# FUNDAMENTAL COCYCLES OF TILING SPACES

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Abstract. We study continuous cocycles defined on the set of planar tilings with values in discrete groups. Following Schmidt we show that for generalized domino tilings, L-tiles, and some systems of paths there exists a fundamental cocycle, i. e. we find a cocycle $c_f$, so that all other continuous cocycles $c$ are cohomologous to a homomorphic image of $c_f$.

## 1. Introduction and Definitions

Fundamental cocycles for higher-dimensional subshifts of finite type were introduced by Schmidt in [4]. The question of which higher-dimensional subshifts of finite type have such cocycles is central to our understanding of cohomological properties. In this work we find fundamental cocycles for a class of two-dimensional subshifts of finite type. We also investigate for those subshifts the relation between cocycles and hole-filling.

In the following sections we will consider tilings of the plane by a fixed set of square tiles. These tiles should always be aligned to the lattice (that is, the corners of the tiles should be lattice points). A Wang tile system is a set of unit squares where each edge has a colour and the set of colours of horizontal edges is disjoint from the set of colours of vertical edges. The allowed tilings have to satisfy that two squares which have an edge in common should have the same colour at this edge. The set of all allowed tilings of $\mathbb{R}^2$ is a shift of finite type which we call the tiling space $X$. As usual, we use in this shift of finite type the subspace topology of the product topology. That means a basis of the topology is given by the cylinder-sets, each of which consists of those tilings which are equal to a given tiling in a fixed finite rectangle. Let $\sigma$ be the usual shift-action on the tiling space.

A continuous cocycle of $\sigma$ on $X$ with values in a discrete group $G$ is a continuous function

$$c : \mathbb{Z}^2 \times X \to G$$

which satisfies

$$c(\mathbf{m}+\mathbf{n}, x) = c(\mathbf{m}, \sigma_{\mathbf{n}} x) c(\mathbf{n}, x) \tag{1}$$

for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^2$ and all $x \in X$ (see for instance [3]). One can think of this formula as a form of path-independence. For instance the equation

$$c\big((0,1), \sigma_{(1,0)} x\big) c\big((1,0), x\big) = c\big((1,0), \sigma_{(0,1)} x\big) c\big((0,1), x\big) \tag{2}$$

means that one can either go first to the right and then up or the other way round if one writes the corresponding cocycle-values in the right order.

An example of a continuous cocycle is a coboundary which has the form $c(\mathbf{n}, x) = b(\sigma_{\mathbf{n}} x) b(x)^{-1}$ for a continuous function $b : X \to G$. One can

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also obtain a new cocycle $c'$ from an existing one $c$ by modifying it with a continuous function $b : X \to G$ in the following way

$$c'(\mathbf{n}, x) = b(\sigma_{\mathbf{n}} x) c(\mathbf{n}, x) b(x)^{-1}. \tag{3}$$

Two cocycles $c$ and $c'$ are called cohomologous if there exists a continuous function $b$ satisfying (3). A fundamental cocycle $c_f$ with values in the discrete group $\Gamma$ is a continuous cocycle for which any continuous cocycle $c$ with values in any discrete group $G$ is cohomologous to a homomorphic image $\eta \circ c_f$ where $\eta : \Gamma \to G$ is a group homomorphism. More intuitively one can say that a fundamental cocycle has all the information which any cocycle can have. For some examples we will show that there exist fundamental cocycles.

As $c(\mathbf{n}, x) = \mathbf{n} \in \mathbb{Z}^2$ is always a cocycle, if there is a fundamental cocycle there exists a homomorphism $\phi : \Gamma \to \mathbb{Z}^2$. The kernel $\ker(\phi)$ of this map expresses somehow the complexity of the dynamical system. In the known examples (see [4]) this kernel is either finite or $\mathbb{Z}$. Here we want to give examples where this kernel has a different structure. We will find dynamical systems where this kernel is, for instance, the free group with finitely many generators.

One way to construct a continuous cocycle for the shift $\sigma$ on a tiling space $X$ is via Conway's tiling group $\Gamma$ (see [1] or [5]). This group is defined by the generators which are the colours of the edges of the tiles and, for each tile $W$, the relation $d^{-1}c^{-1}ba = \mathbf{1}$, where $a, b, c, d$ are the colours of the tile

$$W = \begin{smallmatrix} & c & \\ d & & b \\ & a & \end{smallmatrix} \; .$$

The tiling cocycle $c_\Gamma$ is defined by the two functions

$$\begin{aligned} c_\Gamma\big((1,0), x\big) &= \text{bottom colour of } x_{(0,0)} \\ c_\Gamma\big((0,1), x\big) &= \text{left colour of } x_{(0,0)}. \end{aligned}$$

Now the relation $d^{-1}c^{-1}ba = \mathbf{1}$ is just a reformulation of Equation (2).

We extend the cocycle to all $\mathbf{n} \in \mathbb{Z}^2$, for example in the following way

$$c_\Gamma(\mathbf{n}, x) = c_\Gamma\big((0, n_2), \sigma_{(n_1,0)} x\big) c_\Gamma\big((n_1, 0), x\big),$$

where for positive $n_1$ we define

$$\begin{aligned} c_\Gamma\big((n_1, 0), x\big) = c_\Gamma\big((1,0), \sigma_{(n_1-1,0)} x\big) c_\Gamma\big((1,0), \sigma_{(n_1-2,0)} x\big) \dots \\ \dots c_\Gamma\big((1,0), \sigma_{(1,0)} x\big) c_\Gamma\big((1,0), x\big) \end{aligned} \tag{4}$$

and for negative $n_1$

$$c_\Gamma\big((n_1, 0), x\big) = c_\Gamma\big((-n_1, 0), \sigma_{(n_1,0)} x\big)^{-1}.$$

The function $c_\Gamma\big((0, n_2), x\big)$ is defined in an analogous way. The proof that this gives a cocycle uses the relations of the tiling group.

In connection with the tiling group we also investigate whether or not a given roughly rectangular *hole* in a tiling can be filled by using the tiles. If we go around the hole and write down the consecutive colours from right to left, then the resulting value in the tiling group must be equal to the identity of the group if the hole allows a tiling. This is because if the hole can be tiled then this gives a point $x \in X$ and the calculated value is equal to $c_\Gamma\big((0,0),x\big)$. If the hole is tile-able after enlarging the hole then this algebraic condition is also satisfied, because the cocycle value around the enlarged hole is conjugate to the original value.

In the examples we also search for criteria for tile-ability of such holes. An easy example for this is the system defined by the two one-dimensional tiles $[0,3]$ and $[0,5]$. The Wang-tiles of those tiles are intervals of length one with a colour on each side. If we write the colours of those intervals as an ordered pair (left colour, right colour), we have the following pairs:

$$(0,1),(1,2),(2,0),(2,3),(3,4),(4,0).$$

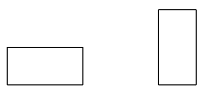
Using these intervals or the original intervals one can of course tile each hole whose length has a representation as a sum of multiples of 3 and 5. It is easy to see that each interval which is sufficiently large has such a representation. However, it is not true that every interval has a representation. So it is more convenient to ask for a condition which is sufficient for large intervals or for a condition which is sufficient if one allows first an enlarging of the hole. This will also occur in our two-dimensional tilings. The enlargement of the hole is in some two-dimensional tilings even more important because the boundary sometimes prevents a tiling although after an enlarging the hole is tile-able.

When we speak below of the tiling of a given region, the tiles have to cover the region and have to be contained in the region, so we mean an exact tiling. The term over tiling should mean that the tiles cover the region but some parts of them can extend beyond the region.

In Section 2 we will consider dominoes and generalized dominoes. Here we find a fundamental cocycle but we will prove as well that there cannot exist a cocycle which gives a sufficient condition for tile-ability of holes. In Section 3 we examine the tiling system of paths in the plane. For those systems there are examples of both behaviors. Some systems of paths have a characterization of tile-ability of holes and some do not, although all of them have a fundamental cocycle. In Section 4 we will consider the L-tiles, find a necessary and sufficient condition for tile-ability of holes and a fundamental cocycle.

## 2. The Dominoes

In [4, Section 6] Schmidt studied the dominoes which are given by the following tiles



and the corresponding dynamical system. Theorem 6.7 in [4] gives a fundamental cocycle for the domino shift. We want to generalize this theorem to all systems which are defined by two rectangular tiles but with the restriction that the system is topologically mixing. For algorithms regarding the question of tile-ability for some of these systems see also [2].

