MARKOV PARTITIONS AND HOMOCLINIC POINTS OF ALGEBRAIC $\mathbb{Z}^d$-ACTIONS

MANFRED EINSIEDLER AND KLAUS SCHMIDT

Abstract. We prove that a general class of expansive $\mathbb{Z}^d$-actions by automorphisms of compact, abelian groups with completely positive entropy has 'symbolic covers' of equal topological entropy. These symbolic covers are constructed by using homoclinic points of these actions. For $d = 1$ we adapt a result of Kenyon and Vershik in [7] to prove that these symbolic covers are, in fact, sofic shifts. For $d \geq 2$ we are able to prove the analogous statement only for certain examples, where the existence of such covers yields finitary isomorphisms between topologically nonisomorphic $\mathbb{Z}^2$-actions.

1. Introduction

Markov partitions of hyperbolic dynamical systems are a powerful technical tool and provide a crucial link between smooth and symbolic dynamics (cf. [4], [20]). The first explicit examples of Markov partitions were obtained for hyperbolic toral automorphisms in [1], and the construction of such partitions reflecting the algebraic structure of toral automorphisms continues to be an area of active investigation: Markov partitions of certain toral automorphisms related to two-sided beta-expansions of Pisot numbers appear in [3] and [13], and [7] describes a general method for finding sofic covers of an arbitrary irreducible hyperbolic toral automorphism A by using certain sets of algebraic integers in the number field generated by the eigenvalues of A. A similar proof appeared subsequently in [8].

Markov partitions are also beginning to play a role in the investigation of certain $\mathbb{Z}^2$-actions, e.g. of coupled map lattices (cf. [12] and [5]). However, since two-dimensional shifts of finite type tend to be considerably more complicated than the classical one-dimensional ones, the advantage of constructing a Markov partition for a $\mathbb{Z}^2$-action is a priori less obvious than for a $\mathbb{Z}$-action unless the resulting shift of finite type has a particularly simple form, as it does in [12], [5] and in the examples in Section 5. With this proviso Markov partitions of $\mathbb{Z}^2$-action can be very useful, e.g. for the construction of invariant measures or of finitary isomorphisms (cf. the Corollaries 5.1 and 5.2 and Remark 5.1).

In order to make precise the notion of a Markov (or, more generally, sofic) partition of a continuous $\mathbb{Z}^d$-action T on a compact space X we assume that $d \geq 1$, A a finite set (the *alphabet*), and that ρ is the shift-action of $\mathbb{Z}^d$ on $A^{\mathbb{Z}^d}$, defined by

$$(\rho^{\mathbf{n}} z)_{\mathbf{m}} = z_{\mathbf{m}+\mathbf{n}} \tag{1.1}$$

Published in: Proceedings of the Stekhlov Institute of Mathematics, Vol. 216, 1997, 259–279.

for every $\mathbf{n} \in \mathbb{Z}^d$ and $z = (z_{\mathbf{m}}) \in A^{\mathbb{Z}^d}$. If $Z \subset A^{\mathbb{Z}^d}$ a closed, shift-invariant set we write ρ_Z for the restriction of ρ to Z. A closed, shift-invariant set $Z \subset A^{\mathbb{Z}^d}$ is a *shift of finite type* if there exists a finite set $F \subset \mathbb{Z}^d$ with

$$Z = \{z \in A^{\mathbb{Z}^d} : \pi_F(\rho^{\mathbf{n}} z) \in \pi_F(Z) \text{ for every } \mathbf{n} \in \mathbb{Z}^d\}, \tag{1.2}$$

where $\pi_F \colon A^{\mathbb{Z}^d} \longmapsto A^F$ is the projection of each $z \in A^{\mathbb{Z}^d}$ onto its coordinates in F (cf. [16]–[17]). By changing the alphabet A, if necessary, we can always assume that

$$F = \{0,1\}^d \subset \mathbb{Z}^d. \tag{1.3}$$

A closed, shift-invariant set $Z \subset A^{\mathbb{Z}^d}$ is *sofic* if there exists a finite set A', a shift of finite type $Z' \subset A'^{\mathbb{Z}^d}$, and a continuous, surjective map $\phi \colon Z' \longmapsto Z$ with $\phi \cdot \rho'^{\mathbf{n}}(z) = \rho^{\mathbf{n}} \cdot \phi(z)$ for every $z \in Z'$ and $\mathbf{n} \in \mathbb{Z}^d$, where ρ' is the shift-action (1.1) of $\mathbb{Z}^d$ on $A'^{\mathbb{Z}^d}$.

Now suppose that $T \colon \mathbf{n} \mapsto T^{\mathbf{n}}$ is a continuous $\mathbb{Z}^d$-action on a compact space X, A a finite set, ρ the shift-action of $\mathbb{Z}^d$ on $A^{\mathbb{Z}^d}$ and $Y \subset A^{\mathbb{Z}^d}$ closed, shift-invariant subset. Then Y (or, more precisely, (Y, ρ_Y)) is a *symbolic cover* of (X, T) if there exists a continuous, surjective map $\phi \colon Y \longmapsto X$ with

$$\phi \cdot \rho_Y^{\mathbf{n}} = T^{\mathbf{n}} \cdot \phi \tag{1.4}$$

for every $\mathbf{n} \in \mathbb{Z}^d$. For later use we set

$$Y' = \{y \in Y : \{y\} = \phi^{-1}(\{\phi(y)\})\}, \tag{1.5}$$

note that $Y' \subset Y$ is a ρ_Y-invariant G_δ-set, and that the restriction $\phi|_{Y'}$ of ϕ to Y is a homeomorphism of Y' onto the G_δ-set $\phi(Y') \subset X$.

A symbolic cover $Y \subset A^{\mathbb{Z}^d}$ of (X,T) is of *equal entropy* if the topological entropies $h(\rho_Y)$ and $h(T)$ coincide, *finite* if ϕ is bounded-to-one, and a *symbolic representation* of (X,T) if $\nu(Y') = 1$ for every ρ-invariant probability measure with maximal entropy on Y (cf. (1.5)). A symbolic cover or representation Y of (X,T) is *sofic* or of *finite type* if Y is sofic or of finite type. If $Y \subset A^{\mathbb{Z}^d}$ is a symbolic representation of (X,T) which is sofic or of finite type, and if $[a]_{\mathbf{0}} = \{y = (y_{\mathbf{n}}) : y_{\mathbf{0}} = a\}$ for every $a \in A$, then $\mathcal{P} = \{\phi([a]_{\mathbf{0}}) : a \in A\}$ is called a *sofic* or *Markov partition* of (X,T) ($\mathcal{P}$ is, of course, not a partition of X).

In this note we restrict ourselves to $\mathbb{Z}^d$-actions by automorphisms of compact, abelian groups; in order to simplify terminology we call such a $\mathbb{Z}^d$-action *algebraic*. Suppose that $d \geq 1$, and that α is an algebraic $\mathbb{Z}^d$-action which is expansive, has completely positive entropy, and which is of the special form described in Proposition 2.1 (the last restriction is not really necessary for our investigation, but offers considerable notational and technical simplification; for a single automorphism $A \in GL(n, \mathbb{Z})$ of the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ it means that A is the companion matrix of its characteristic polynomial). In [10] it was shown that every such $\mathbb{Z}^d$-action has a 'fundamental' homoclinic point w^Δ which generates all other homoclinic points of the $\mathbb{Z}^d$-action. By using this fundamental homoclinic point we can write the $\mathbb{Z}^d$-action α as a topological factor of the shift-action σ of $\mathbb{Z}^d$ on the full shift $\mathcal{V} = \{0, \dots, N\}^{\mathbb{Z}^d}$ for some $N \geq 1$, and we write $\xi \colon \mathcal{V} \longmapsto X$ for the continuous factor map (cf. Corollary 2.1). In Theorem 3.1 we construct a closed, shift-invariant subset $\mathcal{V}^* \subset \mathcal{V}$ which is a symbolic cover of equal entropy of (X,α). In particular, if $d=1$, then the map $\xi\colon \mathcal{V}^* \longmapsto X$ is bounded-to-one.

For $d=1$ one can adapt the ideas of Kenyon and Vershik in [7] to prove that $\mathcal{V}^*$ is a sofic shift (Theorem 4.1), but for $d>1$ we are currently unable to prove that the resulting symbolic cover $\mathcal{V}^*$ of (X,α) has such nice properties. In Section 5 we present two examples of $\mathbb{Z}^2$-actions for which $\mathcal{V}^*$ is a symbolic representation of (X,α) which is of finite type and sofic, respectively. These examples are of independent interest, since they yield nontrivial finitary isomorphisms between certain topologically inequivalent $\mathbb{Z}^2$-actions by automorphisms of compact, abelian groups.

The use of homoclinic points to construct sofic or Markov partitions offers two advantages.

(1) For a hyperbolic toral automorphism α of the form described in Proposition 2.1 the scalar multiples of the fundamental homoclinic point provide an alphabet of the sofic cover of α which is perhaps more (but certainly not completely) canonical than the alphabet used in [7].
(2) The construction of the symbolic cover of equal entropy resulting from this fundamental homoclinic point extends *without any change* not only to automorphisms of solenoids, but also to the expansive algebraic $\mathbb{Z}^d$-actions with completely positive entropy described in Proposition 2.1.

Homoclinic points of hyperbolic diffeomorphisms can be used to prove very strong orbit-shadowing and specification properties which are intimately connected with hyperbolicity and local product structure (cf. [2] and Chapter 18 in [6]), as well as with the construction of Markov partitions. For expansive algebraic $\mathbb{Z}^d$-actions with completely positive entropy there is no local product structure if $d>1$; however, the shadowing and specification properties remain true (cf. Theorem 5.2 in [10]), and the examples in Section 5 suggest that the link between hyperbolic and symbolic $\mathbb{Z}$-actions survives the transition from $d=1$ to $d>1$ at least under some circumstances.

2. A class of algebraic $\mathbb{Z}^d$-actions and their homoclinic points

In this section we describe a class of $\mathbb{Z}^d$-actions which form the 'building blocks' from which *all* expansive $\mathbb{Z}^d$ actions by automorphisms of compact, abelian groups with completely positive entropy are constructed (cf. [15] and [18]).

Let $d\ge 1$, $\mathbb{T}=\mathbb{R}/\mathbb{Z}$, and let σ be the shift-action of $\mathbb{Z}^d$ on $\mathbb{T}^{\mathbb{Z}^d}$, defined as in (1.1) by

$$(\sigma^{\mathbf{m}}x)_{\mathbf{n}} = x_{\mathbf{m}+\mathbf{n}}$$

for every $\mathbf{m}\in\mathbb{Z}^d$ and $x=(x_{\mathbf{n}})\in\mathbb{T}^{\mathbb{Z}^d}$. We denote by $\mathfrak{R}_d=\mathbb{Z}[u_1^{\pm1},\dots,u_d^{\pm1}]$ the set of Laurent polynomials with integer coefficients in the variables $u_1,\dots,u_d$ and write each $h\in\mathfrak{R}_d$ as

$$h=\sum_{\mathbf{n}\in\mathbb{Z}^d} h_{\mathbf{n}}u^{\mathbf{n}} \tag{2.1}$$

with $h_{\mathbf{n}} \in \mathbb{Z}$ and $u^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$ for every $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$, where $h_{\mathbf{n}} \neq 0$ for only finitely many $\mathbf{n} \in \mathbb{Z}^d$. For every $h = \sum_{\mathbf{n}\in\mathbb{Z}^d} h_{\mathbf{n}} u^{\mathbf{n}} \in \mathfrak{R}_d$ and $x \in \mathbb{T}^{\mathbb{Z}^d}$ we set

$$h(\sigma)(x) = \sum_{\mathbf{n}\in\mathbb{Z}^d} h_{\mathbf{n}} \sigma^{\mathbf{n}} x \tag{2.2}$$

and observe that $\ker(h(\sigma))$ is a closed, shift-invariant subgroup of $\mathbb{T}^{\mathbb{Z}^d}$. The following proposition was proved in [15], [11], [23] and [14] (cf. Theorems 6.5, 18.1, 19.5 and 20.8 in [18]).

