 1/10$ sufficiently small such that, for all $\varepsilon < \varepsilon_1$, we have $|u_\varepsilon| \le \frac{19}{20}$ in the ball $B_\varepsilon(q)$. Using that $W(u) = \frac{1}{4}(1-u^2)^2$, we deduce

$$\varepsilon^{s-n} \cdot \varepsilon^{-s} \int_{B_\varepsilon(q)} W(u_\varepsilon)\, dV \ge \theta > 0$$

for some positive absolute constant θ.

Let now U be the s-extension of u_ε in $\widetilde{M} = M \times [0,\infty)$. The previous lower bound on the potential energy in $B_\varepsilon(q)$ leads to

$$\varepsilon^{s-n}\widetilde{\mathcal{E}}_\varepsilon(U) = \varepsilon^{s-n}\left(\frac{\beta_s}{2}\int_{\widetilde{B}^+_\varepsilon(q,0)} z^{1-s}|\nabla U|^2\, dV dz + \varepsilon^{-s}\int_{B_\varepsilon(q)} W(u_\varepsilon)\, dV\right) \ge \theta\,.$$

Applying the monotonicity formula of Theorem 2.23, we deduce that

$$\rho^{s-n}\widetilde{\mathcal{E}}_\rho(U) \ge \theta/C_1\,, \quad \text{for all } \varepsilon \le \rho \le R_{\mathrm{mon}}\,, \tag{84}$$

for some $C_1 = C_1(n)$, where R_{mon} is the radius given by the monotonicity formula and can be taken to be $R_{\mathrm{mon}} = \mathrm{inj}_M(q)/4$ – see Remark 2.24 – and thus by hypothesis is $R_{\mathrm{mon}} \ge 1/8$. In particular, for all $\varepsilon < \rho < 1/8$, by Lemma 3.11 with $a = \frac{\theta}{4CC_1}$ we have

$$\begin{aligned}
\frac{\theta}{C_1} &\le \rho^{s-n}\widetilde{\mathcal{E}}_\rho(U) \\
&\le \frac{C}{a}\rho^{s-n}\int_{\widetilde{B}^+_{2\rho}} z^{1-s}|\nabla U|^2\, dV dz + Ca + C\varepsilon^s\rho^{-n}\mathcal{E}^{\mathrm{Pot}}_{B_{2\rho}}(u_\varepsilon) \\
&= C\rho^{s-n}\int_{\widetilde{B}^+_{2\rho}} z^{1-s}|\nabla U|^2\, dV dz + C\varepsilon^s\rho^{-n}\mathcal{E}^{\mathrm{Pot}}_{B_{2\rho}}(u_\varepsilon) + \frac{\theta}{4C_1} \\
&\le C\rho^{s-n}\int_{\widetilde{B}^+_{4\rho}} z^{1-s}|\nabla U|^2\, dV dz + C\varepsilon^s + \frac{\theta}{4C_1}\,,
\end{aligned}$$

where in the last line we have used again Lemma 3.11 with $a = 1$.

Hence, for all $\varepsilon < \varepsilon_0 := \min\{\varepsilon_1, (\frac{\theta}{4CC_1})^{1/s}\}$ depending only on n and s (and this will be our final choice of ε_0), by Lemma B.3 with $k = 1$ we have

$$\begin{aligned}
\frac{\theta}{2C_1} &\le C\rho^{s-n}\int_{\widetilde{B}^+_{4\rho}} z^{1-s}|\nabla U|^2\, dV dz \\
&\le \frac{C}{r^s} + C\left(\rho^{-n}\int_{B_{4r\rho}(q)} |1+u_\varepsilon|\right)^{1-s}\left(\rho^{1-n}\int_{B_{4r\rho}(q)} |\nabla u_\varepsilon|\right)^s,
\end{aligned}$$

for all $r \ge 1$ provided $4r\rho \le 1/2$. Now, at this point choose $r = \left(\frac{\theta}{4CC_1}\right)^{-1/s}$ and ρ such that $4r\rho = 1/2$, and note that this is allowed as long as $1 = 8r\rho \ge 8r\varepsilon =: C_0\varepsilon$, where $C_0 = C_0(n,s)$. This gives

$$\frac{\theta}{4C_1} \le C\omega_0^{1-s}\left(\int_{B_{1/2}(q)} |\nabla u_\varepsilon|\right)^s,$$

and by the BV estimate of Theorem 1.25 we reach contradiction if the density ω_0 is too small. This concludes the proof of the first part of the proposition.

□

From the proof we can extract an additional auxiliary result.

Proposition 3.20. *Let $u : M \to (-1,1)$ be a solution of (7) in $B_R(p) \subset M$ with Morse index $m_{B_R(p)}(u) \le m$, and suppose that M satisfies the flatness assumption $\mathrm{FA}_2(M,g,R,p,\varphi)$. Then, there exist positive constants C_0 and ε_0, depending only on n, s, and m, such that the following holds: whenever $\varepsilon \le \varepsilon_0$ and $R \ge C_0\varepsilon$, if for some $q \in B_{R/2}(p)$ we have $|u_\varepsilon(q)| \le \frac{9}{10}$ then*

$$\int_{B_{R/2}(q)} |\nabla u_\varepsilon|\, dV \ge c_0 R^{n-1}, \tag{85}$$

for some $c_0 = c_0(n,s,m)$. This fact will be useful in the proof of Proposition 3.22 below.

Proof. It follows by simply repeating the proof of Proposition 1.28 above, from when we found a point $q \in B_{1/2}(p)$ with $|u_\varepsilon(q)| \le \frac{9}{10}$ to the very last line, and using that $|u_\varepsilon| \le 1$ to estimate the density $\int_{B_{1/2}(q)} |1+u_\varepsilon|$ from above instead of using the bound ω_0. ☐

3.2.5 Decay of $\mathcal{E}^{\mathrm{Pot}}$ – Proof of Theorem 1.27

Lemma 3.21. *Let $s \in (0,1)$, $p \in M$, and assume that $\mathrm{FA}_2(M,g,p,R,\varphi)$ holds; recall Definition (1.21). Let $u_\varepsilon : M \to (-1,1)$ be a solution of (7) in $B_R(p)$. Then, there exist positive constants $C = C(n,s)$ such that, if $\varepsilon < 1$ and $1 - |u_\varepsilon| \le \frac{1}{10}$ in $\varphi(\mathcal{B}_R(0))$, then*

$$0 \le 1 - |u_\varepsilon| \le C(\varepsilon/R)^s \quad \text{in } \varphi(\mathcal{B}_{R/2}(0)).$$

Proof. Since the statement is scaling invariant, let us assume w.l.o.g. $R = 1$. Suppose in addition that $\frac{9}{10} \le u_\varepsilon \le 1$ in $\varphi(\mathcal{B}_1(0))$; the case $-1 \le u_\varepsilon \le -\frac{9}{10}$ can be reduced to the previous by the even symmetry of W (i.e. replacing u_ε by $-u_\varepsilon$). Then, the function $v := 1 - u_\varepsilon$ satisfies

$$(-\Delta)^{s/2} v = -\varepsilon^{-s} u_\varepsilon (1 - u_\varepsilon) \le -\frac{1}{2}\varepsilon^{-s} v \quad \text{in } \varphi(\mathcal{B}_1(0)). \tag{86}$$

Fix now a smooth function $\xi \in C^\infty(\mathbb{R}^n)$ such that $\chi_{\mathcal{B}_{4/5}(0)} \le 1 - \xi \le \chi_{\mathcal{B}_1(0)}$ and consider the function $\overline{\xi} := \xi \circ \varphi^{-1}$ defined on M, considered identically one outside $\varphi(\mathcal{B}_1(0))$. Then , since $\mathrm{FA}_2(M,g,p,1,\varphi)$ holds, by Lemma 2.9 for every $q \in \varphi(\mathcal{B}_{1/2}(0))$ we have

$$\big|(-\Delta)^{s/2}\overline{\xi}\,\big|(q) \le \int_M |\overline{\xi}(q) - \overline{\xi}(p)| K_s(p,q)\, dV_p \le \int_{M\setminus\varphi(\mathcal{B}_{4/5}(0))} K_s(p,q)\, dV_p \le C_0,$$

for some constant $C_0 \ge 1$ that depends only on n and s. Consider now $w := 2\overline{\xi} + 2C_0\varepsilon^s$, we have in $\varphi(\mathcal{B}_1(0))$ that

$$(-\Delta)^{s/2} w \ge -C_0 = -2C_0\varepsilon^s \cdot \frac{1}{2}\varepsilon^{-s} \ge -\frac{1}{2}\varepsilon^{-s} w.$$

Hence, recalling (86) and since $w \ge 2 \ge v$ in $M \setminus \varphi(\mathcal{B}_1(0))$ we get, by the maximum principle, that $w \ge v$ on M. Hence, using that $\xi \equiv 0$ in $\varphi(\mathcal{B}_{1/2}(0))$, we have shown that $2C_0\varepsilon^s \ge v$ in $\varphi(\mathcal{B}_{1/2}(0))$, as desired. ☐

The following proposition shows the quantitative convergence to zero, as $\varepsilon \searrow 0$, of the potential energy of finite index solutions to the A-C equation (7). The statement and proof are inspired by the ones of Proposition 6.2 in [20], which deals with stable solutions of the fractional Allen-Cahn equation in $\mathbb{R}^n$. We moreover simplify the proof in [20], by using the lower bound (85) on the BV norm that we have obtained as a byproduct of (the proof of) Proposition 1.28.

Proposition 3.22. *Let $s \in (0,1)$, $p \in M$, and assume that the flatness assumption $\mathrm{FA}_2(M,g,p,R,\varphi)$ holds. Let $u_\varepsilon : M \to (-1,1)$ be a solution of (7) in $B_R(p) \subset M$ with Morse index $m_{B_R(p)}(u_\varepsilon) \le m$. Then, there exist constants C and ε_0, depending only on n, s, and m, such that for all $\varepsilon \le \varepsilon_0$:*

$$\varepsilon^{-s} \int_{B_{R/2}(p)} W(u_\varepsilon)\, dV \le C R^{n-s} (\varepsilon/R)^\beta,$$

where $\beta := \min\left(\frac{1-s}{2}, s\right) > 0$.

Proof. Given $q \in B_{R/2}(p)$, let

$$r_q := \max(\min\left(\tfrac{R}{16}, \tfrac{1}{2}\mathrm{dist}(q, \{|u| \le \tfrac{9}{10}\})\right), C_0\varepsilon),$$

where $C_0 > 0$ is a large enough constant, depending only on n and s, to be chosen later. Note that if $R/16 \le C_0\varepsilon$, then

$$\int_{B_{R/2}(p)} (\varepsilon/R)^{-s} W(u_\varepsilon)\, dV \le (16C_0)^s (\max_{[-1,1]} W) |B_{R/2}(p)| \le CR^n \le CC_0^\beta (\varepsilon/R)^\beta R^n.$$

Thus, we may (and do) assume that $R/16 > C_0\varepsilon$. In particular, $r_q \in [C_0\varepsilon, R/16]$ for all $q \in B_{R/2}(p)$. Now, we claim that

$$\int_{B_{4r_q}(q)} |\nabla u_\varepsilon| \ge c(r_q)^{n-1} \quad \text{whenever } r_q < R/16\,, \tag{87}$$

for some constant $c = c(n, s, m)$. Indeed, if $r_q < R/16$ then there exists $q' \in \{|u_\varepsilon| \le \frac{9}{10}\} \cap B_{9R/16}(p)$ such that $\mathrm{dist}(q, q') \le 2r_q$. Then, choosing ε_0 small and C_0 big (depending only on n, s and m) according to the constants in Proposition 1.28, by (85) we have

$$\int_{B_{4r_q}(q)} |\nabla u_\varepsilon| \ge \int_{B_{2r_q}(q')} |\nabla u_\varepsilon| \ge c(2r_q)^{n-1}\,,$$

as desired.

We now produce a covering of $B_{R/2}(p)$ by some of the balls $\{B_{r_q}(q)\}_{q \in B_{R/2}(p)}$ as follows. Given $k \le -5$, let $X_k := \{q \in B_{R/2}(p) \,:\, r_q \in (2^k R, 2^{k+1} R]\}$ and let $\{q_j^k\}_{j \in \mathcal{J}_k}$ be a maximal subset of X_k with the property that the balls $B_{r(q_j^k)/4}(q_j^k)$ are disjoint, where $r(q_j^k) := r_{q_j^k}$. It then follows, by the very definition of X_k, that

$$X_k \subset \bigcup_{j \in \mathcal{J}_k} B_{r(q_j^k)}(q_j^k)$$

and that the family of quadruple balls

$$\big\{B_{4r(q_j^k)}(q_j^k)\big\}_{j \in \mathcal{J}_k}$$

has (dimensional) finite overlapping.[16] Note also that since $R/16 > C_0\varepsilon$ we have $\lfloor \log_2(C_0\varepsilon/R) \rfloor \le -5$ and, by construction, the union of the sets X_k when k runs on $\{\lfloor \log_2(C_0\varepsilon/R) \rfloor \le k \le -5\}$ covers all of $B_{R/2}(p)$.

Now, on the one hand, by the BV estimate of Theorem 1.25 we have $\int_{B_{3R/4}(p)} |\nabla u_\varepsilon| \le CR^{n-1}$, and this yelds

$$\#\mathcal{J}_k \le C2^{-k(n-1)}\,, \tag{88}$$

for all $k \le -5$. Indeed, this follows using that the balls $\big\{B_{4r(q_j^k)}(q_j^k)\big\}_{j \in \mathcal{J}_k}$ have finite overlapping and are all contained in $B_{3/4R}(p)$.

On the other hand, we claim that Lemma 3.21 yields

$$\int_{B_{r_x}(x)} (\varepsilon/R)^{-s} W(u_\varepsilon)\, dV \le C \int_{B_{r_x}(x)} (\varepsilon/R)^{-s} (1 - |u_\varepsilon|)^2\, dV \le C(\varepsilon/R)^{-s} \left(\frac{\varepsilon}{r_x}\right)^\alpha r_x^n$$

for any given $\alpha \in [0, 2s]$. Indeed, note that if $r_q = C_0\varepsilon$ the previous estimate is trivial, while if $r_q > C_0\varepsilon$ then $r_q \le \frac{1}{2}\mathrm{dist}(q, \{|u| \le \frac{9}{10}\})$ and hence we may apply Lemma 3.21 (recall that $r_q \ge C_0\varepsilon \ge \varepsilon$).

[16] That is, every point $q \in \big\{B_{4r(q_j^k)}(q_j^k)\big\}_{j \in \mathcal{J}_k}$ belongs to at most $N = N(n)$ of these balls. This is easy to check: if q belongs to N of such balls, we would have the existence of N points q_j^k in $B_{4 \cdot R2^{k+1}}(p)$ such that the balls $B_{\frac{1}{4}R2^k}(q_j^k)$ are disjoint and contained in $B_{9 \cdot R2^k}(p)$. Then, comparing the volumes and using that $\mathrm{FA}_2(M, g, p, R, \varphi)$ holds gives a dimension bound on N.

Therefore, choosing $\alpha := \min\left(\frac{1+s}{2}, 2s\right) \in (0,1)$ we obtain —using (88)— that

$$
\begin{aligned}
\int_{B_{R/2}(p)} (\varepsilon/R)^{-s} W(u_\varepsilon)\, dV &\le C \sum_{k=\lfloor \log_2(C_0\varepsilon/R)\rfloor}^{-5} \sum_{j\in\mathcal{J}_k} \int_{B_{r(q_j^k)}(q_j^k)} (\varepsilon/R)^{-s} W(u_\varepsilon)\, dV \\
&\le C \sum_{k=\lfloor \log_2(C_0\varepsilon/R)\rfloor}^{-5} \sum_{j\in\mathcal{J}_k} (\varepsilon/R)^{-s} \left(\frac{\varepsilon}{r_{q_j^k}}\right)^{\alpha} r^n_{q_j^k} \\
&\le C \sum_{k=\lfloor \log_2(C_0\varepsilon/R)\rfloor}^{-5} (\varepsilon/R)^{-s} \left(\frac{\varepsilon}{2^k R}\right)^{\alpha} (2^{k+1}R)^n \,\#\mathcal{J}_k \\
&\le C \sum_{k=\lfloor \log_2(C_0\varepsilon/R)\rfloor}^{-5} (\varepsilon/R)^{\alpha-s} R^n 2^{k(n-\alpha)} 2^{-k(n-1)} \\
&\le C R^n (\varepsilon/R)^{\alpha-s} \sum_{k=-\infty}^{-5} (2^k)^{1-\alpha} \\
&\le C R^n (\varepsilon/R)^{\beta},
\end{aligned}
$$

as we wanted to show. □

With the above estimates at hand, the proof of Theorem 1.27 is straightfoward.

Proof of Theorem 1.27. The bound on the Sobolev part of the energy is a direct consequence of the BV estimate of Theorem 1.25 and the interpolation inequality Proposition 2.22, and the bound on the Potential part is exactly the statement of Proposition 3.22 above. □

3.3 Strong convergence to a limit interface – Proof of Theorem 1.30

With the estimates for Allen-Cahn solutions of Section 3.2 at hand, we can finally prove Theorem 1.30.

Proof of Theorem 1.30. We split the proof according to the different statements in the Theorem.

Step 1. Convergence in $H^{s/2}(M)$.

Since M is compact, there is a small radius $R = R(M) > 0$ so that the flatness assumption $\mathrm{FA}_2(M, g, R, p, \varphi_p)$ holds for every $p \in M$; see Remark 1.22. We can then apply the BV estimate of Theorem 1.25 to get a bound on the BV norm $[u_{\varepsilon_j}]_{BV(B_{R/2}(p))}$ independently of $p \in M$. For any $\sigma \in (0,1)$, the interpolation result of Proposition 2.22 together with the comparability between $K_\sigma(\varphi_p(x), \varphi_p(y))$ and $\frac{1}{|x-y|^{n+\sigma}}$ (see Lemma 2.11) gives then the bound

$$
\iint_{B_{R/2}(p)\times B_{R/2}(p)} |u_{\varepsilon_j}(p) - u_{\varepsilon_j}(q)|^2 K_\sigma(p,q)\, dV_p\, dV_q \le C(n,\sigma)\,,
$$

valid for any $p \in M$. Combining this with (29) of Proposition 2.6 (with $\alpha = 0$), we see that

$$
\iint_{B_{R/4}(p)\times M} |u_{\varepsilon_j}(p) - u_{\varepsilon_j}(q)|^2 K_\sigma(p,q)\, dV_p\, dV_q \le C(n,\sigma)\,,
$$

which after covering M with finitely many such balls of radius $R/4$ shows that

$$
\|u_{\varepsilon_j}\|_{H^{\sigma/2}(M)} \le C(M,\sigma)\,. \tag{89}
$$

In particular, we can choose some fixed $\sigma > s$. Then, the (standard) compactness of the inclusion[17] $H^{\sigma/2}(M) \hookrightarrow H^{s/2}(M)$ shows that a subsequence converges strongly in $H^{s/2}(M)$ to a limit function $u_0 \in$

[17]The compactness of this inclusion is well known on balls of $\mathbb{R}^n$. This immediately gives one way of showing it for compact manifolds as well, after covering them with a finite number of small coordinate balls and using the same estimations for the kernel as in the present proof.

$H^{s/2}(M)$. Moreover, after extracting a further subsequence (that we do not relabel), we also assume that the convergence holds almost everywhere on M.

Step 2. Convergence of the Potential energies $\mathcal{E}^{\rm Pot}_M(u_{\varepsilon_j})$ to zero and structure of u_0.

Again as in Step 1, covering M with a finite number of balls of radius R so that $\mathrm{FA}_2(M, g, R, p, \varphi_p)$ holds for all $p \in M$, applying Proposition 3.22 to each ball of the covering we get (for j large) that

$$\mathcal{E}^{\rm Pot}_M(u_{\varepsilon_j}) \le C(M, s, m)\varepsilon_j^{\beta}\,, \tag{90}$$

which shows that $\mathcal{E}^{\rm Pot}_M(u_{\varepsilon_j}) \to 0$ as $j \to \infty$ (since then $\varepsilon_j \to 0$).

The fact that the limit function is of the form $u_0 = \chi_E - \chi_{E^c}$ for a set $E \subset M$ follows: since we just proved that $\varepsilon_j^{-s}\int_M W(u_{\varepsilon_j}) \to 0$ as $j \to \infty$, of course $\int_M W(u_{\varepsilon_j}) \to 0$ as well. By Fatou's Lemma we deduce that $\int_M W(u_0) = 0$, which shows that the limit u_0 can only take the values ± 1. Hence $u_0 = \chi_E - \chi_{E^c}$ for some measurable set $E \subset M$, which is actually a set of finite perimeter since the u_{ε_j} satisfy uniform BV estimates. The fact that (17)-(19) hold, after choosing the representative of E for which every point of E with density 1 belongs to its interior and every point of density 0 belongs to its complement, follows from the convergence in $L^1(M)$ and the density estimate of Proposition 1.28.

Step 3. Convergence of the level sets to ∂E in the Hausdorff distance.

This is a direct consequence of Lemma 3.21 and the density estimate in Proposition 1.28. Fix $c \in (-1, 1)$. Arguing by contradiction, assume one could find points $p_j \in \{u_{\varepsilon_j} \ge c\}$ and $q_j \in E$ with $d(p_j, q_j) \ge r > 0$ and $B_{r/2}(p_j) \cap E = \varnothing$, for some small $r > 0$. By compactness for a subsequence there is $p_\circ$ such that $p_j \to p_\circ$, and in particular $B_{r/4}(p_\circ) \subset E^c$. This implies (up to subsequences, that we do not relabel) that $\lim_{j\to\infty} u_{\varepsilon_j} = -1$ a.e. in $B_{r/4}(p_\circ)$. By the density estimate of Proposition 1.28 then $u_{\varepsilon_j} \le -\frac{9}{10}$ in $B_{r/8}(p_\circ)$ for all j sufficiently large, and with Lemma 3.21 this implies $u_{\varepsilon_j} \le -1 + C(\varepsilon_j/r)^s$. This contradicts $u_{\varepsilon_j}(p_j) \ge c > -1$ for j large, and concludes the proof.

Step 4. The limit set E is stationary for the fractional perimeter.

Claim: Let X be a vector field of class C^∞ on M, and let $\psi^t := \psi^t_X$ denote its flow at time t. Putting $u_{\varepsilon_j,t}(p) := u_{\varepsilon_j}(\psi^{-t}(p))$, then

$$\frac{d^\ell}{dt^\ell}\mathcal{E}^{\rm Sob}_M(u_{\varepsilon_j,t}) \to \frac{d^\ell}{dt^\ell}\mathrm{Per}_s(\psi^t(E)) \quad \text{and} \quad \frac{d^\ell}{dt^\ell}\mathcal{E}^{\rm Pot}_M(u_{\varepsilon_j,t}) \to 0\,, \quad \text{as } j \to \infty\,. \tag{91}$$

Proof of the claim. The change of variables in (39) gives that

$$\frac{d^\ell}{dt^\ell}\mathcal{E}^{\rm Sob}_M(u_{\varepsilon_j,t}) = \iint |u_{\varepsilon_j}(p) - u_{\varepsilon_j}(q)|^2 \frac{d^\ell}{dt^\ell}\Big[K_s(\psi^t(p), \psi^t(q))\, J_t(p) J_t(q)\Big] dV_p\, dV_q\,,$$

and likewise

$$\frac{d^\ell}{dt^\ell}\mathrm{Per}_s(\psi^t(E)) = \iint |u_0(p) - u_0(q)|^2 \frac{d^\ell}{dt^\ell}\Big[K_s(\psi^t(p), \psi^t(q))\, J_t(p) J_t(q)\Big] dV_p\, dV_q\,.$$

We can rewrite the former as

$$\frac{d^\ell}{dt^\ell}\mathcal{E}^{\rm Sob}_M(u_{\varepsilon_j,t}) = \iint |u_{\varepsilon_j}(p) - u_{\varepsilon_j}(q)|^2 K_s(p,q) \frac{\frac{d^\ell}{dt^\ell}\Big[K_s(\psi^t(p), \psi^t(q))\, J_t(p) J_t(q)\Big]}{K_s(p,q)} dV_p\, dV_q\,.$$

Since $u_{\varepsilon_j} \to u_0$ in $H^{s/2}(M)$ by Step 1, we immediately see that

$$A_j := |u_{\varepsilon_j}(p) - u_{\varepsilon_j}(q)|^2 K_s(p,q) \to |u_0(p) - u_0(q)|^2 K_s(p,q) =: A \quad \text{in } L^1(M \times M)\,.$$

On the other hand, (40) and (41) show that the fixed function $B := \frac{\frac{d^\ell}{dt^\ell}\left[K_s(\psi^t(p),\psi^t(q))\,J_t(p)J_t(q)\right]}{K_s(p,q)}$ belongs to $L^\infty(M\times M)$. Therefore, $A_jB \to AB$ in $L^1(M\times M)$ as well, which gives the first part of the claim.

