The shape of asymptotic dependence

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November 17, 2010

Abstract

Multivariate risk analysis is concerned with extreme observations. If the underlying distribution has a unimodal density then both the decay rate of the tails and the asymptotic shape of the level sets of the density are of importance for the dependence structure of extreme observations. For heavy-tailed densities, the sample clouds converge in distribution to a Poisson point process with a homogeneous intensity. The asymptotic shape of the level sets of the density is the common shape of the level sets of the intensity. For light-tailed densities, the asymptotic shape of the level sets of the density is the limit shape of the sample clouds. This paper investigates how the shape changes as the rate of decrease of the tails is varied while the copula of the distribution is preserved. Four cases are treated: a change from light tails to light tails, from heavy to heavy, heavy to light and light to heavy tails.

1 Introduction

Sample clouds evoke densities rather than distribution functions. Here a sample cloud is a finite set of independent observations from a multivariate distribution, treated as a geometric object, such as the set of points on a computer screen for a bivariate sample. Shape is important, the precise scale not.

The classic models such as the multivariate Gaussian distribution and the Student t distributions have continuous unimodal densities, provided the distribution is non-degenerate. These densities are determined by a bounded set, the ellipsoid which describes the shape of the level sets of the density, and by the rate of decrease. In risk analysis one is interested in extreme observations, and it is the asymptotic shape of the level sets and the rate of decrease of the tails of the density that are important. Let us illustrate these two components with a few simple examples.

A *homothetic density* has all level sets of the same shape, scaled copies of some given set. It is completely determined by the set and by the density generator which determines the decay along any ray. Altering the density on compact sets does not affect the asymptotic behaviour. So assume that the density is asymptotic to a homothetic density and impose conditions on the rate of decrease along rays - regular variation, or exponential decay - to ensure a simple asymptotic description of the tails, and also of the shape of large sample clouds (the two limits are related as will be explained later). Within this rather restricted setting of multivariate probability distributions with continuous unimodal densities and level sets with limit shape, we have a simple theory to describe extremes and say something about the asymptotic dependence structure. For a Student t density with spherical level sets the sample clouds, properly scaled, converge to a Poisson point process whose intensity has spherical level sets and decreases like a negative power along rays. For the standard Gaussian density the sample clouds converge onto a ball.

There are more advanced theories where instead of scalar normalizations one uses linear transformations, and where one drops the assumption of a density. The mean measure ρ of the limiting Poisson point process then is an excess measure on $\mathbb{R}^d \setminus \{\mathbf{0}\}$ (see [1], p.19). It gives finite mass to closed halfspaces which do not contain the origin, and satisfies a one-parameter set of symmetries: $e^{tA}\rho = e^t\rho$, $t \in \mathbb{R}$, where e^{tA} is a continuous one-parameter group of linear contractions. In this paper normalizations are scalar and hence the mean measure of the limit Poisson point process is *homogeneous*:

$$\rho(rA) = \rho(A)/r^\lambda, \qquad r > 0 \tag{1.1}$$

for all Borel sets A. The measure ρ is infinite, but the complement of any centered ball has finite mass.

For heavy-tailed dfs F, the scaled sample clouds may converge to a Poisson point process N with mean measure ρ as in (1.1). If ρ has a continuous positive density h then by homogeneity all level sets $\{h > c\}$ have the same shape. They are scaled copies of a bounded open star-shaped set D which contains the origin and which has a continuous boundary. Let $\mathcal{D}$ denote the class of such sets. The function h is the intensity of the point process N. It is completely determined by the set $\{h > 1\} \in \mathcal{D}$ and a parameter $\lambda > 0$ since by homogeneity of ρ it satisfies $h(r\mathbf{w}) = h(\mathbf{w})/r^{\lambda+d}$. The condition $\lambda > 0$ ensures that h is integrable over the complement of the open unit ball B, and hence N almost surely has finitely many points on the complement of centered balls. A continuous density f whose level sets $\{f > c\}$ asymptotically have shape D will lie in the domain of attraction of the measure ρ with density h if on any ray it is asymptotic to $cL(r)/r^{\lambda+d}$ for some slowly varying function L where c depends on the direction. The set of such densities is denoted by $\mathcal{F}_\lambda$. The densities $f \in \mathcal{F}_\lambda$ may be regarded as generalizations of the spherically symmetric Student t density f_λ with λ degrees of freedom. The asymptotic power decrease $c_\lambda/r^{\lambda+d}$ of f_λ is replaced by a regularly varying function $cL(r)/r^{\lambda+d}$; the spherical level sets are replaced by level sets which asymptotically have the shape D for some $D \in \mathcal{D}$. In risk management both the shape D and the parameter λ play a role.

For light-tailed densities there is a similar extended model. The central place is taken by the standard Gaussian density. Here too there is a one-parameter family, the spherically symmetric Weibull densities $g_\theta(\mathbf{x}) = c_\theta e^{-r^\theta/\theta}$, $r = \|\mathbf{x}\|$, for $\theta > 0$. One can now introduce the class $\mathcal{G}_\theta$ of continuous densities g asymptotic to a homothetic function whose level sets are scaled copies of a set $D \in \mathcal{D}$, and where g decreases like $ce^{-\psi(r)}$ along rays, with $\psi(r)$ a continuous function which varies regularly with exponent θ. The tails of g decrease rapidly. That implies that sample clouds tend to have a definite shape. For $g \in \mathcal{G}_\theta$ the sample clouds, properly scaled, converge onto the closure of the set D. In general, one may consider light-tailed distributions whose scaled sample clouds converge onto a compact set E. The set E then is star-shaped, but its boundary need not be continuous. The set E may even have empty interior. For light-tailed densities the limit shape D is quite robust. If we multiply the standard Gaussian density $e^{-r^2/2}/2\pi$ by a function like $c(1 + x^6)e^{r \sin x^2 y^2}$ the new function is integrable and will be a probability density for an appropriate choice of $c > 0$. The auxiliary factor fluctuates wildly, but the new density will have level sets which are asymptotically circular.

The theory so far is geometric. It does not depend on the coordinates. In the light-tailed case the asymptotics are described by a compact star-shaped set E; in the heavy-tailed case by a homogeneous measure ρ. In both cases there is a class of continuous densities whose asymptotic behaviour is determined by a bounded open star-shaped set $D \in \mathcal{D}$, and a positive parameter θ or λ describing the rate of decrease of the tails. The parameter determines the severity of the extremes; the shape tells us where these extremes are more likely to occur. For heavy tails it is the parameter λ which is of greater interest; for light tails the shape becomes increasingly important since new extremes are likely to occur close to the boundary.