The two rectangles $R_i (i = 1,2)$ have the dimensions $m_i \times n_i$. We call the colours on the horizontal and the vertical edges $H$ and $V$ respectively. We also have to fix colours for the edges in the interior of the tiles in order to define a system of Wang tiles. We do this as indicated in the $2 \times 2, 3 \times 3$ example below:

```
                ┌H┬H┬H┐
                V  v1  v2  V
   ┌H┬H┐       ├H┼H┼b2┤
   V  v1  V     V  v1  v2  V
   ├H┼a1┤      ├H┼H┼b1┤
   V  v1  V     V  v1  v2  V
   └H┴H┘       └H┴H┴H┘
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Each tiling of $\mathbb{R}^2$ with the above Wang tiles corresponds to a tiling with the $2 \times 2$- and $3 \times 3$-rectangles and the other way round.

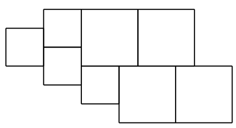
Write $k \perp l$ to mean that $k$ and $l$ are coprime.

**Lemma 2.1.** *The dynamical system corresponding to two rectangles as tiles is topologically mixing if and only if $m_1 \perp m_2$ and $n_1 \perp n_2$.*

*Proof.* Assume that $k > 1$ is the greatest common divisor of $m_1$ and $m_2$. We want to show that the dynamical system is not mixing. To show this we take one of the rectangles and define the open set $O$ to be the set of tilings of the plane such that the fixed rectangle occurs at position $(0, 0)$. If the system is mixing there would be a $L$ such that $O \cap \sigma_{(-l,0)}(O) \neq \emptyset$ for $l \geq L$. However, for $l = ak + 1 \geq L$ it is easy to see that the intersection must be empty.

Now we assume that the condition is satisfied and we want to show that the system is mixing. We have to take two non-empty open sets $O_1, O_2$, where we can assume that the sets are defined by fixed patterns around the position $(0, 0)$. We want to construct a tiling of a vertical strip with finite horizontal width such that the left and the right boundary are flat and that the pattern for $O_1$ occurs at $(0, 0)$. If we can do the same for the pattern of $O_2$, we can move the two strips away from each other such that the distance is big. Then the distance is a sum of positive multiples of $m_1$ and $m_2$ as the two integers are coprime. Therefore one can easily tile the gap between. This proves mixing since one can do the same for horizontal strips.

Take a point $x \in O_1$. To construct a tiling of a vertical strip we have to find a finite part of the tiling $x$ which contains the pattern and allows us to construct the tiling of the strip. This can be done for instance if the finite part is such that each horizontal part of the boundary has a length which is a sum of positive multiples of $m_1$ and $m_2$.



In the above picture this condition is satisfied but if one removes the rightmost rectangle then the condition would fail. If we already have such a region then we can put the two rectangles above (resp. below) all the horizontal parts corresponding to how the length is written as a sum of positive multiples of $m_1$ and $m_2$ and get a tiling of the vertical strip.

To find such a finite part of $x$ we have to find a path going around the pattern of $O_1$ which splits into four paths. We start somewhere to the right and above the pattern of $O_1$ at a point which is on an outer edge of a rectangle in $x$. First we want to go down and to the right always on the outer edges of the rectangles in $x$. There exists a natural number $N$ such that each $n \geq N$ is a sum of positive multiples of $m_1$ and $m_2$. The rule which forces our path is to go right until we reach a point where this is impossible or our actual horizontal step has reached a length of $N$ where we go down when possible. The path eventually leads under the lowest part of the pattern, then we stop and change our rule. From here on we want to go to the left and down. We go to the left whenever possible and down when not. Eventually we are left to the left end of the pattern and again we change our rule. Now we want to go up and to the left. Our rule is now to go left until our horizontal step has reached a length of $N$ or left is not possible, then we go up. Eventually we are above the upper end of the pattern, then we start going to the right when possible or up. If we hit our path we stop. If we miss our path we will get above our previous starting point, stop and start again in our starting point to go to the left when possible or up and hit the path.

The above path encloses a finite part of $x$ which contains the pattern corresponding to $O_1$, is exactly tiled by rectangles and satisfies the constraints about the horizontal steps which concludes our proof. □

To calculate the tiling group we again look at our Wang tiles and we see that each colour in the interior of the rectangles can be expressed in terms of $H$ and $V$. The two rectangles then correspond to the two relations $H^{m_1}V^{n_1} = V^{n_1}H^{m_1}$ and $H^{m_2}V^{n_2} = V^{n_2}H^{m_2}$.

In the following $C_k * C_l$ denotes the non-abelian free product of the cyclic groups $C_k \cong \mathbb{Z}/k\mathbb{Z}$ and $C_l$ which is given by the set of all alternating products of elements in the two groups. We also write the cyclic groups multiplicatively, where $H$ (resp. $V$) stands for the generator of the first (resp. second) cyclic group in the free product. The map $\eta : C_k * C_l \to C_k \times C_l$ denotes the canonical map to the direct product, where the elements of the first group are multiplied regardless of their positions to get the first component, and the same for the second group.

The following proposition generalizes [2, Section 3.1].

**Proposition 2.2.** *Let $m_1 \perp m_2$ and $n_1 \perp n_2$, so that the dynamical system corresponding to the two rectangles is topologically mixing. The tiling group $\Gamma$ of the system is isomorphic to*

$$\Delta \subseteq \mathbb{Z}^2 \times (C_{m_1} * C_{n_2}) \times (C_{m_2} * C_{n_1})$$

*where $\Delta$ is defined to be those elements $\big((k_1, k_2), \gamma_1, \gamma_2\big)$ which satisfy*

$$\begin{aligned} \eta(\gamma_1) &= \Big(H^{k_1 \bmod m_1}, V^{k_2 \bmod n_2}\Big) \text{ and} \\ \eta(\gamma_2) &= \Big(H^{k_1 \bmod m_2}, V^{k_2 \bmod n_1}\Big). \end{aligned}$$

*The map which gives an isomorphism is defined by*

$$\begin{aligned} \phi : H &\mapsto \big((1,0), H, H\big) \text{ and} \\ \phi : V &\mapsto \big((0,1), V, V\big). \end{aligned}$$

*Proof.* First we want to prove that $\phi$ is well-defined, i. e. that $\phi(\Gamma)) \subseteq \Delta$ and that the two relations

$$H^{m_i}V^{n_i}H^{-m_i}V^{-n_i} = \mathbf{1} \ (i = 1, 2)$$

are respected by $\phi$. As $\Delta$ is a subgroup and $\phi(H), \phi(V) \in \Delta$ we already know that $\phi(\Gamma) \subseteq \Delta$. The easy calculation

$$\phi(H)^{m_1}\phi(V)^{n_1}\phi(H)^{-m_1}\phi(V)^{-n_1} = \big((0,0), \mathbf{1}\cdot V^{n_1}\cdot \mathbf{1}\cdot V^{-n_1}, H^{m_1}\cdot \mathbf{1}\cdot H^{-m_1}\cdot \mathbf{1})\big) = \mathbf{1}$$

and a second one like this show that $\phi$ is well-defined. To prove that $\phi$ is an isomorphism we use a chain of normal subgroups in $\Gamma$ and in $\Delta$.

It is easy to see that $H$ commutes with $V^{n_1n_2}$ and $V$ with $H^{m_1m_2}$ as $m_1 \perp m_2$ and $n_1 \perp n_2$. Therefore $V^{n_1n_2}$ and $H^{m_1m_2}$ are in the center of the group. Let $G_1$ be the subgroup generated by those two elements. $G_1$ is a normal subgroup and is mapped injectively to the subgroup of $\Delta$ generated by $\big((m_1m_2, 0), \mathbf{1}, \mathbf{1}\big)$ and $\big((0, n_1n_2), \mathbf{1}, \mathbf{1}\big)$.

The quotient $\Delta/\phi(G_1)$ is isomorphic to

$$(C_{m_1} * C_{n_2}) \times (C_{m_2} * C_{n_1}) \tag{5}$$

as the first coordinates of the elements in $\Delta$ can be calculated from the $\eta$-images of the others modulo a point in $\phi(G_1)$ by using the Chinese remainder theorem.