For the second part of the claim, the fact that

$$\frac{d^\ell}{dt^\ell}\mathcal{E}^{\rm Pot}_M(u_{\varepsilon_j,t}) \to 0 \quad \text{as } j\to\infty$$

follows from combining (43) and (90). ☐

Now that we have shown the claim, the fact that E is stationary for the fractional perimeter follows from considering a vector field X of class C^∞ on M, and applying the claim (with $l=1$ and $t=0$) and the stationarity of u_{ε_j} for the Allen-Cahn energy:

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Per}_s(\psi^t(E)) = \lim_{j\to\infty}\frac{d}{dt}\bigg|_{t=0}\left[\mathcal{E}^{\rm Sob}_M(u_{\varepsilon_j,t}) + \mathcal{E}^{\rm Pot}_M(u_{\varepsilon_j,t})\right] = 0\,.$$

Step 5. E has Morse index at most m (recall Definition 1.7).

To check this, consider $(m+1)$ vector fields $X_0,\dots,X_m$ of class C^∞ on M.

Letting $a := (a_0,a_1,\dots,a_m)\in\mathbb{R}^{m+1}$ and $X[a] = a_0X_0+\dots+a_mX_m$, we can define the quadratic form $Q_{\varepsilon_j}(a) := \frac{d^2}{dt^2}\Big|_{t=0}\mathcal{E}(u_{\varepsilon_j}\circ\psi^{-t}_{X[a]})$, which we can write as $Q_{\varepsilon_j}(a) = Q^{kl}_{\varepsilon_j}a_ka_l$ for some a_k, a_l. From (91) and the polarization identity for a quadratic form, it is immediate to see that $Q^{kl}_{\varepsilon_j}\to Q^{kl}_0$ as $j\to\infty$, where $Q_0(a) := \frac{d^2}{dt^2}\Big|_{t=0}\mathrm{Per}_s(\psi^t_{X[a]}(E)) = Q^{kl}_0 a_ka_l$.
Now, since the u_{ε_j} have Morse index $\le m$, by definition we know that for every j there must exist some $a^j\in\mathbb{S}^m$ such that

$$Q_{\varepsilon_j}(a^j) = \frac{d^2}{dt^2}\bigg|_{t=0}\mathcal{E}(u_{\varepsilon_j}\circ\psi^{-t}_{X[a^j]}) \ge 0\,; \tag{92}$$

the convergence of the coefficients $Q^{kl}_{\varepsilon_j}$ to Q^{kl}_0 then immediately shows that $Q_0(a)\ge 0$ for some $a\in\mathbb{S}^m$ as well.

☐

The rest of this section is devoted to proving that the sets constructed as limits of solutions to the Allen-Cahn equation (and which were shown to be critical points of the the fractional perimeter under inner variations) are actually viscosity solutions to the NMS equation.

Lemma 3.23 (Palatucci-Savin-Valdinoci [82])**.** *There exists a unique increasing function $v_\circ : \mathbb{R}\to(-1,1)$ with $v_\circ(0)=0$ that solves $(-\Delta)^s(v_\circ)+W'(v_\circ)=0$ in $\mathbb{R}$.*

Remark 3.24. Let A be any symmetric positive definite matrix with $A_{in}=\delta_{in}$, $1\le i\le n$. Defining $v_{\varepsilon,\tau}(x) = v_\circ(\varepsilon^{-1}(x_n-\tau))$ (where $v_\circ$ is the function from Lemma 3.23) we have

$$\alpha_{n,s}\int_{\mathcal{B}_\varrho}\frac{(v_{\varepsilon,\tau}(x)-v_{\varepsilon,\tau}(y))}{|A(x-y)|^{n+s}}|A|dy + \varepsilon_j^{-s}W'(v_{\varepsilon,\tau}) = 0,$$

where $|A|$ denotes the determinant of A

Remark 3.25. We will implicitly use many times the following fact. Let $\varphi : \mathcal{B}_{r_\circ}(0)\to M$ is a diffeomorphism onto its image with $\varphi(0)=p$, and let $F\subset M$ be a measurable set. Then, the limit

$$\lim_{r\downarrow 0}\int_{M\setminus B_r(p)}(\chi_F-\chi_{F^c})(q)K(p,q)\,dV_q$$

exists if and only if the limit

$$\lim_{r\downarrow 0}\int_{M\setminus\varphi(\mathcal{B}_r(0))}(\chi_F-\chi_{F^c})(q)K(p,q)\,dV_q$$

exists, and if they do exist they coincide. This is not due to cancellations and can be seen as follows: for $r\le r_\circ$ sufficiently small, by Lemma 2.11 we can estimate

$$\begin{aligned}\Big|\int_{M\setminus B_r(p)}(\chi_F-\chi_{F^c})(q)K(p,q)\,dV_q-\int_{M\setminus\varphi(\mathcal{B}_r(0))}(\chi_F-\chi_{F^c})(q)K(p,q)\,dV_q\Big|\\ \le C\int_{B_r(q)\Delta\varphi(\mathcal{B}_r(0))}\frac{1}{d(q,p)^{n+s}}\,dV_q\le\frac{C}{r^{n+s}}\mathrm{Vol}\big(B_r(q)\triangle\varphi(\mathcal{B}_r(0))\big)\to 0\end{aligned}$$

as $r\to 0^+$, since there holds $\mathrm{Vol}\big(B_r(q)\triangle\varphi(\mathcal{B}_r(0))\big)=O(r^{n+2})$ for small r.

Proposition 3.26. *Assume that u_{ε_j} are solutions to the A-C equation (7) on M, with parameters $\varepsilon_j\to 0$ and Morse index $m(u_{\varepsilon_j})\le m$, and that moreover $u_{\varepsilon_j}\to u_0:=\chi_E-\chi_{E^c}$ in $H^{s/2}(M)$. Then ∂E is a viscosity solution of the NMS equation in the following sense: whenever $p\in\partial E$, and $\varphi:\mathcal{B}_{\varrho_\circ}(0)\to V$ is a diffeomorphism from $\mathcal{B}_{\varrho_\circ}$ to an open neighborhood $V\subset M$ of p satisfying $\varphi(0)=p$ and $V^+:=\varphi(\mathcal{B}^+_{\varrho_\circ})\subset E$ (where we denote $\mathcal{B}^+_r:=\mathcal{B}_r\cap\{x_n>0\}$) we have*

$$\lim_{r\downarrow 0}\int_{M\setminus B_r(p)}(\chi_F-\chi_{F^c})(q)K(p,q)\,dV_q\le 0,\qquad \textit{for } F=V^+\cup(E\setminus V). \tag{93}$$

Proof. We suppose by contradiction that for some p and $\varphi:\mathcal{B}_{\varrho_\circ}\to V$ as in the statement of the proposition we had

$$\lim_{r\downarrow 0}\int_{M\setminus B_r(p)}(\chi_F-\chi_{F^c})(q)K(p,q)\,dV_q\ge 2\delta>0,\qquad \text{for } F=V^+\cup(E\setminus V). \tag{94}$$

Our goal is now to obtain a contradiction.

Let us make the following useful observation that we will use several times throughout the proof. Let $\psi:=\mathcal{B}_\varrho\to W\subset V$ be another diffeomorphism with $\psi(0)=p$ such that $\varphi(\mathcal{B}^+_{\varrho_\circ})\cap W\subset\psi(\mathcal{B}^+_\rho)$. Put $G=\psi(\mathcal{B}^+_\varrho)\cup(E\setminus\psi(\mathcal{B}_\varrho))$; then,

$$(\chi_G-\chi_{G^c})(q)\ge(\chi_F-\chi_{F^c})(q)\quad\text{for all } q\in M.$$

Hence, the integral (94) only grows when replacing F by G. In particular, this applies to "restrictions of domain", such as $\psi=\varphi|_{\mathcal{B}_\varrho}$ for any $\rho<\varrho_\circ$.

Step 1. Let us first observe that (setting $x=(x',x_n)$) we can replace F by

$$F_t:=\varphi(\{x\in\mathcal{B}_{\varrho_\circ}:x_n>t|x'|^2\})\cup(E\setminus V)$$

in (94), for $t>0$ sufficiently small, provided we also replace 2δ by δ. Indeed, it is easy to see (using the local bounds for K in Lemma 2.11) that

$$f(t):=\lim_{r\downarrow 0}\int_{M\setminus B_r(p)}(\chi_{F_t}-\chi_{F_t^c})(q)K(p,q)\,dV_q$$

is continuous in t. Since $f(0)\ge 2\delta$, for $t>0$ sufficiently small, we will still have $f(t)\ge\delta>0$.

Next, fix $t=t_\circ>0$ small and choose "Fermi coordinates" adapted to the hypersurface $\Gamma:=\varphi(\{x_n=t_\circ|x'|^2\})$ around p. More precisely: there exists a diffeomephism $\psi:\mathcal{B}_{\varrho_1}\to W=\psi(\mathcal{B}_{\varrho_1})$, with $\psi(0)=p$ and $W\subset V$ open neighborhood of p, such that for all $x\in\mathcal{B}_{\varrho_1}$,

$$d(\psi(x),\Gamma)=\begin{cases}x_n & \text{if } x_n\ge 0\\ -x_n & \text{if } x_n\le 0\end{cases}$$

and $\psi(\mathcal{B}^+_{\varrho_1}) = W \cap \varphi(\mathcal{B}_{\varrho_\circ} \cap \{x_n > t|x'|^2\})$.

Moreover, since $G := \psi(\mathcal{B}^+_{\varrho_1}) \cup (E \setminus W)$ contains F_t we have

$$\lim_{r\downarrow 0} \int_{M\setminus B_r(p)} (\chi_G - \chi_{G^c})(q)K(p,q)\,dV_q \geq \delta > 0\,. \tag{95}$$

Also, by construction we have

$$\psi(\mathcal{B}_{\varrho_1} \cap \{x_n \geq -c|x'|^2\}) \subset E, \tag{96}$$

where $c > 0$ depends on $t_\circ$.

Step 2. We now perform a key computation in coordinates. Let us now choose a smooth cutoff function $\eta : \mathbb{R}^n \to \mathbb{R}_+$ satisfying $\chi_{\mathcal{B}_1} \leq \eta \leq \chi_{\mathcal{B}_2}$ and put:

$$\eta_\varrho(x) := \eta(x/\varrho) \quad \text{and} \quad \bar{\eta}_\varrho := \eta_\varrho \circ \psi^{-1}. \tag{97}$$

Let $K(x,y) = K(\varphi(x), \varphi(y))$ be the expression in coordinates ψ^{-1} of the kernel $K(p,q)$ for $p,q \in W$, that is, for $x,y \in \mathcal{B}_1$. Let $g_{ij} : \mathcal{B}_{\varrho_1} \to \mathbb{R}^{n^2}$ denote the components of the metric in the coordinates ψ^{-1}. Since ψ^{-1} are Fermi coordinates, we have

$$g_{ni} = g_{in} = \delta_{ni}, \quad 1 \leq i \leq n \tag{98}$$

This will be crucially used later.

Fix $\varrho \in (0, \varrho_1/2)$ small to be chosen later. By (95), we have for $G_\varrho := \psi(\mathcal{B}^+_\varrho) \cup (E \setminus \psi(\mathcal{B}_\varrho))$ and $H := \{x_n > 0\} \subset \mathbb{R}^n$

$$\begin{aligned}\lim_{r\downarrow 0} \int_{\mathcal{B}_{2\varrho}\setminus\mathcal{B}_r} (\chi_H - \chi_{H^c})(y)K(0,y)\eta_\varrho(y)\sqrt{|g|}(y)\,dy &= \lim_{r\downarrow 0} \int_{M\setminus B_r(p)} (\chi_{G_\varrho} - \chi_{G^c_\varrho})(q)K(p,q)\bar{\eta}_\varrho(q)\,dV_q \\ &\geq \delta - \int_{M\setminus\psi(\mathcal{B}_\varrho)} (\chi_E - \chi_{E^c})(q)K(p,q)(1-\bar{\eta}_\varrho)(q)\,dV_q.\end{aligned}$$

Notice that $\mathcal{B}_r(0)$ is not the same as $\varphi^{-1}(B_r(p))$, however as it will become clear from the proof below, the limits as $r \downarrow 0$ of the corresponding integrals give the same value.

Let us also write

$$K(x,y)\sqrt{|g|}(y) = \frac{\alpha_{n,s}}{|A(x)(x-y)|^{n+s}}\sqrt{|g|}(x) + \widehat{K}(x,y),$$

where $A(x)$ is the nonnegative definite symmetric square root of the matrix $(g_{ij}(x))$. Notice that, thanks to (98), we have $A_{ni}(x) = A_{in}(x) = \delta_{ni}$ for all $1 \leq i \leq n$. Also, by Proposition 2.6 the kernel $\widehat{K}(x,y)$ is not singular, in the sense that

$$|\widehat{K}(x,y)| \leq C(1 + |x-y|^{-n-s+1}).$$

We thus have

$$\begin{aligned}\alpha_{n,s} \lim_{r\downarrow 0} \int_{\mathcal{B}_{\varrho_1}\setminus\mathcal{B}_r} \frac{(\chi_H - \chi_{H^c})\eta_\varrho(y)}{|A(0)(0-y)|^{n+s}}\sqrt{|g|}(0)dy &\geq \delta - \int_{M\setminus\psi(\mathcal{B}_\varrho)} (\chi_E - \chi_{E^c})(q)K(p,q)(1-\bar{\eta}_\varrho)(q)\,dV_q \\ &\quad - \int_{\mathcal{B}_{\varrho_1}} (\chi_H - \chi_{H^c})(y)\widehat{K}(0,y)\eta_\varrho(y)\,dy\end{aligned} \tag{99}$$

Let us now recall the assumption that $u_{\varepsilon_j} \to u_0 = \chi_E - \chi_{E^c}$, and let us define for x in a neighbourhood of 0

$$f_j(x) = -\int_{M\setminus\psi(\mathcal{B}_\varrho)} u_{\varepsilon_j}(q)K(\psi(x),q)(1-\bar{\eta}_\varrho)(q)\,dV_q - \int_{\mathcal{B}_{2\varrho}} (u_{\varepsilon_j} \circ \psi)(y)\widehat{K}(x,y)\eta_\varrho(y)\,dy$$

and

$$f_\infty(x) = -\int_{M\setminus\psi(\mathcal{B}_\varrho)} (\chi_E - \chi_{E^c})(q)K(\psi(x),q)(1-\bar{\eta}_\varrho)(q)\,dV_q - \int_{\mathcal{B}_{2\varrho}} (\chi_E - \chi_{E^c})(\psi(y))\widehat{K}(x,y)\eta_\varrho(y)\,dy$$

Define also

$$I(\varrho) := \int_{\mathcal{B}_{2\varrho}} (\chi_E - \chi_{E^c})(\psi(y))\widehat{K}(0,y)\eta_\varrho(y)\,dy - \int_{\mathcal{B}_{2\varrho}} (\chi_H - \chi_{H^c})(y)\widehat{K}(0,y)\eta_\varrho(y)\,dy$$

Fixing $\varrho > 0$ small enough, we will have $|I(\varrho)| \le \delta/4$. Then, is not difficult to show (using the kernel bounds of Proposition 2.6 and of Lemma 2.11) that $f_j(x) \to f_\infty(x)$ uniformly for all x in a neighborhood of 0, and that f_∞ is continuous in a neighborhood of 0. As a consequence, we have $|f_j(x) - f_\infty(0)| < \delta/4$ for all $x \in \mathcal{B}_{r_\circ}(0)$ and $j \ge j_\circ$, for some $j_\circ$.

On the other hand, recall that $(-\Delta)^s u_{\varepsilon_j} + \varepsilon_j^{-s} W'(u_{\varepsilon_j}) = 0$ in M. Hence, in particular

$$\lim_{r\downarrow 0} \int_{M\setminus B_r(\psi(x))} \big(u_{\varepsilon_j}(\psi(x)) - u_{\varepsilon_j}(q)\big)K(p,q)\,dV_q + \varepsilon_j^{-s}W'(u_{\varepsilon_j}(\psi(x))) = 0$$

for all $x \in \mathcal{B}_{r_\circ}(0)$. Proceeding similarly the previous equation can be rewritten as

$$\alpha_{n,s} \int_{\mathcal{B}_\varrho} \frac{\big((u_{\varepsilon_j}\circ\psi)(x) - (u_{\varepsilon_j}\circ\psi)(y)\big)\eta_\varrho(y)}{|A(x)(x-y)|^{n+s}} \sqrt{|g|}(x)\,dy + \varepsilon_j^{-s}W'(u_{\varepsilon_j}(\psi(x))) = f_j(x)\,. \tag{100}$$

Notice also that (99) can be rewriten as

$$\alpha_{n,s} \lim_{r\downarrow 0} \int_{\mathcal{B}_{2\varrho}\setminus\mathcal{B}_r} \frac{(\chi_H - \chi_{H^c})\eta_\varrho(y)}{|A(0)(0-y)|^{n+s}} \sqrt{|g|}(0)\,dy \ge \delta + f_\infty(0) + I(\varrho)\,.$$

We now define $v_{\varepsilon,\tau}(x) = v_\circ\big(\varepsilon^{-1}(x_n - \tau)\big)$, where $v_\circ : \mathbb{R} \to (-1,1)$ is the function from Lemma 3.23. In view of Remark 3.24, we have for $x \in \mathcal{B}_{r_\circ}(0)$, j large, and $|\tau|$ sufficiently small,

$$\alpha_{n,s} \int_{\mathcal{B}_{2\varrho}} \frac{\big(v_{\varepsilon_j,\tau}(x) - v_{\varepsilon_j,\tau}(y)\big)\eta_\varrho(y)}{|A(x)(x-y)|^{n+s}} \sqrt{|g|}(x)\,dy + \varepsilon_j^{-s}W'(v_{\varepsilon_j,\tau}) \le \frac{\delta}{4}.$$

This implies that whenever $x \in \mathcal{B}_{r_\circ}$, j sufficiently large, and $|\tau|$ sufficiently small

$$\alpha_{n,s} \int_{\mathcal{B}_{2\varrho}} \frac{\big(v_{\varepsilon_j,\tau}(x) - v_{\varepsilon_j,\tau}(y)\big)\eta_\varrho(y)}{|A(x)(x-y)|^{n+s}} \sqrt{|g|}(x)\,dy + \varepsilon_j^{-s}W'(v_{\varepsilon_j,\tau}) \le f_j(x) - \frac{\delta}{4}. \tag{101}$$

In other words, we have shown that $v_{\varepsilon_j,\tau}$ is a strict subsolution of (100).

Step 3. We now reach the desired contradiction. Fix now $\theta \in \left(0, \frac{1}{100}\right)$ sufficiently small (to be chosen) and let

$$\xi_\theta(t) := \begin{cases} -1+\theta & \text{if } t \in [-1, -1+\theta], \\ t & \text{if } t \in [-1+\theta, 1-\theta], \\ 1-\theta & \text{if } t \in [1-\theta, 1]. \end{cases}$$

By the Hausdorff convergence of the level sets of u_{ε_j} which we have proved in Step 3 at page 49, for any $t \in [-1+\theta, 1-\theta]$ the set $\{x \in \mathcal{B}_{2\varrho} : (u_{\varepsilon_j}\circ\varphi) \ge t\}$ converges in Hausdorff distance towards $\psi^{-1}(E) \supset \{x \in \mathcal{B}_{2\varrho} \ : \ x_n \le -c|x'|^2\}$. Hence, for every fixed $\tau > 0$ we have, for all j sufficiently large,

$$\xi_\theta \circ u_{\varepsilon_j} \circ \psi \ge \xi_\theta \circ v_{\varepsilon_j,\tau} \quad \text{ in } \overline{\mathcal{B}_{2\varrho}}\,. \tag{102}$$

Let us define:

$$\tau_j := \min\left\{\tau \in \mathbb{R} \mid (102) \text{ holds for } j\right\}.$$

Notice that by definition of τ_j there is $x_j \in \overline{\mathcal{B}_{2\varrho}} \cap \{|u_{\varepsilon_j}| \le 1-\theta\} \cap \{|v_{\varepsilon_j,\tau}| \le 1-\theta\}$. By the previous Hausdoff convergence property of level sets we it must be $x_j \to 0$ and $\tau_j \to 0$ as $j \to \infty$.

Let us show that, if θ is chosen sufficiently small, we have

$$u_{\varepsilon_j}\circ\psi \ge v_{\varepsilon_j,\tau_j} \quad \text{in } \mathcal{B}_{r_\circ/2}\,. \tag{103}$$

Indeed, thanks to (100)-(101) the difference

$$w := u_{\varepsilon_j}\circ\psi - v_{\varepsilon_j,\tau_j}$$

satisfies

$$\mathcal{L}w(x) := \alpha_{n,s}\int_{\mathcal{B}_\varrho}\frac{(w(x)-w(y))\eta_\varrho(y)}{|A(x)(x-y)|^{n+s}}\sqrt{|g|}(x)\,dy \ge \frac{\delta}{4}+\varepsilon_j^{-s}\big(W'(v_{\varepsilon_j,\tau_j})-W'(u_{\varepsilon_j}\circ\psi)\big)(x) \quad \text{in } \mathcal{B}_{r_\circ}. \tag{104}$$

Notice that since (102) holds for $\tau=\tau_j$ we have $w = v_{\varepsilon_j,\tau_j}-(u_{\varepsilon_j}\circ\varphi)\ge -\theta$ in $\mathcal{B}_{2\varrho}$.

Assume now by contradiction that $\inf_{\mathcal{B}_{r_\circ/2}} w<0$. Recall (97) and define

$$\overline{\eta}_t = -\theta + t\eta_{r_\circ/2}\,.$$

and let $t_*\in[0,\theta)$ be the supremum of the $t\ge 0$ such that $w\ge \overline{\eta}_t$ in $\mathcal{B}_{2\varrho}$. By construction there exists $x_*\in\mathcal{B}_{r_\circ}$ such that

$$(w-\overline{\eta}_{t_*})(x_*)=0 \quad \text{while} \quad w-\overline{\eta}_{t_*}\ge 0 \quad \text{in } \mathcal{B}_{2\varrho}.$$

Now evaluating the integro-differential operator $\mathcal{L}$ (whose kernel is supported in $B_{2\varrho}$; see (104)) at the point x_* we obtain

$$C\theta \ge \mathcal{L}\overline{\eta}_{t_*}(x_*) \ge \mathcal{L}w(x_\circ) \ge \frac{\delta}{4}+\varepsilon_j^{-s}\big(W'(v_{\varepsilon_j,\tau_j})-W'(u_{\varepsilon_j}\circ\varphi)\big)(x_\circ)\ge\frac{\delta}{4},$$

Notice that $W''>0$ in the interval $[u_{\varepsilon_j}\circ\varphi(x_\circ), v_{\varepsilon_j,\tau_j}(x_\circ)]$ because (102) holds for $\tau=\tau_j$, and hence either $u_{\varepsilon_j}\circ\varphi(x_\circ)\ge 1-\theta$ or $v_{\varepsilon_j,\tau_j}(x_\circ)\le -1+\theta$. Therefore, choosing $\theta>0$ sufficiently small so that $C\theta<\delta/4$ we reach a contradiction. Hence, we have proved that $w\ge 0$ and (103) holds.

Finally, take j large so that $x_j\in\mathcal{B}_{r_\circ/4}$ (recall that $x_j\to 0$ as $j\to\infty$). Using that $w\ge 0$ in $\mathcal{B}_{r_\circ/2}$, $w(x_j)=0$, and $w\ge-\theta$ in $\mathcal{B}_{2\varrho}\setminus\mathcal{B}_{r_\circ}$ and evaluating $\mathcal{L}w$ at the point $x_j\in\mathcal{B}_{r_\circ/4}$ we obtain, similarly as before

$$C(r_\circ)\theta \ge -\alpha_{n,s}\int_{\mathcal{B}_\varrho}\frac{w(y)\eta_\varrho(y)}{|A(x_j)(x_j-y)|^{n+s}}\sqrt{|g|}(x_j)\,dy = \mathcal{L}w(x_j)\ge\frac{\delta}{4}.$$

Chosing $\theta>0$ sufficiently small, we obtain a contradiction and this completes the proof. □

Theorem 1.30 motivates the definition of $\mathcal{A}_m(M)$ given in the introduction, see Definition 1.15.

The "surfaces" Σ belonging to the class $\mathcal{A}_m(M)$ enjoy some properties additionally to those already described in Theorem 1.30 and Proposition 3.26. We record them in the remark below.