Proposition 2.1. *Let $f \in \mathfrak{R}_d$ be a Laurent polynomial, and let $\alpha = \alpha_f$ be the restriction to*

$$X = X_f = \ker(f(\sigma)) \subset \mathbb{T}^{\mathbb{Z}^d}$$

of the shift-action σ of $\mathbb{Z}^d$ on $\mathbb{T}^{\mathbb{Z}^d}$. The following conditions are equivalent.

(1) *α is expansive;*
(2) *$V_{\mathbb{C}}(f) \cap \mathbb{S}^d = \varnothing$, where $\mathbb{S} = \{s \in \mathbb{C} : |s| = 1\}$ and*

$$V_{\mathbb{C}}(f) = \{c \in (\mathbb{C} \smallsetminus \{0\})^d : f(c) = 0\}.$$

If α is expansive then it is Bernoulli with respect to the normalised Haar measure λ_X of X.

Denote by $\|\cdot\|_1$ and $\|\cdot\|_\infty$ the norms on the Banach spaces $\ell^1(\mathbb{Z}^d, \mathbb{R})$ and $\ell^\infty(\mathbb{Z}^d, \mathbb{R})$ and write

$$\ell^1(\mathbb{Z}^d, \mathbb{Z}) \subset \ell^1(\mathbb{Z}^d, \mathbb{R}), \qquad \ell^\infty(\mathbb{Z}^d, \mathbb{Z}) \subset \ell^\infty(\mathbb{Z}^d, \mathbb{R})$$

for the subgroups of integer-valued functions. By viewing every Laurent polynomial $h = \sum_{\mathbf{n}\in\mathbb{Z}^d} h_{\mathbf{n}} u^{\mathbf{n}} \in \mathfrak{R}_d$ as an element $(h_{\mathbf{n}}) \in \ell^1(\mathbb{Z}^d, \mathbb{Z})$ we can identify $\mathfrak{R}_d$ and $\ell^1(\mathbb{Z}^d, \mathbb{Z})$.

Consider the surjective map $\eta\colon \ell^\infty(\mathbb{Z}^d, \mathbb{R}) \longmapsto \mathbb{T}^{\mathbb{Z}^d}$ given by

$$\eta(v)_{\mathbf{n}} = v_{\mathbf{n}} \pmod 1 \tag{2.3}$$

for every $v = (v_{\mathbf{n}}) \in \ell^\infty(\mathbb{Z}^d, \mathbb{R})$ and $\mathbf{n} \in \mathbb{Z}^d$. Let $\bar{\sigma}$ be the shift-action of $\mathbb{Z}^d$ on $\ell^\infty(\mathbb{Z}^d, \mathbb{R})$, defined as in (1.1) by

$$(\bar{\sigma}^{\mathbf{m}} v)_{\mathbf{n}} = v_{\mathbf{m}+\mathbf{n}}$$

for every $\mathbf{m} \in \mathbb{Z}^d$ and $v = (v_{\mathbf{n}}) \in \ell^\infty(\mathbb{Z}^d, \mathbb{R})$, and set, for every $h = \sum_{\mathbf{n}\in\mathbb{Z}^d} h_{\mathbf{n}} u^{\mathbf{n}} \in \mathfrak{R}_d$ and $v \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$,

$$\begin{aligned} \tilde{h} &= \sum_{\mathbf{n}\in\mathbb{Z}^d} h_{\mathbf{n}} u^{-\mathbf{n}}, \\ h(\bar{\sigma})(v) &= \sum_{\mathbf{n}\in\mathbb{Z}^d} h_{\mathbf{n}} \bar{\sigma}^{\mathbf{n}} v. \end{aligned} \tag{2.4}$$

Then

$$\begin{aligned} hh' &= \tilde{h}(\bar{\sigma})(h'), \\ \eta \cdot h(\bar{\sigma}) &= h(\sigma) \cdot \eta \end{aligned} \tag{2.5}$$

for every $h, h' \in \mathfrak{R}_d$ (cf. (2.2)). Following [15] or Theorem 6.5 in [18] we can characterise the expansiveness of $\alpha = \alpha_f$ in Proposition 2.1 also in terms of the kernel of $f(\bar{\sigma})$.

Proposition 2.2. *Let $f \in \mathfrak{R}_d$ be a Laurent polynomial and let $\alpha = \alpha_f$ be the $\mathbb{Z}^d$-action on $X = X_f$ defined in Proposition 2.1. The following conditions are equivalent.*

(1) *α is expansive;*
(2) $\ker(f(\bar{\sigma})) = \{0\} \subset \ell^\infty(\mathbb{Z}^d, \mathbb{R})$.

For the remainder of this section we fix a Laurent polynomial $f \in \mathfrak{R}_d$ such that the $\mathbb{Z}^d$-action $\alpha = \alpha_f$ on $X = X_f = \ker(f(\sigma))$ in the Propositions 2.1–2.2 is expansive and hence mixing.

According to the proof of Lemma 4.5 in [10] there exists a unique element $w^\Delta \in \ell^\infty(\mathbb{Z}^d, \mathbb{R})$ with the property that

$$f(\bar{\sigma})(w^\Delta)_{\mathbf{n}} = \begin{cases} 1 & \text{if } \mathbf{n} = \mathbf{0}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

The point w^Δ also has the property that there exist constants $c_1 > 0, 0 < c_2 < 1$ with

$$\left|w^\Delta_{\mathbf{n}}\right| \le c_1 c_2^{\|\mathbf{n}\|} \tag{2.7}$$

for every $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$, where

$$\|\mathbf{n}\| = \max_{i=1,\dots,d} |n_i| \tag{2.8}$$

(cf. Proposition 2.2 in [10]). In particular,

$$\|w^\Delta\|_1 = \sum_{\mathbf{n}\in\mathbb{Z}^d} |w^\Delta_{\mathbf{n}}| < \infty \tag{2.9}$$

From the properties of w^Δ it is clear that

$$\bar{\xi}(v) = \sum_{\mathbf{n}\in\mathbb{Z}^d} v_{\mathbf{n}} \bar{\sigma}^{\mathbf{n}} w^\Delta \tag{2.10}$$

is a well-defined element of $\ell^\infty(\mathbb{Z}^d, \mathbb{R})$ for every $v \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$, and we denote by

$$\bar{\xi} \colon \ell^\infty(\mathbb{Z}^d, \mathbb{Z}) \longmapsto \ell^\infty(\mathbb{Z}^d, \mathbb{R}), \qquad \xi = \eta \cdot \bar{\xi} \colon \ell^\infty(\mathbb{Z}^d, \mathbb{Z}) \longmapsto X \tag{2.11}$$

the resulting group homomorphisms.

Proposition 2.3. *For every $v \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$,*

$$\begin{gathered} f(\bar{\sigma})(\bar{\xi}(v)) = \bar{\xi}(f(\bar{\sigma})(v)) = v, \\ \|\bar{\xi}(v)\|_\infty \le \|w^\Delta\|_1 \|v\|_\infty, \\ \|v\|_\infty \le \|f\|_1 \|\bar{\xi}(v)\|_\infty. \end{gathered} \tag{2.12}$$

Furthermore, $\xi \colon \ell^\infty(\mathbb{Z}^d, \mathbb{Z}) \longmapsto X$ is a surjective group homomorphism and

$$\begin{gathered} \xi \cdot \bar{\sigma}^{\mathbf{n}} = \alpha^{\mathbf{n}} \cdot \xi \text{ for every } \mathbf{n} \in \mathbb{Z}^d, \\ \ker(\xi) = f(\bar{\sigma})(\ell^\infty(\mathbb{Z}^d, \mathbb{Z})), \\ \ker(\xi) \cap \ell^1(\mathbb{Z}^d, \mathbb{Z}) = f(\bar{\sigma})(\ell^1(\mathbb{Z}^d, \mathbb{Z})) = \tilde{f}\mathfrak{R}_d. \end{gathered} \tag{2.13}$$

Proof. The statements (2.12) are immediate consequences of (2.6) and (2.9)–(2.10). In order to see that ξ is surjective we fix $x \in X$ and choose an element $w = (w_{\mathbf{n}}) \in \eta^{-1}(\{x\}) \subset \ell^\infty(\mathbb{Z}^d, \mathbb{R})$ with $|w_{\mathbf{n}}| \le \frac{1}{2}$ for every $\mathbf{n} \in \mathbb{Z}^d$. Since $f(\sigma)(x) = 0_X$ according to the definition of X, $v = f(\bar{\sigma})(w) \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$, and we claim that $\xi(v) = x$.

Indeed, $f(\bar{\sigma})(\bar{\xi}(v')) = v'$ by (2.12) for every $v' \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$, and by applying this to $v = f(\bar{\sigma})(w)$ we obtain that

$$f(\bar{\sigma})(w - \bar{\xi}(v)) = 0.$$

Proposition 2.2 guarantees that $w = \bar{\xi}(v)$ and hence that $\eta(w) = \xi(v) = x$.

The first equation in (2.13) is clear from the definition of ξ. If $v \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ satisfies that $\xi(v) = 0_X$, then $w = \bar{\xi}(v) \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ and $f(\bar{\sigma})(w) = f(\bar{\sigma})(\bar{\xi}(v)) = v$ by (2.12). Hence $v \in f(\bar{\sigma})(\ell^\infty(\mathbb{Z}^d, \mathbb{Z}))$, as claimed. Conversely, every $v \in f(\bar{\sigma})(\ell^\infty(\mathbb{Z}^d, \mathbb{Z}))$ is of the form $v = f(\bar{\sigma})(w)$ for some $w \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$, and $\xi(v) = \xi(f(\bar{\sigma})(w)) = f(\sigma)(\xi(w)) = 0_X$. Finally, if $v \in \ker(\xi) \cap \ell^1(\mathbb{Z}^d, \mathbb{Z}) = \ker(\xi) \cap \mathfrak{R}_d$, then $h = \bar{\xi}(v) \in \ell^1(\mathbb{Z}^d, \mathbb{Z}) = \mathfrak{R}_d$ and $f(\bar{\sigma})h = \tilde{f}h = v \in \tilde{f}\mathfrak{R}_d$. □

From (2.7)–(2.11) it is clear that the restriction of $\bar{\xi}$ to every bounded subset of $\ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ is continuous in the weak*-topology. The following corollary yields a bounded subset $\mathcal{V} \subset \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ with $\xi(\mathcal{V}) = X$.