Let $G_2$ be the subgroup generated by $H^{m_2}, V^{n_1}$. We have to show that $G_2$ is a normal subgroup. For instance $V^aH^{m_2}V^{-a}$ must be an element of $G_2$. There exists an integer $k$ such that $n_2|kn_1 - 1$. As $V^{n_2}$ commutes with $H^{n_2}$ we have

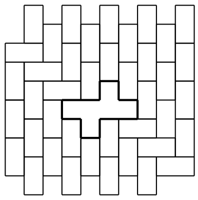
$$V^aH^{m_2}V^{-a} = V^aV^{an_2\left(\frac{kn_1-1}{n_2}\right)}H^{m_2}V^{-an_2\left(\frac{kn_1-1}{n_2}\right)}V^{-a} = V^{akn_1}H^{m_2}V^{-akn_1} \in G_2.$$

The same can be done for $H^aV^{n_1}H^{-a}$, which proves that $G_2$ is a normal subgroup. Assume $\phi(\gamma) = \mathbf{1}$ for $\gamma \in G_2/G_1$. Then we write $\gamma$ as an alternate product of powers of $H^{m_2}$ and of $V^{n_1}$ where we cancel the factors of the form $(H^{m_2})^{m_1}$ or $(V^{n_1})^{n_2}$. If the product is non-empty then $\phi(\gamma)$ is non-trivial as no cancellation in $C_{m_1} * C_{n_2}$ takes place. Therefore $\phi|_{G_2/G_1}$ is injective. As $m_1 \perp m_2$ and $n_1 \perp n_2$ the image $\phi(G_2/G_1)$ is $(C_{m_1} * C_{n_2}) \times \{\mathbf{1}\}$.

If we take $\Gamma$ modulo $G_2$ we see that the group we get is defined only by the relations $H^{m_2} = V^{n_1} = \mathbf{1}$. Therefore it is isomorphic to $C_{m_2} * C_{n_1}$ which is the quotient of the group in (5) modulo $\phi(G_2/G_1)$. This proves that $\phi$ is bijective. □

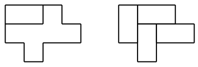
For the proof that the tiling cocycle is a fundamental cocycle we will need a result which says roughly that the tiling group describes the shift space quite well. For instance, if the tiling cocycle characterized tile-ability, this would be sufficient. Unfortunately, the tiling cocycle for the dominoes does not characterize tile-ability of rectangular holes which can be seen in the following example due to Sam Lightwood.

**Example 2.3.** *If we look at the ordinary dominoes with dimensions* $2 \times 1$ *and* $1 \times 2$ *then the following pattern*



*extended to infinity gives us a hole which is not tile-able by the dominoes.*

*First we would like to point out that the tiling cocycle of the path around the hole vanishes, so that our algebraic condition is satisfied. This can be seen in two ways. If we take only the shape of the hole and add a domino, the cocycle value corresponding to the shape of the region will not change. By doing this as shown below*



*the hole becomes tile-able, so the cocycle value has to vanish. The second way to see this is a better analysis of the tiling group* (*see* [4, Section 6]). *For this tiling system the tiling cocycle value corresponding to a hole counts the difference between the number of white and black squares inside the hole if one colours the plane like a chess board. This difference vanishes in our example.*

*Let us consider a hole which is constructed by removing dominoes in the above tiling. The original and also the new tiling contain two regular patterns above and below the series of horizontal dominoes. If we remove finitely many dominoes out of this pattern, the only way to tile the uncovered space is to rebuild the pattern. Therefore one comes eventually to the original tiling, where the hole is not tile-able.*

*Together we have now that the tiling cocycle vanishes for this hole, but it is not tile-able even after enlarging the hole.*

This example shows that the tiling cocycle is somehow not good enough to characterize tile-ability and one would ask for better cocycles. But we will show that the tiling cocycle is a fundamental cocycle, so there is no cocycle which can characterize tile-ability of holes.

For convenience we state here a completely elementary fact about the tile-ability of some rectangles using our rectangles.

**Lemma 2.4.** *The rectangles*

$$[0, km_1 + lm_2] \times [0, n_1 n_2] \text{ and } [0, m_1 m_2] \times [0, kn_1 + ln_2]$$

*for* $k, l \geq 0$ *are tile-able by using the rectangles* $m_1 \times n_1$ *and* $m_2 \times n_2$.

We will now state and prove the proposition which will show that the tiling group describes the shift space well.

**Proposition 2.5.** *Let* $k$ *be a positive integer and* $Y$ *be the set of tilings of the strip* $[0, km_1 m_2] \times \mathbb{R}$ *by the two rectangles such that*

$$c_f\big((km_1 m_2, 0), y\big) = H^{km_1 m_2}.$$

*The shift space* $Y$ *is topologically transitive.*

In the following proof the term weight of the boundary refers to the cocycle value of the path which follows the boundary but without the possible simplifications. We also write in contrast to Equation (1) the weight from the left to the right.

*Proof.* Let $G_1 = C_{m_1} * C_{n_2}$ and $G_2 = C_{m_2} * C_{n_1}$ be the second resp. the third factor of the group introduced in Proposition 2.2. For the proof of transitivity we have to look at the various shapes of the upper boundary of finite tilings. We will show that each boundary occurs as a bottom boundary of a tiling with the upper boundary being flat. By symmetry this suffices to establish topological transitivity.

We choose the boundary following the outer edges of the tiles such that each horizontal step has a length which is representable as a sum of positive multiples of $m_1$ and $m_2$. To find such a boundary we start on the left of the strip and search for a way to the right. If possible we go to the right, if not we go up until it is possible again to go to the right. This path gives us a boundary which satisfies the constraints because the tiles just below the horizontal step give us the representation. We also assume without loss of generality that the path describing the boundary starts and ends at the same vertical height.

In the following we split the original boundary into its horizontal and vertical steps. We call a step $G_i$-trivial if the corresponding value in $G_i$ is vanishing. We call a part of the boundary $G_i$-trivial if each horizontal and vertical step of this part is $G_i$-trivial. Our assumption is that the value corresponding to the boundary is trivial. If it is in addition $G_1$-trivial and $G_2$-trivial then each horizontal (resp. vertical) step has a length which is a multiple of $m_1m_2$ (resp. $n_1n_2$). But then it is easy to find a tiling which ends with a flat boundary above due to Lemma 2.4. Therefore we want to add some rectangles above the original boundary until the boundary is $G_1$-trivial and afterwards until it is $G_2$-trivial.

For an easier understanding of the following it is useful to keep track of Example 2.6. We prove the existence of the $G_1$-trivial boundary by induction on the number of horizontal and vertical steps which are not $G_1$-trivial. The weight of the boundary looks like

$$H^{a_0} \cdot V^{b_1} \cdot H^{a_1} \cdot \dots \cdot V^{b_k} \cdot H^{a_k} \cdot V^{b_{k+1}} \tag{6}$$

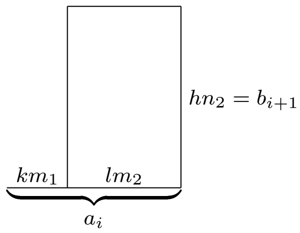
with $a_0, b_{k+1} \geq 0$ and $a_i, b_i > 0$ for $0 < i \leq k$. The factors $H^{a_i}$ and $V^{b_i}$ correspond to the horizontal and vertical steps of the boundary respectively. As the value is vanishing in $G_1$ there is a possible cancellation. There are two possibilities for the first cancellation: some non-trivial factors of the form $V^{b_i}$ (or of the form $H^{a_i}$) are separated by trivial factors of the form $V^{b_j}$ or $H^{a_j}$. The collection of the trivial factors corresponds to a part of the boundary which is $G_1$-trivial per definition. We know that this part consists of horizontal (resp. vertical) steps of a length which is a multiple of $m_1$ (resp. $n_2$). Therefore we can use the rectangle $m_1 \times n_1$ to raise this part of the boundary by a multiple of $n_1$.

Assume $V^{b_i}$ and $V^{b_j}$ with $i < j$ are non-trivial and separated by trivial factors. The raising described above ends with a boundary whose weight looks like