Remark 3.27. Every $\Sigma=\partial E\in\mathcal{A}_m(M)$ also satisfies that if $\mathrm{FA}_2(M,g,R,p,\varphi)$ is satisfied, then the following hold:

(1) **BV and energy estimate.** For some $C=C(n,s,m)>0$ there holds

$$\mathrm{Per}(E;B_{R/2}(p))\le CR^{n-1} \quad \text{and} \quad \mathrm{Per}_s(E;B_{R/2}(p))\le CR^{n-s}.$$

(2) **Density estimate.** For some positive constant ω_0, which depends only on n, s and m, we have that if $R^{-n}|E\cap B_R(p)|\le\omega_0$ then $|E\cap B_{R/2}(p)|=0$.

Indeed, by Definition 1.15 of $\mathcal{A}_m(M)$ we can find a sequence u_{ε_j}, made of A-C solutions with Morse index $\le m$ and parameters $\varepsilon_j\to 0$, converging to E in $L^1(M)$, and also in $H^{s/2}(M)$ thanks to Theorem 1.30. Then, property (1) follows from the lower semicontinuity of the BV norm under L^1 convergence and the convergence of Sobolev energies under strong $H^{s/2}$ convergence, together with the fact that the u_{ε_j} satisfy uniform BV and Sobolev estimates themselves by Theorems 1.25 and 1.27. Similarly, property (2) follows from the L^1 convergence and the density estimates of Proposition 1.28 satisfied by the u_{ε_j} themselves.

3.4 The Yau conjecture for nonlocal minimal surfaces – Proof of Theorem 1.9

We can now combine the existence and convergence results in the previous sections to prove the Yau conjecture for nonlocal minimal surfaces.

Proof of Theorem 1.9. Fix $\mathfrak{p} \in \mathbb{N}$. Theorem 1.20 gives the existence, for all $\varepsilon \in (0, \varepsilon_{\mathfrak{p}})$, of a solution $u_{\varepsilon,\mathfrak{p}}$ to the fractional Allen-Cahn equation with Morse index $m(u_{\varepsilon,p}) \leq p$ and energy bounds

$$C^{-1}\mathfrak{p}^{s/n} \leq (1-s)\mathcal{E}_M^{\varepsilon}(u_{\varepsilon,\mathfrak{p}}) \leq C\mathfrak{p}^{s/n}. \tag{105}$$

Thanks to the convergence result in Theorem 1.30, we can find a subsequence $\{\varepsilon_j\}_j$ such that the $u_{\varepsilon_j,\mathfrak{p}}$ converge in $H^{s/2}(M)$ to a limit function

$$u_{0,p} = \chi_{E^{\mathfrak{p}}} - \chi_{M\setminus E^{\mathfrak{p}}},$$

where $\partial E^{\mathfrak{p}}$ is an s-minimal surface and $\partial E^{\mathfrak{p}} \in \mathcal{A}_{\mathfrak{p}}(M)$ by definition. Moreover, by (105) and the strong convergence of the Allen-Cahn energies stated in Theorem 1.30, we deduce that the fractional perimeter of $E^{\mathfrak{p}}$ satisfies the bounds

$$C^{-1}\mathfrak{p}^{s/n} \leq (1-s)\mathrm{Per}_s(E^{\mathfrak{p}}) \leq C\mathfrak{p}^{s/n}. \tag{106}$$

In particular, the fractional perimeter of the $E^{\mathfrak{p}}$ goes to infinity as $\mathfrak{p} \to \infty$, whence we conclude that the family $\{E^{\mathfrak{p}}\}_{p\in\mathbb{N}}$ is infinite. We remark that, unlike in [60] or [76], there is no multiplicity phenomenon (thanks to the strong convergence as $\varepsilon \to 0$) and we can distinguish the limit surfaces by their fractional perimeters. ☐

4 Regularity and rigidity results

4.1 Blow-up procedure

The goal of this subsection is to explicit how to perform blow-ups of (sequences of) s-minimal surfaces around points with flatness assumptions, proving strong convergence results for the blowing-up sequence to a Euclidean limit surface in a manner similar to Section 3.3.

Definition 4.1 (Blow-up sequence). Let $(M_j, g^{(j)})$ be a sequence of closed manifolds of dimension n, and let $p_j \in M_j$ be points such that M_j satisfies the flatness assumption $\mathrm{FA}_3(M_j, g^{(j)}, 1, p_j, \varphi_j)$. Suppose in addition that $g^{(j)}_{kl}(0) = \delta_{kl}$, i.e. that the metric of M_j with respect to the chart φ_j^{-1} at the point 0 is the Euclidean metric.
For each j, let ∂E_j be an s-minimal surface in M_j, satisfying uniform BV estimates in the sense that there is some C_0 independent of j such that

$$\mathrm{Per}(\varphi_j^{-1}(E_j); B_r(x)) \leq C_0 r^{n-1} \quad \text{for all } x \in \mathcal{B}_{1/2} \text{ and } r \in (0, 1/4),$$

where we put $\varphi_j^{-1}(E_j) := \{y \in \mathcal{B}_1 : \varphi_j(y) \in E_j\}$.
Given $r_j \searrow 0$, a sequence of subsets of $\mathbb{R}^n$ of the form

$$F_j := \frac{1}{r_j}\varphi_j^{-1}(E_j) \subset \mathcal{B}_{1/r_j} \subset \mathbb{R}^n$$

(for some M_j, p_j, E_j as above) is called a blow-up sequence.

Remark 4.2. F_j is a blow-up sequence if and only if there exist $(\widehat{M}_j, \widehat{g}^{(j)})$, $\widehat{p}_j \in \widehat{M}_j$, and $R_j \nearrow \infty$ such that

- $\mathrm{Per}(F_j; B_r(x)) \leq C_0 r^{n-1}$ for all $x \in \mathcal{B}_{R_j/2}$ and $r \in (0, R_j/4)$;
- $\mathrm{FA}_3(\widehat{M}_j, \widehat{g}^{(j)}, R_j, \widehat{p}_j, \widehat{\varphi}_j)$ holds and $\widehat{g}^{(j)}_{kl}(0) = \delta_{kl}$, where $\widehat{g}^{(j)}_{kl} = \widehat{g}^{(j)}((\widehat{\varphi}_j)_*(e_k), (\widehat{\varphi}_j)_*(e_l))$ denotes the metric in coordinates;

- For each j there is an s-minimal surface $\partial\widehat{E}_j$ in $\widehat{M}_j$ such that $F_j = \widehat{\varphi}_j^{-1}(E_j)$.

Proof of the remark. This follows from putting $\widehat{M}_j = M_j$, $\widehat{p}_j = p_j$, $\widehat{g}^{(j)} = \frac{1}{r_j^2} g^{(j)}$ and $R_j = \frac{1}{r_j}$ in Definition 4.1 and considering the scaling properties stated in Remark 1.24. $\square$

We record some auxiliary results. The notation $K_{\widehat{M}_j}$ will be used instead of K_s when we want to explicit which manifold the kernel K_s is being considered on.

Proposition 4.3. *Let $F_j \subset \mathbb{R}^n$ be a blow-up sequence, with associated $(\widehat{M}_j, \widehat{g}^{(j)})$, $\widehat{E}_j \subset \widehat{M}_j$ and $R_j \to \infty$ as in Remark 4.2. The following hold:*

(i) The components $\widehat{g}_{kl}^{(j)}$ of the metric of $\widehat{M}_j$ (using the chart parametrization $\widehat{\varphi}_j$) converge locally uniformly to the Euclidean ones, in the sense that given $R_0 > 0$,

$$\sup_{x \in \mathcal{B}_{R_0}} \left|\widehat{g}_{kl}^{(j)}(x) - \delta_{kl}\right| \to 0 \quad \text{as } j \to \infty.$$

(ii) The kernel $K_{\widehat{M}_j}$ converges locally uniformly to the Euclidean one, in the sense that given $R_0 > 0$,

$$\sup_{(x,y) \in \mathcal{B}_{R_0} \times \mathcal{B}_{R_0}} \left| \frac{K_{\widehat{M}_j}(\widehat{\varphi}_j(x), \widehat{\varphi}_j(y))}{\frac{\alpha_{n,s}}{|x-y|^{n+s}}} - 1 \right| \to 0 \quad \text{as } j \to \infty.$$

Proof. The first part follows from the definition of the flatness assumptions and the fact that $R_j \to \infty$.

As for part *(ii)*, it is a consequence of Proposition 2.6. Precisely, it follows from putting $R = R_j$ and $z = 1 - x$ in (26) of Proposition 2.6. $\square$

Lemma 4.4. *Let $F_j \subset \mathbb{R}^n$ be a blow-up sequence, with associated $(\widehat{M}_j, \widehat{g}^{(j)})$, $\widehat{E}_j \subset \widehat{M}_j$ and $R_j \to \infty$ as in Remark 4.2. Put*

$$K_j(x,y) := K_{\widehat{M}_j}(\widehat{\varphi}_j(x), \widehat{\varphi}_j(y))$$

and (for a fixed $\rho < R_j/4$)

$$\mathrm{Per}_s^{(j)}(F_j; \mathcal{B}_\rho) := \frac{1}{4} \iint_{(B_{R_j} \times B_{R_j}) \setminus (B_\rho^c \times B_\rho^c)} |u_j(x) - u_j(y)|^2 K_j(x,y) \sqrt{g^{(j)}}(x) \sqrt{g^{(j)}}(y)\, dx\, dy\,,$$

where $u_j := \chi_{F_j} - \chi_{F_j^c}$.
Given a vector field $X \in C_c^\infty(B_\rho; \mathbb{R}^n)$, define $X_j := (\widehat{\varphi}_j)_* X$ and extend it by zero to a vector field on $\widehat{M}_j$. The following hold:

(1) Let $0 \le \ell \le 3$. Then

$$\left| \frac{d^\ell}{dt^\ell} \Big(\mathrm{Per}_s^{\widehat{M}_j}(\psi_{X_j}^t(\widehat{E}_j); \widehat{\varphi}_j(\mathcal{B}_\rho)) - \mathrm{Per}_s^{(j)}(\psi_X^t(F_j); \mathcal{B}_\rho) \Big) \right| \le \frac{C_X}{R_j^s}\,.$$

(2) If $\chi_{F_j} \to \chi_F$ in $H^{s/2}_{\mathrm{loc}}(\mathbb{R}^n)$, then for $0 \le \ell \le 2$

$$\frac{d^\ell}{dt^\ell} \mathrm{Per}_s^{(j)}(\psi_X^t(F_j); \mathcal{B}_\rho) \to \frac{d^\ell}{dt^\ell} \mathrm{Per}_s^{\mathbb{R}^n}(\psi_X^t(F); \mathcal{B}_\rho)\,.$$

Proof. We begin by proving (1). Let $v_j^t := \chi_{\psi^t_{X_j}(\widehat{E}_j)} - \chi_{\psi^t_{X_j}(\widehat{E}_j^c)}$ and $u_j^t := \chi_{\psi^t_X(F_j)} - \chi_{\psi^t_X(F_j^c)}$.
By splitting the domain of the corresponding integral and then passing to coordinates, we can write

$$\begin{aligned}
\mathrm{Per}_s^{\widehat{M}_j}(\psi^t_{X_j}(\widehat{E}_j); \widehat{\varphi}_j(\mathcal{B}_\rho)) &= \frac{1}{4}\iint_{(\widehat{\varphi}_j(\mathcal{B}_{R_j})\times\widehat{\varphi}_j(\mathcal{B}_{R_j}))\setminus(\widehat{\varphi}_j(\mathcal{B}_\rho)^c\times\widehat{\varphi}_j(\mathcal{B}_\rho)^c)} |v_j^t(p)-v_j^t(q)|^2\, K_{\widehat{M}_j}(p,q)\, dV_p\, dV_q \\
&\quad + \frac{1}{2}\iint_{\widehat{\varphi}_j(\mathcal{B}_\rho)\times\widehat{\varphi}_j(\mathcal{B}_{R_j})^c} |v_j^t(p)-v_j^t(q)|^2\, K_{\widehat{M}_j}(p,q)\, dV_p\, dV_q \\
&= \frac{1}{4}\iint_{(\mathcal{B}_{R_j}\times\mathcal{B}_{R_j})\setminus(\mathcal{B}_\rho^c\times\mathcal{B}_\rho^c)} |u_j^t(x)-u_j^t(y)|^2 K_j(x,y)\sqrt{g^{(j)}(x))}\sqrt{g^{(j)}(y))}\, dx\, dy \\
&\quad + \frac{1}{2}\iint_{\widehat{\varphi}_j(\mathcal{B}_\rho)\times\widehat{\varphi}_j(\mathcal{B}_{R_j})^c} |v_j^t(p)-v_j^t(q)|^2\, K_{\widehat{M}_j}(p,q)\, dV_p\, dV_q \\
&= \mathrm{Per}_s^{(j)}(\psi^t_X(F_j); \mathcal{B}_\rho) + \frac{1}{2}\iint_{\widehat{\varphi}_j(\mathcal{B}_\rho)\times\widehat{\varphi}_j(\mathcal{B}_{R_j})^c} |v_j^t(p)-v_j^t(q)|^2\, K_{\widehat{M}_j}(p,q)\, dV_p\, dV_q\,.
\end{aligned}$$

From this computation, changing variables with the flow as in (39) and then passing to coordinates in the first variable we can compute

$$\begin{aligned}
&\left|\frac{d^\ell}{dt^\ell}\Big(\mathrm{Per}_s^{\widehat{M}_j}(\psi^t_{X_j}(\widehat{E}_j); \widehat{\varphi}_j(\mathcal{B}_\rho)) - \mathrm{Per}_s^{(j)}(\psi^t_X(F_j); \mathcal{B}_\rho)\Big)\right| = \\
&\qquad = \frac{1}{2}\left|\frac{d^\ell}{dt^\ell}\iint_{\widehat{\varphi}_j(\mathcal{B}_\rho)\times\widehat{\varphi}_j(\mathcal{B}_{R_j})^c} |v_j^t(p)-v_j^t(q)|^2\, K_{\widehat{M}_j}(p,q)\, dV_p\, dV_q\right| \\
&\qquad = \frac{1}{2}\left|\iint_{\widehat{\varphi}_j(\mathcal{B}_\rho)\times\widehat{\varphi}_j(\mathcal{B}_{R_j})^c} |v_j(p)-v_j(q)|^2\, \frac{d^\ell}{dt^\ell}\Big[K_{\widehat{M}_j}(\psi^t_{X_j}(p),q)\, J_t(p)\Big]\, dV_p\, dV_q\right| \\
&\qquad \le C\left|\iint_{\mathcal{B}_\rho\times\widehat{\varphi}_j(\mathcal{B}_{R_j})^c} \frac{d^\ell}{dt^\ell}\Big[K_{\widehat{M}_j}(\widehat{\varphi}_j(\psi^t_X(x)),q)\, J_t(p)\Big]\, dx\, dV_q\right|.
\end{aligned}$$

Bounding the derivatives in time of the Jacobian $J_t(p)$ by a constant, and using (29) with $R = R_j$ to bound the integral in q, we conclude the result in (1).

To see (2), let $R > \rho$ and put $f_j(t) := \mathrm{Per}_s^{(j)}(\psi^t_X(F_j); \mathcal{B}_\rho)$. Changing variables with the flow (as above) and splitting the domain of the integral, for $R < R_j$ we can write

$$\begin{aligned}
&\frac{d^\ell}{dt^\ell}\mathrm{Per}_s^{(j)}(\psi^t_X(F_j); \mathcal{B}_\rho) \\
&= \frac{1}{4}\iint_{(\mathcal{B}_R\times\mathcal{B}_R)\setminus(\mathcal{B}_\rho^c\times\mathcal{B}_\rho^c)} |u_j(x)-u_j(y)|^2 \frac{d^\ell}{dt^\ell}\Big[K_j(\psi^t_X(x),\psi^t_X(y))\sqrt{g^{(j)}}(\psi^t_X(x))\sqrt{g^{(j)}}(\psi^t_X(y))J_t(x)J_t(y)\Big]\, dx\, dy \\
&\quad + \frac{1}{2}\iint_{\mathcal{B}_\rho\times(\widehat{\varphi}_j(\mathcal{B}_{R_j})\setminus\widehat{\varphi}_j(\mathcal{B}_R))} |u_j(x)-u_j(\varphi^{-1}(q))|^2 \frac{d^\ell}{dt^\ell}\Big[K_{\widehat{M}_j}(\widehat{\varphi}_j(\psi^t_X(x)),q)\sqrt{g^{(j)}}(\psi^t_X(x))J_t(x)\Big]\, dx\, dV_q\,.
\end{aligned}$$

Let $0 \le \ell \le 3$. Thanks to the flatness assumptions and (29) of Proposition 2.6, we can bound

$$\Big|\iint_{\mathcal{B}_\rho\times(\widehat{\varphi}_j(\mathcal{B}_{R_j})\setminus\widehat{\varphi}_j(\mathcal{B}_R))}|u_j(x)-u_j(\varphi^{-1}(q))|^2\frac{d^\ell}{dt^\ell}\Big[K_{\widehat{M}_j}(\widehat{\varphi}_j(\psi^t_X(x)),q)\sqrt{g^{(j)}}(\psi^t_X(x))J_t(x)\Big]\,dx\,dV_q\Big|$$
$$\le C\iint_{\mathcal{B}_\rho\times(\widehat{M}_j\setminus\widehat{\varphi}_j(\mathcal{B}_R))}\left|\frac{d^\ell}{dt^\ell}\Big[K_{\widehat{M}_j}(\widehat{\varphi}_j(\psi^t_X(x)),q)\sqrt{g^{(j)}}(\psi^t_X(x))J_t(x)\Big]\right|\,dx\,dV_q$$
$$\le\frac{C}{R^s}\,. \tag{107}$$

On the other hand, by the flatness assumptions and (37) in Proposition 2.15 we have that, for $t\in(-T,T)$ and j large enough so that $R<R_j/4$,

$$\iint_{(\mathcal{B}_R\times\mathcal{B}_R)\setminus(\mathcal{B}^c_\rho\times\mathcal{B}^c_\rho)}|u_j(x)-u_j(y)|^2\left|\frac{d^\ell}{dt^\ell}\Big[K_j(\psi^t_X(x),\psi^t_X(y))\sqrt{g^{(j)}}(\psi^t_X(x))\sqrt{g^{(j)}}(\psi^t_X(y))J_t(x)J_t(y)\Big]\right|\,dx\,dy$$
$$\le C_T\iint_{\mathcal{B}_R\times\mathcal{B}_R)}|u_j(x)-u_j(y)|^2\frac{\alpha_{n,s}}{|x-y|^{n+s}}\,dx\,dy$$
$$\le C_T\,,$$

where in the last line we combined the fact that F_j has bounded classical perimeter in $B_{R_j/4}$ with the interpolation result in Proposition 2.22.

This shows that the functions $\frac{d^\ell}{dt^\ell}f_j(t)$ are locally uniformly bounded for $0\le\ell\le 3$; in particular, for $0\le\ell\le 2$ we deduce that the $\frac{d^\ell}{dt^\ell}f_j(t)$ are locally uniformly bounded and moreover have a uniform modulus of continuity, thus by Arzelà-Ascoli they subsequentially converge locally uniformly. By standard single-variable calculus, to conclude our desired result it then suffices to show that $f_j(t)$ converges pointwise to $g(t):=\mathrm{Per}_s^{\mathbb{R}^n}(\psi^t_X(F);\mathcal{B}_\rho)$, which we shall do now.

Denote $u^t:=\chi_{\psi^t_X(F)}-\chi_{\psi^t_X(F^c)}$. We can then write

$$\begin{aligned}g(t)&=\mathrm{Per}_s^{\mathbb{R}^n}(\psi^t_X(F);\mathcal{B}_\rho)\\&=\frac14\iint_{(\mathcal{B}_R\times\mathcal{B}_R)\setminus(\mathcal{B}^c_\rho\times\mathcal{B}^c_\rho)}|u(x)-u(y)|^2\frac{\alpha_{n,s}}{|\psi^t_X(x)-\psi^t_X(y)|^{n+s}}J_t(x)J_t(y)\,dx\,dy\\&\quad+\frac12\iint_{\mathcal{B}_\rho\times\mathcal{B}^c_R}|u(x)-u(y)|^2\frac{\alpha_{n,s}}{|\psi^t_X(x)-y|^{n+s}}J_t(x)\,dx\,dy\,.\end{aligned}$$

Clearly

$$\iint_{\mathcal{B}_\rho\times\mathcal{B}^c_R}|u(x)-u(y)|^2\frac{\alpha_{n,s}}{|\psi^t_X(x)-y|^{n+s}}J_t(x)\,dx\,dy\to 0\quad\text{as } R\to\infty\,,$$

since the integrand is absolutely integrable by (29) in Proposition 2.6. Together with (107), given $\varepsilon>0$, we deduce that there exists an $R>\rho$ (depending only on ρ and ε) such that the aforementioned terms are both smaller than $\varepsilon/2$ for all j large enough. From this fact and a simple triangle inequality, we find that

$$\begin{aligned}&\left|\mathrm{Per}_s^{(j)}(\psi^t_X(F_j);\mathcal{B}_\rho)-\mathrm{Per}_s^{\mathbb{R}^n}(\psi^t_X(F);\mathcal{B}_\rho)\right|\\&\le\frac14\iint_{(\mathcal{B}_R\times\mathcal{B}_R)\setminus(\mathcal{B}^c_\rho\times\mathcal{B}^c_\rho)}|u_j(x)-u_j(y)|^2\cdot\\&\qquad\cdot\left|K_j(\psi^t_X(x),\psi^t_X(y))\sqrt{g^{(j)}}(\psi^t_X(x))\sqrt{g^{(j)}}(\psi^t_X(y))-\frac{\alpha_{n,s}}{|\psi^t_X(x)-\psi^t_X(y)|^{n+s}}\right|J_t(x)J_t(y)\,dx\,dy\\&\quad+\frac14\Big|\iint_{(\mathcal{B}_R\times\mathcal{B}_R)\setminus(\mathcal{B}^c_\rho\times\mathcal{B}^c_\rho)}\Big(|u_j(x)-u_j(y)|^2-|u(x)-u(y)|^2\Big)\frac{\alpha_{n,s}}{|\psi^t_X(x)-\psi^t_X(y)|^{n+s}}J_t(x)J_t(y)\,dx\,dy\Big|\\&\quad+\varepsilon\,.\end{aligned}$$

Regarding the first term, thanks to Proposition 4.3 and (37) in Proposition 2.15 it can be bounded as follows:

$$\begin{aligned}
&\iint_{(\mathcal{B}_R\times\mathcal{B}_R)\setminus(\mathcal{B}_\rho^c\times\mathcal{B}_\rho^c)} |u_j(x)-u_j(y)|^2\cdot \\
&\qquad\cdot\left|K_j(\psi_X^t(x),\psi_X^t(y))\sqrt{g^{(j)}}(\psi_X^t(x))\sqrt{g^{(j)}}(\psi_X^t(y)) - \frac{\alpha_{n,s}}{|\psi_X^t(x)-\psi_X^t(y)|^{n+s}}\right| J_t(x)J_t(y)\,dx\,dy \\
&= \iint_{(\mathcal{B}_R\times\mathcal{B}_R)\setminus(\mathcal{B}_\rho^c\times\mathcal{B}_\rho^c)} |u_j(x)-u_j(y)|^2\frac{\alpha_{n,s}}{|\psi_X^t(x)-\psi_X^t(y)|^{n+s}}\cdot \\
&\qquad\qquad\cdot\left|\frac{K_j(\psi_X^t(x),\psi_X^t(y))\sqrt{g^{(j)}}(x)\sqrt{g^{(j)}}(y)}{\frac{\alpha_{n,s}}{|\psi_X^t(x)-\psi_X^t(y)|^{n+s}}}-1\right| J_t(x)J_t(y)\,dx\,dy \\
&\le o_j(1)\iint_{\mathcal{B}_R\times\mathcal{B}_R} |u_j(x)-u_j(y)|^2\frac{\alpha_{n,s}}{|\psi_X^t(x)-\psi_X^t(y)|^{n+s}}\,dx\,dy \\
&\le o_j(1)\,C_T\iint_{\mathcal{B}_R\times\mathcal{B}_R} |u_j(x)-u_j(y)|^2\frac{\alpha_{n,s}}{|x-y|^{n+s}}\,dx\,dy\,,
\end{aligned}$$

where $o_j(1)\to 0$ as $j\to\infty$. This implies that the whole expression goes to zero, since the factor $\iint_{\mathcal{B}_R\times\mathcal{B}_R}|u_j(x)-u_j(y)|^2\frac{\alpha_{n,s}}{|x-y|^{n+s}}\,dx\,dy$ can be bounded by a constant independent of j: indeed, for j large enough so that $R<R_j/4$, the F_j satisfy uniform perimeter estimates in $\mathcal{B}_R$ by assumption (see Remark 4.2), and thus also uniform fractional energy estimates by interpolation (see Proposition 2.22).