Now introduce coordinates. Points in the sample clouds are d-tuples of random variables, $\mathbf{Z} = (Z_1, \ldots, Z_d)$. By deleting some of the coordinates the sample is projected onto the lower dimensional space spanned by the remaining coordinates. If we only retain the ith coordinate we have a one-dimensional sample cloud. In the light-tailed situation this univariate cloud converges onto the set E_i, the projection of E onto the ith coordinate. The set E_i is an interval $E_i = [-c_i^-, c_i^+]$ with $c_i^\pm \geq 0$ since E is star-shaped. The d-dimensional coordinate box $[-\mathbf{c}^-, \mathbf{c}^+]$ fits nicely around the limit set E. If E is the closure of a set $D \in \mathcal{D}$, the $2d$ constants $c_i^\pm$ are strictly positive. If desired, one may then scale the sample clouds such that $c_d^+ = 1$. In the heavy-tailed case the univariate sample clouds converge to a one-dimensional Poisson point process on $\mathbb{R} \setminus \{0\}$. The mean measure of this point process is the marginal ρ_i of the homogeneous measure ρ. By the homogeneity property (1.1),

$$\rho_i(-\infty, -t) = a_i^-/t^\lambda, \qquad \rho_i(t, \infty) = a_i^+/t^\lambda, \qquad t > 0.$$

Here too the balance constants are strictly positive if ρ has a continuous positive density, and one may choose the scaling constants for the sample clouds such that $a_d^+ = 1$. The balance constants $a_i^\pm$ reflect the balance in the upper and lower tails of the margins f_i of the underlying density $f \in \mathcal{F}_\lambda$. There is a slowly varying function $L(t)$ such that $f_i(\pm t) \sim a_i^\pm L(t)/t^{\lambda+1}$, $i = 1, \ldots, d$, for $t \to \infty$.

In both cases one may apply a coordinatewise semi-linear transformation to the observations $\mathbf{Z}_1, \mathbf{Z}_2, \ldots$ of the underlying distribution to obtain new i.i.d. observations $\tilde{\mathbf{Z}}_1, \tilde{\mathbf{Z}}_2, \ldots$ with components

$$\tilde{Z}_{ni} = b_i^- Z_{ni} \mathbf{1}_{[Z_{ni}<0]} + b_i^+ Z_{ni} \mathbf{1}_{[Z_{ni}\geq 0]}, \qquad i = 1, \ldots, d, \; n \geq 1,$$

with $b_i^\pm = 1/(c_i^\pm)^{1/\theta}$ or $= 1/(a_i^\pm)^{1/\lambda}$ (where we assume strictly positive balance constants). The new limit measure $\tilde{\rho}$ has equal and symmetric margins; the new limit set $\tilde{E}$ has projections $E_i = [-1, 1]$ for $i = 1, \ldots, d$. By a similar procedure one may alter the parameter λ or θ. If the vectors $\mathbf{Z}_1, \mathbf{Z}_2, \ldots$ come from a density f with margins $f_i(\pm t) \sim L(t)/t^{\lambda+1}$ for $t \to \infty$ then by an appropriate coordinatewise transformation one obtains new i.i.d. vectors $\tilde{\mathbf{Z}}_n$ with components $\tilde{Z}_{ni}$ which have a standard Cauchy density $\tilde{f}_i(t) = 1/\pi(1 + t^2)$, $i = 1, \ldots, d$. Indeed, the transformation to Cauchy margins may be written down explicitly: $\tilde{Z}_{ni} = \tan(\pi F_i(Z_{ni}) - \pi/2)$, where F_i is the df of Z_{ni}. The points $\mathbf{W}$ of the limit Poisson point process N are transformed into points $\tilde{\mathbf{W}} = J^\lambda(\mathbf{W})$ of the new limit point process $\tilde{N}$ by a power transformation

$$\tilde{W}_i = W_i^\lambda \mathbf{1}_{[W_i \geq 0]} - |W_i|^\lambda \mathbf{1}_{[W_i < 0]}, \qquad i = 1, \dots, d.$$

The mean measure $\tilde{\rho}$ is the image of ρ under the coordinatewise power transformation J^λ. Similarly, one may transform the observations $\mathbf{X}_1, \mathbf{X}_2, \dots$ from a light-tailed density g with parameter θ and limit set E with projections $E_i = [-1, 1]$ into i.i.d. observations $\tilde{\mathbf{X}}_n$ with standard Gaussian components $\tilde{X}_{ni}$. The new sample clouds converge onto $\tilde{E} = J^{\theta/2}E$. If $\mathbf{X}$ has i.i.d. components with Weibull density $b_\theta e^{-|t|^\theta/\theta}$, the limit shape is the closure of the unit ball B_θ in ℓ_θ-norm and $\tilde{E}$ is the closure of the Euclidean ball B.

Since we use coordinatewise transformations, the copula of the underlying distribution is not affected. The vectors $\tilde{\mathbf{Z}}_n$ and $\mathbf{Z}_n$ have the same copula. So have $\tilde{\mathbf{X}}_n$ and $\mathbf{X}_n$. Coordinatewise transformations are widely used in multivariate extreme value theory (EVT), and for heavy tails the results agree with EVT where it is standard usage to assume that the vectors have positive components with Fréchet margins e^{-1/t^λ} with parameter $\lambda = 1$.

The results above are formulated as Theorems 3.2 and 4.1 below. The main focus of the paper however is on continuous densities whose level sets have asymptotic shape $D \in \mathcal{D}$, in particular densities in $\mathcal{F}_\lambda$ and $\mathcal{G}_\theta$. The shape D may be regarded as a geometric expression of the asymptotic dependence. It is natural to ask how the copula changes as one varies the exponent λ or θ or if one goes from heavy tails to light tails or vice versa, but retains the shape D of the level sets. Since we find it difficult to specify "change in the copula", we change our point of view and address the question: How does the shape D change if one alters the tails of the density but retains the copula?