$$\cdots \cdot V^{b_i+kn_1} \cdot (\text{trivial factors}) \cdot V^{b_j-kn_1} \cdot \cdots$$

and we choose $k$ such that $b_i + kn_1$ is divisible by $n_2$. Therefore the factor $V^{b_i+kn_1}$ is $G_1$-trivial and we have reduced the number of non-trivial factors.

If the non-trivial factors $H^{a_i}$ and $H^{a_j}$ are separated by some trivial factors, we write the first exponent as a sum $a_i = km_1 + lm_2$ with $k, l \geq 0$. As $H^{a_i}$ is not $G_1$-trivial we have $l > 0$. If the next horizontal step lies below this one (that means if $b_{i+1} < 0$) then we raise this step by the height of a multiple of $n_1 n_2$ by filling the enclosed area with rectangles (see Lemma 2.4) such that the new $b_{i+1} > 0$. As $V^{b_{i+1}}$ is trivial in $G_1$, we know $n_2 | b_{i+1}$. Therefore we can fill the rectangle

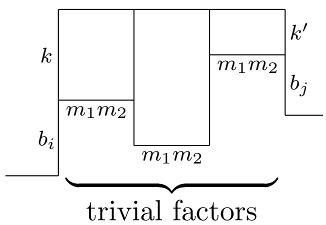



with the $m_2 \times n_2$-rectangle. The changed boundary has now a weight of the form

$$\cdots \cdot H^{a_i - lm_2} \cdot V^{b_{i+1}} \cdot H^{a_{i+1}+lm_2} \cdot \cdots.$$

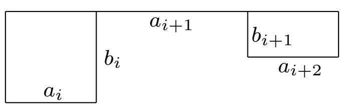
The new exponent $a_i' = a_i - lm_2 = km_1$ shows that the corresponding factor $H^{a_i'} = \mathbf{1} \in G_1$. If $j = i + 1$ we are done because we have again reduced the number of non-trivial factors. If $j > i+1$ we repeat the above modifications until $lm_2$ is finally added to $a_j$.

Now our assumption is that the boundary is $G_1$-trivial and its corresponding cocycle value is trivial in $G_2$. As above we prove the existence of a $G_1$-trivial boundary which is also $G_2$-trivial by induction. Again we search for the first cancellation of non-trivial factors in the product in (6) viewed in $G_2$. Assume that $V^{b_i}$ and $V^{b_j}$ with $i < j$ are non-trivial in $G_2$ and are separated by trivial factors. Then we add tiles such that the boundary looks like



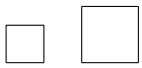

which reduces the number of non-trivial factors if we choose the height $k$ such that the first vertical step has a length $b_i + k$ which is divisible by $n_1 n_2$.

If $H^{a_i}$ and $H^{a_j}$ are non-trivial and separated by trivial factors, we can simplify the boundary corresponding to the trivial factors like in the picture above with $k$ being a multiple of $n_1 n_2$. Therefore the trivial factors simplify to $V^{b_i+k} H^{lm_1 m_2} V^{b_j+k}$ and we can assume that $j = i + 2$. In the picture

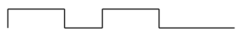


the rectangles are tile-able as $n_1 n_2 | b_i, b_{i+1}$. As the changed boundary is still $G_1$-trivial and has now less $G_2$-nontrivial factors we have finished the proof that we can reach a boundary which is both $G_1$-trivial and $G_2$-trivial. □

**Example 2.6.** *We again use the two tiles* $2 \times 2$ *and* $3 \times 3$.



*We take the boundary*



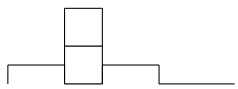
*of a possible tiling below and want to construct a flat boundary above and a tiling of the region between. We want to use the proof of Proposition 2.5 so we first check if the assumption is satisfied. In the group* $G_1 = C_2 * C_3$ *we calculate*

$$VH^3V^{-1}H^2VH^3V^{-1}H^4 = VH^3V^{-1}VH^3V^{-1} = VH^6V^{-1} = \mathbf{1}$$

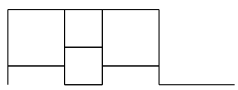
*and in* $G_2 = C_3 * C_2$ *we have*

$$VH^3V^{-1}H^2VH^3V^{-1}H^4 = VV^{-1}H^2VV^{-1}H^4 = H^6 = \mathbf{1}.$$

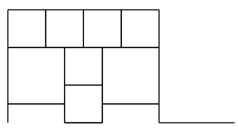
*So in the first group the cancellation starts at* $V^{-1}H^2V$ *and we only have to use the* $2 \times 2$*-rectangle*



*to get to the term* $V^3H^2V^{-3}$ *instead which is* $G_1$*-trivial. Now* $H^3$ *left and right of this term are non-trivial and we can use the* $3 \times 3$*-rectangle to get the picture.*

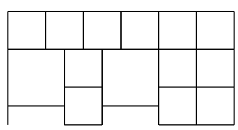


*Here* $V^4H^8V^{-4}$ *is the term which simplifies next. Therefore we put again the* $2 \times 2$*-rectangles on top.*



*We have achieved a* $G_1$*-trivial boundary with the weight* $V^6H^8V^{-6}H^4$.

*Our last step is to achieve a* $G_2$*-trivial boundary. The only non-trivial factors are* $H^8$ *and* $H^4$ *which are separated by* $V^{-6}$. *We can fill the rectangle with the edges* $V^{-6}$ *and* $H^4$ *and obtain the tiling*



*whose boundary is* $G_1$*-trivial and* $G_2$*-trivial. In fact it is a 'flat' boundary.*

Now we turn our attention to cocycles and prove that the tiling cocycle is a fundamental cocycle.

**Theorem 2.7.** *The tiling cocycle for the system corresponding to two rectangles* $m_1 \times n_1$ *and* $m_2 \times n_2$ *with* $m_1 \perp m_2$ *and* $n_1 \perp n_2$ *is fundamental.*

*Proof.* We write $c_f$ for the tiling cocycle and $\Gamma$ for the tiling group. Let $c : \mathbb{Z}^2 \times X \to G$ be a second continuous cocycle. We have to find a homomorphism $\eta$ and a function $b : X \to G$ with

$$c(\mathbf{n}, x) = b\big(\sigma_{\mathbf{n}}(x)\big)\eta\big(c_f(\mathbf{n}, x)\big)b(x)^{-1}$$

for all $\mathbf{n} \in \mathbb{Z}^2$ and $x \in X$.

As the two functions $c\big((1,0),\cdot\big)$ and $c\big((0,1),\cdot\big)$ are continuous they depend only on a fixed part of the tiling. More precisely there exists a natural number $r$ such that $c\big((1,0),x\big) = c\big((1,0),y\big)$ and $c\big((0,1),x\big) = c\big((0,1),y\big)$ whenever the two tiling $x$ and $y$ coincide on the square $[-r,r]^2$. We call this number $r$ a sight radius.

For the purpose of finding the homomorphism $\eta$ it would be good to produce a situation where the unknown function $b$ is constant. This will be achieved by looking at tilings $x$ and tuples $\mathbf{n} \in \mathbb{Z}^2$ with the property that $x$ and $\sigma_{\mathbf{n}}x$ coincide on a large square. For this we call the periodic pattern consisting of the first rectangle and aligned such that the corner of one of the rectangle is in $(0,0)$ the standard pattern $S$.

We will assume for convenience that $m_1m_2|r$ and $n_1n_2|r$. We always use the pattern $S$ big enough such that at least the area $[-r,r]^2$ is covered by rectangles. The set $O_S$ is defined as the open set of tilings which look like $S$ in the square $[-r,r]^2$.

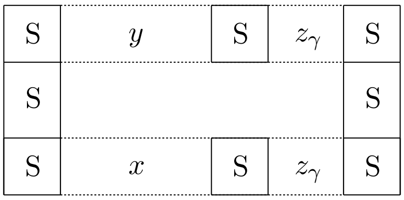
Let $x \in X$ be such that $x \in O_S$ and $\sigma_{(n,0)}(x) \in O_S$ for some $n \geq 1$, i. e. that the standard pattern occurs in $x$ at the positions $(0,0)$ and $(n,0)$. We would like to set

$$\eta\Big(c_f\big((n,0),x\big)\Big) = c\big((n,0),x\big). \tag{7}$$

In order to prove that this is well-defined we have to look at two $x, y$ with the same tiling cocycle value. But then the two horizontal paths which appear have the same length $n$. We look at the rectangles in $x$ and separately for them in $y$ which hit the strip $[-r, n+r] \times [-r,r]$ and get two pictures like


where the rectangular tiles may stand out of the dashed line. In order to get to a situation where we can apply Proposition 2.5 we fix for all elements $\gamma \in \Gamma$ which appears in this way a point $z_\gamma$ with $z_\gamma \in O_S \cap \sigma_{(0,l)}^{-1}O_S$ such that $c_f\big((0,l),z_\gamma\big)\gamma = H^{n+l}$ and $m_1m_2|n+l$. If we now glue the two stripes for $x$ and $z_\gamma$ such that the right standard pattern of $x$ is exactly the left standard pattern for $z_\gamma$ then the tiling cocycle value from the left standard pattern for $x$ to the right one for $z_\gamma$ is equal to $H^{n+l}$. Now we can view this long strip as a finite part of a tiling of the strip $[r, n+l-r] \times [-r,r]$ where we removed the two standard pattern left and right. This point satisfies the assumption in Proposition 2.5 so we can find a finite tiling above our strip inside $[r, n+l-r] \times \mathbb{R}$ which ends with a flat boundary above. If necessary we add by using Lemma 2.4 and $m_1m_2|n+l-2r$ additional rectangles above the boundary such that the total height from the $x$-axes to the top boundary is divisible by $n_1n_2$.