As for the second term, we can write

$$\begin{aligned}
&\left|\iint_{(\mathcal{B}_R\times\mathcal{B}_R)\setminus(\mathcal{B}_\rho^c\times\mathcal{B}_\rho^c)}\Big(|u_j(x)-u_j(y)|^2-|u(x)-u(y)|^2\Big)\frac{\alpha_{n,s}}{|\psi_X^t(x)-\psi_X^t(y)|^{n+s}}J_t(x)J_t(y)\,dx\,dy\right| = \\
&=\left|\iint_{(\mathcal{B}_R\times\mathcal{B}_R)\setminus(\mathcal{B}_\rho^c\times\mathcal{B}_\rho^c)}\Big(|u_j(x)-u_j(y)|^2-|u(x)-u(y)|^2\Big)\frac{\alpha_{n,s}}{|x-y|^{n+s}}\cdot\right. \\
&\qquad\qquad\left.\cdot\frac{\frac{\alpha_{n,s}}{|\psi_X^t(x)-\psi_X^t(y)|^{n+s}}J_t(x)J_t(y)}{\frac{\alpha_{n,s}}{|x-y|^{n+s}}}\,dx\,dy\right|. \qquad (108)
\end{aligned}$$

Since $u_j\to u$ in $H^{s/2}_{\mathrm{loc}}(\mathbb{R}^n)$ by assumption, one immediately sees that

$$A_j(x,y) := \Big(|u_j(x)-u_j(y)|^2-|u(x)-u(y)|^2\Big)\frac{\alpha_{n,s}}{|x-y|^{n+s}}\to 0 \quad\text{in } L^1_{\mathrm{loc}}\,.$$

On the other hand,

$$B(x,y) := \frac{\frac{\alpha_{n,s}}{|\psi_X^t(x)-\psi_X^t(y)|^{n+s}}J_t(x)J_t(y)}{\frac{\alpha_{n,s}}{|x-y|^{n+s}}}$$

is a fixed function in L^∞_{loc} by (37) in Proposition 2.15. Thus $A_jB\to 0$ in L^1_{loc}, and this means that (108) goes to 0 as $j\to\infty$ as well.

Putting everything together, we deduce that

$$\limsup_{j\to\infty}\left|\mathrm{Per}_s^{(j)}(\psi_X^t(F_j);\mathcal{B}_\rho)-\mathrm{Per}_s^{\mathbb{R}^n}(\psi_X^t(F);\mathcal{B}_\rho)\right|\le\varepsilon\,;$$

since ε was arbitrary, we conclude that

$$f_j(t)=\mathrm{Per}_s^{(j)}(\psi_X^t(F_j);\mathcal{B}_\rho)\to\mathrm{Per}_s^{\mathbb{R}^n}(\psi_X^t(F);\mathcal{B}_\rho)=g(t)\,.$$

As explained before, this gives the desired result. □

The main result of this section is the following:

Theorem 4.5 (Convergence to a limit)**.** *Let $F_j \subset \mathbb{R}^n$ be a blow-up sequence. Then, there exists a Euclidean s-minimal surface $F \subset \mathbb{R}^n$ such that a subsequence of the $v_j := \chi_{F_j} - \chi_{(\mathbb{R}^n \setminus F_j)}$ converges to $v := \chi_F - \chi_{(\mathbb{R}^n \setminus F)}$ in $H^{s/2}_{\mathrm{loc}}(\mathbb{R}^n)$.*

Proof. We divide the proof in two steps.

Step 1. Convergence to a limit set F.

Fix a radius R. For j large enough so that $R < R_j/4$, the F_j satisfy a uniform BV estimate in $\mathcal{B}_R$, as indicated in the third bullet of Proposition 4.3. As in Step 1 of the proof of Theorem 1.30 (see page 48), a bound on the BV norm implies that a subsequence of the $v_j = \chi_{F_j} - \chi_{F_j^c}$ converges strongly in $H^{s/2}(\mathcal{B}_R)$ norm. Iterating the same reasoning on increasingly large balls and using a diagonal selection argument, we can find a subsequence (still denoted by v_j) converging in each the norms $H^{s/2}(\mathcal{B}_k)$, $k \in \mathbb{N}$, to a limit function $v = \chi_F - \chi_{F^c}$.

Step 2. Proof that F is stationary for the fractional perimeter.

Fix an arbitrary Euclidean vector field $X \in C^\infty_c(\mathcal{B}_\rho)$, for some $\rho > 0$, and let ψ^t_X denote its flow at time t. Since the F_j are a blow-up sequence, let $\widehat{E}_j \subset \widehat{M}_j$ and $R_j \to \infty$ be those given by Remark 4.2. For j large enough so that $\rho < R_j$, define $X_j = (\widehat{\varphi}_j)_*(X)$; extending it by 0, we obtain a vector field X_j defined on all of $\widehat{M}_j$. Since $\partial \widehat{E}_j$ is an s-minimal surface in $\widehat{M}_j$, $\frac{d}{dt}\big|_{t=0} \mathrm{Per}_s^{\widehat{M}_j}(\psi^t_{X_j}(\widehat{E}_j); \widehat{\varphi}_j(\mathcal{B}_\rho)) = 0$. Lemma 4.4 gives then that

$$\left| \frac{d}{dt}\right|_{t=0} \mathrm{Per}_s^{\mathbb{R}^n}(\psi^t_X(F); \mathcal{B}_\rho) \Bigg| = \lim_{j\to\infty} \left| \frac{d}{dt}\right|_{t=0} \mathrm{Per}_s^{(j)}(\psi^t_X(F_j); \mathcal{B}_\rho) \Bigg| \le \lim_{j\to\infty} \frac{C_X}{R_j^s} = 0$$

as desired. □

We will next prove that the convergence in the theorem also holds in the Hausdorff distance sense. First, we show that the assumptions in Definition 4.1 imply uniform density estimates.

Lemma 4.6. *Let (M, g) be a closed manifold of dimension n, satisfying the flatness assumption* $\mathrm{FA}_3(M, g, R, p, \varphi)$*. Suppose in addition that $g^{(j)}_{kl}(0) = \delta_{kl}$, i.e. the metric of M with respect to the chart φ^{-1} at the point 0 is the Euclidean metric.*
Let E be an s-minimal surface in M, satisfying a uniform BV estimate in the sense that there is some C_0 such that

$$\mathrm{Per}(\varphi^{-1}(E); \mathcal{B}_r(x)) \le C_0 r^{n-1} \quad \textit{for all } x \in \mathcal{B}_{R/2} \textit{ and } r \in (0, R/4)\,.$$

Then there exists a positive constant $\omega_0 = \omega_0(n, s, C_0)$ such that if

$$r^{-n} |E \cap B_r(q)| \le \omega_0$$

for some $q \in \varphi(\mathcal{B}_{R/2})$ and $r \in (0, R/8)$, then

$$|E \cap B_{r/2}(q)| = 0\,.$$

Proof. Notice that since the statement is scaling-invariant, it suffices to prove it for $R = 1$. We also recall that the stationarity of E implies that it satisfies the monotonicity formula of Theorem 2.23 (with potential $F \equiv 0$). Observe also that up to modifying E on a set of measure zero we can assume that its topological boundary coincides with its essential boundary. We then proceed as follows:

Step 1. Positive density of the extended energy at every boundary point.

Since $\varphi^{-1}(E)$ is a set of finite perimeter in $\mathcal{B}_{1/2}$, De Giorgi's structure theorem for sets of finite perimeter gives that if $x \in \partial\varphi^{-1}(E) \cap \mathcal{B}_{1/2}$ is in the reduced boundary, given $r_j \to 0$ the sequence of sets $H_j = \frac{1}{r_j}(\varphi^{-1}(E) - x)$ converges in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ to a half-space H passing through 0.

For a fixed x as above, defining $M_j = M$, $E_j = E$, $p_j = \varphi(x)$, $r_j = \frac{1}{j}$, and $\varphi_j(y) = \varphi(x + A(x)y)$, where $A(x)$ is a matrix chosen so that the metric of M is the identity at 0 in the coordinates given by φ_j, the associated $F_j := \frac{1}{r_j}\varphi_j^{-1}(E) \subset \mathbb{R}^n$ are a blow-up sequence (in the sense of Definition 4.1). Thus, by Theorem 4.5 they converge in $L^1_{\rm loc}$ to a limit F. On the other hand, $F_j = A(x)^{-1}\frac{1}{r_j}(\varphi^{-1}(E) - x) = A(x)^{-1}H_j$, so that in fact $F = A(x)^{-1}H$ and thus it is also a hyperplane passing through 0.

Let N_j denote the rescaled manifold $(M, \frac{1}{r_j^2}g)$, and write $u_j = \chi_E - \chi_{E^c}$, viewed as a function on N_j. Write U_j for its Caffarelli-Silvestre extension to $N_j \times \mathbb{R}_+$, and V for the Caffarelli-Silvestre extension of $\chi_F - \chi_{F^c}$ to $\mathbb{R}^n \times \mathbb{R}_+$. By the lower semicontinuity of the extended Sobolev energy under a blow-up, seen for example arguing as in Step 2 in the proof of Lemma 4.16,

$$\liminf_{j\to\infty} \int_{\widetilde{B}_1^{N_j}(p_j,0)} |\widetilde{\nabla} U_j(p,z)|^2\, dV_p\, z^{1-s}dz \geq \int_{\widetilde{\mathcal{B}}_1} |\widetilde{D} V(x,z)|^2 dx\, z^{1-s}dz = c(n,s) > 0\,. \tag{109}$$

If U denotes the Caffarelli-Silvestre extension of $u = \chi_E - \chi_{E^c}$ (viewed as a function on M) to $M \times \mathbb{R}_+$, by scaling (recall that N_j is just $(M, \frac{1}{r_j^2}g)$) the inequality (109) can be written as

$$\liminf_{j\to\infty} \frac{1}{r_j^{n-s}} \int_{\widetilde{B}^M_{r_j}(\varphi(x),0)} |\widetilde{\nabla} U(p,z)|^2 dV_p\, z^{1-s}dz \geq c(n,s) > 0\,.$$

In words, we have found that E has extended energy density uniformly bounded from below by a constant $c(n,s)$ at $p = \varphi(x)$, for every reduced boundary point p as above. On the other hand, the reduced boundary is dense in the essential boundary, as one can see for example by the isoperimetric inequality. Then, since we have shown that the above lower bound holds at all reduced boundary points $p \in \partial E$, by the upper semicontinuity of the extended energy density (proved as in case of classical minimal surfaces, using the monotonicity formula of Theorem 2.23) it actually holds at every $p \in \partial E$.

Step 2. Conclusion.

Assume that there are $q \in \varphi(\mathcal{B}_{1/2})$ and $r \in (0, 1/8)$ such that $r^{-n}|E \cap B_r(q)| \leq \omega_0$ but $|E \cap B_{r/2}(q)| > 0$; if ω_0 is small enough, then automatically also $|E^c \cap B_{r/2}(q)| > 0$. By the isoperimetric inequality, this implies that $\partial E \cap B_{r/2}(q) \neq \emptyset$ as well.

Let $p \in \partial E \cap B_{r/2}(q)$; we can now argue as in the proof of Proposition 1.28, which showed density estimates in the case of solutions of the Allen-Cahn equation. First, a uniform lower bound as in (84) holds in our case for all $0 \leq \rho \leq R_{\rm mon}$, thanks to combining Step 1 and the monotonicity formula. We then apply the interpolation Lemma B.3, after which the BV estimate assumption allows us to conclude the argument as in the proof of Proposition 1.28. □

Proposition 4.7. *In the conclusions of Theorem 4.5, the convergence also holds locally in the Hausdorff distance sense.*

Proof. Let E_j be as in Remark 4.2, so that $F_j = \widehat{\varphi}_j^{-1}(E_j)$. Applying Lemma 4.6 and the flatness assumption on the metric we find that the F_j satisfy density estimates in $\mathcal{B}_{R_j/16}$, with $R_j \to \infty$. The local convergence in the Hausdorff distance follows then arguing by contradiction, simply due to local L^1 convergence to F and the density estimates. □

4.2 Properties of blow-ups of Allen-Cahn limits

We define the class of all surfaces which are blow-up limits of sets in $\mathcal{A}_m$ (recall Definitions 1.15 and 4.1).

Definition 4.8. A set $\partial F \subset \mathbb{R}^n$ is said to be in the class $\mathcal{A}_m^{Blow-up}$ if it is a blow-up limit of sets in $\mathcal{A}_m$. This means that there exist $\Sigma_j = \partial E_j \in \mathcal{A}_m(M_j)$ and $r_j \to 0$ such that the associated $F_j = r_j^{-1}\varphi_j^{-1}(E_j)$ are a blow-up sequence converging to F in $L^1_{\rm loc}(\mathbb{R}^n)$ as $j \to \infty$ (by Theorem 4.5 and Proposition 4.7, the convergence can then be upgraded to be in $H^{s/2}_{\rm loc}(\mathbb{R}^n)$ and locally in the Hausdorff distance sense).

Remark 4.9. Since $\partial E_j \in \mathcal{A}_m(M_j)$, the assumption in Definition 4.1 that the sets F_j satisfy the classical perimeter estimates is automatically satisfied if the rest of assumptions are, thanks to (1) in Remark 3.27.

We now prove a precise almost-stability inequality for sets in $\mathcal{A}_m^{Blow-up}$, which will be used in the next section. We begin by showing its counterpart for Allen-Cahn solutions.

Lemma 4.10. *Assume that M satisfies the flatness assumptions $\mathrm{FA}_1(M, g, 1, p, \varphi)$, and let $u_\varepsilon : M \to \mathbb{R}$ be a solution to the Allen-Cahn equation in $\varphi(\mathcal{B}_1)$ with Morse index at most m. Let $A_1, ..., A_{m+1} \subset \varphi(\mathcal{B}_{1/2})$ be $(m+1)$ open sets, with pairwise distances denoted by $D_{ij} := \mathrm{dist}(A_i, A_j)$, and for every $1 \le i < j \le m+1$ choose any positive weights $\lambda_{ij} > 0$. Then, in at least one of the A_i there holds that*

$$\mathcal{E}''(u_\varepsilon)[\xi, \xi] \ge -C\|\xi\|^2_{L^1(A_i)} \left(\sum_{j<i} \frac{1}{\lambda_{ji}} D_{ij}^{-(n+s)} + \sum_{j>i} \lambda_{ij} D_{ij}^{-(n+s)} \right) \quad \forall \xi \in C^1_c(A_i)\,,$$

for some $C = C(n, s, m)$.

Proof. The statement is a more precise version of Lemma 3.10, and the proof proceeds similarly. Using (9), we compute the second variation at u_ε for linear combinations of $m+1$ test functions ξ_i, supported each in the corresponding A_i, getting

$$\begin{aligned}
&\mathcal{E}''(u_\varepsilon)[a_1\xi_1 + a_2\xi_2 + \ldots + a_{m+1}\xi_{m+1}, a_1\xi_1 + a_2\xi_2 + \ldots + a_{m+1}\xi_{m+1}] \\
&\quad = a_1^2\mathcal{E}''(u)[\xi_1, \xi_1] + \ldots + a_{m+1}^2\mathcal{E}''(u)[\xi_{m+1}, \xi_{m+1}] \\
&\quad + 2a_1a_2 \iint_{A_1 \times A_2} (\xi_1(p) - \xi_1(q))(\xi_2(p) - \xi_2(q))K_s(p, q)\, dV_p\, dV_q \\
&\quad + \ldots \\
&\quad + 2a_m a_{m+1} \iint_{A_m \times A_{m+1}} (\xi_m(p) - \xi_m(q))(\xi_{m+1}(p) - \xi_{m+1}(q))K_s(p, q)\, dV_p\, dV_q\,.
\end{aligned}$$

Thanks to the flatness assumptions and Lemma 2.11, we have that $K_s(\varphi(x), \varphi(y)) \le \frac{C}{|x-y|^{n+s}}$, for some $C = C(n, s)$ and for $(\varphi(x), \varphi(y)) \in A_i \times A_j$. Recall that the supports of ξ_i and ξ_j are the disjoint subsets $A_i, A_j \subset \varphi_j(\mathcal{B}_{1/2})$. Then, the term containing the double integral over $A_i \times A_j$ with $i < j$ can be bounded as follows:

$$\begin{aligned}
2a_ia_j &\iint_{A_i \times A_j} (\xi_i(p) - \xi_i(q))(\xi_j(p) - \xi_j(q))K_s(p, q)\, dV_p dV_q \\
&= -2a_ia_j \iint_{A_i \times A_j} \xi_i(p)\xi_j(q)K(p, q)\, dV_p\, dV_q \\
&\le 2|a_ia_j|CD_{ij}^{-(n+s)}\|\xi_i\|_{L^1(A_i)}\|\xi_j\|_{L^1(A_j)} \\
&\le \lambda_{ij}a_i^2CD_{ij}^{-(n+s)}\|\xi_i\|^2_{L^1(A_i)} + \frac{C}{\lambda_{ij}}a_j^2D_{ij}^{-(n+s)}\|\xi_j\|^2_{L^1(A_j)}\,,
\end{aligned}$$

where we have applied Young's inequality in the last line. Substituting this into the second variation expression gives

$$\begin{aligned}
&\mathcal{E}''(u)[a_1\xi_1 + a_2\xi_2 + \ldots + a_{m+1}\xi_{m+1}, a_1\xi_1 + a_2\xi_2 + \ldots + a_{m+1}\xi_{m+1}] \\
&\quad \le \sum_{i=1}^{m+1} a_i^2 \left[\mathcal{E}''(u)[\xi_i, \xi_i] + C\|\xi_i\|^2_{L^1(A_i)} \left(\sum_{j<i} \frac{1}{\lambda_{ji}} D_{ij}^{-(n+s)} + \sum_{j>i} \lambda_{ij} D_{ij}^{-(n+s)} \right) \right].
\end{aligned}$$

The condition that the Morse index is at most m implies that the expression cannot be < 0 for all $(a_1, \ldots, a_{m+1}) \ne 0$. Hence, we find that there must exists some i such that

$$\mathcal{E}''(u)[\xi_i, \xi_i] \ge -C\|\xi_i\|^2_{L^1(A_i)} \left(\sum_{j<i} \frac{1}{\lambda_{ji}} D_{ij}^{-(n+s)} + \sum_{j>i} \lambda_{ij} D_{ij}^{-(n+s)} \right)$$

holds for all $\xi_i \in C^1_c(A_i)$, and this concludes the proof. ∎

From this we will obtain the desired almost-stability inequality for blow-up sets.

Lemma 4.11. *Let $F \in \mathcal{A}_m^{Blow-up}$. Let $X_1, X_2, ..., X_{m+1}$ be smooth vector fields on $\mathbb{R}^n$ with disjoint compact supports $A_1, A_2, ..., A_{m+1}$, and denote $D_{k\ell} := \mathrm{dist}(A_k, A_\ell)$. For $1 \le i < \ell \le m+1$, choose positive weights $\lambda_{i\ell} > 0$. Then, for at least one of the i (depending on F) we have that*

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}_s(\psi^t_{X_i}(F); A_i) \ge -C\|X_i\|^2_{L^\infty} \mathrm{diam}(A_i)^{2(n-1)} \left(\sum_{\ell<i} \frac{1}{\lambda_{\ell i}} D_{i\ell}^{-(n+s)} + \sum_{\ell>i} \lambda_{i\ell} D_{i\ell}^{-(n+s)} \right), \tag{110}$$

where $C = C(n, s, m)$.

Proof. Since $F \in \mathcal{A}_m^{Blow-up}$, from the definition and proceeding as in Remark 4.2 there exist $(\widehat{M}_j, \widehat{g}^{(j)})$, $\widehat{p}_j \in \widehat{M}_j$, and $R_j \nearrow \infty$ satisfying the assumptions in the Remark and $\widehat{E}_j \in \mathcal{A}_m(\widehat{M}_j)$ such that the associated $F_j = \widehat{\varphi}_j^{-1}(\widehat{E}_j)$ converge to F in the appropriate sense. Fix one such j; since $\widehat{E}_j \in \mathcal{A}_m(\widehat{M}_j)$, by definition there exists a sequence $\{u_k\}_{k\in\mathbb{N}}$ of solutions to Allen-Cahn on $\widehat{M}_j$, with parameters $\varepsilon_k \to 0$, converging to $\chi_{\widehat{E}_j} - \chi_{\widehat{E}_j^c}$ in $L^1(\widehat{M}_j)$ as $k \to \infty$. By Lemma 4.10, given k we can find an index $i(k)$, $1 \le i(k) \le m+1$, such that the inequality in the Lemma is true for u_k on $\widehat{\varphi}_j(A_{i(k)}) \subset \widehat{M}_j$. We select an index i so that the inequality is valid for a whole subsequence of the u_k (which we do not relabel), so that

$$\mathcal{E}''(u_k)[\xi_i, \xi_i] \ge -C\|\xi_i\|^2_{L^1(\widehat{\varphi}_j(A_i))} \left(\sum_{\ell<i} \frac{1}{\lambda_{\ell i}} \widehat{D}_{i\ell}^{-(n+s)} + \sum_{\ell>i} \lambda_{i\ell} \widehat{D}_{i\ell}^{-(n+s)} \right) \tag{111}$$

for all $\xi_i \in C^1_c(\widehat{\varphi}_j(A_i))$ and $k \in \mathbb{N}$. Here $\widehat{D}_{i\ell} = \mathrm{dist}(\widehat{\varphi}_j(A_i), \widehat{\varphi}_j(A_\ell))$.
Put $X_{i,j} := (\widehat{\varphi}_j)_* X_i$, and extend it by zero outside its domain of definition to a vector field on all of $\widehat{M}_j$. Selecting $\xi_i = \nabla_{X_{i,j}} u_k$, we arrive at

$$\begin{aligned} \frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{E}(u_k \circ \psi^{-t}_{X_{i,j}}) &= \mathcal{E}''(u_k)[\nabla_{X_{i,j}} u_k, \nabla_{X_{i,j}} u_k] \\ &\ge -C\|\nabla_{X_{i,j}} u_k\|^2_{L^1(\widehat{\varphi}_j(A_i))} \left(\sum_{\ell<i} \frac{1}{\lambda_{\ell i}} \widehat{D}_{i\ell}^{-(n+s)} + \sum_{\ell>i} \lambda_{i\ell} \widehat{D}_{i\ell}^{-(n+s)} \right). \end{aligned} \tag{112}$$

Thanks to the BV estimate of Theorem 3.13 and the flatness assumption on the metric, we can bound

$$\|\nabla_{X_{i,j}} u_k\|^2_{L^1(\widehat{\varphi}_j(A_i))} \le C\|X_{i,j}\|^2_{L^\infty} \mathrm{diam}(\widehat{\varphi}_j(A_i))^{2(n-1)} \le C\|X_i\|^2_{L^\infty} \mathrm{diam}(A_i)^{2(n-1)},$$

and also

$$\widehat{D}_{i,l} = \mathrm{dist}(\widehat{\varphi}_j(A_i), \widehat{\varphi}_j(A_l)) \le C \mathrm{dist}(A_i, A_l) = C D_{i,l}\,.$$

Substituting this into (112), and using that by (91) there holds

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{E}(u_k \circ \psi^{-t}_{X_{i,j}}) \to \frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}_s^{\widehat{M}_j}(\psi^t_{X_{i,j}}(\widehat{E}_j); \widehat{\varphi}_j(A_i)) \quad \text{as } k \to \infty,$$

we obtain that

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}_s^{\widehat{M}_j}(\psi^t_{X_{i,j}}(\widehat{E}_j); \widehat{\varphi}_j(A_i)) \ge -C\|X_i\|^2_{L^\infty} \mathrm{diam}(A_i)^{2(n-1)} \left(\sum_{\ell<i} \frac{1}{\lambda_{\ell i}} D_{i\ell}^{-(n+s)} + \sum_{\ell>i} \lambda_{i\ell} D_{i\ell}^{-(n+s)} \right).$$

On the other hand, by Lemma 4.4 we have that

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}_s^{\mathbb{R}^n}(\psi^t_{X_i}(F); A_i) = \lim_j \frac{d^2}{dt^2}\bigg|_{t=0} \left[\mathrm{Per}_s^{(j)}(\psi^t_{X_i}(F_j); A_i) \right] = \lim_j \frac{d^2}{dt^2}\bigg|_{t=0} \left[\mathrm{Per}_s^{\widehat{M}_j}(\psi^t_{X_{i,j}}(\widehat{E}_j); \widehat{\varphi}_j(A_i)) \right],$$

which then proves (110). □

4.3 Classification of blow-up limits

The main result of this section is the following classification result:

Theorem 4.12 (Classification result). *Let $s \in (0,1)$ and $3 \le n < n_s^*$. Let $\mathcal{F}$ be any family of sets of $\mathbb{R}^n$ satisfying the following properties:*

*(1) **Stationarity.** Every set $E \in \mathcal{F}$ is an s-minimal surface, in the sense of Definition 1.5.*

*(2) **BV estimate.** There is $C_\circ$ such that for every $E \in \mathcal{F}$, $x \in \mathbb{R}^n$, and $R > 0$ we have*

$$\mathrm{Per}(E; B_R(x)) \le C_\circ R^{n-1}\,.$$

*(3) **Viscosity solution of the NMS[18] equation.** If $x_0 \in \partial E$ and E admits an interior (resp. exterior) tangent ball at x_0, then $\int \frac{\chi_E(y) - \chi_{E^c}(y)}{|x_0-y|^{n+s}} dy \le 0$ (resp. ≥ 0).*

*(4) **Almost-stability in one out of $(m+1)$ disjoint sets.** There exists some (fixed) $m \in \mathbb{N}$ such that the following holds. Let $X_1, X_2, ..., X_{m+1}$ be smooth vector fields with disjoint compact supports $A_1, A_2, ..., A_{m+1}$, and denote $D_{kl} := \mathrm{dist}(A_k, A_l)$. For $1 \le i < l \le m+1$, choose positive weights $\lambda_{il} > 0$. Then, given $E \in \mathcal{F}$, for at least one of the i (depending on E) we have that*

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Per}_s(\psi^t_{X_i}(E); A_i) \ge -C\|X_i\|^2_{L^\infty} \mathrm{diam}(A_i)^{2(n-1)} \Big(\sum_{\ell<i} \frac{1}{\lambda_{\ell i}} D_{i\ell}^{-(n+s)} + \sum_{\ell > i} \lambda_{i\ell} D_{i\ell}^{-(n+s)}\Big)\,, \tag{113}$$

where $C = C(\mathcal{F})$.