In answering this question we compare two densities with the same copula but with different rates of decrease in the tails. What happens to the asymptotic shape of the level sets and to the sample clouds if we change the margins? We distinguish four cases: 1) changing from light-tailed margins to light-tailed ones; 2) from heavy to heavy; 3) from heavy to light; and 4) from light to heavy tails. The copula is kept constant. The analysis is presented in Sections 3–6.

In the next section we give precise definitions, review some results on the limit behaviour of sample clouds, and introduce meta transformations. The paper ends with our conclusions.

1.1 Notation

Two positive continuous functions f and g are *asymptotic* and we write $f \sim g$ if $g(\mathbf{z}_n)/f(\mathbf{z}_n) \to 1$ for every sequence $\mathbf{z}_n$ for which $\|\mathbf{z}_n\| \to \infty$. The functions are *weakly asymptotic* and we write $f \asymp g$ if there exists a constant $M > 1$ such that $f/M \leq g \leq Mf$. We write B for the open unit ball in the Euclidean norm $\|\mathbf{z}\|$, and ∂A for the boundary of the set A. Thus ∂B is the unit sphere. $\mathcal{R}_\theta$ denotes the set of continuous functions f defined on $[0, \infty)$ which vary regularly in infinity with exponent θ, i.e. f is positive eventually and $f(tx)/f(t) \to x^\theta$ for $t \to \infty$ and $x > 0$. The class $\mathcal{D}_d$ of bounded open star-shaped sets in $\mathbb{R}^d$ and the set $\mathcal{F}_\lambda$ of continuous positive densities asymptotic to a homothetic function $f_*(n_D)$ with $f_* \in \mathcal{R}_{-(\lambda+d)}$ and $D \in \mathcal{D}_d$ will be defined in Section 2.

2 Preliminaries

This section contains definitions of certain concepts: star-shaped set and its gauge function, sample cloud, homothetic function and its generator, homogeneous measure, von Mises function and its scale function, and meta density. We briefly review the relation between the asymptotic behaviour of multivariate densities and of sample clouds, convergence in distribution and convergence onto a set. Meta densities will play a basic role in our investigation on the relation between shape (of level sets and sample clouds) and copulas. More detailed information may be found in [1] and [2].

2.1 Definitions and basics

A set E in $\mathbb{R}^d$ is *star-shaped* if it contains the origin and if $\mathbf{x} \in E$ implies $r\mathbf{x} \in E$ for $0 < r < 1$. We define $\mathcal{D} = \mathcal{D}_d$ to be the set of all bounded open star-shaped sets D in $\mathbb{R}^d$ whose boundary is continuous. With such a set D we associate the *gauge function* n_D. This is the unique function which satisfies the two conditions

$$D = \{n_D < 1\}, \qquad n_D(r\mathbf{x}) = rn_D(\mathbf{x}), \qquad r \geq 0. \tag{2.1}$$

If D is convex and $-D = D$ then the gauge function is a norm and D the open unit ball in this norm. A bounded open star-shaped set D has a continuous boundary ∂D if and only if the gauge function is continuous. A continuous positive function f_0 on $\mathbb{R}^d$ is *homothetic* with shape set $D \in \mathcal{D}$ if the level sets $\{f_0 > c\}$ are scaled copies of D for $0 < c < \sup f_0$. One may use the gauge function (like the Euclidean norm $\|\mathbf{z}\|$) to write down explicit expressions for homothetic functions: $f_0(\mathbf{z}) = f_*(n_D(\mathbf{z}))$ for $\mathbf{z} \in \mathbb{R}^d$. The function f_* is called the *generator* of the function f_0. We shall always assume that f_* is a continuous, strictly decreasing, positive function on $[0, \infty)$. This implies that f_0 is continuous and positive on $\mathbb{R}^d$. In order to obtain interesting asymptotics we assume that f_* varies regularly with exponent $-\lambda - d$ with $\lambda > 0$ (to ensure a finite integral) or that f_* varies rapidly.

Write $\mathcal{F}_\lambda$ for the set of all continuous densities f asymptotic to $f_*(n_D)$ with $f_* \in \mathcal{R}_{-\lambda-d}$ and $D \in \mathcal{D}$. Such a density has *heavy tails*. Its asymptotics are described by a function $h : \mathbb{R}^d \setminus \{\mathbf{0}\} \to (0, \infty)$ of the form

$$h(\mathbf{w}) = 1/n_D(\mathbf{w})^{\lambda+d} = \eta(\omega)/r^{\lambda+d}, \qquad r = \|\mathbf{w}\| > 0,\ \omega = \mathbf{w}/r. \tag{2.2}$$

Here η is a continuous positive function on the unit sphere ∂B. The relation between η and the boundary ∂D is simple:

$$r\omega \in \partial D \qquad \Longleftrightarrow \qquad \eta(\omega) = 1/r^{\lambda+d}.$$

The function h is the intensity of a Poison point process N with mean measure ρ. This measure ρ is a Radon measure on $\mathbb{R}^d \setminus \{\mathbf{0}\}$. It is *homogeneous with exponent* $-\lambda$, see (1.1). If such a homogeneous measure ρ has a continuous positive density, the density has the form (2.2). The homogeneity condition (1.1) implies that the margins ρ_i, $i = 1, \ldots, d$, are Radon measures on $\mathbb{R}$ with density $c_i^- \lambda/|t|^{\lambda+1}$ for $t < 0$ and $c_i^+ \lambda/t^{\lambda+1}$ for $t > 0$. (Apply (1.1) to $A = \{x_i \leq -1\}$ or to $A = \{x_i \geq 1\}$.) Let $f \in \mathcal{F}_\lambda$. Then (cf. Proposition 4.3)

$$h_t(\mathbf{w}) := \frac{f(t\mathbf{w})}{f_*(t)} \to h(\mathbf{w}) = \frac{1}{n_D(\mathbf{w})^{\lambda+d}}, \qquad t \to \infty,\ \mathbf{w} \neq 0. \tag{2.3}$$

Pointwise convergence follows from regular variation of f_*. An application of Potter's bounds (see [6]) yields $\mathbf{L}^1$ convergence on the complement of centered balls. Choose t_n such that $t_n^d f_*(t_n) = 1/n$. Then h_{t_n} is the density of a measure ρ_{t_n}, of mass n. This measure is the mean measure of the sample cloud

$$N_n = \{\mathbf{Z}_1/t_n, \ldots, \mathbf{Z}_n/t_n\}, \tag{2.4}$$

where $\mathbf{Z}_1, \mathbf{Z}_2, \dots$ are independent observations from the density f. By definition a *sample cloud* is a scaled random sample. The $\mathbf{L}^1$ convergence in (2.3) on the complement of centered balls implies weak convergence $\rho_{t_n} \to \rho$ on the complement of centered balls and also convergence of the sample clouds: $N_n \Rightarrow N$ weakly on the complement of centered balls. Here N is the Poisson point process with mean measure ρ on $\mathbb{R}^d \setminus \{\mathbf{0}\}$ and $\Rightarrow$ denotes convergence in distribution.

