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D MATH & RiskLab

Four Theorems and a Financial Crisis

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Talk based on the following paper:

Four Theorems and a Financial Crisis

B. DAS, P. EMBRECHTS & V. FASEN

Outline

1 Which Crisis ?

- 2007-2009 Subprime (Credit) Crisis?
- 2010-20xy Government Bond – Euro Crisis?
- Other?

- Current and past crises demand for a critical rethinking on where we stand with Financial Mathematics.

- A very small percentage of the (mainly investment) banking world has been creating products which were both questionable from a societal point of view as well as highly complicated from a mathematical point of view.

- Overall theme of the talk: MODEL UNCERTAINTY

2 Of Finance and Alchemy

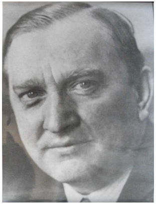
- "The world of finance is the only one in which people still believe in the possibility of turning iron into gold".

  - P.E., 1999

- The financial alchemist's SORCEROR'S STONE
  - Asset-backed securities, like synthetic CDOs.
  - for example ABACUS 2700-AC1.

- " So by financial alchemy, assets can be transmuted from garbage to gold - and therefore, requires less capital."

  - BRAITHWAITE, The Financial Times, October 25, 2011



STEFAN BANACH

ALFRED TARSKI

Theorem (Banach and Tarski [1924])
Given any two bounded sets A and B in the three-dimensional space $\mathbb{R}^3$, each having non-empty interior, one can partition A into finitely many (at least five) disjoint parts and rearrange them by rigid motions (translation, rotation) to form B.

Version 1

Given a three-dimensional solid ball (of gold, say), then it is possible to cut this ball in finitely many pieces and reassemble these to form two solid balls, each identical in size to the first one.

Version 2

Any solid, a pea, say, can be partitioned into a finite number of pieces, then reassembled to form another solid of any specified shape, say the Sun. For this reason, *Banach-Tarski Theorem* is often referred to as "The Pea and the Sun Paradox".

Correct Interpretation

- AXIOM OF CHOICE and NON-MEASURABILITY
- The theorem does NOT hold in $\mathbb{R}^2$, it does for $\mathbb{R}^d$, $d \geq 3$.

3 To diversify . . . or not?

AN EXAMPLE

- OPRISK: The risk of losses resulting from inadequate or failed internal processes, people and systems, or external events. Included is legal risk, excluded are strategic/business and reputational risk.

- Under the New Basel Capital Accord (Basel II/III) banks are required to set aside capital for the specific purpose of offsetting OpRisk.

OpRisk Pillar 1: Loss Distribution Approach (LDA)

- Operational losses $L_{i,j}$ are separately modeled in eight business lines (rows) and by seven risk types (columns) in the 56-cell Basel matrix.

- Marginal risks have non-homogeneous distributions.

- Pillar 1 in LDA based on $VaR_{0.999}^{1\ \text{year}}$, i.e., a 1 in 1000 year event.

	RT_1	...	RT_j	...	RT_7	
BL_1						$\rightarrow$ VaR_1
$\vdots$						$\vdots$
BL_i			$L_{i,j}$			$\rightarrow$ VaR_i
$\vdots$						$\vdots$
BL_8						$\rightarrow$ VaR_8

- As indicated in Basel II – add (comonotonicity): $VaR_+ = \sum_{i=1}^{8} VaR_i$.
- Diversify: $VaR_{reported} = (1 - \delta)VaR_+$, $0 < \delta < 1$ (often $\delta \in [0.1, 0.3]$).
- Is this a reliable estimate of the total Value-at-Risk ?

Risk measures

- A risk measure $\rho(X)$: Risk capital required for holding the position X.
- Axiomatics: coherent/convex risk measures.
- Examples:
  - Value-at-Risk: $\text{VaR}_\alpha(X)$:

$$\mathbb{P}(X > \text{VaR}_\alpha(X)) = 1 - \alpha.$$

    * Nice properties for elliptical models (MVN).
    * Problems with non-convexity for heavy-tailed or very skewed risks or special dependence.

– Expected shortfall: $\mathbb{ES}_\alpha(X)$:

$$\begin{aligned}\mathbb{ES}_\alpha(X) &= \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(X)du \\ &= \mathbb{E}(X|X > \mathrm{VaR}_\alpha(X)) \quad \text{for } F_X \text{ continuous.}\end{aligned}$$

* Needs $\mathbb{E}|X| < \infty$. $(X \in L_1)$

* Has convexity property: always admits diversification.