We do the same for $y$ but this time we search for a flat boundary below of the strip of $y$ and with the same restriction of the height. If we now put the tiling from $y$ above the one from $x$ and add left and right of this strip the standard pattern then we get something like




where the height between the two original horizontal strip is divisible by $n_1 n_2$. If we use the cocycle $c$ we get the relation

$$gc\big((l,0), z_\gamma\big)c\big((n,0), x)\big) = c\big((l,0), z_\gamma\big)c\big((n,0), y\big)g$$

where $g$ is the cocycle value for the path upwards. As we can do the same with the doubled height, the same equation holds also for $g^2$. Therefore we have proved that $c\big((n,0), x)\big) = c\big((n,0), y\big)$.

It is now easy to check that

$$\eta(\gamma\delta) = \eta(\gamma)\eta(\delta) \tag{8}$$

for $\gamma$ and $\delta$ appearing as a cocycle value for a horizontal path from the standard pattern to itself. We extend our definition by setting

$$\eta(H^{-km_1m_2}\gamma) = \eta(H^{m_1m_1})^{-k}\eta(\gamma)$$

for $k \geq 0$ and $\gamma$ like above. This is a valid definition because for

$$H^{-km_1m_2}\gamma = H^{-lm_1m_2}\delta$$

we get

$$\begin{aligned} H^{lm_1m_2}\gamma &= H^{km_1m_2}\delta \\ \eta(H^{m_1m_1})^l\eta(\gamma) &= \eta(H^{m_1m_1})^k\eta(\delta) \text{ and} \\ \eta(H^{m_1m_1})^{-k}\eta(\gamma) &= \eta(H^{m_1m_1})^l\eta(\delta). \end{aligned}$$

As $H^{m_1m_2}$ is in the center of $\Gamma$ this extended definition still satisfies Equation (8).

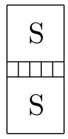
Let $\beta \in \Gamma$ be such that the exponents of $V$ in $\beta$ sum up to zero. We would like to find $k \geq 0$ and $\gamma$ like above such that $\beta = H^{-km_1m_2}\gamma$ and $\eta(\beta)$ is defined. As $H^{m_1m_2}$ is in the center of $\Gamma$ we can increase the exponents of $H$ somewhere in $\beta$ by increasing $k$ simultaneously. Therefore we can assume that $\gamma$ starts and ends with $H^r$ and each exponent of $H$ appearing in the product between is representable by a sum of positive multiples of $m_1$ and $m_2$. But for a $\gamma$ with this properties it is easy to find a point $x \in X$ and an integer $n \geq 1$ with $x, \sigma_{(n,0)}x \in O_S$ and $c_f\big((n,0), x\big) = \gamma$. So we have already defined $\eta(\beta)$ for all "horizontal" elements $\beta \in \Gamma$.

What we have to do next is to define $\eta(V)$. We can not do this directly, because we still want to work with paths starting and ending in the standard pattern. So we define

$$\eta(V^{n_1}) = c\big((0, n_1), x_S\big)$$

where $x_S$ is the periodic point corresponding to the standard pattern $S$. As $x_S$ has vertical period $n_1$ this path satisfies the above restraint.

Let $y$ be a point of the following form




where between the two standard patterns there is a layer of the $m_2 \times n_2$-rectangles. We define

$$\eta(V^{2r+n_2}) = c\big((0, 2r+n_2), y\big)$$

as the cocycle value for the path from the bottom standard pattern to the top one.

It is now easy to check that both definitions are compatible with the definitions of $\eta$ for 'horizontal' elements. If $\gamma \in \Gamma$ corresponds to a horizontal path, the same is true for $V^{n_1}\gamma V^{-n_1}$ and we have

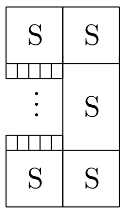
$$\eta(V^{n_1}\gamma V^{-n_1}) = \eta(V^{n_1})\eta(\gamma)\eta(V^{n_1})^{-1}$$

which can be seen immediately by looking at the corresponding paths in a point $x$ which defines $\eta(\gamma)$. One can do the same for $\eta(V^{2r+n_2})$.

As $n_1|r$ we get that $n_1(2r+n_2)$ is the least common multiple of $n_1$ and $2r+n_2$. We have to check if

$$\eta(V^{n_1})^{2r+n_2} = \eta(V^{2r+n_2})^{n_1}.$$

For the proof of this we take a point of the following form



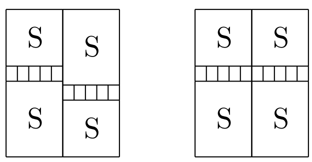

where the left column is chosen such that the cocycle value from the bottom standard pattern to the top one is $\eta(V^{2r+n_2})^{n_1}$. So we get

$$\eta(H^{2r})\eta(V^{2r+n_2})^{n_1} = \eta(V^{n_1})^{2r+n_2}\eta(H^{2r}),$$

and if we use the same loop in the point $x_S$ we get

$$\eta(H^{2r})\eta(V^{n_1})^{2r+n_2} = \eta(V^{n_1})^{2r+n_2}\eta(H^{2r}).$$

The commutativity law for $\eta(V^{n_1})$ and $\eta(V^{2r+n_2})$ can be proved similarly by using the patterns:




So the definition

$$\eta(V) = \eta(V^{n_1})^a \eta(V^{2r+n_2})^b,$$

where $a$ and $b$ are such that $an_1 + b(2r+n_2) = 1$, extends $\eta$ to a group homomorphism from $\Gamma$ to $G$.

Now we can define $c'(\mathbf{n},x) = \eta\big(c_f(\mathbf{n},x)\big)$ and have two cocycles $c$ and $c'$ with values in the same group $G$. So the assumptions of the next proposition are satisfied and this concludes the proof. □

The proof of Theorem 2.7 is completed by the following Proposition.

**Proposition 2.8.** *Let $X$ be a topologically mixing two-dimensional shift of finite type, $c$ and $c'$ be two continuous cocycles with values in a discrete group and $r$ be its sight radius. Fix a point $S$ and define $O_S$ to be the set of tilings which look like $S$ in the square $[-r,r]^2$. Assume that for all $x \in X$ and $n \in \mathbb{Z}$ with $x, \sigma_{(n,0)}(x) \in O_S$ the equation*

$$c\big((n,0),x\big) = c'\big((n,0),x\big) \tag{9}$$

*holds. In addition we have the same for two particular patterns $y_1, y_2$ and two numbers $l_1 \perp l_2$ as vertical distances. Then the two cocycles are cohomologous.*

*Proof.* We define two functions $b, b' : X \to G$ as follows. Take an arbitrary point $x \in X$. As the shift is topologically mixing we know that there is a point $z^+(x) \in X$ which looks exactly like $x$ in the square $[-r,r]^2$ and like the standard pattern in $[-r+n(x), r+n(x)] \times [-r,r]$ for some nonnegative $n(x)$. For $x \in O_S$ we choose $n(x) = 0$. We choose for each point $y$ which looks like $x$ in $[-r,r]^2$ the same $z^+(y)$, therefore the definitions

$$\begin{aligned} b(x) &= c\big((n(x),0), z^+(x)\big) \\ b'(x) &= c'\big((n(x),0), z^+(x)\big) \end{aligned}$$

gives us two continuous functions on $X$ with values in $G$. We can do the same for a non positive $m(x)$ and define the two functions $d, d' : X \to G$ via the corresponding tiling $z^-(x)$:

$$\begin{aligned} d(x) &= c\big((m(x),0), z^-(x)\big)^{-1} \\ d'(x) &= c'\big((m(x),0), z^-(x)\big)^{-1} \end{aligned}$$

For the definitions of the functions $b, b'$ only the right half of the tiling $z^+(x)$ is important and for the functions $d, d'$ only the left half of $z^-(x)$.

Take a positive $k$ and any $x \in X$. We define $y = \sigma_{(k,0)}(x)$. Now Equation (9) and the tiling below (which is build from the various parts as labeled)

| | | | | | | |
|---|---|---|---|---|---|---|
| S | $z^-(x)$ | $x$ | | $y$ | $z^+(y)$ | S |

leads to the equation

$$b(y)c\big((k,0),x\big)d(x) = b'(y)c'\big((k,0),x\big)d'(x). \tag{10}$$

The same argument with

| | | | | |
|---|---|---|---|---|
| S | $z^-(x)$ | $x$ | $z^+(x)$ | S |

gives us

$$b(x)d(x) = b'(x)d'(x). \tag{11}$$

If we multiply (10) from the right with the inverse of (11) then we get

$$b(y)c\big((k,0),x\big)b^{-1}(x) = b'(y)c'\big((k,0),x\big)b'^{-1}(x).$$

If we now define the function $e(x) = b(x)^{-1}b'(x)$ then we conclude that

$$c\big((k,0),x\big) = e(\sigma_{(k,0)}(x))c'\big((k,0),x\big)e^{-1}(x). \tag{12}$$

Therefore we have already showed the desired equation for horizontal paths.

We define

$$c''(\mathbf{n},x) = e(\sigma_{\mathbf{n}}(x))c'(\mathbf{n},x)e^{-1}(x).$$

We will show that $c$ and $c''$ are equal. We have already the equality along horizontal paths. From our assumptions we also know that $c$ and $c''$ are equal for $y_i$ along $\mathbf{n} = (0,l_i)$ because for points in $O_S$ the value $e$ vanishes.