Corollary 2.1. *For every $h = \sum_{\mathbf{n} \in \mathbb{Z}^d} h_{\mathbf{n}} u^{\mathbf{n}} \in \mathfrak{R}_d$ we set*

$$h^+ = \sum_{\mathbf{n} \in \mathbb{Z}^d} \max(0, h_{\mathbf{n}}) u^{\mathbf{n}}, \qquad h^- = -\sum_{\mathbf{n} \in \mathbb{Z}^d} \min(0, h_{\mathbf{n}}) u^{\mathbf{n}},$$
$$\|h^+\|_1' = \max(\|h^+\|_1 - 1, 0), \qquad \|h^-\|_1' = \max(\|h^-\|_1 - 1, 0),$$
$$\|h\|_1^* = \|h^+\|_1' + \|h^-\|_1'.$$

Then the set

$$\mathcal{V} = \{v \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z}) : 0 \le v_{\mathbf{n}} \le \|f\|_1^* \text{ for every } \mathbf{n} \in \mathbb{Z}^d\} \tag{2.14}$$

satisfies that $\xi(\mathcal{V}) = X$.

Proof. We use the notation employed for the proof of Proposition 2.3 and choose an element $w \in \eta^{-1}(\{x\})$ with $0 \le w_{\mathbf{n}} < 1$ for every $\mathbf{n} \in \mathbb{Z}^d$. If $v = f(\bar{\sigma})(w) \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ then $\xi(v) = x$ and $-\|f^-\|_1' \le v_{\mathbf{n}} \le \|f^+\|_1'$ for every $\mathbf{n} \in \mathbb{Z}^d$. Hence $\xi(\mathcal{V}') = X$ with

$$\mathcal{V}' = \{v \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z}) : -\|f^-\|_1' \le v_{\mathbf{n}} \le \|f^+\|_1' \text{ for every } \mathbf{n} \in \mathbb{Z}^d\}.$$

Finally we denote by $y \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ the constant element with $y_{\mathbf{n}} = \|f^-\|_1'$ for every $\mathbf{n} \in \mathbb{Z}^d$ and obtain that $\xi(\mathcal{V}' + y) = \xi(\mathcal{V}') + \xi(y) = X$. Since $\mathcal{V}' + y = \mathcal{V}$ we have proved the corollary. □

The bound appearing in (2.14) need not be optimal, as the following examples show.

Example 2.1. (1) Let $d = 2$ and $f = 3 - u_1 - u_2 \in \mathfrak{R}_2$. Then $\|f\|_1^* = 3$, but the set

$$\mathcal{W} = \{v \in \ell^\infty(\mathbb{Z}^2, \mathbb{Z}) : 0 \le v_{\mathbf{n}} \le 2\} \subsetneq \mathcal{V} \tag{2.15}$$

also satisfies that $\xi(\mathcal{W}) = X$.

In order to prove that $\xi(\mathcal{W}) = X$ we define $\mathcal{V}$ by (2.14) and set, for every $n \geq 0$, $Q_n = \{-n, \dots, n\}^2 \subset \mathbb{Z}^2$. Fix $n \geq 0$ and put, for every $v \in \mathcal{V}$,

$$h^v_{\mathbf{n}} = \begin{cases} 1 & \text{if } \mathbf{n} \in Q_n \text{ and } v_{\mathbf{n}} = 3, \\ 0 & \text{otherwise,} \end{cases}$$

$$h^v = \sum_{\mathbf{n} \in \mathbb{Z}^2} h^v_{\mathbf{n}} u^{\mathbf{n}}.$$

Fix $v \in \mathcal{V}$ and define inductively $v^{(0)} = v$ and $v^{(m+1)} = v^{(m)} - \tilde{f} h^{v^{(m)}}$ for every $m \geq 0$, where $\tilde{f}$ is defined in (2.4). It is clear that there exists an $M \geq 0$ with $v^{(m)} = v^{(M)}$ for every $m \geq M$, since $\sum_{\mathbf{n} \in Q_n} v^{(m+1)} < \sum_{\mathbf{n} \in Q_n} v^{(m)}$ whenever $m \geq 0$ and $v^{(m+1)} \neq v^{(m)}$.

We put $w = v^{(M)}$, note that $0 \leq w_{\mathbf{n}} \leq 2$ for every $\mathbf{n} \in Q_n$, and claim that $0 \leq w_{\mathbf{n}} \leq 8$ for every $\mathbf{n} \in \mathbb{Z}^2 \smallsetminus Q_n$. Indeed, if ν is the measure on $\mathbb{Z}^2$ defined by

$$\nu(\{\mathbf{m}\}) = \begin{cases} 1/3 & \text{if } \mathbf{m} \in \{(-1,0), (0,-1)\}, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$w_{\mathbf{k}} \leq v_{\mathbf{k}} + 3 \sum_{\{\mathbf{m} \in Q_n : v_{\mathbf{m}} = 3\}} \sum_{l \geq 1} \nu^{*l}(\{\mathbf{k} + \mathbf{m}\})$$

for every $\mathbf{k} \in \mathbb{Z}^2 \smallsetminus Q_n$, where ν^{*l} is the l-th convolution power of ν. As

$$\sum_{\mathbf{m} \in \mathbb{Z}^2} \sum_{l \geq 1} \nu^{*l}(\{\mathbf{m}\}) = \sum_{l \geq 1} (2/3)^l = 2$$

we obtain that $0 \leq w_{\mathbf{n}} \leq 8$ for every $\mathbf{n} \in \mathbb{Z}^2 \smallsetminus Q_n$.

The elements v and w differ by an element of $\mathfrak{R}_d = \ell^1(\mathbb{Z}^2, \mathbb{Z}) \subset \ell^\infty(\mathbb{Z}^2, \mathbb{Z})$ which is a multiple of $\tilde{f}$. Hence $v - w \in f(\overline{\sigma})(\ell^\infty(\mathbb{Z}^2, \mathbb{Z}))$, and Proposition 2.3 guarantees that $\xi(w) = \xi(v)$.

We have proved the following: for every $n \geq 0$ the set

$$\begin{aligned} \mathcal{W}_n = \{w \in \ell^\infty(\mathbb{Z}^2, \mathbb{Z}) : 0 \leq w_{\mathbf{n}} \leq 2 &\text{ for } \mathbf{n} \in Q_n, \\ 0 \leq w_{\mathbf{n}} \leq 8 &\text{ for } \mathbf{n} \in \mathbb{Z}^2 \smallsetminus Q_n\} \end{aligned}$$

satisfies that $\xi(\mathcal{W}_n) = X$.

For every $x \in X$ and $n \geq 1$, the set

$$\begin{aligned} C_n(x) &= \xi^{-1}(\{x\}) \cap \mathcal{W}_n \\ &\subset \mathcal{W}' = \{v \in \ell^\infty(\mathbb{Z}^2, \mathbb{Z}) : 0 \leq v_{\mathbf{n}} \leq 8 \text{ for every } \mathbf{n} \in \mathbb{Z}^2\} \end{aligned}$$

is nonempty and compact in the weak*-topology, and $C_n(x) \supset C_{n+1}(x)$. The intersection

$$C(x) = \bigcap_{n \geq 1} \mathcal{C}_n(x) = \xi^{-1}(\{x\}) \cap \mathcal{W}$$

is thus nonempty for every $x \in X$, which proves that $\xi(\mathcal{W}) = X$.

(2) The argument in example (1) yields the following more general result. *Suppose that $d \geq 1$, and that $f = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} u^{\mathbf{n}} \in \mathfrak{R}_d$ is a Laurent polynomial*

with the property that there exists a $\mathbf{m} \in \mathbb{Z}^d$ *with* $f_{\mathbf{m}} > \sum_{\mathbf{n}\in\mathbb{Z}^d\smallsetminus\{\mathbf{m}\}} |f_{\mathbf{n}}|$ *and* $f_{\mathbf{n}} \le 0$ *whenever* $\mathbf{m} \ne \mathbf{n} \in \mathbb{Z}^d$*. Then* α_f *is expansive, and the set*

$$\mathcal{W} = \{v \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z}) : 0 \le v_{\mathbf{n}} < f_{\mathbf{m}} \text{ for every } \mathbf{n} \in \mathbb{Z}^d\} \tag{2.16}$$

satisfies that $\xi(\mathcal{W}) = X_f$*.*

3. Covers with equal entropy

Let $f \in \mathfrak{R}_d$ be a Laurent polynomial such that the $\mathbb{Z}^d$-action $\alpha = \alpha_f$ on $X = X_f$ is expansive (cf. Propositions 2.1–2.2), and define $\mathcal{V} \subset \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ and $\xi \colon \mathcal{V} \longmapsto X$ as in Corollary 2.1. Then $\xi(\mathcal{V}) = X$, but ξ may be far from injective on $\mathcal{V}$. The purpose of this section is to find closed, shift-invariant subsets $\mathcal{W} \subset \mathcal{V}$ with $\xi(\mathcal{W}) = X$ such that the restriction of $\bar\sigma$ to $\mathcal{W}$ has the same topological entropy as α.

Let $\mathcal{W} \subset \mathcal{V}$ be a closed, shift-invariant subset with $\xi(\mathcal{W}) = X$, and consider the equivalence relations

$$\begin{aligned}\mathbf{R}_{\mathcal{W}} &= \{(v, v') \in \mathcal{W} \times \mathcal{W} : \xi(v) = \xi(v')\},\\ \Delta_{\mathcal{W}} &= \{(v, v') \in \mathcal{W} \times \mathcal{W} : v - v' \in \ell^1(\mathbb{Z}^d, \mathbb{Z}) = \mathfrak{R}_d\},\\ \Delta'_{\mathcal{W}} &= \{(v, v') \in \mathcal{W} \times \mathcal{W} : v - v' \in f(\bar\sigma)(\ell^1(\mathbb{Z}^d, \mathbb{Z})) = \tilde f \mathfrak{R}_d\}\\ &= \mathbf{R}_{\mathcal{W}} \cap \Delta_{\mathcal{W}}.\end{aligned} \tag{3.1}$$

The inverse lexicographic order on $\mathbb{Z}^d$ can be used to define a total order $\prec$ on $\mathfrak{R}_d = \ell^1(\mathbb{Z}^d, \mathbb{Z})$ by setting $h \prec 0$ if and only if $h_{\mathbf{m}} < 0$ for the lexicographically smallest $\mathbf{m} \in \mathbb{Z}^d$ with $h_{\mathbf{m}} \ne 0$, and by saying that $h \prec h'$ whenever $h - h' \prec 0$. The order $\prec$ on $\mathfrak{R}_d$ induces a total order (again denoted by $\prec$) on the equivalence classes of $\Delta_{\mathcal{W}}$: if $v \in \mathcal{W}$ and $v', v'' \in \Delta_{\mathcal{W}}(v)$ then $v' - v'' \in \ell^1(\mathbb{Z}^d, \mathbb{Z}) = \mathfrak{R}_d$, and $v' \prec v''$ if $v' - v'' \prec 0$.

Theorem 3.1. *Let* $f \in \mathfrak{R}_d$ *be a Laurent polynomial such that the* $\mathbb{Z}^d$*-action* $\alpha = \alpha_f$ *on* $X = X_f$ *is expansive* (*cf. Propositions* 2.1–2.2), *and define* $\mathcal{V} \subset \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ *and* $\xi \colon \mathcal{V} \longmapsto X$ *as in Corollary* 2.1*. Then there exists a closed, shift-invariant subset* $\mathcal{V}^* \subset \mathcal{V}$ *with the following properties.*

1. *If* $v \in \mathcal{V}^*$ *and* $v' \in \Delta'_{\mathcal{V}}(v)$ *then* $v \preceq v'$*; in particular,* $\mathcal{V}^*$ *intersects each equivalence class* $\Delta'_{\mathcal{V}}(v)$*,* $v \in \mathcal{V}$*, in at most one point;*
2. $\xi(\mathcal{V}^*) = X$*.*

Proof. The set $\mathfrak{R}_d^+ = \{h \in \mathfrak{R}_d : h \succ 0\}$ is an additive and multiplicative semigroup and is in particular invariant under multiplication by $u^{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^d$. Furthermore, since $\alpha_f = \alpha_{-f}$, we may replace f by $-f$, if necessary, and assume that $\tilde f \in \mathfrak{R}_d^+$ (cf. (2.4)).