*(5) **Completeness under scalings and $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ limits**. If $E \in \mathcal{F}$, then any translation, dilation and rotation of E is in $\mathcal{F}$ as well. Moreover, if E_i is a sequence of elements of $\mathcal{F}$ and $E_i \to E_\infty$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then $E_\infty \in \mathcal{F}$ as well.*

*(6) **Cones with $n-2$ translation-invariant directions are half-spaces.** If $E \in \mathcal{F}$ is a cone and there is a linear $(n-2)$-dimensional subspace $L \subset \mathbb{R}^n$ such that $E + x = E$ for all $x \in L$, then ∂E must be a hyperplane.*

Then, every $E \in \mathcal{F}$ which is not equal (up to null sets) to $\mathbb{R}^n$ or $\varnothing$ must be a half-space.

An important property follows from (1) and (2) above:

Lemma 4.13. *Let $\mathcal{F}$ be a family of sets of $\mathbb{R}^n$ satisfying properties (1) and (2) in Theorem 4.12. Then, any set $E \in \mathcal{F}$ also satisfies density estimates, meaning that there exists a positive constant $\omega_0 = \omega_0(n, s, C_0)$ such that if*

$$R^{-n}|E \cap B_R(q)| \le \omega_0$$

for some $q \in \varphi(\mathcal{B}_{1/2})$ and $R \in (0, 1/8)$, then

$$|E \cap B_{R/2}(q)| = 0\,.$$

Moreover, if $E_i \in \mathcal{F}$ and $E_i \to E_\infty$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then they also converge to E_∞ locally in the Hausdorff distance sense.

Proof. Same as for Lemma 4.6 and Proposition 4.7. ☐

We will need the following result, which is obtained by combining the $C^{1,\alpha}$ improvement of flatness theorem in [25] and the $C^{1,\alpha}$-to-C^∞ bootstrap result for nonlocal minimal graphs in [10].

Theorem 4.14 ([10, 25]). *Let $s \in (0,1)$. Then, there exists $\sigma > 0$, depending on n and s, such that the following holds: let $E \subset \mathbb{R}^n$ and $x \in \partial E$, and assume that*

(i) The set E is a viscosity solution of the NMS equation in $\mathcal{B}_r(x)$, in the sense of Proposition 3.26.

[18] Meaning "nonlocal minimal surface".

(ii) *The boundary ∂E is included in a σ-flat cylinder in $\mathcal{B}_r(x)$, that is*

$$\partial E \cap \mathcal{B}_r(x) \subset \{y \in \mathbb{R}^n : |e \cdot (y - x)| \le \sigma r\},$$

for some direction $e \in \mathbb{S}^{n-1}$.

Then ∂E is a C^∞ graph in the direction e in $\mathcal{B}_{r/2}(x)$, with uniform estimates. In particular, its second fundamental form $\mathrm{II}_{\partial E}$ satisfies

$$\sup_{y \in \partial E \cap \mathcal{B}_{r/2}(x)} |\mathrm{II}_{\partial E}|(y) \le \frac{C}{r}, \tag{114}$$

with $C = C(n, s)$.

We will also need the following intuitive lemma, to be read as "cones with finite Morse index are stable outside the origin", and which will be proved after Theorem 4.12.

Lemma 4.15. *Let $E \subset \mathbb{R}^n$ be a cone with $\mathrm{Per}_s(E; \mathcal{B}_1(0)) < +\infty$. Assume that E is stationary for the s-perimeter, in the sense of Definition 1.5, and that it satisfies property (4) in the statement of Theorem 4.12. Then E is stable in $\mathbb{R}^n \setminus \{0\}$.*

We will now prove Theorem 4.12.

Proof of Theorem 4.12. Let E_∞ be a blow-down limit of E, i.e. a limit of a sequence $E_i = \frac{1}{r_i}E$, with $r_i \to \infty$ (by property (5), such a limit exists and it is also a member of $\mathcal{F}$). Then E_∞ is a cone: Let U, U_∞ and U_i denote the Caffarelli-Silvestre extensions of $u := \chi_E - \chi_{E^c}$, $u_\infty := \chi_{E_\infty} - \chi_{E^c_\infty}$ and $u_i := \chi_{E_i} - \chi_{E_i^c}$, respectively. Using the notation $\Phi_V(r) := r^{s-n} \int_{\tilde{B}_r^+(0,0)} z^{1-s} |\nabla V(x,z)|^2 dx\, dz$, by convergence of the extended energies[19] and scaling we have that

$$\Phi_{U_\infty}(r) = \lim_i \Phi_{U_i}(r) = \lim_i \Phi_U(r r_i)\,.$$

By the monotonicity of Φ_E, which we know since E is an s-minimal surface by property (1) and thus satisfies Theorem 2.23, the limit $\lim_{R\to\infty} \Phi_U(R)$ exists, and by property (2) and the interpolation result in Lemma 134 it is a finite constant. The equality above then shows that $\Phi_{U_\infty}(r)$ is equal to this constant independently of r. Since E_∞ is also an s-minimal surface (by properties (1) and (5)), the last paragraph in Theorem 2.23 gives that E_∞ is a cone.

We will now prove that E_∞ is in fact a hyperplane; by the local Hausdorff convergence of the $E_i = \frac{1}{r_i}E$ to E_∞ (see Lemma 4.13 above), E then satisfies the hypotheses of Theorem 4.14 for every $r > 0$, and therefore by (114) E then needs to be a hyperplane as well (since $\mathrm{II}_{\partial E}$ vanishes).

First, Lemma 4.15 states that E_∞ is stable outside the origin. If it also were smooth outside the origin, the assumption that $3 \le n < n_s^*$ would imply that ∂E_∞ is a hyperplane and we would finish the proof.
If, arguing by contradiction, there is instead some point $x_1 \neq 0$ where E_∞ is not smooth, we need to apply a dimension reduction argument: blowing-up around x_1, we obtain a new cone $E_1 \in \mathcal{F}$ which is now translation invariant along some direction; after a rotation, we can write $E_1 = \widetilde{E}_1 \times \mathbb{R}$, and this is allowed by property (5).

We claim that E_1 cannot be smooth outside the origin. First, if that were the case, E_1 would be a hyperplane, since $3 \le n < n_s^*$. Now, the blow-up rescalings of E_∞ around x_1 converge locally in the Hausdorff distance sense to E_1 by Lemma 4.13, and they are viscosity solutions of the NMS equation by property (3). If E_1 were indeed a hyperplane, the assumption in the improvement of flatness Theorem 4.14 would be satisfied for the blow-up rescalings of E_∞ around x_1 (for large enough indices in the sequence), thus E_∞ would be smooth in a neighborhood of x_1, contradiction.

Now that we know that E_1 is not smooth outside the origin, we can iterate the argument with this new cone: Since $E_1 = \widetilde{E}_1 \times \mathbb{R}$ is not smooth outside the origin, there is some point $x_2 \in \mathbb{R}^{n-1} \setminus \{0\}$ where $\widetilde{E}_1$ is not smooth. Hence, we can blow-up again around $(x_2, 0)$ and obtain a new cone E_2 which is now

[19] Follows easily from convergence in $H^{s/2}_{\mathrm{loc}}(\mathbb{R}^n)$. The latter is proved, thanks to property (2), as in Step 1 in the proof of Theorem 4.5.

translation invariant with respect to two orthogonal directions. After a rotation, $E_2 = \widetilde{E}_2 \times \mathbb{R}^2$. Moreover, E_2 cannot be smooth outside the origin, by the same improvement of flatness argument we applied to E_1. Iterating this reasoning $n-2$ times, we end up with a cone which is translation invariant in $n-2$ orthogonal directions, i.e. of the form $\widetilde{E} \times \mathbb{R}^{n-2}$ after a rotation, and which is not smooth outside the origin. This is not possible by property (6), and therefore we reach a contradiction. □

We now give the proof of Lemma 4.15.

Proof of Lemma 4.15. Consider an annular region of the form $A_0 = B_1 \setminus B_{R_0}$ with $0 < R_0 < 1$, centered at the origin. It suffices to show that E is stable in A_0, by the arbitrariness of R_0 and the dilation invariance of E.

The strategy is the following. Let X be a vector field supported on the annulus A_0. Let $A_1, ..., A_{m+1}$ be $(m+1)$ rescaled copies of A_0 of the form $A_i = RA_{i-1} = R^i A_0$, with $R > 0$ sufficiently large so that they are disjoint. Likewise, consider the $(m+1)$ rescaled vector fields $X_i := R^i X(x/R^i)$, which are supported in the respective A_i. Since E satisfies property (4) in the statement of Theorem 4.12, we know that the almost-stability inequality (113) will hold in at least one of the A_i. Moreover, since E is dilation invariant, we will be able to translate this information back into A_0, and taking R arbitrarily large we will find that E is actually stable on A_0 and conclude the proof.

Define $u := \chi_E - \chi_{E^c}$, and let ψ_X^t denote the flow of the vector field X at time t. Observe that $u_i^t := \chi_{\psi_{X_i}^t(E)} - \chi_{\psi_{X_i}^t(E)^c}$ is the composition of $u = \chi_E - \chi_{E^c}$ with the flow of $X_i = R^i X(x/R^i)$, which is given by $\psi_{X_i}^t = R^i \psi_X^t(x/R^i)$. By the dilation-invariance of the cone E, and hence of u_∞, we have that

$$u_i^t(x) = u(R^i \psi_X^t(x/R^i)) = u(\psi_X^t(x/R^i)) = u^t(x/R^i);$$

the scaling property of the fractional Sobolev energy then gives

$$\mathrm{Per}_s(\psi_{X_i}^t(E); A_i) = \mathcal{E}_{A_i}^{\mathrm{Sob}}(u_i^t) = \mathcal{E}_{A_i}^{\mathrm{Sob}}(u^t(x/R^i)) = R^{i(n-s)} \mathcal{E}_{A_0}^{\mathrm{Sob}}(u^t) = R^{i(n-s)} \mathrm{Per}_s(\psi_X^t(E); A_0)\,,$$

so that in particular

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}_s(\psi_{X_i}^t(E); A_i) = R^{i(n-s)} \frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}_s(\psi_X^t(E); A_0)\,. \tag{115}$$

Now, by assumption we know that the almost-stability inequality (113) will be satisfied in one of the A_i. Combined with (115), we obtain that

$$R^{i(n-s)} \frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}_s(\psi_X^t(E); A_0) \geq -C \|X_i\|_{L^\infty}^2 \mathrm{diam}(A_i)^2 \Big(\sum_{\ell<i} \frac{1}{\lambda_{\ell i}} D_{i\ell}^{-(n+s)} + \sum_{\ell>i} \lambda_{i\ell} D_{i\ell}^{-(n+s)}\Big)\,,$$

where λ_{ij} are positive weights and $D_{ij} = \mathrm{dist}(A_i, A_j)$. We can bound

$$\|X_i\|_{L^\infty}^2 \mathrm{diam}(A_i)^{2(n-1)} = \|X\|_{L^\infty}^2 (R^i)^2 (R^i \mathrm{diam}(A_0))^{2(n-1)} \leq C_{X,A_0} R^{2ni}\,.$$

We also observe that

$$D_{i,l} = \mathrm{dist}(R^i A_0, R^l A_0) \geq c R^{\max\{i,l\}}$$

for some small c, for all R sufficiently large depending on A_0.

Substituting into the inequality we obtained and dividing both sides by $R^{i(n-s)}$, we get

$$\begin{aligned}
\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}_s(\psi_X^t(E); A_0) &\geq -C_{X,A_0} R^{i(n+s)} \Big(\sum_{\ell<i} \frac{1}{\lambda_{\ell i}} R^{-i(n+s)} + \sum_{\ell>i} \lambda_{i\ell} R^{-\ell(n+s)}\Big) \\
&= -C_{X,A_0} \Big(\sum_{\ell<i} \frac{1}{\lambda_{\ell i}} + \sum_{\ell>i} \lambda_{i\ell} R^{-(\ell-i)(n+s)}\Big)\,.
\end{aligned}$$

Now, choosing the positive weights as $\lambda_{ij} = R^{\frac{n+s}{2}}$ for every pair $i < j$, we obtain

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Per}_s(\psi_X^t(E); A_0) \geq -C_{X,A_0}\Big(\sum_{\ell<i} R^{-\frac{n+s}{2}} + \sum_{\ell>i} R^{-(\ell-i-\frac{1}{2})(n+s)}\Big)\,,$$

so that all the powers of R become strictly negative. Letting $R \to \infty$, we deduce that

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Per}_s(\psi_X^t(E); A_0) \geq 0$$

as desired. □

We will in particular apply Theorem 4.12 to the class $\mathcal{A}_m^{Blow-up}$. The next lemma proves that property (6) in the assumptions holds for this class.

Lemma 4.16. *Let $n \geq 3$. Assume that some nontrivial cone $E \subset \mathbb{R}^n$ belongs to $\mathcal{A}_m^{Blow-up}$ and is of the form $\widetilde{E} \times \mathbb{R}^{n-2}$ for some cone $\widetilde{E} \subset \mathbb{R}^2$. Then, ∂E is a hyperplane.*

Proof. We divide the proof into two steps.

Step 1. Let us show the following claim: assume that $\mathrm{FA}_1(M, g, 1, p, \varphi)$ holds, and $u : M \to (-1, 1)$ is a solution of Allen-Cahn with parameter $\varepsilon \in (0, 1)$ in $B_1(p)$ (equivalently a critical point of $\mathcal{E}_{B_1(p)}$) that is Λ-almost stable in $B_1(p)$ (see Definition 3.9). Let $U : M \times \mathbb{R}_+ \to (-1, 1)$ be the Caffarelli-Silvestre extension of u.

Then, for some constant $C = C(n, s, \Lambda) > 0$ we have:

$$\int_{\widetilde{B}^+_{1/2}(p,0)} \mathcal{A}^2(U)\, dV\, z^{1-s} dz \leq C\,,$$

where

$$\mathcal{A}^2(U) := (|\nabla^2 U|^2 - |\nabla|\nabla U||^2)\chi_{\{|\nabla u|>0\}} = \Big(|\nabla^2 U|^2 - \nabla^2 U\Big(\tfrac{\nabla U}{|\nabla U|}, \tfrac{\nabla U}{|\nabla U|}\Big)\Big)\chi_{\{|\nabla u|>0\}} \geq 0\,.$$

Here $\nabla^2 U$ denotes the "horizontal" Hessian of $U(\cdot, z)$ —i.e. for z fixed— with respect to g.

Indeed since u is Λ-almost stable, for all $\xi \in C^1(\widetilde{B}_1^+(p, 0))$ with support contained in $\overline{\widetilde{B}^+_{3/4}(p, 0)}$ and trace ξ_0 on $z = 0$, we have :

$$\begin{aligned}\widetilde{\mathcal{E}}_1''(U)[\xi, \xi] &= \beta_s \int_{\widetilde{B}_1^+} z^{1-s}|\nabla\xi|^2\, dVdz + \varepsilon^{-s}\int_{B_1} W''(u_\varepsilon)\xi_0^2\, dV\\ &\geq \mathcal{E}''_{B_1}(u)[\xi_0, \xi_0] \geq -\Lambda\|\xi_0\|^2_{L^1(B_1)}\,,\end{aligned}$$

where $\widetilde{B}_1^+$ and B_1 are brief notations for $\widetilde{B}_1^+(p, 0)$ and $B_1(p)$.

Thus, testing the above almost stability inequality with a test function that is product $\xi = c\eta$ we obtain (with a simple integration by parts similar to [21, Proof of Theorem 1.3]):

$$\begin{aligned}&\int_{B_1} c\big(\beta_s(y^{1-s}\partial_y)c(\,\cdot\,, 0^+) - \varepsilon^{-s}W''(u)\big)\eta^2 dV\\ &\qquad\leq \int_{\widetilde{B}_1^+} (c^2 z^{1-s}|\widetilde{\nabla}\eta|^2 - c\widetilde{\mathrm{div}}(z^{1-s}\widetilde{\nabla}c)\eta^2)dVdz + \Lambda\Big(\int_{B_1}|c\eta|dV\Big)^2.\end{aligned} \tag{116}$$

Taking the horizontal gradient ∇ of $\beta_s(y^{1-s}\partial_y)U(\,\cdot\,, 0^+) - \varepsilon^{-s}W'(u) = 0$ on $y = 0$, and computing the scalar product with ∇u, we obtain

$$\beta_s(y^{1-s}\partial_y)|_{z=0^+}(\nabla U)\cdot\nabla u - \varepsilon^{-s}W''(u)|\nabla u|^2 = 0\,.$$

Using $\beta_s(z^{1-s}\partial_z)|_{z=0^+}(\nabla U)\cdot\nabla u = \frac{\beta_s}{2}(z^{1-s}\partial_z)|_{z=0^+}|\nabla U|^2 = \frac{\beta_s}{2}|\nabla U|(\beta_s z^{1-s}\partial_z)|_{z=0^+}|\nabla U|$ we obtain that $c = |\nabla U|$ makes the left hand side of (116) vanish. Hence, for this choice of c we obtain

$$0 \le \int_{\widetilde{B}_1^+} (c^2 z^{1-s}|\widetilde{\nabla}\eta|^2 - c\widetilde{\mathrm{div}}(z^{1-s}\widetilde{\nabla}c)\eta^2)dVd + \Lambda\Big(\int_{B_1}|c\eta|dV\Big)^2. \tag{117}$$

Notice that

$$c\widetilde{\mathrm{div}}(z^{1-s}\widetilde{\nabla}c) = z^{1-s}c\Delta c + c\partial_z(z^{1-s}\partial_z c) = \big(\tfrac{1}{2}\Delta(c^2) - |\nabla c|^2\big)z^{1-s} + c\partial_z(z^{1-s}\partial_z c).$$

Now, since $c = |\nabla U|$, the Bochner identity —applied to each "horizontal slice" $M\times\{z\}$ of $\widetilde{M}$— yields

$$\tfrac{1}{2}\Delta(c^2) = \nabla U\cdot\nabla(\Delta U) + |\nabla^2 U|^2 + \mathrm{Ric}(\nabla u,\nabla u).$$

Since, by the equation defining the extension, $z^{1-s}\Delta U = -\partial_z(z^{1-s}\partial_z U)$, we obtain

$$z^{1-s}\nabla U\cdot\nabla(\Delta U) = -\nabla U\cdot\partial_z(z^{1-s}\partial_z\nabla U).$$

But explicit computation shows that

$$\begin{aligned}|\nabla U|\partial_z(z^{1-s}\partial_z|\nabla U|) &= |\nabla U|\partial_z\left(\frac{z^{1-s}\partial_z(\frac{1}{2}|\nabla U|^2)}{|\nabla U|}\right)\\ &= \nabla U\partial_z(z^{1-s}\cdot\partial_z\nabla U)) + z^{1-s}\Big(|\partial_z\nabla U|^2 - \big(\partial_z|\nabla U|\big)^2\Big)\\ &\ge \nabla U\cdot\partial_z(z^{1-s}\partial_z\nabla U).\end{aligned}$$

Hence, estimating $\mathrm{Ric}(\nabla U,\nabla U)\ge -C|\nabla U|^2$, we deduce that

$$c\widetilde{\mathrm{div}}(z^{1-s}\widetilde{\nabla}c) \ge z^{1-s}\big(|\nabla^2 U|^2 - |\nabla|\nabla U||^2 - C|\nabla U|^2\big)\chi_{\{|\nabla u|>0\}}.$$

Inserting this in (117), we reach

$$\int_{\widetilde{B}_1^+} z^{1-s}\mathcal{A}^2(u)\eta^2 dVdz \le \int_{\widetilde{B}_1^+}|\nabla U|^2 z^{1-s}(|\widetilde{\nabla}\eta|^2 + C\eta^2)dVdz + \Lambda\Big(\int_{B_1}|\nabla u||\eta|dV\Big)^2.$$

From this we conclude the claim in Step 1, fixing a cuttoff satisfying $\chi_{\widetilde{B}^+_{1/2}} \le \eta \le \chi_{\widetilde{B}^+_{3/4}}$ and using the estimates for $\int_{\widetilde{B}^+_{3/4}} z^{1-s}|\nabla U|^2 dVdz$ and $\int_{B_{3/4}}|\nabla u|dV$ proved in Section 3.2.3. In particular, Lemma B.3 with $R=1, k=0$ and the fact that Λ-almost stability implies a uniform BV estimates, as recorded in Proposition 3.13.

Step 2. Recall now that, as recorded in Remark 4.2, if E belongs to $\mathcal{A}_m^{Blow-up}$ we have sequences of:

- closed manifolds (M_j, g^{M_j});
- points $p_j\in M_j$ and scales $R_j\uparrow\infty$ for which $\mathrm{FA}_3(M_j, g^{M_j}, R_j, p_j, \varphi_j)$ holds and $g^{M_j}_{k\ell}(0) = \delta_{k\ell}$.
- solutions of Allen-Cahn $u_j : M_j \to (-1,1)$ with parameters $\varepsilon_j\downarrow 0$ and Morse index bounded by m such that $(u_j\circ\varphi_j)\to u_\circ := \chi_E - \chi_{E^c}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Let $U_j : \widetilde{M}_j \to (-1,1)$ be the extensions of the u_j and and observe that $U_j \rightharpoonup U_\circ$ in weakly in $L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$, where $U_\circ$ is the (unique, bounded) Caffarelli-Silvestre extension of $u_\circ$ to $\mathbb{R}^{n+1}_+$. Actually, thanks to Theorem 4.5, one could prove local strong convergence in the weighted Hilbert space $H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+; |z|^{1-s}dxdz)$, although (much rougher) weak L^1_{loc} will suffice here.

Notice also that in the local coordinates φ_j^{-1} we will have $g^{M_j}_{k\ell}\to\delta_{k\ell}$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$), since $R_j\to\infty$. Hence by standard elliptic estimates $U_j\circ\widetilde{\varphi}_j \to U_\circ$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n\times(0,+\infty))$ (up to subsequence), where $\widetilde{\varphi}_j(x,z) = (\varphi_j(x), z)$.