For an arbitrary $x \in X$ the mixing property implies that there exists a $k \geq 1$ and a point $z \in X$ which looks like $y_i$ in $[-r,r] \times [-r,l_i+r]$ and like $\sigma_{(k,0)}(x)$ in $[k-r,k+r] \times [-r,l_i+r]$. The cocycle equation for $c$ and $c''$ applied to $z$ yields $c\big((0,l_i),x\big) = c''\big((0,l_i),x\big)$ because we know $c$ and $c''$ are equal along the other three sides of the rectangle between $(0,0)$ and $(k,l_i)$. Because $l_1$ and $l_2$ generate $\mathbb{Z}$ we get the equality of $c$ and $c''$. □

## 3. Paths

In this section we will consider systems of paths in the plane. We will look both at directed and undirected coloured paths which will be allowed either to cross each other or not. In order to describe the corresponding system in more detail we define the following set of tiles.

The tile in

$$\mathcal{E} = \left\{ \begin{array}{ccc} \ulcorner & H & \urcorner \\ V & & V \\ \llcorner & H & \lrcorner \end{array} \right\}$$

stands for an empty space. The tiles in

$$\mathcal{P}^d_a = \left\{ \begin{array}{ccc} \ulcorner & a_u & \urcorner \\ V & \uparrow & V \\ \llcorner & a_u & \lrcorner \end{array}, \begin{array}{ccc} \ulcorner & a_d & \urcorner \\ V & \downarrow & V \\ \llcorner & a_d & \lrcorner \end{array}, \begin{array}{ccc} \ulcorner & H & \urcorner \\ a_r & \rightarrow & a_r \\ \llcorner & H & \lrcorner \end{array}, \begin{array}{ccc} \ulcorner & H & \urcorner \\ a_l & \leftarrow & a_l \\ \llcorner & H & \lrcorner \end{array}, \begin{array}{ccc} \ulcorner & H & \urcorner \\ a_l & \nwarrow & V \\ \llcorner & a_u & \lrcorner \end{array}, \dots \right\}$$

describe directed paths with colour $a$ where the subscript of $a$ in the tiles always indicates the direction of the path crossing this edge. In addition to the above tiles we allow paths of different colour to share a unit square. This means that the set

$$\mathcal{N}^d_{a,b} = \left\{ \begin{array}{ccc} \ulcorner & b_r & \urcorner \\ a_l & \nwarrow\nwarrow & b_u \\ \llcorner & a_u & \lrcorner \end{array}, \dots \right\}$$

for two colours $a, b$ is allowed. Here we only list the cases where the paths with the different colours are not intersecting each other.

If we have a tiling of the plane by using the tiles in $\mathcal{E}$, $\mathcal{P}^d_a$ and $\mathcal{N}^d_{a,a}$ then this corresponds exactly to a collection of paths in the plane with colour $a$. For two colours we get collections of paths in the plane, where each path has one of the two colours. But the paths with colour $a$ are not crossing paths with colour $b$.

We can also allow paths of different colours to cross each other, this gives us for two colours $a, b$ the set

$$\mathcal{C}^d_{a,b} = \left\{ \begin{array}{ccc} \ulcorner & a_u & \urcorner \\ b_r & \overset{\uparrow}{\rightarrow} & b_r \\ \llcorner & a_u & \lrcorner \end{array}, \begin{array}{ccc} \ulcorner & a_d & \urcorner \\ b_r & \underset{\downarrow}{\rightarrow} & b_r \\ \llcorner & a_d & \lrcorner \end{array}, \dots \right\}.$$

If we allow for two colours $a, b$ all the tiles we have had up to now, we get a tiling space where each tiling corresponds to a collection of paths with two colours and the paths can cross each other.

For the undirected case we can analogously define the set

$$\mathcal{P}_a^u = \left\{ \begin{array}{ccc} \ulcorner & a_h & \urcorner \\ V & | & V \\ \llcorner & a_h & \lrcorner \end{array}, \begin{array}{ccc} \ulcorner & H & \urcorner \\ a_v & — & a_v \\ \llcorner & H & \lrcorner \end{array}, \begin{array}{ccc} \ulcorner & H & \urcorner \\ a_v & \urcorner & V \\ \llcorner & a_h & \lrcorner \end{array}, \dots \right\}$$

and similarly $\mathcal{N}_{a,b}^u$ and $\mathcal{C}_{a,b}^u$.

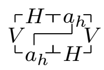
We can now form the four different tiling systems by taking the appropriate unions. For each finite set of colours $\mathcal{A}$ we can define

$$\begin{aligned} \mathcal{T_A}^d &= \mathcal{E} \cup \bigcup_{a \in \mathcal{A}} \mathcal{P}_a^d \cup \bigcup_{a,b \in \mathcal{A}} \mathcal{N}_{a,b}^d, \\ \mathcal{T_A}^{dc} &= \mathcal{E} \cup \bigcup_{a \in \mathcal{A}} \mathcal{P}_a^d \cup \bigcup_{a,b \in \mathcal{A}} \mathcal{N}_{a,b}^d \cup \bigcup_{a,b \in \mathcal{A}} \mathcal{C}_{a,b}^d, \\ \mathcal{T_A}^u &= \mathcal{E} \cup \bigcup_{a \in \mathcal{A}} \mathcal{P}_a^u \cup \bigcup_{a,b \in \mathcal{A}} \mathcal{N}_{a,b}^u, \\ \mathcal{T_A}^{uc} &= \mathcal{E} \cup \bigcup_{a \in \mathcal{A}} \mathcal{P}_a^u \cup \bigcup_{a,b \in \mathcal{A}} \mathcal{N}_{a,b}^u \cup \bigcup_{a,b \in \mathcal{A}} \mathcal{C}_{a,b}^u. \end{aligned}$$

For instance $\mathcal{T_A}^u$ defines the system of undirected paths with colours in $\mathcal{A}$ which are not allowed to cross.

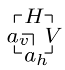
**Proposition 3.1.** *The tiling group $\Gamma_{\mathcal{A}}^u$ of the tiles in $\mathcal{T_A}^u$ is isomorphic to $\mathbb{Z}^2 \times \prod_{a \in \mathcal{A}}^* C_2$, where $C_2$ is the cyclic group of order two and $\prod^*$ denotes the free product. Similarly we have $\Gamma_{\mathcal{A}}^{uc} \equiv \mathbb{Z}^2 \times C_2^{\mathcal{A}}$, $\Gamma_{\mathcal{A}}^d \equiv \mathbb{Z}^2 \times \prod_{a \in \mathcal{A}}^* \mathbb{Z}$ and $\Gamma_{\mathcal{A}}^{dc} \equiv \mathbb{Z}^2 \times \mathbb{Z}^{\mathcal{A}}$.*

*Proof.* We know from the tile in $\mathcal{E}$ that $H$ and $V$ are commuting. The first tiles in $\mathcal{P}_a^u$ show that $V$ commutes with $a_h$ and $H$ with $a_v$. The finite tiling

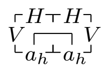



shows that $H$ also commutes with $a_h$.

We define $a = a_h H^{-1}$. The tile




shows that $a = a_v V^{-1}$. The equation $a^2 = \mathbf{1}$ can be seen from

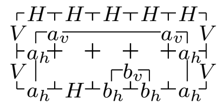



Therefore we have for each colour $a \in \mathcal{A}$ a corresponding element of order two in the tiling group. The colours commute with $H$ and $V$, but satisfy no non-trivial relation between them except that they all have order two. This proves the first case; the others are similar. □

Our next step is to prove that the tiling group has enough information to characterize the transitive components of the corresponding one-dimensional shift. So the next statement replaces Proposition 2.5.

**Proposition 3.2.** *Let $Y$ be the set of tilings in the strip $[0, n] \times \mathbb{R}$ by coloured paths (directed or not, allowed to cross or not) with the property that the tiling cocycle value from the left border of this strip to the right one is equal to $H^n$. The one-dimensional shift of finite type $Y$ is topologically mixing.*

*Proof.* Similarly as in the proof of Proposition 2.5 we can search the position of first cancellation of the cocycle value and simplify the boundary there. In the undirected case without crossing this would work for instance for the boundary given by $a_h, H, b_h, b_h, a_h$ like this:




□

As our systems are clearly mixing the proof of Theorem 2.7 now also proves the next theorem.

**Theorem 3.3.** *For any finite set of colours $\mathcal{A}$ the tiling cocycle for each of the above system of paths with colours in $\mathcal{A}$ is fundamental.*

We also want to check whether the tiling cocycle characterizes tile-ability of the holes in the different systems. The rest of this section is joint work with Sam Lightwood and answers this question.

**Example 3.4.** *We first consider the case of directed paths. We will see that the cocycle cannot characterize tile-ability if there is only one colour, so the same happens for more colours. In the picture*

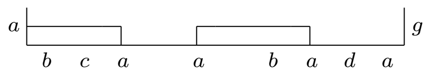
*all the paths are going down and you should consider the infinite pattern by continuing the picture to all directions. It is easy to see that the hole is not tile-able as four paths should go through the dotted line. But the same argument applies if you enlarge the hole. As the algebraic condition is satisfied we see that the cocycle does not characterize tile-ability.*