If the generator f_* varies rapidly the asymptotics are different. Again let $\mathbf{Z}_1, \mathbf{Z}_2, \dots$ be independent observations from a density f asymptotic to $f_*(n_D)$. Then $h_t(\mathbf{w}) := f(t\mathbf{w})/f_*(t)$ tends to ∞ uniformly on compact sets in D and tends to zero uniformly on the complement of any open set U containing the closure of D. This convergence to zero on U^c also holds in $\mathbf{L}^1$. Hence the measures ρ_t with density h_t satisfy $\rho_t(\mathbf{p} + \epsilon B) \to \infty$, $t \to \infty$, for each point $\mathbf{p}$ in the closure of D and all $\epsilon > 0$, and $\rho_t(U^c) \to 0$. Choose t_n such that $\rho_{t_n}(\mathbb{R}^d) = n$. Then ρ_{t_n} is the mean measure of the sample cloud N_n in (2.4), and these sample clouds *converge onto* the compact set $E = \mathrm{cl}(D)$: For $\mathbf{p} \in E$, $\epsilon > 0$, U any open set containing E, and any integer $m \geq 1$

$$\mathbb{P}\{N_n(\mathbf{p} + \epsilon B) \geq m\} \to 1, \qquad \mathbb{P}\{N_n(U^c) = 0\} \to 1.$$

Typical heavy-tailed densities in $\mathcal{F}_\lambda$ are multivariate Student densities with λ degrees of freedom and spherical, elliptical or cubical level sets. In the light-tailed case one may think of generator functions of the form

$$f_*(t) = e^{-\varphi(t)} = at^b e^{-pt^\theta}, \qquad t \geq t_0, \qquad a, p, \theta > 0.$$

These functions with Weibull-type tails vary rapidly. They also have the property that the exponent φ varies regularly with exponent $\theta > 0$, and that f_* is a *von Mises function* with *scale function* $a(t) = 1/\varphi'(t)$:

$$f_* = e^{-\varphi}, \qquad \varphi \in C^2[0, \infty),\ \varphi'(t) > 0,\ a'(t) \to 0, \qquad t \to \infty. \tag{2.5}$$

Von Mises functions have simple exponential asymptotic behaviour (see e.g. [9]):

$$f_*(t + a(t)v)/f_*(t) \to e^{-v}, \qquad v \in \mathbb{R}, \qquad t \to \infty. \tag{2.6}$$

Convergence in (2.6) holds in $\mathbf{L}^1$ on halflines $[c, \infty)$ for all $c \in \mathbb{R}$. The von Mises condition for a df F to lie in the maximum domain attraction of the Gumbel distribution is $(1 - F(t))f'(t)/f(t)^2 \to -1$ for $t \to \infty$. This gives (2.5) for $\varphi = -\log(1 - F)$.

2.2 Meta distributions

It is possible to construct a multivariate df G with Gaussian margins and the copula of a heavy-tailed multivariate elliptical Student t distribution with df F.

For any two continuous strictly increasing dfs F_0 and G_0 on $\mathbb{R}$ there exists a unique increasing transformation K_0 such that $G_0 = F_0 \circ K_0$. Obviously $K_0 = F_0^{-1} \circ G_0$. Let the vector $\mathbf{Z} = (Z_1, \ldots, Z_d)$ have df F with continuous strictly increasing margins F_i, and let G_i be continuous strictly increasing univariate dfs. Write $G = F \circ K$, where K is the coordinatewise increasing transformation

$$K : \mathbf{x} \mapsto \mathbf{z} = (K_1(x_1), \ldots, K_d(x_d)), \qquad K_i = F_i^{-1} \circ G_i, \qquad i = 1, \ldots, d.$$

Then G is the *meta distribution with margins G_i based on the df F*. The transformation K is called the *meta transformation.* If $\mathbf{X}$ has df G then $\mathbf{Z} = K(\mathbf{X})$ has df F. The distributions F and G have the same copula. In most of our applications the margins have continuous positive densities.

Proposition 2.1. *If F has a continuous strictly positive density f with continuous margins f_i, and if the univariate dfs $G_1, \ldots, G_d$ have continuous positive densities g_i, then the meta df $G = F \circ K$ with margins G_i based on F has a continuous strictly positive density g. Moreover,*

$$\frac{g(x_1, \ldots, x_d)}{g_1(x_1) \cdots g_d(x_d)} = \frac{f(z_1, \ldots, z_d)}{f_1(z_1) \cdots f_d(z_d)}, \qquad \mathbf{z} = K(\mathbf{x}) \in \mathbb{R}^d. \tag{2.7}$$

For the proof of this result and more information on meta distributions we refer to [2].

We shall use the notation F and f for the original df and its density and denote the margins by F_i and f_i. The meta df G based on F is specified by its margins G_i.

We can now become concrete. Define $\mathbf{Z}$ to have df F with density f asymptotic to $f_*(n_D)$ for a set $D \in \mathcal{D}$, where the generator f_* varies regularly or rapidly. Choose a continuous symmetric unimodal positive density g_0 with tails which vary regularly or rapidly. Construct the meta density g with margins g_0 based on f. Then f and g have the same copula. What is the asymptotic shape of the level sets of g? What is the asymptotic behaviour of the sample clouds from the density g?