Now, for all $j \gg 1$ sufficiently large, take the $m+1$ balls $\{B_1(\varphi_j(3ie_3))\}_{i\le m}$, $i = 0, 1, \ldots, m$. By property (4) —i.e. almost stability in one out of $(m+1)$ disjoint sets— U_j will be almost stable in one of them. We may assume without loss of generality (up to translation and subsequence) that it actually is $B_1(\varphi_1(0))$, and then Step 1 gives

$$\int_{\varphi_j(\mathcal{B}^+_{1/2})} \mathcal{A}^2(U_j)dV_j\, z^{1-s}dz \le C\,.$$

After passing to the limit (using that $U_j \circ \varphi_j \to U_\circ$ in $C^2_{\rm loc}$) we obtain, for every $\delta > 0$:

$$\int_{\mathcal{B}^+_{1/2-\delta}\cap\{z>\delta\}} \left(|D^2U_\circ|^2 - D^2U_\circ\left(\tfrac{DU_\circ}{|DU_\circ|}, \tfrac{DU_\circ}{|DU_\circ|}\right)\right)\chi_{\{|DU_\circ|>\delta\}}\, dx\, z^{1-s}dz \le C.$$

On the other hand, since E is a nontrivial cone, it is easy to show (e.g. by a simple scaling reasoning using that $U_\circ$ is 0-homogeneous while the integrand is not identically zero since E is nontrivial) that

$$\lim_{\delta\downarrow 0} \int_{\mathcal{B}^+_{1/2-\delta}\cap\{z>\delta\}} \left(|D^2U_\circ|^2 - D^2U_\circ\left(\tfrac{DU_\circ}{|DU_\circ|}, \tfrac{DU_\circ}{|DU_\circ|}\right)\right)\chi_{\{|DU_\circ|>\delta\}}\, dx\, z^{1-s}dz \to \infty\,,$$

and this is a contradiction. □

The properties proved so far for the family $\mathcal{F} = \mathcal{A}_m^{Blow-up}$ (see Definition 4.8) show that it satisfies the hypotheses of Theorem 4.12, whence we deduce

Corollary 4.17 (**Blow-ups of limit surfaces of Allen-Cahn are hyperplanes**). *Let $s \in (0,1)$ and $3 \le n < n_s^*$. Then, any nonempty ∂F in $\mathcal{A}_m^{Blow-up}$ is a hyperplane.*

Proof. It follows from applying Theorem 4.12 to the class $\mathcal{F} = \mathcal{A}_m^{Blow-up}$. Properties (1) and (5) in the assumptions of Theorem 4.12 follow immediately from (the proof of) Theorem 4.5. Property (2) follows from (1) in Remark 3.27 and the lower semicontinuity of the BV seminorm under L^1-convergence. Properties (4) and (6) have been proved, respectively, in Lemma 4.11 and Lemma 4.16. Finally, property (3) —namely that blow-ups are viscosity solutions of the NMS equation in $\mathbb{R}^n$— follows easily from Proposition 3.26 and the convergence of boundaries in Hausdorff distance under a blow-up (Proposition 4.7), using the convergence of the kernels in Proposition 4.3 part *(ii)*. See [25] and [27]. □

4.4 Uniform regularity and separation in low dimensions – Proof of Theorem 1.16

In this section we will prove Theorem 1.16, which stated that sets in $\mathcal{A}_m(M)$, i.e. the limits of Allen-Cahn solutions on M with index at most m, are smooth with *uniform regularity* and *separation* estimates in low dimensions.

We will need the following improvement of flatness theorem for sets which are viscosity solutions of the NMS equation in a Riemannian manifold, proved in [81] more generally assuming boundedness of the nonlocal mean curvature, and which extends the result in [25] to the setting of ambient Riemannian manifolds.

Theorem 4.18 ([81]). *Let $s \in (0,1)$ and $0 < \alpha < s$. Then, there exists $\sigma > 0$, depending on n,s and α, such that the following holds. Let (M, g) be an n-dimensional Riemannian manifold. Take $p \in M$, and assume that the flatness assumption* $\mathrm{FA}_1(M, g, r, p, \varphi)$ *holds, with $r \le 1$. Let $E \subset M$ with $p \in \partial E$, and assume that*

(i) The set E is a viscosity solution of the NMS equation in $\varphi(B_r(0))$, in the sense of Proposition 3.26.

(ii) The boundary $\varphi^{-1}(\partial E)$ is included in a σ-flat cylinder in $\mathcal{B}_{r/2}(0)$, that is

$$\varphi^{-1}(\partial E) \cap \mathcal{B}_r(0) \subset \{|e\cdot x| \le \sigma r\},$$

for some direction $e \in \mathbb{S}^{n-1}$.

Then $\varphi^{-1}(\partial E)$ is a single $C^{1,\alpha}$ graph in the direction e in $\mathcal{B}_{r/2}(0)$, with uniform estimates.

Proof of Theorem 1.16. We will first show that E is trapped (in the coordinates given by φ^{-1}) in a very flat cylinder, as recorded in the next claim:

Claim: Let $\sigma > 0$. Then there exists a uniform constant $R_\sigma = R_\sigma(m, s, \sigma)$ and a unit vector $e \in \mathbb{S}^{n-1}$ such that

$$-\sigma R_\sigma \le y \cdot e \le \sigma R_\sigma \quad \text{for all } y \in \varphi^{-1}(\partial E) \cap \mathcal{B}_{R_\sigma}. \tag{118}$$

Proof of the Claim: Fix $\sigma > 0$; the proof will be by contradiction and blow-up. Let $R_j = 1/j$. If the Claim were false, then for every $j \in \mathbb{N}$ there would exist closed manifolds M_j satisfying the flatness assumptions $\mathrm{FA}_3(M_j, g, 1, p_j, \varphi_j)$, and some sets $E_j \in \mathcal{A}_m(M_j)$ so that $p_j \in \partial E_j$ but such that (118) is not satisfied for any unit vector (with E_j, R_j and φ_j in place of E, R_σ and φ).

Consider, then, the blow-up sequence $F_j = \frac{1}{R_j}\varphi_j^{-1}(E_j)$. By Proposition 4.7, a subsequence of the F_j converges (in particular) locally in the Hausdorff distance sense to a limit set $F \in \mathcal{A}_m^{Blow-up}$. Moreover, since $0 \in F_j$, we see that $0 \in F$ as well.

Now, from the classification result of Corollary 4.17, we know that ∂F is in fact a hyperplane passing through the origin. The local Hausdorff convergence of the $\partial F_j = \frac{1}{R_j}\varphi_j^{-1}(\partial E_j)$ to the hyperplane ∂F implies then that the condition

$$-\sigma \le y \cdot e \le \sigma \quad \text{for all } y \in \tfrac{1}{R_j}\varphi_j^{-1}(\partial E_j) \cap \mathcal{B}_1$$

will be satisfied for all j large enough in the subsequence and for e the normal vector to the limit hyperplane. Rescaling this condition by a factor R_j, we obtain exactly that (for j large) the E_j satisfy (118) with E_j, R_j and φ_j, contradiction. This finishes the proof of the claim. □

Now that the claim is known to be true, choosing σ in it to be the constant in Theorem 4.18 (recall that sets in $\mathcal{A}_m(M)$ are viscosity solutions of the NMS equation by Proposition 3.26, and that our notion of viscosity solution in 3.26 is equivalent to the one used in [81] to obtain Theorem 4.18), we obtain that $\varphi^{-1}(\partial E) \cap \mathcal{B}_{R_\sigma/2}$ is a single graph with uniform $C^{1,\alpha}$ estimates.

□

4.5 Dimension reduction – Proof of Theorem 1.11

This section proves Theorem 1.11. We will call singular points those points $x \in \partial E$ where ∂E cannot be described as a $C^{1,\alpha}$ graph around x, and we will denote by $\mathrm{sing}(\partial E)$ or $\mathrm{sing}(E)$ the (closed) set of all the singular points of ∂E. We state here a more general result about regularity for s-minimal surfaces which are limits of Allen-Cahn and immediately show how it proves Theorem 1.11.

Theorem 4.19. *Let $s \in (0,1)$. Let (M, g) be a closed Riemannian manifold of dimension $n \ge 3$, and let $\partial E \in \mathcal{A}_m(M)$. Then:*

- *If $n < n_s^*$, then ∂E is a C^∞ hypersurface.*
- *If $n = n_s^*$, ∂E is a C^∞ hypersurface outside of a discrete set.*
- *If $n > n_s^*$, then ∂E is a C^∞ hypersurface outside of a closed set of Hausdorff dimension at most $n - n_s^*$.*

We readily deduce:

Proof of Theorem 1.11. The surfaces $\Sigma^{\mathfrak{p}} = \partial E^{\mathfrak{p}}$ in Theorem 1.9 belong to $\mathcal{A}_{\mathfrak{p}}(M)$ by construction (see Section 3.4 for the proof of Theorem 1.9). Therefore, Theorem 4.19 applies to them, which gives Theorem 1.11. □

Theorem 4.19 will be proved after two preliminary lemmas.

Lemma 4.20. *Let $\partial E \in \mathcal{A}_m^{Blow-up}$ and let $x \in \mathrm{sing}(\partial E)$. Choose $r_j \to 0$; then, the blow-up sequence $\frac{1}{r_j}(E - x)$ converges in L^1_{loc} and locally in the Hausdorff distance sense to a singular cone $C_\infty \in \mathcal{A}_m^{Blow-up}$ which is stable in $\mathbb{R}^n \setminus \{0\}$. If moreover x is an accumulation point of $\mathrm{sing}(\partial E)$, then r_j can be chosen so that C_∞ has a singular point on ∂B_1 (thus an entire line of singular points).*

Proof. Recall that $\mathcal{A}_m^{Blow-up}$ satisfies the properties in the statement of Theorem 4.12, see the proof of Corollary 4.17. The convergence of the $F_j := \frac{1}{r_j}(E - x)$ to a cone C_∞ in the appropriate sense follows then as in the beginning of the proof of Theorem 4.12, and Lemma 4.15 gives the stability of C_∞ outside the origin.

If C_∞ were non-singular (i.e. if C_∞ were a half-space), then the Hausdorff convergence of the $F_j = \frac{1}{r_j}(E - x)$ to C_∞ on $\mathcal{B}_1$ would imply that the assumption of the improvement of flatness result of Theorem 4.14 is satisfied by E on a small ball centered at x. Hence ∂E would be a $C^{1,\alpha}$ hypersurface around x, and this would contradict the assumption that x is a singular point. Moreover, in case x is a limit point of a sequence $x_j \in \mathrm{sing}(\partial E)$, choosing $r_j := \mathrm{dist}(x, x_j)$ the $F_j = \frac{1}{r_j}(E - x)$ have singular points at 0 and at $\frac{1}{r_j}(x_j - x) \in \partial\mathcal{B}_1$. Selecting a subsequence j_k such that the x_{j_k} converge to a limit point $x' \in \partial\mathcal{B}_1$, the improvement of flatness argument above shows that the limit cone of the F_{j_k} must have a singular point at $x' \in \partial\mathcal{B}_1$. □

Lemma 4.21. *Let $C \subset \mathbb{R}^n$ be a cone in $\mathcal{A}_m^{Blow-up}$, with $n \geq n_s^*$. Then $\mathcal{H}^t(\mathrm{sing}(C)) = 0$ for all $t > n - n_s^*$. Moreover, in the case $n = n_s^*$, C is smooth outside the origin.*

Proof. Fix $t > n - n_s^*$, and assume for contradiction that $\mathcal{H}^t(\mathrm{sing}(C)) > 0$ (or that C is not smooth outside the origin in the case $n = n_s^*$).

Claim. If $n > n_s^*$, there exists $x \in \mathrm{sing}(C) \cap \partial\mathcal{B}_1$ such that, blowing up around x, we find a cone of the form $\widetilde{C} \times \mathbb{R}$ (up to a rotation) with $\mathcal{H}^{t-1}(\mathrm{sing}(\widetilde{C})) > 0$.

Proof of the claim. Since we are assuming that $\mathcal{H}^t(\mathrm{sing}(C)) > 0$, there must exist some point $x \in \mathrm{sing}(C) \cap \partial\mathcal{B}_1$ of positive $\mathcal{H}^t_\infty$-density, in the sense that (with the appropriate constant normalization) there exists a sequence $r_j \to 0$ such that $\mathcal{H}^t_\infty(\mathrm{sing}(C) \cap \mathcal{B}_{r_j}(x)) \geq r_j^s$ for all j. Consider the blow-up sequence $C_j = \frac{1}{r_j}(C - x)$; by Lemma 4.20, a subsequence will converge locally in the Hausdorff distance sense to a limit cone C_∞ which (since C is itself already a cone) is of the form $C_\infty = \widetilde{C}_\infty \times \mathbb{R}$, up to performing a rotation. Assume by contradiction that $\mathcal{H}^{t-1}(\mathrm{sing}(\widetilde{C}_\infty)) = 0$, or equivalently that $\mathcal{H}^t(\mathrm{sing}(C_\infty)) = 0$.
Now, given any fixed finite cover by open sets of $\mathrm{sing}(C_\infty) \cap \bar{\mathcal{B}}_1$, for j large enough the $\mathrm{sing}(C_j) \cap \mathcal{B}_1$ are also contained in the cover: otherwise, we would have a subsequence $y_j \in \mathrm{sing}(C_j)$ converging to some $y \in (C_\infty \setminus \mathrm{sing}(C_\infty)) \cap \bar{\mathcal{B}}_1$, so that by Hausdorff convergence the C_j would be contained (for j large enough) in an arbitrarily flat piece of slab around the y_j (thanks to the regularity of C_∞ at y); by the improvement of flatness result of Theorem 4.14, the y_j would be regular points as well, a contradiction. By arbitrariness of the finite open cover of $\mathrm{sing}(C) \cap \bar{\mathcal{B}}_1$, the assumption that $\mathcal{H}^t(\mathrm{sing}(\widetilde{C}_\infty)) = 0$ and the definition of $\mathcal{H}^t_\infty$ lead us to deduce that $\mathcal{H}^t_\infty(\mathrm{sing}(C_j) \cap \mathcal{B}_1)$ converges to zero. Scaling back (recall that $C_j = \frac{1}{r_j}(C - x)$), we find that for some j large enough $\mathcal{H}^t_\infty(\mathrm{sing}(C) \cap \mathcal{B}_{r_j}(x)) \leq \frac{1}{2} r_j^t$, a contradiction with how x was chosen. □

With the claim at hand, the proof now continues as follows.
In the case $n > n_s^*$, since we assumed $t > n - n_s^*$, the claim can be easily further iterated up to $(n - n_s^*)$ times. This leads to the existence, in the class $\mathcal{A}_m^{Blow-up}$, of a cone of the form $\widetilde{C} \times \mathbb{R}^{n-n_s^*}$ with $\widetilde{C} \subset \mathbb{R}^{n_s^*}$ and $\mathcal{H}^{t-(n-n_s^*)}(\mathrm{sing}(\widetilde{C})) > 0$. In particular, $\widetilde{C}$ is not smooth outside the origin.
In the case $n = n_s^*$, we are already assuming by contradiction that $\widetilde{C} := C \subset \mathbb{R}^{n_s^*}$ is not smooth outside the origin.
The rest of the proof is now common for both cases. Let $y \in \mathbb{R}^{n_s^*}$ be such that $y \in \mathrm{sing}(\widetilde{C}) \cap \partial\mathcal{B}_1$. Blowing-up around $(y, 0) \in \mathbb{R}^n$, we obtain a new cone which is translation invariant with respect to an additional orthogonal direction. Moreover, this new cone will not be smooth outside the origin either, since otherwise the definition of n_s^* would imply that it is a half-space, and then the Hausdorff convergence and the improvement of flatness result in Theorem 4.14 would give that $\widetilde{C}$ is smooth around y. Iterating this argument, we obtain in the end a cone in $\mathcal{A}_m^{Blow-up}$ which is translation invariant with respect to $n - 2$ directions and which is not a half-space by the improvement of flatness argument we have been repeatedly using. Lemma 4.16 then gives a contradiction, concluding the proof. □

Proof of Theorem 4.19. Let $\partial E \in \mathcal{A}_m(M)$, with M of dimension $n \geq 3$. We distinguish between the three cases depending on n_s^*:

- Assume $n_s^* > n$. At every $p \in \partial E$, the flatness assumptions $\mathrm{FA}_3(M, g, R_0, p, \varphi_p)$ will be satisfied for some $R_0 > 0$ (recall (d) in Remark 1.24), so that we can apply Theorem 1.16 after scaling and conclude the $C^{1,\alpha}$ regularity (in fact, with quantitative estimates) of ∂E around p.
- Assume $n_s^* < n$. Fix any $t > n - n_s^*$; by Theorem 4.5 and the arguments in Lemma 4.20, given any $q \in \mathrm{sing}(E)$ we can blow-up around q and find a cone C_q. Applying Lemma 4.21, we deduce that $\mathcal{H}^t(\mathrm{sing}(C_q)) = 0$.
  Now, assume for contradiction that $\mathcal{H}^t(\mathrm{sing}(E)) > 0$. We can then apply the same argument as in the Claim in the proof of Lemma 4.21, but with $\mathrm{sing}(E)$ instead of $\mathrm{sing}(C) \cap \partial\mathcal{B}_1$; this shows the existence of a point $q \in \mathrm{sing}(E)$ such that, blowing-up around q, we would find a cone with $\mathcal{H}^t(\mathrm{sing}(C_q)) > 0$, thus reaching a contradiction.
- Assume $n_s^* = n$. Suppose that $q \in \mathrm{sing}(E)$ is an accumulation point. By Theorem 4.5 and the arguments in Lemma 4.20, we can blow-up around q and find a cone C_q which is not smooth outside the origin. Lemma 4.21 then gives a contradiction.

This proves that E is $C^{1,\alpha}$ outside of a set of the desired size. The fact that $C^{1,\alpha}$ s-minimal surfaces are smooth (C^∞) is proved in [55]. □

4.6 The De Giorgi and Bernstein conjectures in the finite Morse index case – proof of Theorems 1.32 and 1.31

We will now first prove Theorem 1.32. We will need the following result, which is a consequence of an improvement of flatness theory for phase transitions in the "genuinely nonlocal" regime, meaning that the order s of the operator is strictly less than 1.

Theorem 4.22 (Theorem 1.2 in [47]). *Let $n \geq 2$, $s \in (0,1)$, and $W(u) = \frac{1}{4}(1-u^2)^2$. Let $u : \mathbb{R}^n \to (-1,1)$ be a solution of $(-\Delta)^{s/2}u + W'(u) = 0$ in $\mathbb{R}^n$.*

Assume that there exists a function $a : (1,\infty) \to (0,1]$ such that $a(R) \to 0$ as $R \to +\infty$ and such that, for all $R > 0$, we have

$$\{e_R \cdot x \leq -a(R)R\} \subset \{u \leq -\tfrac{4}{5}\} \subset \{u \leq \tfrac{4}{5}\} \subset \{e_R \cdot x \leq a(R)R\} \quad \text{in } \mathcal{B}_R\,, \tag{119}$$

for some $e_R \in \mathbb{S}^{n-1}$ which may depend on R.

Then, $u(x) = \phi(e \cdot x)$ for some direction $e \in \mathbb{S}^{n-1}$ and an increasing function $\phi : \mathbb{R} \to (-1,1)$.

Proof of Theorem 1.32. Define the sequence of blow-down rescalings $u_R(x) := u(Rx)$, which are also solutions to the Allen-Cahn equation but with parameter ε/R. By the convergence result of Theorem 1.30 (which can be proved on $\mathbb{R}^n$ exactly as on a closed manifold M, up to the selection of a diagonal subsequence to have convergence on each of the balls $\mathcal{B}_k$ for $k \in \mathbb{N}$), there is some sequence $R_j \to \infty$ such that $u_{R_j}(x) = u(R_j x)$ converges to some limit $u_\infty = \chi_E - \chi_{E^c}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.
In particular, E belongs to the class $\mathcal{A}_m(\mathbb{R}^n)$ of sets $E \subset \mathbb{R}^n$ such that there exists a sequence of functions $u_j : \mathbb{R}^n \to (-1,1)$ which are solutions to the Allen-Cahn equation (7) on $\mathbb{R}^n$, with Morse index $m(u_j) \leq m$ for all j, and parameters $\varepsilon_j \to 0$, such that $u_j \to u_0 := \chi_E - \chi_{E^c}$ in $L^1(\mathbb{R}^n)$. The properties in the hypotheses of Theorem 4.12 can also be proved for this class, exactly as they were proved for (blow-up limits of) limits as $\varepsilon \to 0$ of Allen-Cahn on a closed manifold M as recorded in Corollary 4.17. Indeed, all the necessary results have been stated with local assumptions, other than the use of the kernel $K_s(x,y)$ which becomes $\alpha_{n,s}|x-y|^{-n-s}$ on $\mathbb{R}^n$, and in fact several proofs could be made simpler thanks to working on $\mathbb{R}^n$. Thus, applying Theorem 4.12 to the class $\mathcal{A}_m(\mathbb{R}^n)$, we deduce that $E \in \mathcal{A}_m(\mathbb{R}^n)$ is in fact a half-space. Finally, the local convergence in the Hausdorff distance sense of the aggregations of level sets of $u_{R_j} = u(R_j\,\cdot)$ to ∂E, see Theorem 1.30, shows after rescaling back that u satisfies the asymptotic flatness hypothesis in Theorem 4.22, and therefore it is a 1-D solution. □

We turn now to the proof of Theorem 1.31, the finite Morse index analog for s-minimal surfaces of class C^2 of the Bernstein conjecture. We recall that this result is false for classical minimal surfaces, since for example the catenoid in $\mathbb{R}^3$ is a complete minimal surface with Morse index 1, and that even under the assumption of stability this result is only known up to dimension 4 despite stable classical minimal cones being known to be hyperplanes up to dimension 7. See the Introduction for more details.
To prove Theorem 1.31, we will again apply the classification result of Theorem 4.12. For that reason we introduce the following definition:

Definition 4.23. We say that a set $E \subset \mathbb{R}^n$ belongs to the class $\mathcal{A}'_m(\mathbb{R}^n)$ if there exists a sequence of sets $E_j \subset \mathbb{R}^n$ with $E_j \to E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that:

(i) the boundaries ∂E_j are $(n-1)$-dimensional manifolds of class C^2;

(ii) E_j are critical sets for the s-perimeter in $\mathbb{R}^n$ with Morse index $\leq m$ in the weak sense.

Proposition 4.24. *Let $n \geq 3$. Then the family $\mathcal{A}'_m(\mathbb{R}^n)$ satisfies the properties in the hypotheses of Theorem 4.12.*

Proof. For E an s-minimal surface of class C^2 and Morse index at most m in $\mathbb{R}^n$:

- A uniform BV estimate holds, which one can see by considering the proof for stable s-minimal surfaces in [40] and adapting it to the finite Morse index case using the ideas in the present paper, or by directly interpreting the arguments in the present paper for s-minimal surfaces of class C^2 and Morse index at most m in $\mathbb{R}^n$ instead of for solutions of the Allen-Cahn equation. More precisely, the C^2 assumption allows one to use the geometric formula on $\mathbb{R}^n$ for the second variation of the fractional perimeter (see Theorem 6.1 and Remark 6.2 in [51]), instead of the second-variation formula for the Allen-Cahn energy, with which one first proves an almost-stability inequality like the one in Lemma 3.10. Afterwards, one shows the BV estimate arguing as in Section 3.2.3.

- The surface E is a viscosity solution of the NMS equation, since it is stationary and of class C^2 (in particular, its nonlocal mean curvature is well defined and is equal to 0 at every boundary point).

- The almost-stability inequality (4) is proved exactly like the one in Lemma 4.10 considering the formula for the second variation of the fractional perimeter and test functions $\xi_i = X_i \cdot \nu_{\partial E}$ instead, where $\nu_{\partial E}$ denotes the outer normal vector.

If we then consider E to be any element of $\mathcal{A}'_m(\mathbb{R}^n)$, not necessarily of class C^2, by definition we can approximate it with E_i satisfying the above. This readily shows that E inherits the properties of the E_i, which proves (1)-(5) for the class $\mathcal{A}'_m(\mathbb{R}^n)$. Regarding property (6), concerning the classification of cones in $\mathcal{A}'_m(\mathbb{R}^n)$ with $n-2$ directions of translation invariance, it is proved as in Lemma 7.7 in [19], the latter dealing with limits of Allen-Cahn solutions in the stable case and in $\mathbb{R}^n$. Instead of the inequality (7.17) in [19], which is stated for solutions of Allen-Cahn, one considers the second variation formula for the fractional perimeter. The almost-stability inequality (4) together with the assumption of having at least one direction of translation invariance (recall that we are assuming $n \geq 3$) then results in an inequality analogous to (7.17) in [19], with an additional term coming from the assumption of almost-stability (instead of stability) but which is immediately seen to be bounded thanks to the BV estimate. □

Proof of Theorem 1.31. Thanks to Proposition 4.24, we can apply Theorem 4.12 to the family $\mathcal{A}'_m(\mathbb{R}^n)$ and conclude the result. □

A Proofs of the heat kernel estimates

Proof of Lemma 2.7. Put $H(x,y,t) := H_M(\varphi(x), \varphi(y), t)$. Let $g_{ij} \in C^0(\mathcal{B}_1(0))$ be the metric coefficients in the chart φ^{-1}, choose $\xi \in C^0_c(\mathcal{B}_1(0))$ such that $\chi_{\mathcal{B}_{3/4}(0)} \leq \xi \leq \chi_{\mathcal{B}_1(0)}$ and put $g'_{ij} := g_{ij}\xi + \delta_{ij}(1-\xi)$. By assumption we have $\left|\overline{g}'_{ij}(x)v^iv^j - |v|^2\right| \leq \frac{1}{100}|v|^2$ for all $x, v \in \mathbb{R}^n$. Morevoer, $g'_{ij} \equiv g_{ij}$ inside $\mathcal{B}_{3/4}(0)$.