For every $h \in \mathfrak{R}_d^+$ we set

$$\mathcal{V}_h = \mathcal{V} \smallsetminus (\mathcal{V} + \tilde f h),$$

where $\mathcal{V} + \tilde f h = \{v + \tilde f h : v \in \mathcal{V}\}$. Since $\tilde f h$ has only finitely many nonzero coordinates, $\mathcal{V}_h$ is a closed subset of $\mathcal{V}$ and

$$\mathcal{V}^* = \bigcap_{h \in \mathfrak{R}_d^+} \mathcal{V}_h = \mathcal{V} \smallsetminus \bigcup_{h \in \mathfrak{R}_d^+} (\mathcal{V} + \tilde f h) \tag{3.2}$$

is a closed, shift-invariant subset of $\mathcal{V}$.

From the definition of $\mathcal{V}^*$ it is clear that $v - \tilde{f}h \notin \mathcal{V}$ whenever $v \in \mathcal{V}^*$ and $h \in \mathfrak{R}_d^+$. In particular, if $v \in \mathcal{V}^*$ and $v' \in \Delta'_{\mathcal{V}}(v)$, then $v' \succeq v$, which proves (1). In order to verify (2) we choose an enumeration $\{h^{(1)}, h^{(2)}, \dots\}$ of $\mathfrak{R}_d^+$, consider the sets

$$\mathcal{V}_n^* = \bigcap_{k=1}^{n} \mathcal{V}_{h^{(k)}} = \mathcal{V} \smallsetminus \bigcup_{k=1}^{n} (\mathcal{V} + \tilde{f}h^{(k)}),\ n \geq 1,$$

and claim that $\xi(\mathcal{V}_n^*) = X$ for every $n \geq 1$.

Indeed, fix $n \geq 1$ and $v \in \mathcal{V}$. We denote by

$$\mathcal{S}(h) = \{\mathbf{n} \in \mathbb{Z}^d : h_{\mathbf{n}} \neq 0\} \tag{3.3}$$

the *support* of an element $h \in \mathfrak{R}_d$, put $S_n = \bigcup_{k=1}^{n} \mathcal{S}(\tilde{f}h^{(k)})$, and observe that the set

$$\{v' \in \Delta'_{\mathcal{V}}(v) : \mathcal{S}(v' - v) \subset S_n\}$$

is finite and contains a smallest element w with respect to the order $\prec$. As $\Delta'_{\mathcal{V}} \subset \mathbf{R}_{\mathcal{V}}$, $\xi(w) = \xi(v)$. Furthermore, $w \in \mathcal{V}_n^*$ since, for every $k = 1, \dots, n$, $w - \tilde{f}h^{(k)} \prec w$ and hence $w - \tilde{f}h^{(k)} \notin \mathcal{V}$. This shows that there exists, for every $v \in \mathcal{V}$, an element $w \in \mathcal{V}_n^*$ with $\xi(w) = \xi(v)$, and Corollary 2.1 guarantees that $\xi(\mathcal{V}_n^*) = X$.

We conclude our proof as in Example 2.1 (1): for every $x \in X$ and $n \geq 1$, the set $C_n(x) = \xi^{-1}(\{x\}) \cap \mathcal{V}_n^*$ is nonempty and closed in the weak*-topology on $\mathcal{V}$. Since $C_n(x) \supset C_{n+1}(x)$ for every $n \geq 1$, the intersection $C(x) = \bigcap_{n\geq 1} C_n(x) = \xi^{-1}(\{x\}) \cap \mathcal{V}^*$ is again nonempty, which proves that $\xi(\mathcal{V}^*) = X$. □

Corollary 3.1. *Let τ be the restriction to $\mathcal{V}^* \subset \mathcal{V} \subset \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ of the shift-action $\bar{\sigma}$ of $\mathbb{Z}^d$ on $\ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ (cf. Theorem* 3.1). *Then $h(\tau) = h(\alpha)$, where $h(\cdot)$ denotes topological entropy.*

Proof. Since $\xi\colon \mathcal{V}^* \longmapsto X$ is surjective and $\xi \cdot \alpha^{\mathbf{n}} = \tau^{\mathbf{n}} \cdot \xi$ for every $\mathbf{n} \in \mathbb{Z}^d$ it is clear that $h(\alpha) \leq h(\tau)$.

Put $Q_n = \{-n, \dots, n\}^d \subset \mathbb{Z}^d$ for every $n \geq 0$. If $A = \{0, \dots, \|f\|_1^*\} \subset \mathbb{Z}$ and $\pi_F\colon \mathcal{V} = A^{\mathbb{Z}^d} \longmapsto A^F$ denotes the projection onto a set of coordinates $F \subset \mathbb{Z}^d$ then

$$\begin{aligned} h(\tau) &= \lim_{n\to\infty} \frac{1}{(2n+1)^d} \log \left|\pi_{Q_n}(\mathcal{V}^*)\right| \\ &= \lim_{n\to\infty} \frac{1}{(2n+1)^d} \sup_{z \in \pi_{Q_{n+m} \smallsetminus Q_n}(\mathcal{V}^*)} \log \left|\pi_{Q_n}(\pi^{-1}_{Q_{m+n}\smallsetminus Q_n}(z))\right| \end{aligned} \tag{3.4}$$

for every $m \geq 1$.

For every $t \in \mathbb{T}$ we set

$$|t| = \min_{\mathbf{n}\in\mathbb{Z}} |t - n| \tag{3.5}$$

The expansiveness of the $\mathbb{Z}^d$-action α on $X \subset \mathbb{T}^{\mathbb{Z}^d}$ guarantees that

$$\varepsilon = \frac{1}{2} \inf_{0\neq x\in X} \sup_{\mathbf{n}\in\mathbb{Z}^d} |x_{\mathbf{n}}| > 0. \tag{3.6}$$

According to (2.7) we can find an integer $M \geq 1$ with

$$\begin{gathered}\sum_{\mathbf{m}\in\mathbb{Z}^d\smallsetminus Q_{M/2}} |w^{\Delta}_{\mathbf{m}}| < \frac{\varepsilon}{10\|f\|_1^*},\\ \max\{\|\mathbf{n}\| : \mathbf{n}\in\mathbb{Z}^d \text{ and } f_{\mathbf{n}}\neq 0\} < M/2.\end{gathered} \tag{3.7}$$

For every $m \geq 0$ and $v \in \mathcal{V}$ we set

$$v^{(m)}_{\mathbf{m}} = \begin{cases} v_{\mathbf{m}} & \text{if } \mathbf{m}\in Q_m,\\ 0 & \text{if } \mathbf{m}\in \mathbb{Z}^d\smallsetminus Q_m.\end{cases}$$

If $v, w \in \mathcal{V}^*$ satisfy that $\pi_{Q_{n+M}\smallsetminus Q_n}(v) = \pi_{Q_{n+M}\smallsetminus Q_n}(w)$ then we claim that the following conditions are equivalent:

(i) $v^{(n+M)} = w^{(n+M)}$,
(ii) $\xi(v^{(n+M)}) = \xi(w^{(n+M)})$,
(iii) $|\xi(v)_{\mathbf{n}} - \xi(w)_{\mathbf{n}}| < \varepsilon$ for every $\mathbf{n}\in Q_n$.

The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. If (iii) is satisfied, then the point $z = (v-w)^{(n)} = (v-w)^{(n+M)}$ satisfies that $|z_{\mathbf{n}}| \leq \|f\|_1^*$ for every $\mathbf{n}\in\mathbb{Z}^d$, and our choice of M guarantees that $|\xi(z)_{\mathbf{n}}| < 6\varepsilon/5$ for every $\mathbf{n}\in\mathbb{Z}^d$. According to the definition of ε this implies that $z = 0$, i.e. that (ii) is satisfied. From (ii) and Proposition 2.3 we conclude that

$$z = v^{(n)} - w^{(n)} \in f(\bar\sigma)(\ell^1(\mathbb{Z}^d,\mathbb{Z})) = \tilde f \mathfrak{R}_d.$$

If $z \succeq 0$ then this means that $(v, v-z)\in\Delta'_{\mathcal{V}}$ and $v - z \preceq v$, and Theorem 3.1 implies that $z = 0$. Similarly we see that $z = 0$ whenever $z \succeq 0$. Hence $z = 0$, which proves (i).

If we combine the equivalent statements (i)–(iii) with (3.7) we see that, for every $n \geq 0$ and $v, w \in \mathcal{V}^*$ with $\pi_{Q_{n+M}\smallsetminus Q_n}(v) = \pi_{Q_{n+M}\smallsetminus Q_n}(w)$, either $v^{(n)} = w^{(n)}$ and $|\xi(v)_{\mathbf{n}} - \xi(w)_{\mathbf{n}}| < \varepsilon/5$ for every $\mathbf{n}\in Q_n$, or $v^{(n)} \neq w^{(n)}$ and $|\xi(v)_{\mathbf{n}} - \xi(w)_{\mathbf{n}}| > 4\varepsilon/5$ for some $\mathbf{n}\in Q_n$. We fix $z\in\pi_{Q_{n+m}\smallsetminus Q_n}(\mathcal{V}^*)$ and choose, for every $w' \in \pi_{Q_n}(\pi^{-1}_{Q_{m+n}\smallsetminus Q_n}(z))$, an element $w\in\mathcal{V}^*$ with $\pi_{Q_n}(w) = w'$. The resulting set $S = \{w : w' \in \pi_{Q_n}(\pi^{-1}_{Q_{m+n}\smallsetminus Q_n}(z))\} \subset \mathcal{V}^*$ satisfies that

$$\max_{\mathbf{n}\in Q_n} |a_{\mathbf{n}} - b_{\mathbf{n}}| > 4\varepsilon/5$$

whenever $a = \xi(v), b = \xi(w)$ with $v, w\in S$, $v\neq w$, and from Proposition 13.7 in [18] we obtain that $h(\tau) \leq h(\alpha)$. Hence the two entropies are equal, as claimed. □

Corollary 3.2. *Let $f\in\mathfrak{R}_1$ be a Laurent polynomial such that the $\mathbb{Z}$-action $\alpha = \alpha_f$ on $X = X_f$ is expansive, and define $\mathcal{V}\subset\ell^\infty(\mathbb{Z}^d,\mathbb{Z})$ and $\xi\colon\mathcal{V}\longmapsto X$ as in Corollary* 2.1. *Then the restriction ξ^* of ξ to the set $\mathcal{V}^*\subset\mathcal{V}$ in* (3.2) *is bounded-to-one.*

Proof. We use the notation of the proof of Theorem 3.1. Set $N = \|f\|_1^*$ and $\mathcal{V} = \{0,\dots,N\}^{\mathbb{Z}}$, and use the continuity of $\xi\colon\mathcal{V}\longmapsto X$ to find an integer K with $|\xi(v)_0 - \xi(v')_0| < \varepsilon/2$ whenever $v, v'\in\mathcal{V}$ and $v_j = v'_j$ for $j = -K,\dots,K$, where $|\cdot|$ and ε are defined in (3.5) and (3.6). By increasing K and multiplying f by a suitable power of $\pm u_1$, if necessary, we may also assume without loss in generality that $f = f_0 + \dots + f_m u_1^m$ with $f_m > 0$, $f_0 \neq 0$ and $m < K$.