3 Light tails to light tails

For a multivariate normal vector with independent components, the level sets of the density are balls. If the margins have a Laplace density, the level sets are tetrahedra. For symmetric Weibull margins $ce^{-|t|^p/p}$, the level sets are open balls in ℓ_p. *Power transformations* J^γ for $\gamma > 0$ are the coordinatewise transformations

$$J^\gamma : \mathbf{x} \mapsto \mathbf{z}, \qquad z_i = x_i^\gamma \mathbf{1}_{[x_i \geq 0]} - |x_i|^\gamma \mathbf{1}_{[x_i < 0]}. \tag{3.1}$$

They form a group: $J^\alpha J^\beta = J^{\alpha\beta}$ and the inverse of J^γ is $J^{1/\gamma}$. Moreover, they map the unit ball in ℓ_q norm into a unit ball in ℓ_p norm with $p = q/\gamma$. Thus $J^\gamma(B_q) = B_{q/\gamma}$ where B_p denotes the open unit ball in ℓ_p since $J^\gamma \mathbf{x} \in B_p$ precisely if $(|x_1|^\gamma)^p + \cdots + (|x_d|^\gamma)^p < 1$, i.e. precisely if $\mathbf{x} \in B_{\gamma p}$. Since vectors with independent components have the same copula, the distributions above are linked by meta transformations $K = (K_0, \ldots, K_0)$. The K are no power transformations. The square of an exponential variable is not one-sided normal. Power transformations do describe the asymptotic relation between level sets of light-tailed densities though. We need a lemma to link the tail behaviour of the marginal densities and dfs.

Lemma 3.1. *Let $e^{-\Psi(s)} = \int_s^\infty e^{-\psi(t)} dt$ for $\psi \in \mathcal{R}_\theta$, $\theta > 0$. There exists $s_0 > 1$ such that*

$$\psi(s) - \log(2s) \leq \Psi(s) \leq \max\{\psi(t) \mid s \leq t \leq s+1\}, \qquad s \geq s_0.$$

Proof The second inequality is obvious. For the first one, write

$$\int_s^\infty e^{-\psi(t)} dt = se^{-\psi(s)} \int_1^\infty e^{-(\psi(rs) - \psi(s))} dr,$$

and observe that the Potter bounds (see [6]) yield an $s_1 > 1$ such that $(\psi(rs) - \psi(s))/\psi(s) \geq \theta \log r$ for $r \geq 2$ and $s \geq s_1$ (since $\min_{r \geq 2}(r^\theta - 1)/\log r > \theta$ and $\log r << r^{\theta/2}$). Write J_n for the integral on the right over the interval $[n, n+1]$. First assume ψ is increasing. Then

$$J_1 \leq 1, \qquad J_m \leq e^{-(\psi(ms) - \psi(s))} \leq e^{-\theta\psi(s)\log m} < 1/m^2, \qquad m > 1,\ \psi(s) > 2/\theta.$$

Hence the integral on the right is bounded by $\pi^2/6 < 2$. If ψ is not monotone, the bound 2 will do. ¶

3.1 Sample clouds

We shall first look at sample clouds. They are more intuitive to work with.

Sample clouds from light-tailed unimodal densities tend to have a definite shape. Assume that the sample clouds, suitably scaled, converge onto a compact set E. Such a limit set is star-shaped (see [7]). If E is the closure of a star-shaped open set $D \in \mathcal{D}$ then it is reasonable to model the underlying distribution by a continuous positive density f which is homothetic, or weakly asymptotic to a homothetic density, or to a unimodal density whose level sets have limit shape D. Now consider the meta density with Gaussian margins based on f. What do the sample clouds from the meta density look like? Can they be scaled to converge onto a limit set, and if so, what is the relation between this limit set and the compact star-shaped set E? The answer depends on the tails of the margins.

Theorem 3.2. *Let $S \in \mathcal{R}_\sigma$ and $T \in \mathcal{R}_\tau$ with $\sigma, \tau > 0$. Let $\mathbf{Z}_1, \mathbf{Z}_2, \dots$ be independent observations from the df F with continuous strictly increasing margins F_i which satisfy*

$$-\log F_i(-t) \sim T(t), \qquad -\log(1 - F_i(t)) \sim T(t), \qquad t \to \infty,\ i = 1, \dots, d. \tag{3.2}$$

Suppose there exists a compact set E and $a_n > 0$ such that the sample clouds $\{\mathbf{Z}_1/a_n, \dots, \mathbf{Z}_n/a_n\}$ converge onto E. Let E_i denote the projection of E onto the ith coordinate, and assume $\max E_d = 1$. Then $E_i = [-1, 1]$ for $i = 1, \dots, d$, and $T(a_n) \sim \log n$. Let $\mathbf{X}_1, \mathbf{X}_2, \dots$ be independent observations from the meta df G with continuous strictly increasing margins G_i which satisfy

$$-\log G_i(-s) \sim S(s), \qquad -\log(1 - G_i(s)) \sim S(s), \qquad s \to \infty,\ i = 1, \dots, d.$$

Let $S(b_n) \sim \log n$. Then the sample clouds $N_n = \{\mathbf{X}_1/b_n, \dots, \mathbf{X}_n/b_n\}$ converge onto the compact star-shaped set $J^\gamma(E)$ with $\gamma = \sigma/\tau$, where J^γ is the power transformation in (3.1).

Proof The equality $E_i = [-1, 1]$ and $T(a_n) \sim \log n$ follow from (3.2) by univariate EVT; see e.g. [9]. Coordinatewise power transformations map rays onto rays, and hence map star-shaped sets into star-shaped sets. Continuity of J^γ ensures that the image $J^\gamma(E)$ is compact. Let K denote the meta transformation with coordinates K_i which satisfy $F_i(K_i) = G_i$. Then we may assume that $\mathbf{X}_n = M(\mathbf{Z}_n)$ for $n = 1, 2, \dots$, where $M = K^{-1}$, see (3.4). The coordinates $M_i(t)$ and $-M_i(-t)$ are asymptotic to $S^{-1} \circ T$ for $t \to \infty$, and vary regularly with exponent $\gamma = \tau/\sigma > 0$. Let $J_n(\mathbf{w}) = M(a_n\mathbf{w})/b_n$. By regular variation of the coordinates, and the choice of a_n and b_n one finds $J_n(\mathbf{w}) \to J^\gamma(\mathbf{w})$ uniformly on compact sets. Moreover, J_n maps the complement of the cube $[-2,2]^d$ into the complement of a cube $[-c,c]^d$ for some $c>1$ eventually. It follows that $J_n(N_n)$ converges onto $J^\gamma(E)$. ¶