Consider the complete Riemannian manifold $M' := (\mathbb{R}^n, g')$ and let $H'(x,y,t)$ denote its associated heat kernel. Then, by Lemma 2.5 we have

$$\frac{c_1}{t^{n/2}} e^{-c_2|x-y|^2/t} \le H'(x,y,t) \le \frac{c_3}{t^{n/2}} e^{-c_4|x-y|^2/t}. \tag{120}$$

Now, since $H'(0,x,t)$ is the heat kernel of the stochastically complete manifold M' we have

$$\int_M H'(0,x,t)\sqrt{|g'|(x)}dx = 1 \quad \text{for all } t > 0. \tag{121}$$

On the other hand, for every fixed $\tau > 0$ set $h(\tau) := \frac{c_3}{\tau^{n/2}} e^{-c_4(1/4)^2/\tau}$. Using (120) we have that $u(x,t) = \big(H'(0,x,t) - h(\tau)\big)^+$, where $(\,\cdot\,)^+$ denotes the positive part, is a subsolution of

$$\begin{cases} u_t \le \frac{1}{\sqrt{|g'|}} \frac{\partial}{\partial x^i}\left(\sqrt{|g'|}(g')^{ij} \frac{\partial}{\partial x^j} u\right) & \text{in } \mathcal{B}_{1/4}(0) \times (0,\tau)\,, \\ u = 0 & \text{in } \partial\mathcal{B}_{1/4}(0) \times (0,\tau)\,. \end{cases}$$

Since $g' \equiv g$ in $\mathcal{B}_{1/4}(0)$, it easily follows (using that both $H(0,x,t)$ and $H'(0,x,t)$ have as initial condition a Dirac delta with respect to the same volume form $\sqrt{|g|}dx$) that $u \le H(0,x,t)$ for all $t \in (0,\tau)$. This gives, for all $t \in (0,\tau)$

$$\int_{\mathcal{B}_{1/4}} (H'(0,x,t) - h(\tau))^+ \sqrt{|g|}dx = \int_{\mathcal{B}_{1/4}} u(x,t)\sqrt{|g|}dx \le \int_{\mathcal{B}_{1/4}} H(0,x,t)\sqrt{|g|}dx$$

On the other hand, using (121) and (120) we obtain that for all $t \in (0,\tau)$

$$\begin{aligned} 1 - C\exp(-c/\tau) &\le 1 - \int_{\mathbb{R}^n\setminus\mathcal{B}_{3/4}(0)} \frac{c_3}{t^{n/2}} e^{-c_4|x|^2/t}\big(1+\tfrac{1}{100}\big)^{n/2} dx \le 1 - \int_{\mathbb{R}^n\setminus\mathcal{B}_{3/4}(0)} H'(0,x,t)\sqrt{|g'|}dx \\ &= \int_{\mathcal{B}_{3/4}(0)} H'(0,x,t)\sqrt{|g|}dx. \end{aligned}$$

Finally, since also $h(\tau) \le C\exp(-c/\tau)$ (notice that we can "absorbe" $\tau^{-n/2}$ in $Ce^{-c/\tau}$ chosing $c > 0$ slightly smaller and a larger C), we obtain the desired estimate

$$1 - C\exp(-c/\tau) \le \int_{\mathcal{B}_{1/4}(0)} H(0,x,t)\sqrt{|g|}dx \le \int_{\varphi(\mathcal{B}_{1/2}(0))} H(p,q,t)dV_q, \quad \forall t \in (0,\tau)\,,$$

and for all $\tau > 0$. The bound by above by 1 of the same quantity follows immediately using that H is a heat kernel, i.e. nonnegative and with total mass bounded by 1. □

Proof of Lemma 2.8. Notice that $u(x,t) := H_M(\varphi(x),q,t)$ satisfies $u_t = Lu$, in $\mathcal{B}_1(0) \times [0,\infty)$ and $u \equiv 0$ at $t = 0$, where

$$Lu = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ij}\frac{\partial u}{\partial x^j}\right) \tag{122}$$

is the Laplace-Beltrami operator with metric g.

Let us show that

$$|u| \le C\exp(-c/t) \quad \text{for } (x,t) \in \mathcal{B}_{3/4}(0) \times [0,\infty), \tag{123}$$

with $C, c > 0$ dimensional constants. This follows from the following standard probabilistic consideration. Fix $x_\circ \in \varphi(\mathcal{B}_{3/4}(0))$. By continuity of sample paths, the probability that a Brownian motion started at $q \in M \setminus \varphi(\mathcal{B}_1(0))$ hits $\varphi(\mathcal{B}_\delta(x_\circ))$ $(0 < \delta \ll 1)$ within time $\le t$ is less that the supremum among $q' \in \varphi(\partial\mathcal{B}_{8/9})$ of the probability that a Brownian motion started at a point q' hits $\varphi(\mathcal{B}_\delta(x_\circ))$ within time $\le t$. This gives

$$u(x_\circ, t) \le \sup_{q' \in \varphi(\partial\mathcal{B}_{8/9})} H_M(\varphi(x_\circ), q', t). \tag{124}$$

Let us now use (124), Lemma 2.7, and the parabolic Harnack inequality as follows to show (123).

For fixed $q' \in \varphi(\partial\mathcal{B}_{8/9})$ set $v(x,t) := H_M(\varphi(x), q', t)$ and consider the rescaled $\widetilde{v}(x,t) := v(x_\circ + rx, t_\circ + r^2 t)$ for $r \in (0, 1/10)$. Then $\widetilde{v} \geq 0$ satisfies a (uniformly) parabolic equation in $\mathcal{B}_1(0) \times (0,1)$ with smooth coefficients (that only improve as r gets smaller). Thus, by the Harnack inequality for every $x \in \mathcal{B}_{1/2}(0)$ and $t \in (1/4, 1/2)$ we have $\widetilde{v}(x,t) \leq C \inf_{\mathcal{B}_{1/2}(0)} \widetilde{v}(\,\cdot\,, 1) \leq C\widetilde{v}(y,1)$ for all $y \in \mathcal{B}_{1/2}(0)$. Integrating

$$\widetilde{v}(0,t) \leq C \int_{\mathcal{B}_{1/2}(0)} \widetilde{v}(y,1)dy = C \int_{\mathcal{B}_{1/2}(0)} v(x_\circ + ry, t_\circ + r^2)dy = Cr^{-n} \int_{\mathcal{B}_{r/2}(x_\circ)} v(z, t_\circ + r^2)dz\,,$$

for some $C = C(n) > 0$. Thus, for all $t \in (t_\circ + r^2/4, t_\circ + r^2/2)$

$$v(x_\circ, t) \leq Cr^{-n} \int_{\mathcal{B}_{r/2}(x_\circ)} v(z, t_\circ + r^2)dz\,.$$

But $\varphi(\mathcal{B}_{r/2}(x_\circ)) \subset M \setminus B_{1/10}(q')$ for every $q' \in \varphi(\partial\mathcal{B}_{8/9}(0))$. Then by Lemma 2.7 we get

$$v(x_\circ, t) \leq Cr^{-n} \int_{M\setminus\varphi(B_{1/10}(q'))} H_M(z, q', t_\circ + r^2)dz \leq Cr^{-n} \exp\left(-\frac{c}{t_\circ + r^2}\right),$$

where $C, c > 0$ depend only on n. Now, for small times $t_\circ \leq 1/100$ choosing $r^2 = t_\circ$ (together with the probabilistic argument above) gives the result, since one can absorb the term $r^{-n} = t_\circ^{-n/2}$ in the exponential up to slightly decreasing the value of c. For non-small times $t_\circ > 1/100$, one can just take $r = 1/10$ and obtain upper bound by a constant as desired. This concludes the proof of (123).

Now, similarly to above, for all $r \in (0, 1/4)$ and $(x_\circ, t_\circ) \in \mathcal{B}_{1/2}(0) \times (0, \infty)$ the rescaled function $\overline{u}(x,t) = u(x_\circ + rx, t_\circ + r^2 t)$ satisfies a (uniformly) parabolic equation with smooth coefficients (since the bounds on every C^k norm of the coefficients only improve as r gets smaller) and, from (123), we have $|\overline{u}| \leq C\exp(-c/t)$ in $\mathcal{B}_1 \times (0,1)$. Hence standard parabolic Schauder estimates give

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\,\overline{u}\right| \leq C\exp(-c/t), \quad \text{for } (x,t) \in \mathcal{B}_{1/2}(0) \times [1/2, 1)\,,$$

for every multi-index α with $|\alpha| \leq \ell$, with $C > 0$ depending only on n and ℓ and $c > 0$ as above.

After scaling back the estimate above we obtain, for all $r \in (0, 1/4]$

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha} u(x_\circ, t_\circ + t)\right| \leq Cr^{-|\alpha|} \exp(-c/r^2), \quad \text{for } (x_\circ, t) \in \mathcal{B}_{1/2}(0) \times [r^2/2, r^2)\,.$$

Then, for "non-small" times $t_\circ \geq 1/16$ we notice that (30) follows taking $r = 1/4$. On the other hand, for small times $t_\circ \in (0, 1/16)$ we obtain (30) taking $r^2 = t_\circ$, bounding $r^{-|\alpha|}$ by $t_\circ^{-\ell/2}$, and absorbing (chosing $c > 0$ smaller and C larger) this negative power of $t_\circ$ in the exponential. □

Proof of Lemma 2.9. It is similar to the proof of Lemma 2.8. Let $\sigma : M \setminus \varphi(\mathcal{B}_1(0)) \to \{+1, -1\}$ be any measurable function to be chosen. Consider

$$u(x,t) := \int_{M\setminus\varphi(\mathcal{B}_1(0))} H_M(\varphi(x), q, t)\sigma(q)dV_q,$$

By Lemma 2.7 — since $H_M \geq 0$ and $\int_M H_M(p,q,t)dV_q \leq 1$ — we obtain

$$|u(x,t)| \leq \int_{M\setminus\varphi(\mathcal{B}_{1/4}(0))} H(\varphi(x), q, t)dV_q \leq C\exp(-c/t), \qquad \forall (x,t) \in \mathcal{B}_{3/4}(0) \times [0,\infty). \tag{125}$$

Notice that in this estimates C and c are positive dimensional constants (and in particular they do not depend on the choice of σ). Also, by the superposition principle u satisfies $u_t = Lu$, in $\mathcal{B}_1(0) \times [0, \infty)$ and $u \equiv 0$ at $t = 0$, where L is as in (122).

Now proceeding exactly as in the proof of Lemma 2.8 we obtain that

$$\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}u\right| \le C\exp(-c/t), \quad \text{for } (x,t)\in \mathcal{B}_{1/2}(0)\times[0,\infty)$$

for $|\alpha|\le \ell$. Now, for any given α, x, and t, we can choose $\sigma : M\setminus\varphi(\mathcal{B}_1(0))\to\{+1,-1\}$ so that

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha}u(x,t) = \int_{M\setminus\varphi(\mathcal{B}_1(0))}\frac{\partial^{|\alpha|}}{\partial x^\alpha}H_M(\varphi(x),q,t)\sigma(q)dV_q = \int_{M\setminus\varphi(\mathcal{B}_1(0))}\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}H_M(\varphi(x),q,t)\right|dV_q,$$

and we are done. □

Proof of Lemma 2.10. Let us show that

$$|H-H'|\le C\exp(-c/t) \quad \text{for } (x,y,t)\in\mathcal{B}_{3/4}(0)\times\mathcal{B}_{3/4}(0)\times[0,\infty), \tag{126}$$

with $C,c>0$ dimensional constants.

Indeed, fix $x_\circ\in\mathcal{B}_{3/4}(0)$ and let us show first that we have

$$\big|(H-H')(x_\circ,y,t)\big|\le C\exp(-c/t) \quad \text{for all } y\in\mathcal{B}_{3/4}(0)\setminus\mathcal{B}_{1/8}(x_\circ) \tag{127}$$

Indeed, the L^∞ estimate of Lemma 2.8 —appropriately rescaled to have $\varphi(B_{1/8}(x_\circ))$ instead of $\varphi(\mathcal{B}_1(0))$— gives

$$H_M(\varphi(x_\circ),\varphi(y),t)\le C\exp(-c/t) \quad \text{for all } y\in\mathcal{B}_{3/4}(0)\setminus\mathcal{B}_{1/8}(x_\circ), \tag{128}$$

and the same estimate with H_M replaced by $H_{M'}$. Hence (126) follows using $|H-H'|\le H+H'$.

Now observing that for all $x_\circ$ as above $u(y,t):=(H-H')(x_\circ,y,t)$ solves the heat equation $(\partial_t-L_y)u=0$ with zero intial condition, (126) easily follows from the maximum principle.

Finally, the estimate for the higher derivatives follows from standard parabolic estimates, noticing that $u(x,y,t):=(H-H')(x,y,t)$ solves

$$\partial_t u=\frac{1}{2}(L_{x,y}u)$$

where

$$L_{x,y}u=\frac{1}{\sqrt{|g(x)|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g(x)|}g^{ij}(x)\frac{\partial}{\partial x^j}u\right)+\frac{1}{\sqrt{|g(y)|}}\frac{\partial}{\partial y^i}\left(\sqrt{|g(y)|}g^{ij}(y)\frac{\partial}{\partial y^j}u\right).$$

is the sum of the Laplace-Beltrami operators with respect to the variables x and y (or, equivalently, the Laplace-Beltrami operator with respect to the product metric in $\mathcal{B}_1(0)\times\mathcal{B}_1(0)$). □

Proof of Proposition 2.12. Notice first that since $H(x,y,t)=H(y,x,t)$ we have

$$H(x,y,t)=t^{-n/2}h\left(\frac{A(x)(y-x)}{\sqrt{t}},x,t\right)=t^{-n/2}h\left(\frac{A(y)(x-y)}{\sqrt{t}},y,t\right).$$

Let $L_xf:=\frac{1}{\sqrt{|g|(x)}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|(x)}g(x)^{ij}\frac{\partial}{\partial x^j}f\right)$ denote the Laplace-Beltrami operator (with respect to x). Direct computation shows:

$$LH=t^{-\frac{n}{2}-1}\left(\sqrt{t}\left(\frac{\partial_i\big(\sqrt{|g|}g^{ij}\big)}{\sqrt{|g|}}\right)(x)A^l_j(y)\frac{\partial}{\partial z^l}h(*)+g^{ij}(x)\big(A^k_iA^l_j\big)(y)\frac{\partial^2}{\partial z^k\partial z^l}h(*)\right),$$

$$\partial_tH=t^{-\frac{n}{2}-1}\left(-\frac{n}{2}h(*)-\frac{1}{2}\frac{\partial}{\partial z^l}h(*)\frac{(A(y)(x-y))^l}{\sqrt{t}}+t\partial_th(*)\right),$$

where

$$(*)\text{ means evaluated at }\left(\frac{A(y)(x-y)}{\sqrt{t}},y,t\right).$$

This leads to the equation for $h = h(z,y,t)$, where we denote $\partial_i := \frac{\partial}{\partial z^i}$ and $\partial_{ij} := \frac{\partial^2}{\partial z^i \partial z^j}$,

$$t\partial_t h = \overline{L}h := a^{ij}(z,y,t)\partial_{ij}h + \big(\sqrt{t}b^i(z,y,t) + \tfrac{z^i}{2}\big)\partial_i h + \frac{n}{2}h,$$

where

$$a^{ij}(z,y,t) := g^{kl}\Big(y+\sqrt{t}z\Big)\big(A^i_k A^j_l\big)(y)$$

and

$$b^i(z,y,t) := \left(\frac{\partial_k\big(\sqrt{|g|}g^{kl}\big)}{\sqrt{|g|}}\right)\Big(y+\sqrt{t}z\Big)A^i_l(y);$$

with initial condition:

$$h(z,y,0^+) = h_\circ(z) = (4\pi)^{-\frac{n}{2}}e^{-|z|^2/4}.$$

(Notice that we defined h so that the its initial condition is independent of y.)

We emphasize that, by the assumption (32), this equation is uniformly elliptic, and the derivatives of a^{ij}, b^i up to order ℓ in the variables z and y are uniformly bounded for times $t \in (0, T_\circ)$ by constants depending only on the constants n and $T_\circ$.

Let us now compute an equation for $\widehat{h} = h - h_\circ$. Since

$$\delta^{ij}\partial_{ij}h_\circ + \tfrac{z^i}{2}\partial_i h + \frac{n}{2}h_\circ = 0,$$

we obtain

$$\begin{aligned} t\partial_t\widehat{h} - \overline{L}\widehat{h} = \overline{L}h_\circ &= (a^{ij}-\delta^{ij})\partial_{ij}h_\circ + \sqrt{t}b^i\partial_i h_\circ \\ &= \big((a^{ij}-\delta^{ij})(z_iz_j-\delta_{ij}) - \sqrt{t}b^i\tfrac{z^j}{2}\delta_{ij}\big)h_\circ\,. \end{aligned}$$

We emphasize that $\widehat{h}$ satisfies the initial condition

$$\widehat{h}(z,y,0) \equiv 0$$

Notice that (since by definition of $A(y)$ is a square rooth of $g(y)$) we have, for all y

$$g^{kl}(y)\big(A^i_k A^j_l\big)(y) = \delta^{ij}$$

and hence, since g^{kl} is smooth,

$$|a^{ij}(z,y,t) - \delta^{ij}| \le C\sqrt{t}$$

Hence, we have

$$|t\partial_t\widehat{h} - \overline{L}\widehat{h}| \le C(1+|z|^2)\sqrt{t}\,h_\circ \tag{129}$$

Let us now find some barrier allowing us to control $\widehat{h}$. We can use as barrier

$$b(z,t) := \sqrt{t}e^{-(1/4+\kappa)|z|^2}$$

Direct computation shows that, for $\sqrt{t} < \theta\kappa$ (so that $a^{ij}\delta_{ij} \ge n - C\theta\kappa$ and $|\delta^{ij} - a^{ij}|z^kz^l\delta_{ik}\delta_{jl} \le C\theta\kappa|z|^2$)

$$\begin{aligned} t\partial_t b - \overline{L}b &= \left(\frac{1}{2} - 4\big(\tfrac{1}{4}-\kappa\big)^2(a^{ij})z^kz^l\delta_{ik}\delta_{jl} + 2\big(\tfrac{1}{4}-\kappa\big)a^{ij}\delta_{ij} + (\sqrt{t}b^i + \frac{z^i}{2})2\big(\tfrac{1}{4}-\kappa\big)z^j\delta_{ij} - \frac{n}{2}\right)b \\ &\ge \left(\frac{1}{2} + \big(\tfrac{1}{4}-\kappa\big)4\kappa|z|^2 - C\theta\kappa|z|^2 - C\kappa - C\theta\kappa|z|\right)b \\ &\ge \big(\tfrac{1}{4} + \tfrac{\kappa}{2}|z|^2\big)b \ge 0, \end{aligned}$$

provided we chose $\theta > 0$ and $\kappa > 0$ sufficiently small.

Since clearly $b \geq \sqrt{t}\, h_\circ$ we obtain that Cb is a supersolution of (129) for $\sqrt{t} < \theta\kappa$. This shows that $|\widehat{h}| \leq Cb$ for all t small enough.

Notice that the estimate $|\widehat{h}| \leq Cb$ (fixing $\kappa > 0$ and $\theta > 0$ small dimensional) shows, in particular, that

$$|\widehat{h}(z, y, t)| \leq C\sqrt{t}\exp(-c|z|^2) \tag{130}$$

holds with $c > 0$ dimensional for all "small" times $t \in (0, \theta^2\kappa^2)$. On the other hand, for "non-small" times $t \geq \theta^2\kappa^2$, the standard heat kernel estimate (120) for H (which holds with c_i dimensional) immediately yields (130) with $\sqrt{t}$ replaced by 1.

In order to bound the derivatives of h with respect to z we notice we notice that, in logarithmic time $\tau = \log t$, the function $h(z, y, e^\tau)$ satisfies, for y fixed, a standard parabolic equation with smooth coefficients in the domain $\mathbb{R}^n \times (-\infty, 0)$. Then, thanks to (130), applying standard parabolic estimates in parabolic cylinders $\{|x - x_\circ| < 2, |\tau - \tau_\circ| < 2\}$ we easily obtain the claimed bounds for all partial derivatives of h with respect to z.

In order to show the regularity in y one can then differentiate the equation with respect to y as many times as needed (the coefficients depend in a very smooth way also in y), and notice that the initial condition will be zero (since $h_\circ$ is independent of y). By standard parabolic regularity arguments (e.g. using a Duhamel-type formula to represent the solutions) we obtain the estimates. $\square$

B Estimates for the extension problem

Lemma B.1. *Let $s \in (0, 2)$ and M satisfy the flatness assumption* $\mathrm{FA}_2(M, g, 2, p, \varphi)$. *Let also $U : \widetilde{B}_2^+(p, 0) \to \mathbb{R}$ be any function solving*

$$\widetilde{\mathrm{div}}(z^{1-s}\widetilde{\nabla} U) = 0 \quad in \quad \widetilde{B}_2^+(p, 0)\,,$$

and let u be its trace on $B_2(p)$. Assume also $U \in L^\infty(\widetilde{B}_2^+(p, 0))$ and $u = 0$ in $B_{3/2}(p)$. Then there exists $C = C(n) > 0$ such that

$$z^{1-s}|\widetilde{\nabla} U(x, z)| \leq \frac{C}{s}\|U\|_{L^\infty(\widetilde{B}^+_{6/5}(p,0))}\,.$$

for every $(q, z) \in \widetilde{B}_1^+(p, 0)$.

Proof. Let C denote a constant that depends only on n. This estimate is proved by a barrier argument. Let $\alpha, \beta > 0$ to be chosen later, and for $q_\circ \in B_{11/10}(p)$ define

$$b_{q_\circ}(q, z) := \frac{\alpha}{2}\big|\varphi^{-1}(q) - \varphi^{-1}(q_\circ)\big|^2 - \beta(z^2 - 2z^s)\,.$$

Denote by Δ_g the Laplace-Beltrami operator of (M, g). Then, by $\mathrm{FA}_2(M, g, 2, p, \varphi)$, for $(x, z) \in \widetilde{B}^+_{6/5}$ there holds $|\Delta_g b_{q_\circ}(q, z)| \leq C$. Moreover

$$(\partial_{zz} + \tfrac{1-s}{z}\partial_z)z^2 = 2s \quad \text{and} \quad (\partial_{zz} + \tfrac{1-s}{z}\partial_z)z^s = 0\,.$$

Hence

$$\widetilde{\mathrm{div}}(z^{1-s}\widetilde{\nabla} b_{q_\circ}) = z^{1-s}\left(\Delta_g b_{q_\circ} + \partial_{zz} b_{q_\circ} + \frac{1-s}{z}\partial_z b_{q_\circ}\right) \leq z^{1-s}(C\alpha - 2s\beta) \leq 0\,,$$

provided we take $\beta = C\alpha/s$.

Since $U = 0$ in $B_{3/2}(p) \times \{0\}$ clearly $|U| \leq b_{q_\circ}(\cdot, 0)$ in $B_{6/5}(p) \times \{0\}$. Moreover, for every $(x, z) \in \partial^+\widetilde{B}^+_{6/5}(p, 0)$ there holds

$$b_{q_\circ}(q, z) \geq C\alpha - \beta(z^2 - 2z^s) \geq C\alpha \geq \|U\|_{L^\infty(\widetilde{B}^+_{6/5}(p,0))} \geq U(x, z)\,,$$

provided we choose $\alpha = C\|U\|_{L^\infty(\widetilde{B}^+_{6/5}(p,0))}$.