If ξ^* maps more than $N^{2(2K+1)}$ points of $\mathcal{V}^*$ to the same element $x \in X$ then there exist, for some $M \geq 2K+1$, points $v, v' \in \mathcal{V}^*$ with $v_j = v'_j$ for $j \in \{-M-2K, \dots, -M+2K\} \cup \{M-2K, \dots, M+2K\}$, $v_i \neq v'_i$ for some $i \in \{-M+2K+1, \dots, M-2K-1\}$, and $\xi^*(v) = \xi^*(v')$. Put

$$w_j = \begin{cases} v_j - v'_j & \text{for } j \in \{-M, \dots, M\}, \\ 0 & \text{otherwise,} \end{cases}$$

and assume without loss in generality that $w \prec 0$ (otherwise interchange v and v'). Our definition of K implies that $|\xi(v)_j - \xi(v-w)_j| < \varepsilon/2$ for $|j| \geq M$ and $|\xi(v')_j - \xi(v-w)_j| = |\xi(v)_j - \xi(v-w)_j| < \varepsilon/2$ for $|j| \leq M$, and the definition of ε shows that $\xi^*(v) = \xi^*(v-w)$. Since $v \in \mathcal{V}^*$, $w \prec 0$ and $v - w \in \mathcal{V}$ this violates the properties of $\mathcal{V}^*$ in Theorem 3.1. Hence ξ^* is at most $N^{2(2K+1)}$-to-one. $\square$

The next corollary is an immediate consequence of the proof of Theorem 3.1.

Corollary 3.3. *Let $f \in \mathfrak{R}_d$ be a Laurent polynomial such that the $\mathbb{Z}^d$-action $\alpha = \alpha_f$ on $X = X_f$ is expansive, and define $\mathcal{V} \subset \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ and $\xi\colon \mathcal{V} \longmapsto X$ as in Corollary* 2.1. *If $\mathcal{W} \subset \mathcal{V}$ is a closed, shift-invariant set with $\xi(\mathcal{W}) = X$, then the set*

$$\mathcal{W}^* = \mathcal{W} \smallsetminus \bigcup_{h \in \mathfrak{R}_d^+} (\mathcal{W} + \tilde{f}h) \tag{3.8}$$

also satisfies that $\xi(\mathcal{W}^*) = X$.

Remarks 3.1. (1) For different sets $\mathcal{W}_i \subset \mathcal{V}$, $i = 1, 2$, the sets $\mathcal{W}_1^*$ and $\mathcal{W}_2^*$ in Corollary 3.3 may be different: indeed, in Example 2.1 the fixed point $w = (w_{\mathbf{n}})$ with $w_{\mathbf{n}} = 3$ for every $\mathbf{n} \in \mathbb{Z}^2$ lies in $\mathcal{V}^* \smallsetminus \mathcal{W}^*$.

(2) By applying the same argument as at the end of Example 2.1 (1) one can show that there exist minimal, closed, shift-invariant subsets $\mathcal{W} \subset \mathcal{V}$ with $\xi(\mathcal{W}) = X$. For such a minimal subset we obviously have that $\mathcal{W}^* = \mathcal{W}$ (cf. Corollary 3.3).

(3) For the hyperbolic toral automorphism $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ the map $\xi\colon \mathcal{V}^* \longmapsto X = \mathbb{T}^2$ described in (2.11) and Theorem 3.1 is essentially identical to the fibadic expansion of real numbers appearing in [21].

4. Sofic covers and covers of finite type

Suppose that $f \in \mathfrak{R}_d$ is a Laurent polynomial such that the $\mathbb{Z}^d$-action $\alpha = \alpha_f$ on the compact, abelian group $X = X_f$ is expansive. We define the surjective map $\xi\colon \ell^\infty(\mathbb{Z}^d, \mathbb{Z}) \longmapsto X$ by (2.11), denote by $\bar{\sigma}$ the shift-action of $\mathbb{Z}^d$ on $\ell^\infty(\mathbb{Z}^d, \mathbb{Z})$, and recall that $\xi \cdot \bar{\sigma}^{\mathbf{n}} = \alpha^{\mathbf{n}} \cdot \xi$ for every $\mathbf{n} \in \mathbb{Z}^d$ (cf. (2.13)). As we saw in Theorem 3.1, one can find an equal entropy symbolic cover $\mathcal{V}^* \subset \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$. We shall prove the following result (for terminology we refer to the Introduction and Section 2).

Theorem 4.1. *Let $f \in \mathfrak{R}_1$ be a Laurent polynomial such that the $\mathbb{Z}$-action $\alpha = \alpha_f$ on $X = X_f$ is expansive. Then (X, α) has a finite cover of finite* *type, and the set* $\mathcal{V}^* \subset \mathcal{V} \subset \ell^\infty(\mathbb{Z},\mathbb{Z})$ *in* (2.14) *and* (3.2) *is a finite sofic cover of* (X,α).

Proof. The proof of Theorem 4.1 is closely modelled on [7]. After multiplying f by $\pm u_1^k$ for some $k \in \mathbb{Z}$ we may assume that $f = f_0 + \dots + f_m u_1^m$ with $f_0 \neq 0$ and $f_m > 0$. According to Corollary 2.1 there exists a positive integer $N \leq \|f\|_1^*$ such that the compact set $\mathcal{W} = \{0,\dots,N\}^{\mathbb{Z}} \subset \mathcal{V} \subset \ell^\infty(\mathbb{Z},\mathbb{Z})$ satisfies that $\xi(\mathcal{W}) = X$ (cf. also Example 2.1). We denote by $\mathcal{H}' \subset \mathfrak{R}_1$ the set of all elements of the form $h = h_{-m+1}u_1^{-m-1} + \dots + h_0 \in \mathfrak{R}_1$ with $|h_k| \leq M = N\|w^\Delta\|_1$ for every $k = -m+1,\dots,0$. By identifying each $h \in \mathcal{H}'$ with the element $(h_{-m+1},\dots,h_0) \in \{-M,\dots,M\}^m$ we may put $\mathcal{H}' = \{-M,\dots,M\}^m$. Finally we introduce an additional element $\mathbf{0}^* \neq \mathbf{0}$ and set $\mathcal{H} = \mathcal{H}' \cup \{\mathbf{0}^*\}$.

The graphs G *and* G^*. Let G be the finite directed graph whose vertex set V_G is equal to $\mathcal{H}$ and whose edges are labelled by elements of $E = \{0,\dots,N\}$. If $h, h' \in V_G = \mathcal{H}$ and $e \in E$ then there exists an edge

$$h \xrightarrow{e} h' \tag{4.1}$$

if and only if at least one of the following conditions are satisfied:

(i) $h = h' = \mathbf{0}^*$,

(ii) $h = \mathbf{0}^*$, $h' = (0,\dots,0,h_0')$, $h_0' > 0$ and $e - f_0h_0 \in E$,

(iii) $h = (h_{-m+1},\dots,h_0), h' = (h'_{-m+1},\dots,h'_0) \in \mathcal{H}'$, $h_{j+1} = h'_j$ for $j = -m+1,\dots,-1$, and

$$\begin{aligned} e - f_m h'_{-m+1} - \dots - f_1 h'_0 - f_0 h_0 \\ = e - f_m h'_{-m+1} - f_{m-1}h_{-m+1} - \dots - f_0 h_0 \in E. \end{aligned} \tag{4.2}$$

If (4.1) is satisfied we call h' an *e-follower* of h. The set of all e-followers of h is denoted by $\mathsf{f}_e(h) \subset V_G$. Finally we denote by G^* the subgraph of G consisting of all vertices and edges which can be reached from the vertex $\mathbf{0}^*$, write V_{G^*} for the vertex set of G^*, and observe that $\mathsf{f}_e(h) \subset V_{G^*}$ for every $h \in G^*$ and $e \in E$.

The graphs Γ *and* Γ^*. We define a second finite directed graph Γ, whose set of vertices V_Γ is the collection $\mathcal{P}(V_{G^*})$ of all subsets of V_{G^*}, and whose edges are again labelled by elements of E. If γ, γ' are vertices of Γ (i.e. $\gamma_1, \gamma_2 \subset V_{G^*}$) and $e \in E$, then there exists an edge

$$\gamma \xrightarrow{e} \gamma' \tag{4.3}$$

if and only if

$$\gamma' = \bigcup_{h\in\gamma} \mathsf{f}_e(h). \tag{4.4}$$

The set $\mathsf{F}_e(\gamma)$ of e-followers of a vertex $\gamma \in V_\Gamma$ is defined as above. Note that there exists, for each $\gamma \in \Gamma$ and $e \in E$, a unique edge of Γ labelled e which starts at γ, and that $\mathbf{0}^* \in \gamma$ for every $\gamma \in \Gamma$ which can be reached from the vertex $\{\mathbf{0}^*\}$.

The graph Γ^* is obtained from Γ by deleting all vertices of Γ which either cannot be reached from $\{\mathbf{0}^*\}$ or which contain the element $\mathbf{0} \in V_G$, as well as all edges of Γ leading into or coming out of such a vertex. The vertex set of Γ^* is denoted by $V_{\Gamma^*} \subset \mathcal{P}(V_G \smallsetminus \{\mathbf{0}\})$.

The shifts of finite type Ω and Ω^*. Each bi-infinite path in Γ can be written as an element of $(V_\Gamma \times E)^{\mathbb{Z}}$ by noting at each time $k \in \mathbb{Z}$ the vertex γ_k and the label e_k of the edge $\gamma_k \xrightarrow{e_k} \gamma_{k+1}$ traversed by the path, and the set $\Omega \subset (V_\Gamma \times E)^{\mathbb{Z}}$ of all bi-infinite paths in Γ is a shift of finite type. We write $\tilde{\sigma}$ for the shift on Ω, define a continuous map $\theta\colon \Omega \longmapsto E^{\mathbb{Z}} = \mathcal{W}$ by sending each path $\omega = (\omega_k) = ((\gamma_k, e_k)) \in \Omega$ to its bi-infinite sequence of edges $w = (e_k) \in \mathcal{W}$, and observe that $\theta \cdot \tilde{\sigma} = \bar{\sigma} \cdot \theta$. The map $\theta\colon \Omega \longmapsto \mathcal{W}$ is *right resolving* (cf. [9]), and $|\theta^{-1}(v)| = |V_\Gamma|$ for every $v \in \mathcal{V}$.