The asymptotic equalities in (3.2) are not very strong. They hold if the marginal densities $f_i(\pm t)$ are asymptotic to Gamma densities $c_i^\pm t^{b_i^\pm} e^{-t}$ where $c_i^\pm > 0$ and $b_i^\pm$ are arbitrary constants. Yet the implications for the limit set are severe. The projections E_i are symmetric and equal. If we replace the condition on the margins F_i by

$$-\log F_i(-t) \sim a_i^- T(t), \qquad -\log(1-F_i(t)) \sim a_i^+ T(t), \qquad s \to \infty,\ i=1,\dots,d,\ a_i^\pm > 0$$

and similar conditions on the margins G_i with constants $b_i^\pm > 0$ we obtain a similar result. For simplicity assume $T=S$. Now $E_i = [-(a_i^-)^{1/\tau}, (a_i^+)^{1/\tau}]$, $i=1,\dots,d$, and the sample clouds from G converge onto $\Lambda_{\mathbf{c}}(E)$, where $\Lambda_{\mathbf{c}}$ is the coordinatewise semilinear transformation

$$\Lambda_{\mathbf{c}} : \mathbf{u} \mapsto \mathbf{w}, \qquad w_i = c_i^- u_i \mathbf{1}_{[u_i<0]} + c_i^+ u_i \mathbf{1}_{[u_i \geq 0]}, \qquad i=1,\dots,d, \qquad c_i^\pm = (b_i^\pm / a_i^\pm)^{1/\tau}. \tag{3.3}$$

The proof is similar.

3.2 Level sets and densities

We now turn our attention to light-tailed densities $f = f_*(n_D)$ and meta densities with light-tailed margins based on f. For instance one could think of Gaussian margins g_0 and a Weibull generator $f_*(t) = ce^{-t^\tau/\tau}$. Do the level sets of the meta density have a limit shape? If so, what is the relation between this limit shape and the original set D? The problem here is that we make assumptions about the structure of the density f, but we need information on the margins of f in order to determine the meta transformation K linking the dfs F and G. Recall

$$G = F \circ K, \qquad \mathbf{Z}_n = K(\mathbf{X}_n), \qquad K : \mathbf{x} \mapsto \mathbf{z} = (K_1(x_1), \dots, K_d(x_d)). \tag{3.4}$$

Under appropriate conditions on the set D and the generator $f_* = e^{-\varphi_*}$, see [4] or Theorem 8.6 in [1], the margins f_i of a continuous positive density $f \sim f_*(n_D)$ satisfy the asymptotic condition:

$$-\log f_i(t) \sim \varphi_*(|t|), \qquad |t| \to \infty, \qquad i=1,\dots,d. \tag{3.5}$$

Rather than imposing conditions on D and f_* we shall make assumptions about the margins. We assume that the marginal densities of f are continuous positive functions, $f_i = e^{-\varphi_i}$, and

$$\varphi_i(-t) \sim a_i^- T(t), \qquad \varphi_i(t) \sim a_i^+ T(t), \qquad t \to \infty, \quad a_i^\pm > 0,\ i = 1, \ldots, d \tag{3.6}$$

for $T \in \mathcal{R}_\tau$ with $\tau > 0$. We make a similar assumption about the margins $g_i = e^{-\psi_i}$ with respect to $S \in \mathcal{R}_\sigma$ with $\sigma > 0$:

$$\psi_i(-s) \sim b_i^- S(s), \qquad \psi_i(s) \sim b_i^+ S(s), \qquad s \to \infty, \quad b_i^\pm > 0,\ i = 1, \ldots, d. \tag{3.7}$$

Theorem 3.3. *Let $D \in \mathcal{D}_d$. Let $S \in \mathcal{R}_\sigma$ and $T \in \mathcal{R}_\tau$ with $\sigma, \tau > 0$. Let $f = e^{-\varphi}$ be a continuous positive density on $\mathbb{R}^d$ with $\varphi \sim T(n_D)$ and with continuous positive margins $f_i = e^{-\varphi_i}$, $i = 1, \ldots, d$. Let $g_i = e^{-\psi_i}$ be continuous positive densities on $\mathbb{R}$. Assume (3.6) and (3.7). The level sets of the meta density $g = e^{-\psi}$ with margins g_i based on f then have asymptotic shape $Q = \Gamma^{-1}(D)$ where Γ is the coordinatewise semi-power transformation in (3.10). The function ψ is asymptotic to $S(n_Q)$.*

Proof The density g by (2.7) has the form

$$g(\mathbf{x}) = f(\mathbf{z}) g_1(x_1) \cdots g_d(x_d)/(f_1(z_1) \cdots f_d(z_d)), \qquad \mathbf{z} = (K_1(x_1), \ldots, K_d(x_d)). \tag{3.8}$$

The marginal meta transformations K_i are determined by the tails of the dfs, $1 - F_i(K_i(s)) = 1 - G_i(s)$ for all s. Hence $1 - F_i = e^{-\Phi_i}$ and $1 - G_i = e^{-\Psi_i}$ gives $\Phi_i(K_i(s)) = \Psi_i(s)$. Lemma 3.1 shows that $\Phi_i(t) \sim a_i^+ T(t)$ and $\Psi_i(s) \sim b_i^+ S(s)$, which implies that $K_i \in \mathcal{R}_\gamma$ for $\gamma = \sigma/\tau$, and actually $K_i(s) \sim c_i^+ R(s)$ with $c_i^+ = (b_i^+/a_i^+)^{1/\tau}$ and $R = T^{-1} \circ S$ where we assume S and T continuous and strictly increasing. Similarly, $K_i(-s) \sim -c_i^- R(s)$ for $s \to \infty$ with $c_i^- = (b_i^-/a_i^-)^{1/\tau}$. Rewrite (3.8) as

$$\psi(\mathbf{x}) = \varphi(K(\mathbf{x})) + \delta_1(x_1) + \cdots + \delta_d(x_d), \qquad \delta_i(s) = (\varphi_i(t) - \Phi_i(t)) - (\psi_i(s) - \Psi_i(s)), \qquad t = K_i(s),$$

since $\Phi_i(t) = \Psi_i(s)$. We claim that $\psi_0 = \varphi \circ K$ satisfies a simple limit relation and that the δ_i may be neglected. By assumption

$$\varphi(t_n \mathbf{w}_n)/T(t_n) \to n_D^\tau(\mathbf{w}), \qquad \mathbf{w}_n \to \mathbf{w}, \qquad t_n \to \infty. \tag{3.9}$$