Hence, with this choice of α and β, $|U| \le b_{q_\circ}$ on the full boundary $\partial\widetilde{B}^+_{6/5}(p,0)$. Since also U solves $\widetilde{\operatorname{div}}(z^{1-s}\widetilde{\nabla}U) = 0$ in $\widetilde{B}^+_{6/5}(p,0)$, by the maximum principle we get

$$|U(q_\circ, z)| \le b_{q_\circ}(q_\circ, z) \le \frac{Cz^s}{s}\|U\|_{L^\infty(\widetilde{B}^+_{6/5}(p,0))}, \tag{131}$$

for $(q_\circ, z) \in \widetilde{B}^+_{11/10}(p,0)$. Moreover, by standard (interior) gradient estimates for uniformly elliptic equations, for all $(q,z) \in \widetilde{B}^+_1(p,0)$ we have

$$|\widetilde{\nabla}U(q,z)| \le \|\widetilde{\nabla}U\|_{L^\infty(\widetilde{B}^+_{z/100}(q,z))} \le \frac{C}{z}\|U\|_{L^\infty(\widetilde{B}^+_{z/50}(q,z))},$$

which, since $\widetilde{B}^+_{z/50}(q,z) \subset \widetilde{B}^+_{11/10}(p,0)$, together with (131) implies

$$|\widetilde{\nabla}U(x,z)| \le \frac{C}{s}z^{s-1}\|U\|_{L^\infty(\widetilde{B}^+_{6/5})},$$

and this concludes the proof. □

Lemma B.2. *Let $s_0 \in (0,2)$, $s \in (s_0, 2)$. Consider the Riemannian manifold $(\mathbb{R}^n, g)$ with $(1 - \frac{1}{100})|v|^2 \le g_{ij}(x)v^iv^j \le (1 + \frac{1}{100})|v|^2$ and $\|g_{ij}\|_{C^{1,1}(\mathbb{R}^n)} \le 1$. Let also $u : \mathbb{R}^n \to [-1,1]$ and $U : \mathbb{R}^n \times \mathbb{R}_+ \to [-1,1]$ be the extension of u (in the sense of Theorem 2.20). Then*

$$\int_{\mathcal{B}^+_1(0,0)} |\widetilde{\nabla}U|^2 z^{1-s}dVdz \le C\iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\mathcal{B}^c_2\times\mathcal{B}^c_2)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\,dxdy, \tag{132}$$

with C depending only on n and s_0.

Proof. We proceed as in [25, Proposition 7.1]. Assume without loss of generality that $\int_{\mathcal{B}_2} u(x)dx = 0$. Let $\xi : \mathbb{R}^n \to \mathbb{R}$ be a cutoff function such that $\xi = 1$ in $\mathcal{B}_{3/2}(0)$ and it is compactly supported in $\mathcal{B}_2(0)$. We write $u = u\xi + u(1-\xi) = u_1 + u_2$ and $U = U_1 + U_2$.

On the one hand, since u_1 is compactly supported we have

$$\begin{aligned}
\beta_s \int_{\mathbb{R}^{n+1}_+} z^{1-s}|\widetilde{\nabla}U_1|^2 dVdz &= \|u_1\|_{H^{s/2}(\mathbb{R}^n,g)} \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\mathcal{B}^c_2\times\mathcal{B}^c_2)} |u_1(x)-u_1(y)|^2 K(x,y)\,dV_x dV_y \\
&\le C\alpha_{n,s}\iint_{\mathcal{B}_2\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\xi^2(x)\,dxdy + C\alpha_{n,s}\iint_{\mathcal{B}_2\times\mathbb{R}^n} |u(x)|^2\frac{|\xi(x)-\xi(y)|^2}{|x-y|^{n+s}}\,dxdy \\
&\le C\alpha_{n,s}\iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\mathcal{B}^c_2\times\mathcal{B}^c_2)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\,dxdy + C\int_{\mathcal{B}_2}|u(x)|^2dx.
\end{aligned}$$

Moreover, using the fractional Poincaré inequality (recall $\int_{\mathcal{B}_2} u(x)dx = 0$):

$$\int_{\mathcal{B}_2}|u(x)|^2dx \le C\alpha_{n,s}\iint_{\mathcal{B}_2\times\mathcal{B}_2}\frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\,dxdy \le C\alpha_{n,s}\iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\mathcal{B}^c_2\times\mathcal{B}^c_2)}\frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\,dxdy =: \mathrm{I}$$

On the other hand (using again $\int_{\mathcal{B}_2} u(y)dy = 0$)

$$\int_{\mathbb{R}^n}\frac{u(x)^2}{(1+|x|^2)^{\frac{n+s}{2}}}dx = \iint\frac{(u(x)-u(y))^2}{(1+|x|^2)^{\frac{n+s}{2}}}\frac{\chi_{\mathcal{B}_2}(y)}{|\mathcal{B}_2|}dxdy - \iint\frac{u(y)^2}{(1+|x|^2)^{\frac{n+s}{2}}}\frac{\chi_{\mathcal{B}_2}(y)}{|\mathcal{B}_2|}dxdy \le C\mathrm{I},$$

and by Holder inequality

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|^2)^{\frac{n+s}{2}}}dx \le \left(\int_{\mathbb{R}^n} \frac{|u(x)|^2}{(1+|x|^2)^{\frac{n+s}{2}}}dx\right)^{1/2} \left(\int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^{\frac{n+s}{2}}}dx\right)^{1/2} \le \mathrm{CI}^{1/2}.$$

Claim. There is $C = C(n) > 0$ such that for $(x,z) \in \mathcal{B}_1^+$

$$z^{1-s}|\widetilde{\nabla} U_2(x,z)| \le C \int_{\mathbb{R}^n} \frac{|u_2(y)|}{(1+|y|^2)^{\frac{n+s}{2}}}dy\,. \tag{133}$$

We postpone the proof of this claim and first see how to conclude the proof of Lemma B.2 with it.

By the claim, if $(x,z) \in \mathcal{B}_1^+$ then

$$z^{1-s}|\widetilde{\nabla} U_2(x,z)| \le C \int_{\mathbb{R}^n} \frac{|u_2(y)|}{(1+|y|^2)^{\frac{n+s}{2}}}dy \le C \int_{\mathbb{R}^n} \frac{|u(y)|}{(1+|y|^2)^{\frac{n+s}{2}}}dy \le \mathrm{CI}^{1/2}$$

But then the inequality

$$\int_{\mathcal{B}_1^+} z^{1-s}|\widetilde{\nabla} U_2|^2 dxdz \le \mathrm{CI}^{1/2} \int_{\mathcal{B}_1^+} |\widetilde{\nabla} U_2| dxdz \le \mathrm{CI}^{1/2}\left(\int_{\mathcal{B}_1^+} z^{1-s}|\widetilde{\nabla} U_2|^2 dxdz\right)^{1/2}\left(\int_{\mathcal{B}_1^+} z^{s-1} dxdz\right)^{1/2},$$

gives

$$\int_{\mathcal{B}_1^+} z^{1-s}|\widetilde{\nabla} U_2|^2 dxdz \le \mathrm{CI},$$

and the lemma follows.

It only remains to prove (133). Let $H_N(x,y,t)$ be the heat kernel of $N := (\mathbb{R}^n, g)$. By (46) the fractional Poisson kernel[20] $\mathbb{P}_N : \mathbb{R}^n \times \mathbb{R}^n \times (0,\infty) \to [0,\infty)$ of N can be represented as

$$\mathbb{P}_N(x,y,z) = \frac{z^s}{2^s\Gamma(s/2)} \int_0^\infty H_N(x,y,t) e^{-\frac{z^2}{4t}} \frac{dt}{t^{1+s/2}}\,,$$

and the solution U_2 to the extension problem with trace u_2 is $U_2(x,z) = \int_{\mathbb{R}^n} \mathbb{P}_N(x,y,z) u_2(y)\, dV_y$. Then, by Lemma 2.5 we have that $\mathbb{P}_N$ is comparable (up to dimensional constants) to the fractional Poisson kernel of $\mathbb{R}^n$ with its standard metric, that is

$$cs\frac{z^s}{(|x-y|^2+z^2)^{\frac{n+s}{2}}} \le \mathbb{P}_N(x,y,z) \le Cs\frac{z^s}{(|x-y|^2+z^2)^{\frac{n+s}{2}}}\,,$$

for some $C, c > 0$ dimensional. Hence, for every $(x,z) \in \widetilde{\mathcal{B}}_{6/5}^+$

$$|U_2(x,z)| \le Cs \int_{\mathbb{R}^n} \frac{|u_2(y)|}{(|x-y|^2+z^2)^{\frac{n+s}{2}}}\, dy \le Cs \int_{\mathbb{R}^n \setminus \mathcal{B}_2} \frac{|u_2(y)|}{|x-y|^{n+s}}\, dy\,.$$

Since $x \in \mathcal{B}_{6/5}$ and $y \in \mathbb{R}^n \setminus \mathcal{B}_2$ there holds $|x-y| \ge \frac{1}{100}\sqrt{1+|y|^2}$, and hence

$$\|U_2\|_{L^\infty(\mathcal{B}_{6/5}^+)} \le Cs \int_{\mathbb{R}^n} \frac{|u_2(y)|}{(1+|y|^2)^{\frac{n+s}{2}}}\, dy\,.$$

From here, the result follows directly by Lemma B.1. $\square$

[20] Which equals $\sigma_{n,s}\frac{z^s}{(|x-y|^2+z^2)^{\frac{n+s}{2}}}$ on $\mathbb{R}^n$ with its standard metric, for some normalization constant $\sigma_{n,s} > 0$.

Lemma B.3. *Let $s_0 \in (0,1)$ and $s \in (s_0, 1)$. Let M satisfy flatness assumptions* $\mathrm{FA}_1(M, g, 1, p, \varphi)$*. Let also $U : \widetilde{B}_1^+(p,0) \to (-1,1)$ be any function solving*

$$\widetilde{\mathrm{div}}(z^{1-s}\widetilde{\nabla} U) = 0\,, \tag{134}$$

and let u be its trace on $B_1(p)$. Then for all $\varrho > 0$, $R \geq 1$, $k \in \mathbb{R}$ and $q \in B_{1/2}(p)$ such that $B_{R\varrho}(q) \subset B_{3/4}(p)$,

$$\varrho^{s-n}\beta_s \int_{\widetilde{B}_\varrho^+(q,0)} z^{1-s}|\widetilde{\nabla} U|^2\, dV dz \leq \frac{C}{R^s} + \frac{C}{1-s}\left(\varrho^{-n}\int_{B_{R\varrho}(q)} |u+k|\, dV\right)^{1-s}\left(\varrho^{1-n}\int_{B_{R\varrho}(q)} |\nabla u|\, dV\right)^{s},$$

where the constant C depends only on n and s_0.

Proof. Let us show that if U are U' two different solutions $\widetilde{B}_1^+(p,0) \to (-1,1)$ of (134) with the same trace u on $B_{3/4}(p)$, then

$$\int_{\widetilde{B}_\varrho^+(q,0)} z^{1-s}\big(|\widetilde{\nabla} U|^2 - |\widetilde{\nabla} U'|^2\big)\, dV dz \leq C\int_{\widetilde{B}_\varrho^+(q,0)} z^{1-s}|\widetilde{\nabla} U'|^2\, dV dz. \tag{135}$$

Indeed, by Lemma B.1 (rescaled) we have $z^{1-s}\big|\widetilde{\nabla}(U-U')\big| \leq C$ in $\widetilde{B}^+_{1/2}(p,0)$. Thus, we obtain

$$\begin{aligned}
\int_{\widetilde{B}_\varrho^+(q,0)} z^{1-s}\big(|\widetilde{\nabla} U|^2 - |\widetilde{\nabla} U'|^2\big)\, dV dz &= \int_{\widetilde{B}_\varrho^+(q,0)} z^{1-s}\big(\widetilde{\nabla}(U-U')\big)\cdot\big(\widetilde{\nabla}(U+U')\big)\, dV dz \\
&\leq C\left(\int_{\widetilde{B}_\varrho^+(q,0)} z^{s-1} dV dz\right)^{\frac12}\left(\int_{\widetilde{B}_\varrho^+(q,0)} z^{1-s}\big(|\widetilde{\nabla} U|^2 + |\widetilde{\nabla} U'|^2\big) dV dz\right)^{\frac12} \\
&\leq \frac12\int_{\widetilde{B}_\varrho^+(q,0)} z^{1-s}\big(|\widetilde{\nabla} U|^2 - |\widetilde{\nabla} U'|^2\big)\, dV dz + C\int_{\widetilde{B}_\varrho^+(q,0)} z^{1-s}|\widetilde{\nabla} U'|^2\, dV dz
\end{aligned}$$

Thus (135) follows.

Now let g_{ij} be the components of the metric in the coordinates φ^{-1}, $\eta \in C_c^\infty(\mathcal{B}_1)$ be a nonnegative smooth cut-off function satisfying $\eta \equiv 1$ in $\mathcal{B}_{3/4}$, and put $g'_{ij} = g_{ij}\eta + \delta_{ij}(1-\eta)$, a metric defined in the whole $\mathbb{R}^n$. Thanks to (135) it is enough to prove the lemma for the manifold $(\mathbb{R}^n, g')$ with $p = 0$ and with U replaced by the (unique!) bounded solution U' of (134) (with respect to the metric g') in all of $\mathbb{R}^n \times \mathbb{R}_+$. But in this case we can use Lemma B.2 (rescaled) and obtain

$$\begin{aligned}
\varrho^{s-n}\int_{\widetilde{B}_\varrho^+} z^{1-s}|\widetilde{\nabla} U'|^2\, dV dz &\leq C\varrho^{s-n}\iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\mathcal{B}_{2\varrho}^c\times\mathcal{B}_{2\varrho}^c)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}} dx dy \\
&\leq C\varrho^{s-n} 2\iint_{\mathcal{B}_{R\varrho}\times\mathcal{B}_{R\varrho}\,\cup\,\mathcal{B}_\varrho\times\mathcal{B}_{R\varrho}^c} \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}} dx dy \\
&\leq C\varrho^{s-n}\left(\iint_{\mathcal{B}_{R\varrho}\times\mathcal{B}_{R\varrho}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}} dx dy + \frac{C\varrho^{n-s}}{R^s}\right),
\end{aligned}$$

where we have used that

$$\iint_{\mathcal{B}_\varrho\times\mathcal{B}_{R\varrho}^c} \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\, dx dy \leq C\int_{\mathcal{B}_\varrho}\left(\int_{\mathcal{B}_{R\varrho}} \frac{1}{(|y|-\varrho)^{n+s}} dy\right) dx \leq \frac{C\varrho^n}{(R\varrho)^s}.$$

We conclude using the interpolation inequality of Proposition 2.22, since the modulus of the Euclidean gradient in $\mathbb{R}^{n+1}$ and the metric gradient $|\widetilde{\nabla} U|$ are comparable. $\square$

Lemma B.4. *Let $s \in (0,1)$, (M,g) satisfies the flatness assumption $\mathrm{FA}_2(M,g,2,p,\varphi)$, and let $\eta \in C_c^2(\mathcal{B}_{3/4}(0))$ be a cutoff function with $\eta = 1$ in $\mathcal{B}_{1/2}(0)$. Define $\eta_0 = \varphi \circ \eta$ and let $\widetilde{\eta}$ solve*

$$\begin{cases} \widetilde{\mathrm{div}}(z^{1-s}\widetilde{\nabla}\widetilde{\eta}) = 0 & \text{in } \widetilde{B}_1^+(p,0)\,, \\ \widetilde{\eta} = 0 & \text{on } \partial^+\widetilde{B}_1(p,0)\,, \\ \widetilde{\eta} = \eta_0 & \text{on } B_1(p)\times\{0\}\,. \end{cases}$$

Then, for all $q \in B_{3/4}(p)$ there holds that

$$\beta_s \left| \big(-z^{1-s}\partial_z\widetilde{\eta}\big)(q,0^+)\right| \le C \quad \textit{and} \quad \beta_s \int_{\widetilde{B}_1^+(p,0)} z^{1-s}|\widetilde{\nabla}\widetilde{\eta}|^2 dV dz \le C\,, \tag{136}$$

for some dimensional $C = C(n) > 0$.

Proof. Let $U_0 \in \widetilde{H}^1(M\times(0,\infty))$—see Definition 2.18—be the unique Caffarelli-Silvestre extension of η_0 (considered on M extended by zero outside $B_1(p)$), in the sense of Theorem 2.20. Since U_0 and $\widetilde{\eta}$ are two different solutions of (134) with the same trace on $B_1(p)$, hence by Lemma B.1 (rescaled) we have $\beta_s z^{1-s}|\widetilde{\nabla}(U_0-\widetilde{\eta})| \le C$ in $\widetilde{B}^+_{3/4}(p)$ for some dimensional C. Hence in $B_{3/4}(p)$ there holds

$$\beta_s \left| \big(-z^{1-s}\partial_z\widetilde{\eta}\big)(\,\cdot\,,0^+)\right| \le \beta_s \left| \big(-z^{1-s}\partial_z U_0\big)(\,\cdot\,,0^+)\right| + C = \left|(-\Delta)^{s/2}\eta_0\right| + C\,,$$

where we have used Theorem 2.20 in the last equality. Now, a dimensional bound for the $\left|(-\Delta)^{s/2}\eta_0\right|$ follows, for $q \in B_{3/4}(p)$, by writing

$$(-\Delta)^{s/2}\eta_0(q) = \int_{B_1(p)} (\eta_0(q)-\eta_0(r))K_s(q,r)\,dV_r + \int_{M\setminus B_1(p)} (\eta_0(q)-\eta_0(r))K_s(q,r)\,dV_r$$

and using Lemma 2.11 and (29) of Proposition 2.6 respectively to bound these two integrals. This concludes the proof of the first estimate in (136).

The second estimate follows from the first one just integrating by parts and using $\mathrm{FA}_2(M,g,1,p,\varphi)$. ☐

Lemma B.5. *Let $v : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function with $\|v\|_{L^\infty(\mathbb{R}^n)} \le C_\circ$ satisfying $|Lv| \le C_\circ$ in $\mathcal{B}_1(0)$, where L is an integro-differential operator of order $s \in (0,1)$ of the integral form*

$$Lu(x) = \int_{\mathbb{R}^n} (u(x)-u(y))K(x,y)\,dxdy\,,$$

and K is a nonnegative kernel comparable to the one of the fractional s-Laplacian, that is satisfying

$$\frac{c}{|y|^{n+s}} \le K(x,x-y) \le \frac{C}{|y|^{n+s}} \quad \forall\, x,y \in \mathbb{R}^n, \tag{137}$$

for some constants $c, C > 0$. Then

$$[v]_{C^\alpha(\mathcal{B}_{1/2}(0))} \le C(n,s)C_\circ\,, \tag{138}$$

for some small positive $\alpha = \alpha(n,s)$.

Proof. The result is a standard consequence of [85, Theorem 5.1]. Let us point out that Theorem 5.1 in [85] would seem to require assumption [85, (2.2)] to hold for all $r > 0$. However, it is clear from its (very short) proof that (138) only requires assumption [85, (2.2)] to be verified at "small" scales $r \in (0,1)$ (and in our setting this can be easily verified using (137)). ☐

C Estimates for the distance function on a Riemannian manifold

Lemma C.1. *Let (M^n, g) be an n-dimensional Riemannian manifold, $p \in M$, $R_0 < \mathrm{inj}_M(p)$ and let K be an upper bound for all the sectional curvatures in $B_{R_0}(p)$. Denote by d the distance function to the point p. Then, for all $R < \min\{R_0, \frac{1}{\sqrt{K}}\}$ there holds in $B_R(p)$:*

$$|\langle \nabla_V (d\nabla d), V\rangle - |V|^2| \leq \sqrt{K}R|V|^2 ,$$

for every vector field V on M.

Proof of Lemma C.1. We can compute

$$\begin{aligned}\langle V, \nabla_V (d\nabla d)\rangle &= \langle V, \langle V, \nabla d\rangle \nabla d\rangle + d\langle V, \nabla_V (\nabla d)\rangle \\ &= \langle V, \nabla d\rangle^2 + d\,\nabla^2 d(V, V) .\end{aligned}$$

On the other hand, the Hessian Comparison theorem—see Lemma 7.1 in [41]—gives that

$$|d\,\nabla^2 d(V, V) - |V - \langle V, \nabla d\rangle \nabla d|^2| \leq d\sqrt{K}|V|^2$$

in $B_R(p)$, whenever $R < \min\{\mathrm{inj}_M(p), \frac{1}{\sqrt{K}}\}$. Moreover, since $|\nabla d|^2 = 1$, we also have that

$$|V - \langle V, \nabla d\rangle \nabla d|^2 = |V|^2 - 2\,\langle V, \nabla d\rangle^2 + \langle V, \nabla d\rangle^2 |\nabla d|^2 = |V|^2 - \langle V, \nabla d\rangle^2.$$

Hence

$$|d\,\nabla^2 d(V, V) + \langle V, \nabla d\rangle^2 - |V|^2| \leq d\sqrt{K}|V|^2 \leq R\sqrt{K}|V|^2$$

holds in $B_R(p)$, as long as $R < \min\{R_0, \frac{1}{\sqrt{K}}\}$, and this conludes the proof. ☐

Lemma C.2. *Let (M^n, g) be an n-dimensional Riemannian manifold, $p \in M$, $R_0 < \mathrm{inj}_M(p)$ and let K be an upper bound for all the sectional curvatures in $B_{R_0}(p)$. Then, there exists $C = C(n) > 0$ such that, for all $R < R_0$, in $B_R(p)$ we have that*

$$|\mathrm{div}(d\nabla d) - n| \leq CKR^2 .$$

Proof. Fix $p \in M$, and denote $d(p, \cdot)$ just by $d(\cdot)$. Observe first that every geodesic σ with $\sigma(0) = p$ and contained in $B_{R_0}(p)$ is uniquely minimizing. For any $R < R_0$ and $x \in B_R(p)$, let $\gamma : [0, d] \to M$ be the normalized geodesic with $\gamma(0) = p$ and $\gamma(d) = x$. Note also that

$$\mathrm{div}(d\nabla d) = |\nabla d|^2 + d\Delta d = 1 + d\Delta d .$$

Consider $\dot\gamma(d) \in T_x M$, and complete it to an orthonormal basis $\{e_1 := \dot\gamma(d), e_2, \ldots, e_n\}$ of $T_x M$. For $i = 2, 3, \ldots, n$, let γ_i be the geodesic with $\gamma_i(0) = x$ and $\dot\gamma_i(0) = e_i$. We can compute

$$\Delta d(x) = \sum_{i=1}^n \nabla^2 d(x)(e_i, e_i) = \sum_{i=1}^n \frac{d^2}{ds^2}\Big|_{s=0} (d \circ \gamma_i) = \sum_{i=2}^n \frac{d^2}{ds^2}\Big|_{s=0} (d \circ \gamma_i) ,$$

where we have used that $\frac{d^2}{ds^2}\Big|_{s=0} (d \circ \gamma) = \frac{d^2}{ds^2}\Big|_{s=0} (d(x) + s) = 0$.
Let J_i be the Jacobi field along γ with $J_i(0) = 0$ and $J_i(d) = e_i$, well defined by uniqueness of geodesics between endpoints. Denote by

$$I(X, Y) = \int_0^d \langle D_t X, D_t Y\rangle - \mathrm{Rm}(\dot\gamma, X, \dot\gamma, Y)\, dt$$

the index form associated to γ on $[0, d]$. Since γ is minimizing along all curves with the same endpoints, for every vector field X on $\gamma([0, d])$ orthogonal to $\dot\gamma$ and with $X(0) = 0$ and $X(d) = e_i$ we must have

$$0 \leq I(J_i - X, J_i - X) = I(J_i, J_i) - 2I(J_i, X) + I(X, X) .$$

Since J_i is a Jacobi field, one can easily check that $I(J_i, X) = I(J_i, J_i)$, hence $I(J_i, J_i) \leq J(X, X)$. Take $X(t) = \frac{t}{d} E_i(t)$, where $E_i(t)$ is the parallel transport of $e_i \in T_x M$ along γ. From the second variation formula for arc length we get

$$\begin{aligned}\frac{d^2}{ds^2}\Big|_{s=0} (d \circ \gamma_i) &= \int_0^d |D_t J_i|^2 - \mathrm{Rm}(\dot{\gamma}, J_i, \dot{\gamma}, J_i)\, dt = I(J_i, J_i) \\ &\leq I(X, X) = \int_0^d |D_t X|^2 - \mathrm{Rm}(\dot{\gamma}, X, \dot{\gamma}, X)\, dt \\ &\leq \int_0^d |D_t X|^2 + K|X|^2\, dt\,,\end{aligned}$$

where we have used that $\sup_{p \in B_{R_0}} |\mathrm{Sec}_p| \leq K$. Thus

$$\frac{d^2}{ds^2}\Big|_{s=0} (d \circ \gamma_i) \leq \int_0^d |D_t X|^2 + K|X|^2\, dt = \int_0^d \frac{1}{d^2} + K\frac{t^2}{d^2}\, dt = \frac{1}{d}\left(1 + K\frac{d^2}{3}\right).$$

Hence

$$d\Delta d = \sum_{i=2}^n \frac{d^2}{ds^2}\Big|_{s=0} (d \circ \gamma_i) \leq n - 1 + K\frac{n-1}{3} d^2\,,$$

or equivalently

$$|\mathrm{div}(d\nabla d)(x) - n| = |d(x)\Delta d(x) + 1 - n| \leq K\frac{n-1}{3} d^2 \leq K\frac{n-1}{3} R^2,$$

and this completes the proof with $C(n) = \frac{n-1}{3} > 0$. □

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