**Example 3.5.** *Let us now consider the case of undirected paths with at least two colours $a, b$ which are not allowed to cross. Here you can essentially use the last picture with the same argument.*

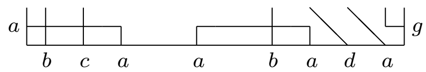
*Give the paths colours. Starting from the middle of the left edge the colours should be $a, b, a, b, b, a, b, a$ in clockwise order. So the algebraic condition is satisfied but the hole is not tile-able because otherwise there would be to many paths crossing the dotted line. If you introduce colours also on all other paths such that colours of two adjacent paths are different then the hole is also not tile-able if you enlarge it. Therefore the cocycle does not give a sufficient condition for tile-ability of holes.*

**Lemma 3.6.** *In the tiling space corresponding to $\mathcal{T}_{\mathcal{A}}{}^{uc}$ the tiling cocycle characterizes tile-ability of holes. In other words, assume you have finitely many colours on the edges of a big rectangle. You can connect them to each other with paths of the right colour if the paths are allowed to cross and the number of edges of a particular colour is even. This can be done in such a way that each unit square does not contain more than two paths.*

*Proof.* Assume that the sides of the rectangular hole are big corresponding to the number of colours. Take one of the sides of the rectangle. We assume it is the one at the bottom. If there is one colour which occurs more than once on this edge or on the two consecutive unit edges of the left and the right side than we take this colour $a$. Now we connect the first occurrence of this $a$ with the second, and so on by using only the first row of the rectangle. This can for instance look like:




After we have done this we connect all the other colours with the top boundary of this strip



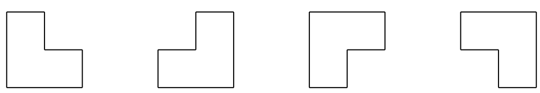

which is always possible because we have connected at least two copies of $a$ to each other. We can repeat this for the other colours as well and conclude that after this process each colour occurs at most $|\mathcal{A}|$ times in the bottom side as the colour $a$ can also appear on the two edges left and right in the next row. We do the same with the other sides as well.

Now we know that each colour does only occur at most $|\mathcal{A}|$ times on each side. As the length of the side is big we can shift the colours to the middle by another manipulation like the one above so that we have many edges on the left and on the right without colours. We repeat this on the other sides of the rectangle. If we now repeat the first process, there are no colours on the left resp. right edge. So this time we can eliminate copies of the same colour.

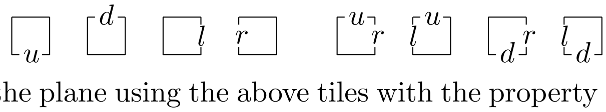
Now each side of the smaller rectangle has for each colour at most one edge with this colour. As the rectangle is still big there is enough space for connecting the remaining coloured edges. □

## 4. The L-Tiles

We define four tiles which we call the L-tiles.



The corresponding shift $X$ consists of all possible tiling of the plane with the above tiles which are aligned to the lattice. This could also be described using the following Wang tiles. The colour of the solid edges are $H$ or $V$ corresponding to the direction of the edge.




(13)

The tilings of the plane using the above tiles with the property that adjacent tiles have the same colour on their common edge correspond in a one-to-one way with the tilings using the L-tiles. This is because the last four tiles in (13) determine exactly two of their neighbor tiles to form the differently oriented L-tiles.

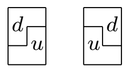
**Proposition 4.1.** *The tiling group of the L-tiles is isomorphic to* $\mathbb{Z}\times\mathbb{Z}\times C_3$ *where* $C_3$ *is the cyclic group of order three and the multiplication is given by*

$$(n_1, n_2, n_3)(m_1, m_2, m_3) = (n_1 + m_2, n_2 + m_2, n_3 + m_3 + n_2 m_1).$$

*The isomorphism maps* $H$ *to* $(1, 0, 0)$ *and* $V$ *to* $(0, 1, 0)$.

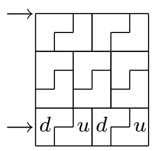
The group in the proposition is a factor of the discrete Heisenberg group. If we write $x^{n_1}y^{n_2}z^{n_3}$ instead of the triple $(n_1, n_2, n_3)$, then it is easy to check that $z$ lies in the center of the group and that $y^{-1}x^{-1}yx = z$.

*Proof.* As before we examine tilings to see algebraic relations. As the two tilings




have the same boundary, we know that the cocycle has the same value when one goes from one side to the other. So we get the equalities $uH = Hu$ and $dH = Hd$. A similar argument shows that $ud = du$.

If we go along the boundary of the above rectangles we see that $V^3$ (resp. $H^3$) commutes with $H^2$ (resp. $V^2$). Therefore $H$ commutes with $V^6$ but as $u = V^{-1}HV$ and $d = VHV^{-1}$ this is also true for $u$ and $d$. In the picture




the arrows have a vertical distance of 6 units, therefore we get

$$H^6 = V^6(uHd)^2V^{-6}.$$

We define

$$\overline{u} = H^{-1}u \text{ and } \overline{d} = H^{-1}d, \tag{14}$$

then the above equality says $(\overline{u}\overline{d})^2 = \mathbf{1}$. The same arguments with another tiling shows that $\overline{d}^3 = \mathbf{1}$. Together we know that $\overline{u} = \overline{d}^{-1}$ has order three.

Completely similar arguments show the same for $\overline{r}$ and $\overline{l}$. From the definitions of these elements it follows that

$$\overline{u} = H^{-1}u = H^{-1}V^{-1}HV = r^{-1}V = (V^{-1}r)^{-1} = \overline{r}^{-1} = \overline{l}.$$

As $H$ and $V$ generate $\Gamma$ we only have to define $\phi(H)$ and $\phi(V)$ as in the proposition stated and to check if all relations are satisfied by the images. Therefore let $\phi$ be the morphism from the free group generated by $V, H$ defined by $\phi(H) = x$ and $\phi(V) = y$. The first four tiles in (13) define the elements $u, d, r, l$ in terms of $V$ and $H$ and do not correspond to relations in $V$ and $H$. Using these definitions we can compute the images of $u, d, r, l$ under $\phi$: This leads to

$$\begin{array}{ll} \phi(u) = xz^{-1} & \phi(d) = xz \\ \phi(l) = yz^{-1} & \phi(r) = yz \end{array}$$

The last four tiles in (13) give relations whose images should be equal to the identity. For example the fifth tile gives the relation

$$\phi(V^{-1}u^{-1}rH) = y^{-1}(xz^{-1})^{-1}(yz)x = z^3 = \mathbf{1}.$$

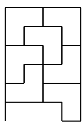
The same hold for the other tiles and therefore the map is well defined on $\Gamma$.

The morphism $\phi$ is of course surjective, it remains to check the injectivity. Take the subgroup $G$ of $\Gamma$ which is generated by $\overline{u}$. We already know that $G$ is cyclic of order three. And as $\overline{u}$ commutes with $H$ and $\overline{l}$ with $V$ we know that $\overline{u} = \overline{l}$ lies in the center of the group. The subgroup $G$ is therefore normal. The restriction $\phi|_G$ maps $\overline{u}$ to $z$ and is injective. On the quotients $\Gamma/G$ and $\mathbb{Z}^2$ the map $\phi$ is also an isomorphism as, for instance, modulo the subgroup $G$ the "horizontal" elements $u$, $d$ and $H$ are all equal. □

As we now know the structure of the tiling group we take a closer look at the shift and its dynamical properties. But first we start with some lemmas which will be useful later.

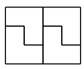