Let $\Omega^* \subset \Omega$ be the set of bi-infinite paths in Γ^* and put

$$\Omega_+ = \{\omega = (\omega_n) \in \Omega : \omega_n = (\{\mathbf{0}^*\}, 0) \text{ for every } n < 0\},$$
$$\Omega_+^* = \Omega^* \cap \Omega_+.$$

Then Ω and Ω^* are the closures of $\bigcup_{n\in\mathbb{Z}} \tilde{\sigma}^n(\Omega_+)$ and $\bigcup_{n\in\mathbb{Z}} \tilde{\sigma}^n(\Omega_+^*)$, respectively, and Ω^* is again a shift of finite type. In the notation of (3.8) we assert that

$$\theta(\Omega^*) = \mathcal{W}^*. \tag{4.5}$$

In order to prove (4.5) we set

$$\mathcal{W}_+ = \{w = (w_n) \in \mathcal{W} : w_n = 0 \text{ for } n < 0\},$$
$$\mathcal{W}_+^* = \mathcal{W}_+ \smallsetminus \bigcup_{h \in \mathfrak{R}_1^+} (\mathcal{W}_+ + \tilde{f}h) \subset \mathcal{W}^*.$$

Then there exists, for every $w \in W_+$, a unique element $\omega \in \Omega_+$ with $\theta(\omega) = w$: if $\omega_n = (\gamma_n, w_n)$ then the elements $\gamma_n \in V_\Gamma$ are determined inductively by the equation

$$\gamma_n = \mathsf{F}_{e_{n-1}}(\gamma_{n-1})$$

for every $n \geq 0$.

Suppose that $\omega \in \Omega_+$ and $\theta(\omega) = w$. If $w \notin \mathcal{W}_+^*$ then there exists a polynomial $h = h_k u_1^k + \cdots + h_{k+L} u_1^{k+L} \in \mathfrak{R}_1^+$ with $k \geq 0$ such that $v = w - g\tilde{f} \in \mathcal{W}_+$. According to the conditions (i)–(iii) characterising (4.1),

$$(0, \dots, 0, h_k) \in \gamma_k = \mathsf{F}_{e_{k-1}}(\gamma_{k-1}),$$
$$(0, \dots, 0, h_k, h_{k+1}) \in \gamma_{k+1} = \mathsf{F}_{e_k}(\gamma_k),$$
$$\vdots$$
$$(h_{k+L}, 0, \dots, 0) \in \gamma_{k+L+m-1} = \mathsf{F}_{e_{k+L+m-2}}(\gamma_{k+L+m-2}),$$
$$\mathbf{0} \in \mathsf{F}_{e_{k+L+m-1}}(\gamma_{k+L+m-1}),$$

which violates the definition of Γ^* in (4.3)–(4.4). Hence $\theta(\Omega_+) \subset \mathcal{W}_+^*$.

Conversely, if $w \in \mathcal{W}_+^*$, then the above argument shows that the unique element $\omega \in \Omega_+$ with $\theta(\omega) = w$ must lie in Ω_+^*. Hence $\theta(\Omega_+^*) = \mathcal{W}_+^*$, and the shift-invariance of $\theta(\Omega^*)$ guarantees that $\theta(\Omega^*) = \mathcal{W}^*$, as claimed in (4.5). In particular, $\mathcal{W}^*$ is a sofic shift, and Corollary 3.2 yields that $\mathcal{W}^*$ is a finite sofic cover of (X, α).

By construction, the map $\theta\colon \Omega^* \longmapsto \mathcal{W}^*$ is bounded-to-one, so that Ω^* is a finite cover of finite type of (X, α). By setting $N = \|f\|_1^*$ we have completed the proof of Theorem 4.1. $\square$

The following example illustrates the proof of Theorem 4.1.

Example 4.1. Let $f = 1 - 3u_1 + u_1^2 \in \mathfrak{R}_1^+$, $X = X_f \subset \mathbb{T}^{\mathbb{Z}}$ and $\alpha = \alpha_f$. The map $\zeta\colon X \longmapsto \mathbb{T}^2$, defined by $\zeta(x) = (x_0, x_1)$ for every $x = (x_n) \in X$, is surjective and satisfies that

$$\zeta \cdot \alpha \cdot \zeta^{-1} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 3 \end{smallmatrix}\right).$$

A direct calculation shows that the point $w^\Delta \in \ell^\infty(\mathbb{Z}, \mathbb{R})$ in (2.6) is given by

$$w_n^\Delta = -\frac{1}{\sqrt{5}} \cdot \left(\frac{3-\sqrt{5}}{2}\right)^{|n|}$$

for every $n \in \mathbb{Z}$. Hence $\|w^\Delta\|_1 = 1$. According to Example 2.1 we may put $\mathcal{W} = \{0,1,2\}^{\mathbb{Z}}$ and $M = N = 2$.

In order to describe the graph G with vertex set $V_G = \{-2,\dots,2\}^2 \cup \{(0,0)^*\}$ and edges labelled by elements of $E = \{0,1,2\}$ we assume that $(i,j),(i',j') \in \{-2,\dots,2\}^2$ and $e \in E$ and draw an edge

$$(i,j) \xrightarrow{e} (i',j')$$

if and only if

$$j = i' \text{ and } e - i + 4j - j' = e - i + 4i' - j' \in E.$$

The graph G has five further edges:

$$(0,0)^* \xrightarrow{0} (0,0)^*,$$
$$(0,0)^* \xrightarrow{1} (0,0)^*,$$
$$(0,0)^* \xrightarrow{2} (0,0)^*,$$
$$(0,0)^* \xrightarrow{1} (0,1),$$
$$(0,0)^* \xrightarrow{2} (0,1).$$

Note that the only edge leading into any vertex of the form $(\pm 2, j')$ are

$$(2,2) \xrightarrow{2} (2,2),$$
$$(-2,-2) \xrightarrow{0} (-2,-2),$$

so that $V_{G^*} \subset \{-1,0,1\}^2$ (cf. Figure 1). From the graph G^* one easily

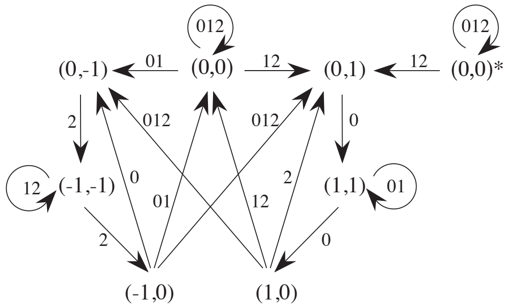

FIGURE 1. The graph G^*

obtains the graph Γ^* in Figure 2. As α is ergodic, there exists an irreducible subshift of finite type $\Omega^{**} \subset \Omega^*$ with $\xi \cdot \theta(\Omega^{**}) = X$, and from Figure 2 we

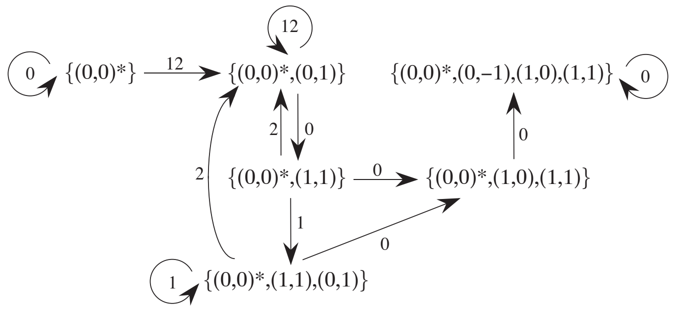

FIGURE 2. The graph Γ^*

see that Ω^{**} is the set of all bi-infinite paths in the graph Γ^{**} in Figure 3. From the existence of the *magic word* '0' in Figure 3 it is clear that that the

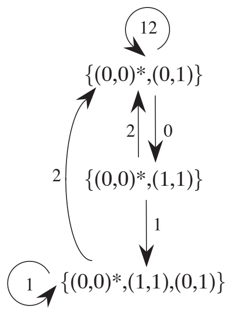

FIGURE 3. The graph Γ^{**}

restriction of θ to the set of doubly transitive points in Ω^{**} is injective.

We claim that the restriction of ξ to the set of doubly transitive points in $\mathcal{W}^*$ is again injective. If $v \in \theta(\Omega^{**}) \subset \mathcal{W}^*$ contains the string '0112' infinitely often both in the past and in the future, then no nonzero $y \in \{0,1,-1\}^{\mathbb{Z}}$ satisfies that $v + f(\bar{\sigma})(y) \in \mathcal{W}^*$, so that $\xi(v) \neq \xi(v')$ for every $v' \in \mathcal{W}^* \smallsetminus \{v\}$. Hence ξ is injective on the set of doubly transitive points in $\mathcal{W}^*$, $\mathcal{W}^*$ is a sofic representation of (X,α), and Ω^* and Ω^{**} are representations of finite type of (X,α).

5. EXAMPLES

We present two examples of sofic representations and representations of finite type of expansive $\mathbb{Z}^2$-actions of the form (X_f,α_f) with $f \in \mathfrak{R}_2$.

5.1. The $\mathbb{Z}^2$-action defined by the polynomial $f = 3 - u_1 - u_2 \in \mathfrak{R}_2$.

Proposition 5.1. *Let $f = 3 - u_1 - u_2 \in \mathfrak{R}_2$, $X = X_f$, $\alpha = \alpha_f$ (cf. Proposition 2.1),*

$$\mathcal{W} = \{0,1,2\}^{\mathbb{Z}^2} \subset \ell^\infty(\mathbb{Z}^2,\mathbb{Z}),$$

$\xi\colon \mathcal{W} \longmapsto X$ the continuous, surjective map (2.11), and let τ be the restriction to $\mathcal{W}$ of the shift-action $\bar{\sigma}$ of $\mathbb{Z}^2$ on $\ell^\infty(\mathbb{Z}^2,\mathbb{Z})$. Then $\mathcal{W}$ is a symbolic representation of finite type of (X,α) and $\nu\xi^{-1} = \lambda_X$, where ν is the equidistributed Bernoulli measure on $\mathcal{W}$.

Proof. From Example 2.1 (1) we know that that $\xi(\mathcal{W}) = X$, and Example 4.6 in [10] shows that the fundamental homoclinic point w^Δ of α in (2.6) is given by

$$w^\Delta_{\mathbf{m}} = \begin{cases} 3^{m_1+m_2-1}\binom{-m_1-m_2}{-m_2} & \text{if } m_1 \leq 0,\, m_2 \leq 0, \\ 0 & \text{otherwise,} \end{cases}$$

for every $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$. In particular, $\|w^\Delta\|_1 = 1$.

Put $Y = \{-2, \dots, 2\}^{\mathbb{Z}^2}$ and assume that $v, w \in \mathcal{W}$ and $\xi(v) = \xi(w)$. Then (2.11) shows that $v - w = f(\bar\sigma)(y)$ for some $y \in \ell^\infty(\mathbb{Z}^2, \mathbb{Z})$, and (2.12) implies that

$$\|y\|_\infty = \|\bar\xi \cdot f(\bar\sigma)(y)\| \leq \|w^\Delta\|_1 \|f(\bar\sigma)(y)\|_\infty = \|f(\bar\sigma)(y)\|_\infty \leq 2.$$

We wish to prove that the set

$$\begin{aligned} N &= \{v \in \mathcal{W} : \xi^{-1}(\xi(v)) \neq \{v\}\} \\ &= \{v \in \mathcal{W} : v + f(\bar\sigma)(y) \in \mathcal{W} \text{ for some nonzero } y \in Y\} \end{aligned}$$

has ν-measure zero. As we can cover N with countably many shifts of the sets

$$\begin{aligned} N_2^+ &= \{v \in \mathcal{W} : \text{there exists a nonzero } y \in Y \text{ with } y_{(0,0)} = \|y\|_\infty = 2 \\ &\qquad\qquad \text{and } v + f(\bar\sigma)(y) \in \mathcal{W}\}, \\ N_2^- &= \{v \in \mathcal{W} : \text{there exists a nonzero } y \in Y \text{ with } -y_{(0,0)} = \|y\|_\infty = 2 \\ &\qquad\qquad \text{and } v + f(\bar\sigma)(y) \in \mathcal{W}\}, \\ N_1^+ &= \{v \in \mathcal{W} : \text{there exists a nonzero } y \in Y \text{ with } y_{(0,0)} = \|y\|_\infty = 1 \\ &\qquad\qquad \text{and } v + f(\bar\sigma)(y) \in \mathcal{W}\}, \\ N_1^- &= \{v \in \mathcal{W} : \text{there exists a nonzero } y \in Y \text{ with } -y_{(0,0)} = \|y\|_\infty = 1 \\ &\qquad\qquad \text{and } v + f(\bar\sigma)(y) \in \mathcal{W}\}, \end{aligned}$$

we only have to prove that the sets $N_2^+, N_2^-, N_1^+, N_1^-$ have ν-measure zero.