Theorem (Delbaen [2009])
Let E be a vector space which is rearrangement invariant and solid, and $\rho : E \to \mathbb{R}$ be a convex risk measure, then $E \subseteq L^1$.

FREDDY DELBAEN

E: vector space of random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

- Rearrangement invariant: If $X \stackrel{d}{=} Y$ and $X \in E$, then also $Y \in E$.
- Solid: If $|Y| \leq |X|$ and $X \in E$, then also $Y \in E$.

. . . in English

- There are no non-trivial, nice (i.e., sub additive or coherent) risk measures on the space of infinite mean risks.

- Diversification is benefitial with $\mathbb{ES}$ which requires $\mathbb{E}|X| < \infty$:

$$\mathbb{ES}_\alpha(X + Y) \leq \mathbb{ES}_\alpha(X) + \mathbb{ES}_\alpha(Y)$$

- this is well known, but asserted and put in a wider context by Delbaen's Theorem.

- VaR does not require a moment condition, so for very heavy tailed risks (e.g., no first moment), often diversification may not be possible.

- Suppose X and Y are i.i.d. risks with d.f. F, such that

$$1 - F(x) \sim x^{-\delta} L(x), \quad (x \to \infty)$$

where $\delta \in (0, 1)$ and L is slowly varying at ∞: $\lim_{t\to\infty} \frac{L(tx)}{L(t)} = 1, x > 0$.

- For α sufficiently close to 0,

$$\text{VaR}_\alpha(X + Y) > \text{VaR}_\alpha(X) + \text{VaR}_\alpha(Y).$$

- So diversification arguments become questionable.

Recent related work and comments

- Aggregation of dependent risks: Embrechts et al. [2009] and Degen et al. [2010].
- In multivariate regular variation setting: Embrechts and Mainik [2012].
- Superadditivity with Normal margins and special dependence structure: Examples in McNeil et al. [2005] and Ibragimov and Walden [2007].
- The realm of econometrics, see Daníelsson [2011], Section 4.4. (Caveat emptor: introduction has some inconsistencies)

4 A tale of tails

FINANCIAL TIMES, APRIL 24, 2009

OF COUPLES AND COPULAS by SAM JONES
In the autumn of 1987, the man who would become the world's most influential actuary landed in Canada on a flight from China.
He could apply the broken hearts maths to broken companies. Li, it seemed, had found the final piece of a risk management jigsaw that banks had been slowly piecing together since quants arrived on Wall Street.

Why did no one notice the formula's Achilles heel?

JOHNNY CASH AND JUNE CARTER

Pricing CDO tranches

Recipe for disaster: The formula that killed Wall Street
by FELIX SALMON, 23RD FEBRUARY 2009,
WIRED MAGAZINE

$$\Pr[T_A < 1, T_B < 1] = \phi_2(\phi^{-1}(F_A(1)), \phi^{-1}(F_B(1), \gamma)$$

- Φ_2 : Bivariate standard Normal cdf with correlation γ.
- Φ^{-1}: quantile function of standard normal.
- $F_A(1)$, $F_B(1)$: default probability of companies A, B within 1 year.

Theorem (Sibuya [1960])
Suppose (X, Y) is a random vector following a bivariate normal distribution with correlation coefficient $\gamma \in [-1, 1)$. Then X and Y are asymptotically independent.

In other words,

$$\lim_{t \to \infty} \mathbb{P}(X > t | Y > t) = 0$$

So regardless of how high a correlation γ we choose, if we go far enough in the tails, extreme events occur fairly independently.

MASAAKI SIBUYA

Some personal recollections

- 28 March 1999
- Columbia-JAFEE Conference on the Mathematics of Finance
  Columbia University, New York.
- 10:00-10:45 P. EMBRECHTS (ETH, Zurich):

  Insurance Analytics: Actuarial Tools in Financial Risk-Management

Why relevant?

1. Paper: P. Embrechts, A. McNeil, D. Straumann (1999) Correlation and Dependence in Risk Management: Properties and Pitfalls. Preprint RiskLab/ETH Zurich.
2. Coffee break: discussion with David Li.