**Lemma 4.2.** *The one-dimensional shift of finite type which consists of all tilings of the bi-infinite strip* $[0,4] \times \mathbb{R}$ *by L-tiles is topologically transitive and has period three. In other words if one has two finite tilings which are over tilings of* $[0,4] \times [1,n]$ *and partial tilings of* $[0,4] \times [0,n+1]$*, then one can move them away from each other such that the gap between them can be filled. The same is true for the tilings of the strip* $[0,8] \times \mathbb{R}$.

*Proof.* For the proof of this lemma one has to look at the possible boundaries of finite tilings. We want to show that each upper boundary allows a tiling above with the other boundary being flat. So there are only 16 cases to be considered, two of them are already flat. The proof is left to the reader: we only look at one of the cases.



As the same holds for the bottom boundaries of a given tiling this proves the topological transitivity.

There exists a tiling with standard bottom and standard top boundary and length three



(15)

so the shift is either periodic with period three or is aperiodic. But the shift can not be aperiodic because the cocycle $c_\Gamma\big((4,0),\cdot\big)$ has at consecutive heights the values

$$\ldots, x^4 z^i, x^4 z^{i+1}, x^4 z^{i+2}, \ldots$$

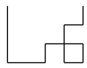
which have exactly the period three.

The strip of width 8 is treated similarly. □

As the proof above resembles an analysis of the game Tetris, the next lemma looks like the proof of the solubility of a slightly changed version of Tetris.

**Lemma 4.3.** *Let* $[0,3] \times [1,\infty] \subseteq R \subseteq [0,4] \times [0,\infty]$ *be a region which is constructed from* $[0,4] \times [0,\infty]$ *by removing some unit squares which are again aligned to the lattice. Then the region* $R$ *can be tiled by L-tiles if at* *most one of the squares* $[2,3]\times[0,1]$ *and* $[3,4]\times[1,2]$ *is removed or together with those two also the square* $[3,4]\times[0,1]$.

We first look at a region $R'$ which does not satisfy the last assumption in the lemma. Here the enclosed square $[3,4]\times[0,1]$ is part of the region.



This region is of course not tile-able by L-tiles so we must avoid such situations.

*Proof.* We leave the details to the reader but one can prove this by a recursion where at each step one fills one row so that in the next row the additional assumption of the lemma is satisfied. This could be done if one always fills the right most square if possible. □

Now we can go back to the two-dimensional shift space and start with an application of the last lemma.

**Proposition 4.4.** *The tiling space of the L-tiles is topologically mixing.*

*Proof.* The definition of mixing says that we have to take two open sets $O_1, O_2$ and prove that their intersection $O_1 \cap \sigma_{-\mathbf{n}}(O_2)$ is nontrivial if $\mathbf{n}$ is large enough. Without loss of generality we take the cylinder sets

$$O_i = \{x : x \text{ is in the rectangle } [0,N]^2 \text{ equal to } z_i\} \text{ with } i = 1,2$$

and look for an intersection. That means we have to find a tiling of the plane which contains the two fixed parts $z_1, z_2$ which describe the cylinder sets. Assume that the first coordinate $n_1$ of $\mathbf{n}$ is positive and large and take two tilings $x \in O_1$ and $y \in O_2$. Take the partial tiling $x_1$ which consists of all L-tiles in $x$ which have nontrivial intersection with $(-\infty, N)\times\mathbb{R}$. This gives a right boundary as it appeared in Lemma 4.3. According to a rotated version of that lemma we can extend this to a tiling $x_2$ of the region $(-\infty, N+4]\times\mathbb{R}$. The same can be done for $y$ but this time we want a tiling $y_2$ of $[-4,+\infty)\times\mathbb{R}$ which is equal to $y$ on the set $[0,+\infty)\times\mathbb{R}$. If we now apply the shift $\sigma_{-\mathbf{n}}(y_2)$ then we have a tiling of $[n_1-4,+\infty)\times\mathbb{R}$. If $n_1 > N+9$ then the distance between $x_2$ and $\sigma_{-\mathbf{n}}(y_2)$ is at least two and the strip between them is therefore tile-able by using the rectangles $2\times 3$ and $3\times 2$ which can be tiled with the L-tiles as we have already seen above. The joined tiling $z \in O_1 \cap \sigma_{-\mathbf{n}}(O_2)$ constructed from $x_2, \sigma_{-\mathbf{n}}(y_2)$ and the tiling of the strip completes the proof. □

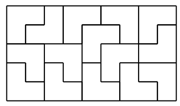
Now we turn our attention to the *holes* mentioned in Section 1 for the case of L-tiles.

**Lemma 4.5.** *A sufficiently large rectangle is tile-able by L-tiles if and only if its area is divisible by three. More precisely if* $M, N \geq 9$ *then the rectangle* $[0,M]\times[0,N]$ *is tile-able by L-tiles if* $3|MN$.

Without the assumption of sufficiently large rectangles the lemma is false because the rectangle $[0,3]\times[0,1]$ is of course not tile-able by L-tiles.

*Proof.* The only if part of the proposition is clear because each L-tiles fills exactly three unit squares.

Assume now the area is divisible by three. Then one side of the rectangle $[0, M] \times [0, N]$ is divisible by three. Without loss of generality we can assume that $3|N$. If $M$ is even then the tiling could be done with the $2 \times 3$-rectangles we have already used. So we only have to look at the case $M = 2m + 1$. As $M \geq 9$ we can cut the rectangle into the two part $[0, 9] \times [0, N]$ and $[9, M] \times [0, N]$. The second rectangle is tile-able because the length $M - 9$ is even. So it remains to tile the first one. If $N$ is even then the same argument holds. So we can assume that $N \geq 9$ is odd and by doing the same cutting argument we only have to look at the rectangle $[0, 9] \times [0, 5]$:



□

Now we allow rectangles with a rough boundary as in Lemma 4.3.

**Proposition 4.6.** *Let* $[1, M-1] \times [1, N-1] \subseteq R \subseteq [0, M] \times [0, N]$ *be a region which is obtained from* $[0, M] \times [0, N]$ *by removing some unit squares which are aligned to the lattice. Assume for each corner square which is part of* $R$ *that one of their neighboring squares is also part of* $R$ *and that* $M, N$ *are sufficiently large. Then there is a tiling of* $R$ *if the area of* $R$ *is divisible by three.*

The additional assumption about $R$ is necessary to avoid situations like the one mentioned before the proof of Lemma 4.3 where a tiling is of course impossible.

*Proof.* One can reduce the problem to Lemma 4.5 by using Lemma 4.3 to flatten the boundary. □

**Corollary 4.7.** *There exists a natural number* $s$ *such that the following is true. Let* $R$ *be any region with* $[1, M-1] \times [1, N-1] \subseteq R \subseteq [0, M] \times [0, N]$ *whose area is divisible by three, and which is obtained from* $[0, M] \times [0, N]$ *by removing some unit squares which are aligned to the lattice. For any tiling* $x$ *of the complement of the region* $R$ *with L-tiles we can define the tiling* $y$ *of all tiles in* $x$ *which are not contained in* $[-s, M+s] \times [-s, N+s]$. *Then there is an extension* $z \in X$ *of the tiling* $y$ *to a tiling of the whole plane.*

*Proof.* This is an easy reformulation of the Proposition 4.6 as for $s > 1$ no problems in the corners are appearing. One has to choose $s$ big enough to ensure that the enlarged hole is big enough for applying Proposition 4.6. □

**Theorem 4.8.** *The tiling cocycle of the L-tiles is a fundamental cocycle.*

*Proof.* The proof of this theorem resembles very much the proof of Theorem 2.7. We also have a statement, which says that our cocycle describes the shift space quite well:

We know roughly speaking that a hole is tile-able if the area $M$ of uncovered squares is divisible by three. If we calculate the cocycle value for a path which goes around the hole, we get $z^M$. So the cocycle value vanishes if and only if there is a tiling of a slightly enlarged hole.

If one defines $\eta$ similarly to (7) in the proof of Theorem 2.7, but with $2(r+s)$ as the width of the strips, one can prove that this map is well defined by using Corollary 4.7. □

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