We start with N_2^+ (the case N_2^- is completely analogous). Take v, y as in the definition of N_2^+ and observe that

$$v_{(0,0)} + 3 \cdot 2 - y_{(1,0)} - y_{(0,1)} \leq 2,$$

which can only be satisfied if $y_{(1,0)} = y_{(0,1)} = 2$ and $v_{(0,0)} = 0$. The same argument holds for the coordinates $(n, 0)$ with $n > 1$. Hence

$$N_2^+ \subset \{v \in \mathcal{W} : v_{(n,0)} = 0 \text{ for every } n \geq 0\}$$

and $\nu(N_2^+) = 0$.

Now take $v \in N_1^+$ and $y \in Y$ as in the definition of N_1^+. As $v + f(\bar\sigma)(y) \in \mathcal{W}$ we get that

$$v_{(0,0)} + (f(\bar\sigma)(y))_{(0,0)} = v_{(0,0)} + 3 - y_{(1,0)} - y_{(0,1)} \leq 2.$$

There are two possibilities: either $y_{(1,0)} = y_{(0,1)} = 1$ and $v_{(0,0)} \in \{0, 1\}$, or one of the two values $y_{(1,0)}, y_{(0,1)}$ is equal to zero and $v_{(0,0)} = 0$. We define inductively a map $p = p_y \colon \mathbb{N} \longmapsto \mathbb{N}^2$ with $p(0) = (0,0)$. If $p(i)$ for is defined for $i < l$, $l \geq 1$, such that $y_{p(i)} = 1$ for $i = 0, \dots, l$ we set $p(l+1) = p(l) + (1,0)$ if $y_{p(l)+(1,0)} = 1$ and $p(l+1) = p(l) + (0,1)$ otherwise.

This shows that there exists, for every point $v \in N_1^+$, an element $y \in Y$ and a map $p = p_y \colon \mathbb{N} \longmapsto \mathbb{N}^2$ with $p(0) = (0,0)$ such that $(f(\bar{\sigma})(y))_{p(i)} \geq 1$, $p(i+1) - p(i) \in \{(1,0),(0,1)\}$, and hence $v_{p(i)} \leq 1$ for every $i \geq 0$.

For every path $p \colon \mathbb{N} \longmapsto \mathbb{N}^2$ with $p(0) = (0,0)$ and $p(i+1) - p(i) \in \{(1,0),(0,1)\}$ for all $i \geq 0$, and for every $l \geq 1$, we can estimate the ν-measure of the sets

$$\begin{aligned} N_p^+(l) = \{v \in \mathcal{W} :& \text{ there exists a } \ y \in Y \ \text{ with } \ v + y \in \mathcal{W} \\ & \text{and } \ p(i) = p_y(i) \ \text{ for every } \ i = 0, \dots, l\}, \\ N_p^-(l) = \{v \in \mathcal{W} :& \text{ there exists a } \ y \in Y \ \text{ with } \ v - y \in \mathcal{W} \\ & \text{and } \ p(i) = p_y(i) \ \text{ for every } \ i = 0, \dots, l\}, \end{aligned}$$

as follows. Let $0 \leq i < l-1$ be an integer such that $p(i+1) = p(i) + (1,0)$. If $v_{p(i)} = 1$ then $(f(\bar{\sigma})(y))_{p(i)} = 1$, $y_{p(i)+(1,0)} = y_{p(i)+(0,1)} = 1$, and $v_{p(i)+(0,1)} \in \{0,1\}$. If $v_{p(i)} = 0$ then $(f(\bar{\sigma})(y))_{p(i)}$ may be equal to 2 and $y_{p(i)+(0,1)}$ may be equal to 0. If, on the other hand, $p(i+1) = p(i) + (0,1)$ then $y_{p(i)+(1,0)} = 0$, $(f(\bar{\sigma})(y))_{p(i)} = 2$ and $v_{p(i)} = 0$. It follows that, if k is the number of horizontal steps of p at the times $0, \dots, l-1$, i.e. the number of $i \in \{0, \dots, l-1\}$ when $p(i+1) = p(i) + (1,0)$, then

$$\nu(N_p(l)) \leq \left(\frac{5}{9}\right)^k \left(\frac{1}{3}\right)^{l-1-k}.$$

As there are exactly $\binom{l-1}{k}$ such paths of length l we obtain that

$$\nu(N_1^+) \leq \lim_{l\to\infty} \sum_{k=0}^{l-1} \binom{l-1}{k} \left(\frac{5}{9}\right)^k \left(\frac{1}{3}\right)^{l-1-k} = \lim_{l\to\infty} \left(\frac{5}{9} + \frac{1}{3}\right)^{l-1} = 0.$$

Similarly we see that $\nu(N_1^-) = 0$.

By using the shift-invariance of ν we conclude that $\nu(N) = 0$ and that $\mathcal{W}$ is a representation of finite type of (X, α). Finally we note that ν and λ_X are the unique measures of maximal entropy for the $\mathbb{Z}^2$-actions τ on $\mathcal{W}$ and λ_X on X (cf. [11]). As $h_\nu(\tau) = h_{\lambda_X}(\alpha) = \log 3$ (the latter again by [11]), we obtain that $\lambda_X = \nu\xi^{-1}$. □

Corollary 5.1. *Let* $f_1 = 3 - u_1 - u_2$, $f_2 = 3 - u_1^{-1} - u_2$, $f_3 = 3 - u_1 - u_2^{-1}$, $f_4 = 3 - u_1^{-1} - u_2^{-1}$, *and define* $X_i = X_{f_i}$, $\alpha_i = \alpha_{f_i}$ *as in Proposition* 2.1 *for* $i = 1, \dots, 4$. *Then the* $\mathbb{Z}^2$*-actions* α_i *and* α_j *are metrically and almost topologically conjugate for* $1 \leq i < j \leq 4$.

Proof. Proposition 5.1 can be adapted to show that each of the actions α_i is metrically and almost topologically conjugate to the full two-dimensional three-shift with equidistributed Bernoulli measure. Hence any two of these actions are metrically and almost topologically conjugate. □

5.2. **The $\mathbb{Z}^2$-action defined by the polynomial** $f = 5 - u_1 - u_1^{-1} - u_2 \in \mathfrak{R}_2$ **.** First we consider the Laurent polynomial $g = u_1^{-1} - 5 + u_1 \in \mathfrak{R}_1$ and denote by $\beta = \alpha_g$ the expansive automorphism of $Y = X_g \cong \mathbb{T}^2$ defined in Proposition 2.1. From Example 2.1 (2) we know that $\xi(\mathcal{V}) = Y$, where $\mathcal{V} = \{0, \dots, 4\}^{\mathbb{Z}} \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ and $\xi \colon \ell^\infty(\mathbb{Z}, \mathbb{Z}) \longmapsto X$ is defined by (2.11).

From Theorem 4.1 we know that the closed, shift-invariant subset $\mathcal{V}^* \subset \mathcal{V}$ defined in (3.2) is a finite-to-one sofic cover of (Y, β), and by modifying the argument of Example 4.1 appropriately one observes that $\mathcal{V}^*$ is, in fact, a sofic representation of (Y, β). By continuing as in Example 4.1 we obtain the directed graph Γ^{**} in Figure 4 and the corresponding shift of finite type $\Omega^{**} \subset \{0, 1, 1', 2, 3, 4\}^{\mathbb{Z}}$.

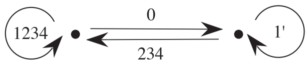

FIGURE 4. The graph Γ^{**}

If we represent the Ω^{**} by the transition matrix

$$\begin{pmatrix} 0&1&0&1&1&1\\ 0&1&0&1&1&1\\ 1&0&1&1&1&1\\ 1&0&1&1&1&1\\ 1&0&1&1&1&1\\ 1&0&1&1&1&1 \end{pmatrix}$$

then the measure ν of maximal entropy of Ω^{**} is given by the stochastic matrix

$$P = \begin{pmatrix} 0&a&0&b&b&b\\ 0&a&0&b&b&b\\ c&0&a&a&a&a\\ c&0&a&a&a&a\\ c&0&a&a&a&a\\ c&0&a&a&a&a \end{pmatrix}, \text{ where } \begin{array}{rcl} a &=& \frac{5}{2} - \frac{1}{2}\sqrt{21}, \\ b &=& -\frac{1}{2} + \frac{1}{6}\sqrt{21}, \\ c &=& -9 + 2\sqrt{21}, \end{array}$$

with its positive left eigenvector

$$\left(-\tfrac{3}{2} + \tfrac{5}{14}\sqrt{21}, 2 - \tfrac{3}{7}\sqrt{21}, \tfrac{1}{2} - \tfrac{1}{14}\sqrt{21}, \tfrac{1}{\sqrt{21}}, \tfrac{1}{\sqrt{21}}, \tfrac{1}{\sqrt{21}}\right).$$

By identifying the symbols 1 and $1'$ of Ω we obtain a sofic shift $\mathcal{V}^{**} \subset \mathcal{V}^* \subset \{0, \dots, 4\}^{\mathbb{Z}} \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ and a measure ν with maximal entropy on $\mathcal{V}^{**}$ which is a sofic representation of (Y, β): $\xi(\mathcal{V}^{**}) = Y$, $\nu\xi^{-1} = \lambda_Y$, ξ is injective ν-a.e. on $\mathcal{V}^{**}$, and $\beta \cdot \xi(v) = \xi \cdot \bar{\sigma}(v)$ for every $v \in \mathcal{U}$, where $\bar{\sigma}$ is the shift on $\mathcal{V}^{**} \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$).