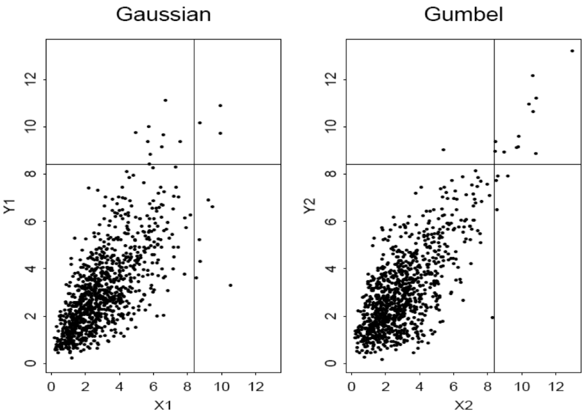


FIGURE 1. 1000 random variates from two distributions with identical Gamma(3,1) marginal distributions and identical correlation $\rho = 0.7$, but *different* dependence structures.

Summary

- Joint Tail dependence is a copula property, whatever the marginals are.
- Under asymptotic independence joint extremes are extremely rare.

Digging deeper ...

- Literature on multivariate regular variation: Laurens de Haan, Sid Resnick, ...
- Recall $X \sim F_X$ is regularly varying with tail index $\delta \geq 0$ if

$$1 - F_X(x) = x^{-\delta} L(x), \quad x > 0$$

Multivariate Regular Variation

- DEFINITION: $(X, Y) \sim F$ is multivariate regularly varying on $\mathbb{E} = [0, \infty]^2 \setminus \{(0, 0)\}$ if $\exists\ b(t) \uparrow \infty$ as $t \to \infty$ and a Radon measure $\nu \neq 0$ such that

$$t\mathbb{P}\left(\frac{(X, Y)}{b(t)} \in \cdot\right) \xrightarrow{v} \nu(\cdot) \qquad (t \to \infty).$$

  Write $(X, Y) \in MRV(b, \nu)$.

- $(X, Y) \sim F$ with Gaussian copula, correlation $\gamma < 1$ and uniform Pareto margins: Sibuya's Theorem (asymptotic independence) + Regular Variation $\Rightarrow$

$$t\mathbb{P}\left(\frac{(X, Y)}{t} \in [0, (x, y)]^c\right) \to \frac{1}{x} + \frac{1}{y} =: \nu([0, (x, y)]^c) \quad x > 0, y > 0$$

- Estimate for any $x > 0, y > 0$ and any $\gamma < 1$,

$$\mathbb{P}(X > x, Y > y) \approx 0.$$

Can we say something more?

- Coefficient of tail dependence [Ledford and Tawn, 1996].

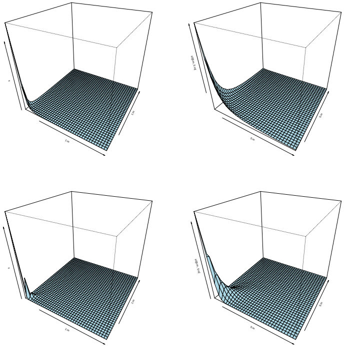
- Hidden regular variation [Resnick, 2002].

- Tail order [Hua and Joe, 2011].

- DEFINITION: Suppose $(X, Y) \in MRV(b, \nu)$. Then $(X, Y) \sim F$ exhibits hidden regular variation on $\mathbb{E}_0 = (0, \infty]^2$ if $\exists\, b_0(t) \uparrow \infty$ as $t \to \infty$ with $\lim_{t\to\infty} b(t)/b_0(t) = \infty$ and a Radon measure $\nu_0 \neq 0$ such that

$$t\mathbb{P}\left(\frac{(X, Y)}{b_0(t)} \in \cdot\right) \xrightarrow{v} \nu_)(\cdot) \qquad (t \to \infty).$$

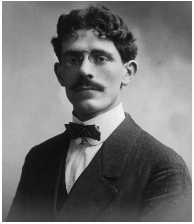
Write $(X, Y) \in HRV(b, b_0, \nu, \nu_0)$.



- Pareto(1) margins with Gaussian copula.
- Left: measure for regular variation, Right: measure for hidden regular variation.
- Top: γ = 0, Bottom: γ = 0.9.

5 Any margin . . . any correlation ?

- Model Uncertainty: a correlation fallacy.
- Simulation of a two-dimensional portfolio with marginal distributions given as $F_1 \equiv LN(0,1)$ and $F_2 = LN(0,4)$ and dependence:
  - Correlation = 80% NO SOLUTION
  - Correlation = 70% NO SOLUTION
  - Correlation = 60% INFINITELY MANY SOLUTIONS
- So understand the model conditions!