Let $\mathbf{u}_n \equiv \mathbf{u}(n) \to \mathbf{u} \in \mathbb{R}^d$ and $r_n \to \infty$. Then

$$\frac{K_i(r_n u_i(n))}{R(r_n)} \to \Gamma_i(u_i) = c_i^+ u_i^\gamma \mathbf{1}_{[u_i \geq 0]} - c_i^- |u_i|^\gamma \mathbf{1}_{[u_i < 0]}, \qquad c_i^\pm = \left(\frac{b_i^\pm}{a_i^\pm}\right)^{1/\tau},\ \gamma = \frac{\sigma}{\tau}. \tag{3.10}$$

Hence $\mathbf{w}_n := K(r_n\mathbf{u}_n)/R(r_n) \to \Gamma(\mathbf{u}) =: \mathbf{w}$, and (3.9) gives

$$\frac{\psi_0(r_n\mathbf{u}_n)}{S(r_n)} = \frac{\varphi(K(r_n\mathbf{u}_n))}{T(R(r_n))} = \frac{\varphi(R(r_n)\mathbf{w}_n)}{T(R(r_n))} \to n_D^\tau(\mathbf{w}) = n_D^\tau(\Gamma(\mathbf{u})) = n_Q^\sigma(\mathbf{u}). \tag{3.11}$$

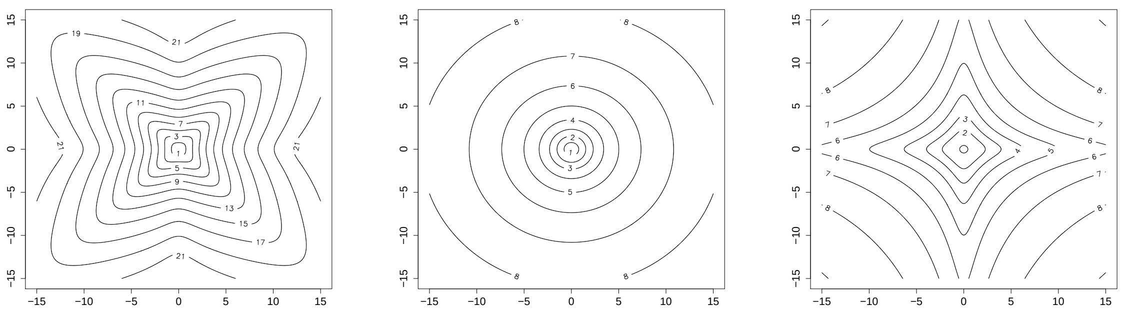
Now let $v_n \to v \in \mathbb{R}$. Then $y_n = K_i(r_nv_n)/R(r_n) \to \Gamma_i(v)$ by (3.10). Hence for $t_n = R(r_n)$

$$S(r_n) = T(t_n), \qquad \psi_i(r_nv_n) - \Psi_i(r_nv_n) = o(S(r_n)), \qquad \varphi_i(t_ny_n) - \Phi(t_ny_n) = o(T(t_n))$$

by Lemma 3.1, (3.6) and (3.7). So limit relation (3.11) also holds for ψ: $\psi \sim S(n_Q)$. The level sets $\{g > e^{-r}\} = \{\psi < r\}$ then have asymptotic shape Q. ¶

4 Heavy tails to heavy tails

Heavy-tailed distributions have a simple asymptotic theory. There is a nice description of the asymptotic structure. For densities $f \in \mathcal{F}_\lambda$, the asymptotic structure is described by the homogeneous limit function $h = 1/n_D^{\lambda+d}$ in (2.3), and hence by the shape D and the parameter λ for given dimension d. Figure 1 below shows what happens to the level sets if we impose Student t margins with μ degrees of freedom on a bivariate spherical Student t density with λ degrees of freedom. The asymptotic shape of the level sets of the new densitiy g is revealed in the limit function $1/n_Q^{\mu+d}$ for g, see (2.3).



(a) $\lambda = 1$, $\mu = 5$ (b) $\lambda = \mu = 1$ (c) $\lambda = 5$, $\mu = 1$

Figure 1: Level sets of bivariate meta densities with standard Student t margins (with μ degrees of freedom) based on the spherical t distribution (with λ degrees of freedom). Levels are powers of 10^{-1}.

Let us first give an overview of the theory. In general, the asymptotic structure is described by a homogeneous measure ρ on $\mathbb{R}^d \setminus \{\mathbf{0}\}$. If $\mathbf{Z}$ with df F lies in the domain of attraction of ρ (see Definition 1 below) then ρ determines the balance conditions of the marginal tails. There is a function $T \in \mathcal{R}_{-\lambda}$ such that

$$F_i(-t)/T(t) \to a_i^-, \qquad (1-F_i(t))/T(t) \to a_i^+, \qquad t \to \infty,\ i = 1, \dots, d, \tag{4.1}$$

where the non-negative constants $a_i^\pm$ are defined by

$$\rho_i(-\infty, -t] = \rho\{w_i \leq -t\} = a_i^-/t^\lambda, \qquad \rho_i[t, \infty) = \rho\{w_i \geq t\} = a_i^+/t^\lambda, \qquad t > 0,\ i = 1, \dots, d. \tag{4.2}$$

These balance conditions not only hold for the components of the vector $\mathbf{Z}$ but for any non-trivial linear combination $Y = \eta\mathbf{Z} = a_1 Z_1 + \cdots + a_d Z_d$:

$$\mathbb{P}\{Y \geq t\}/T(t) \to \rho\{\eta \geq 1\}, \qquad t \to \infty, \tag{4.3}$$

as will be established below. For $f \in \mathcal{F}_\lambda$, the margins f_i satisfy similar balance conditions. Here one may choose $T(t) = t^d f_*(t)$, where f_* is the generator of f. Again, see Proposition 4.2 below, any non-trivial linear combination $Y = \eta\mathbf{Z}$ has a continuous density f_0 which satisfies

$$f_0(t) \sim \rho\{\eta \geq 1\}\lambda T(t)/t, \qquad t \to \infty. \tag{4.4}$$

The condition that D contains the origin ensures that $\rho\{\eta \geq 1\}$ is positive, and so are the $2d$ balance constants $a_i^\pm$.

One of the attractive features of this asymptotic theory is that it is geometric. One can first determine the limit ρ and then choose the coordinates. This geometric point of view has an unexpected consequence. If in the bivariate case the measure ρ lives on two lines through the origin then the components of the vector $\mathbf{Z}$ are asymptotically independent if one chooses these lines as coordinate axes; but if one chooses two other lines as the axes, then ρ lives on two lines $v = au$ and $u = bv$ with a, b non-zero and the components of $\mathbf{Z}$ are mixed comonotonic.