Proposition 5.2. *Let $f = 5 - u_1 - u_1^{-1} - u_2 \in \mathfrak{R}_2$ and let $\alpha = \alpha_f$ be the expansive $\mathbb{Z}^2$-action on $X = X_f$ defined in Proposition* 2.1. *Put*

$$\begin{aligned} \mathcal{U} &= \{v = (v_{\mathbf{n}}) \in \{0, \dots, 4\}^{\mathbb{Z}^2} : (v_{(m,n)}, m \in \mathbb{Z}) \in \mathcal{V}^{**} \text{ for every } n \in \mathbb{Z}\} \\ &\cong (\mathcal{V}^{**})^{\mathbb{Z}}, \end{aligned}$$

and let τ be the shift-action (1.1) *of $\mathbb{Z}^2$ on $\mathcal{U}$. If $\mu = \nu^{\mathbb{Z}}$ is the τ-invariant product measure on $\mathcal{U} \cong (\mathcal{V}^{**})^{\mathbb{Z}}$, and if $\xi \colon \mathcal{U} \longmapsto X$ is the map defined in* (2.11)*, then $\xi(\mathcal{U}) = X$, $\mu\xi^{-1} = \lambda_X$, ξ is injective μ-a.e. on $\mathcal{U}$, and $\alpha^{\mathbf{n}} \cdot \xi(v) = \xi \cdot \tau^{\mathbf{n}}(v)$ for every $v \in \mathcal{V}$. In particular, $(\mathcal{U}, \tau)$ is a sofic representation of (X, α).*

Proof. First we show that $\xi(\mathcal{U}) = X$. From Example 2.1 we know that the set $\mathcal{W} = \{0, .., 4\}^{\mathbb{Z}^2}$ satisfies $\xi(\mathcal{W}) = X$. In order to show that $x \in \xi(\mathcal{U})$ for some fixed $x \in X$ we prove inductively that there exists, for each $n \in \mathbb{N}$, a $v^{(n)} \in \xi^{-1}(\{x\}) \cap \{-1, .., 5\}^{\mathbb{Z}^2}$ with $\pi_{\mathbb{Z}\times[-n,0]}(v^{(n)}) \in \pi_{\mathbb{Z}\times[-n,0]}(\mathcal{U}) = (\mathcal{V}^{**})^{[-n,0]}$. Take an element $w \in \mathcal{W} \cap \xi^{-1}(\{x\})$ and put $w' = \pi_{\mathbb{Z}\times\{0\}}(w) \in \ell^\infty(\mathbb{Z}, \mathbb{N})$. Since $\pi_{\mathbb{Z}\times\{0\}}(X) = Y = X_g \subset \mathbb{T}^{\mathbb{Z}}$ and $\xi(\mathcal{V}^{**}) = Y$ there exists a

$y^{(0)} \in \ell^\infty(\mathbb{Z},\mathbb{Z})$ with $w' + y^{(0)}\tilde{g} \in \mathcal{V}^{**}$ (for notation we refer to the paragraph preceding the statement of this proposition). As

$$0 \le w'_l \le 4 \ \text{ and } \ 0 \le w'_l + (y^{(0)}\tilde{g})_l \le 4$$

for all $l \in \mathbb{Z}$ we obtain from an elementary calculation that $\|y^{(0)}\|_\infty \le 1$. We regard $\tilde{g}$ as an element of $\mathfrak{R}_2 \subset \ell^\infty(\mathbb{Z}^2,\mathbb{Z})$ and set $v^{(0)} = w + y^{(0)}\tilde{f}$.

Suppose that we have found $v^{(l)} \in \xi^{-1}(\{x\}) \cap \{-1,\dots,5\}^{\mathbb{Z}^2}$ for $l = 0,\dots,k$. Put

$$w' = \pi_{\mathbb{Z}\times\{k+1\}}(v^{(k)}) \in \pi_{\mathbb{Z}\times\{k+1\}}(\xi^{-1}(\{x\}))$$

and regard w' as an element of $\{-1,\dots,5\}^{\mathbb{Z}}$. Then the above argument allows us to find an element $y^{(k+1)} \in \ell^\infty(\mathbb{Z},\mathbb{Z})$ with $\|y^{(k+1)}\|_\infty \le 1$ and $w' + y^{(k+1)}\tilde{g} \in \mathcal{V}^{**}$. By setting

$$v^{(k+1)} = v^{(k)} + y^{(k+1)}u_2^{-k+1}\tilde{f} \in \xi^{-1}(\{x\}) \cap \{-1,\dots,5\}^{\mathbb{Z}^2}$$

we have completed the induction step.

Having constructed a sequence $(v^{(n)})$ with the above properties we conclude as in Example 2.1 that the set $\xi^{-1}(\{x\}) \cap \mathcal{U}$ is nonempty for every $x \in X$ and hence that $\xi(\mathcal{U}) = X$.

We furnish the space $\mathcal{U} = (\mathcal{V}^{**})^{\mathbb{Z}}$ with the product measure $\mu = \nu^{\mathbb{Z}}$. Then μ is the unique measure of maximal entropy on $\mathcal{U}$, and a more complicated version of the percolation-type argument in Proposition 5.1 shows the following.

(1) The set

$$N = \{v \in \mathcal{U} : \text{there exists a nonzero } y \in \ell^\infty(\mathbb{Z}^2,\mathbb{Z}) \text{ with } v + y\tilde{f} \in \mathcal{V}\}$$

has μ-measure zero;

(2) If $\mathcal{U}' = \mathcal{U} \smallsetminus N$, then $\xi^{-1}(\xi(\mathcal{U}')) \cap \mathcal{U} = \mathcal{U}'$;

(3) $\mu\xi^{-1} = \lambda_X$.

Modulo some slightly unpleasant details this completes the proof of Proposition 5.2. □

Corollary 5.2. *Let* $f_1 = 5 - u_1 - u_1^{-1} - u_2 \in \mathfrak{R}_2$, $f_2 = 5 - u_1 - u_1^{-1} - u_2^{-1} \in \mathfrak{R}_2$, *and let* $\alpha_i = \alpha_{f_i}$ *be the expansive* $\mathbb{Z}^2$*-action on* $X_i = X_{f_i}$ *defined in Proposition* 2.1, *where* $i = 1,2$. *Then the* $\mathbb{Z}^2$*-actions* α_1 *and* α_2 *are metrically and almost topologically conjugate.*

Proof. Proposition 5.2 and symmetry show that $(\mathcal{U},\tau)$ is a sofic representation of (X_1,α_1) and (X_2,α_2). If we regard $\mathcal{U}$ as a closed, shift-invariant subset of $\ell^\infty(\mathbb{Z}^2,\mathbb{Z})$ as in Proposition 5.2 and define $\xi_i\colon \ell^\infty(\mathbb{Z}^2,\mathbb{Z}) \longmapsto X_i$ by (2.11) for $i = 1,2$. If we consider, for $i = 1,2$, the G_δ $\mathcal{U}'_i \subset \mathcal{U}$ defined by (1.5) with ξ_i replacing ϕ and set $\mathcal{U}' = \mathcal{U}'_1 \cap \mathcal{U}'_2$, $X'_i = \xi_i(\mathcal{U}') \subset X_i$, then $\mu(\mathcal{U}') = 1$, and $\xi_2 \cdot \xi_1^{-1}\colon \xi_1(\mathcal{U}') \longmapsto \xi_2(\mathcal{U}')$ is a Haar measure preserving homeomorphism which sends α_1 to α_2. □

Remark 5.1. In Proposition 5.1 we obtain a symbolic representation of the expansive algebraic $\mathbb{Z}^2$-action α by a two-dimensional full shift. The representation of the $\mathbb{Z}^2$-action α in Proposition 5.2 is in terms of a sofic shift, but it is easy to see that the two-dimensional shift of finite type $(\Omega^{**})^{\mathbb{Z}}$, where Ω^{**} is the shift of finite type represented by the graph Γ^{**} in Figure 4, is again a representation of finite type. In both cases the shift of finite type representing α is of a particularly simple form: it is a cartesian product over $\mathbb{Z}$ of one-dimensional shifts of finite type. One can use these representations to construct, for example, explicit α-invariant probability measures with a variety of specific properties (such as a chosen value for the entropy and/or suitable decay properties).

References

[1] R.L. Adler and B. Weiss, *Entropy, a complete metric invariant of automorphisms of the torus*, Proc. Nat. Acad. Sci. U.S.A. **57** (1967), 1573–1576.
[2] D.V. Anosov, *On a class of invariant sets of smooth dynamical systems*, in: Proceedings of the Fifth International Conference on Nonlinear Oscillations, vol. 2, Mathematics Institute of the Ukrainian Academy of Sciences, Kiev, 1970, 39–45.
[3] A. Bertrand-Mathis, *Développement en base θ, répartition modulo un de la suite $(x\theta^n)_{n\geq 0}$, langages codes et θ-shift*, Bull. Soc. Math. France **114** (1986), 271–323.
[4] R. Bowen, *Markov partitions for axiom A diffeomorphisms*, Amer. J. Math. **92** (1970), 725–747.
[5] M. Jiang, *Equilibrium states for lattice models of hyperbolic type*, Nonlinearity **8** (1994), 631–659.
[6] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, Cambridge, 1995.
[7] R. Kenyon and A. Vershik, *Arithmetic construction of sofic partitions of hyperbolic toral automorphisms*, Preprint: École Normale Supérieure de Lyon, 1995.
[8] S. le Borgne, *Un codage sofique des automorphismes hyperboliques du tore*, C. R. Acad. Sci. Paris Sér. I Math. **323** (1996), 1123–1128.
[9] D. Lind and B. Marcus, *Symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995.
[10] D. Lind and K. Schmidt, *Homoclinic points of algebraic $\mathbb{Z}^d$-actions*, Preprint: Erwin Schrödinger Institute, Vienna, 1996.
[11] D. Lind, K. Schmidt and T. Ward, *Mahler measure and entropy for commuting automorphisms of compact groups*, Invent. Math. **101** (1990), 593–629.
[12] Ya.B. Pesin and Ya.G. Sinai, *Space-time chaos in chains of weakly interacting hyperbolic mappings*, Adv. in Soviet Math. **3** (1991), 165–198.
[13] B.L. Praggastis, *Markov partitions for hyperbolic toral automorphisms*, Ph.D. thesis, University of Washington, 1992.
[14] D.J. Rudolph and K. Schmidt, *Almost block independence and Bernoullicity of $\mathbb{Z}^d$-actions by automorphisms of compact groups*, Invent. Math. **120** (1995), 455–488.
[15] K. Schmidt, *Automorphisms of compact abelian groups and affine varieties*, Proc. London Math. Soc. **61** (1990), 480–496.
[16] K. Schmidt, *Algebraic ideas in ergodic theory*, in: CBMS Lecture Notes, vol. 76, American Mathematical Society, Providence, R.I., 1990.
[17] K. Schmidt, *The cohomology of higher-dimensional shifts of finite type*, Pacific J. Math. **170** (1995), 237–270.
[18] K. Schmidt, *Dynamical Systems of Algebraic Origin*, Birkhäuser Verlag, Basel-Berlin-Boston, 1995.
[19] K. Schmidt, *Tilings, fundamental cocycles and fundamental groups of symbolic $\mathbb{Z}^d$-actions*, Preprint: Erwin Schrödinger Institute, Vienna, 1996.
[20] Ya.G. Sinai, *Markov partitions and U-diffeomorphisms*, Functional Anal. Appl. **2** (1986), 64–89.
[21] A. Vershik, *The fibadic expansion of real numbers and adic transformations*, Preprint, Mittag-Leffler Institute, 1991/92.
[22] A.M. Vershik, *Arithmetic isomorphism of hyperbolic toral automorphisms and sofic shifts*, Funktsional. Anal. i Prilozhen. **26** (1992), 22–27.
[23] T. Ward, *The Bernoulli property for expansive $\mathbb{Z}^2$-actions on compact groups*, Israel J. Math. **79** (1992), 225–249.

Manfred Einsiedler: Mathematics Institute, University of Vienna, Strudlhofgasse 4, A-1090 Vienna, Austria
E-mail address: `manfred@nelly.mat.univie.ac.at`

Klaus Schmidt: Mathematics Institute, University of Vienna, Strudlhofgasse 4, A-1090 Vienna, Austria,
and
Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, A-1090 Vienna, Austria.
E-mail address: `klaus.schmidt@univie.ac.at`