MAURICE FRÉCHET

WASSILY HÖFFDING

Theorem (Höffding [1940, 1941], Fréchet [1957])
Let (X, Y) be a bivariate random vector with finite variances, marginal distribution functions F_X and F_Y and an unspecified joint distribution function F; assume also that X and Y are non-degenerate. The following statements hold.

1. The attainable correlations from any joint model F with the above specifications form a closed interval

$$[\gamma_{\text{min}}, \gamma_{\text{max}}] \subseteq [-1, 1]$$

   with $-1 \leq \gamma_{\text{min}} < 0 < \gamma_{\text{max}} \leq 1$.

2. The minimum correlation γ_{min} is attained if and only if X and Y are countermonotonic; the maximum correlation γ_{max} if and only if X and Y are comonotonic.

3. $\gamma_{\text{min}} = -1$ if and only if X and $-Y$ are of the same type;
   $\gamma_{\text{max}} = 1$ if and only if X and Y are of the same type.

F_X, F_Y	$\gamma_{\max}$	$\gamma_{\min}$
$N(0,\sigma_1^2), N(0,\sigma_2^2), \sigma_1,\sigma_2 > 0$	1	-1
$LN(0,\sigma_1^2), LN(0,\sigma_2^2), \sigma_1,\sigma_2 > 0$	$\frac{e^{\sigma_1\sigma_2}-1}{\sqrt{(e^{\sigma_1^2}-1)(e^{\sigma_2^2}-1)}}$	$\frac{e^{-\sigma_1\sigma_2}-1}{\sqrt{(e^{\sigma_1^2}-1)(e^{-\sigma_2^2}-1)}}$
$\mathrm{Pareto}(\alpha), \mathrm{Pareto}(\beta), \alpha,\beta > 2$	$\frac{\sqrt{\alpha\beta(\alpha-2)(\beta-2)}}{\alpha\beta-\alpha-\beta}$	$\frac{\sqrt{(\alpha-2)(\beta-2)}\left((\alpha-1)(\beta-1)\mathrm{Beta}(1-\frac{1}{\alpha},1-\frac{1}{\beta})-\alpha\beta\right)}{\sqrt{\alpha\beta}}$
$\mathrm{Beta}(1,1), \mathrm{Beta}(\alpha,1), \alpha > 0$	$\frac{\sqrt{3\alpha(\alpha+2)}}{(2\alpha+1)}$	$-\frac{\sqrt{3\alpha(\alpha+2)}}{(2\alpha+1)}$

Table 1: Table of $\gamma_{\max}(F_X,F_Y)$ and $\gamma_{\min}(F_X,F_Y)$ for different pairs of marginal distributions F_X and F_Y.

- The Fréchet-Höffding Theorem holds for Pearson's linear correlation coefficient.

- For other rank-based correlation measures like Spearman's rho or Kendall's tau, for any marginal distribution $\gamma_{\text{max}} = 1$ and $\gamma_{\text{min}} = -1$ is attainable. BUT, Model Uncertainty persists.

- See work by Ludger Rüschendorf and Giovanni Puccetti for further studies. Also Embrechts and Puccetti [2010].

- There is no unique extension of comonotonicity to $d-$dimensions, where X and Y are d-dimensional random vectors. Although multiple definitions are there; see Puccetti and Scarsini [2010].

6 ... Conclusion: the Financial crisis

- Important understand Model Uncertainty.
- Return to a classic: Risk, Uncertainty, and Profit by Knight [1921].
- The Known, the Unknown and the Unknowable [Diebold et al., 2010].
- Mathematical finance today is strong in relating today's prices, but not so much in explaining(predicting) tomorrow's ones (HANS BUHLMANN).
- From a frequency oriented “if" to a severity oriented “what if".

Dimensions of Risk Management

- Dimension 1: Scope.
  - Micro: the individual firm, trading floor, client,. . .
  - Macro: the more global, worldwide system, networks.
- Dimension 2: Time.
  - Short: High Frequency Trading, $<<$ 1 year (or quarter).
  - Medium: Solvency 2/ Basel II/III, $\sim$ 1 year.
  - Long: Social/ life insurance, $>>$ 1 year.
- Dimension 3: Level.
  - Quantitative versus Qualitative.

Going from Micro/ Medium/ Quantitative to the other combinations.

Thank you

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