There is another reason why the asymptotic theory for heavy-tailed distributions is so rich. There is a close link to multivariate EVT. The non-linear projection

$$\mathbf{z} \mapsto \mathbf{z}^+ = (z_1 \vee 0, \dots, z_d \vee 0)$$

maps $\mathbb{R}^d$ onto $[0, \infty)^d$. The image ρ^+ of the homogeneous limit measure ρ under this projection is the exponent measure of the max-stable limit distribution H for the vector $\mathbf{Z}$:

$$F^n(t_n\mathbf{w}) \to H(\mathbf{w}) \qquad n \to \infty, \qquad T(t_n) = 1/n.$$

The exponent measure ρ^+ determines ρ on $(0,\infty)^d$. By an appropriate sign change, replacing $\mathbf{Z}$ by $\Delta(\mathbf{Z})$ for a diagonal matrix with entries ± 1, one can determine ρ on the other $2^d - 1$ orthants. The 2^d limits for the coordinatewise extremes, maxima and minima, determine ρ. (Mass on coordinate planes will show up in the lower dimensional margins.) See [1], Section 17.3. This link allows us to use the *invariance principle* of multivariate EVT. If one applies a coordinatewise strictly increasing continuous transformation of $\mathbb{R}^d$ onto $\mathbb{R}^d$ which transforms the margins of F into univariate dfs G_i whose tails satisfy a balance condition

$$G_i(-s)/S(s) \to b_i^-, \qquad (1 - G_i(s))/S(s) \to b_i^+, \qquad s \to \infty,\ i = 1, \dots, d, \tag{4.5}$$

for $S \in \mathcal{R}_{-\mu}$ with $\mu > 0$, and if all $4d$ balance constants $a_i^\pm$ and $b_i^\pm$ are positive, then the vector $\mathbf{X}$ with the meta df G with margins G_i based on F lies in the domain of attraction of a max-stable limit law, as do the $2^d - 1$ sign-changed vectors $\Delta(\mathbf{X})$. The exponent measures are related by a power transformation; the df G lies in the domain of attraction of a homogeneous measure σ and we may write $\rho = \Gamma(\sigma)$ where $\Gamma = (\Gamma_1, \dots, \Gamma_d)$ is a coordinatewise transformation whose margins Γ_i are determined by the margins σ_i and ρ_i via $\rho_i = \Gamma_i(\sigma_i)$. We find, cf (3.10) with τ replaced by $-\lambda$:

$$b_i^+/s^\mu = \sigma_i[s,\infty) = \rho_i[t,\infty) = a_i^+/t^\lambda \Rightarrow t = \Gamma_i(s) = c_i^+ s^\gamma, \qquad c_i^+ = (a_i^+/b_i^+)^{1/\lambda},\ \gamma = \mu/\lambda.$$

The limiting Poisson point processes for the sample clouds from the dfs F and G are linked by Γ.

4.1 Sample clouds

Let us first say what it means that a probability distribution on $\mathbb{R}^d$ lies in the domain of attraction of a homogeneous measure ρ.

Definition 1. *A measure ρ on $\mathbb{R}^d \setminus \{\mathbf{0}\}$ is* homogeneous of order $-\lambda$ *and we write $\rho \in \mathcal{H}_\lambda$ if $0 < \rho(B^c) < \infty$ and if ρ satisfies (1.1) for all Borel sets A in $\mathbb{R}^d \setminus \{\mathbf{0}\}$. A vector $\mathbf{Z}$ with probability distribution π and df F lies in the domain of attraction of this homogeneous measure ρ and we write $\mathbf{Z} \in \mathcal{A}(\rho)$ or $F \in \mathcal{A}(\rho)$ if $p(r) := \mathbb{P}\{\|\mathbf{Z}\| > r\}$ is positive for all $r > 0$ and*

$$\rho_r := \gamma_r^{-1}(\pi)/p(r) \to \rho \qquad \text{weakly on } \epsilon B^c, \qquad r \to \infty,\ \epsilon > 0, \tag{4.6}$$

where γ_r is the scalar expansion $\gamma_r : \mathbf{z} \mapsto r\mathbf{z}$, and hence $\gamma_r^{-1}(\pi)$ is the distribution of $\mathbf{Z}/r$. $\diamondsuit$

Weak convergence in (4.6) implies for $Y = \eta \mathbf{Z}$ with df F_0 that

$$\mathbb{P}\{Y \geq rs\}/p(r) \to \rho\{\eta \geq s\}/\rho(B^c) = c_0 \rho\{\eta \geq 1\}/s^\lambda, \qquad r \to \infty,\ s > 0, \qquad c_0 = \rho(B^c) = 1.$$

If $\rho\{\eta \geq 1\} = a > 0$ then $(1 - F_0(rs))/(1 - F_0(r)) \to 1/s^\lambda$ and $1 - F_0(r) \sim ap(r)$ for $r \to \infty$ and $s > 0$. Hence, $1 - F_0$ and p vary regularly with exponent $-\lambda$. This establishes (4.3) and hence (4.2).

Theorem 4.1. *Let F be a continuous df in $\mathcal{A}(\rho)$ for a measure $\rho \in \mathcal{H}_\lambda$. Assume (4.1) with $a_i^\pm > 0$. Let $S \in \mathcal{R}_{-\mu}$ and let $G_1, \ldots, G_d$ be strictly increasing continuous univariate dfs which satisfy (4.5) with $b_i^\pm > 0$. Let G be the meta df with margins G_i based on F. Then $G \in \mathcal{A}(\sigma)$ where $\sigma \in \mathcal{H}_\mu$.*

Let $\{\mathbf{W}_1, \mathbf{W}_2, \ldots\}$ be the limiting Poisson point process (with mean measure ρ) for the sample clouds from F. Let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be independent observations from the meta df G. Let $S(s_n) = 1/n$. Then

$$N_n = \{\mathbf{X}_1/s_n, \ldots, \mathbf{X}_n/s_n\} \Rightarrow N = \{\Gamma^{-1}(\mathbf{W}_1), \Gamma^{-1}(\mathbf{W}_2), \ldots\},$$

where Γ is the coordinatewise semipower transformation in (4.7).

Proof This follows from the invariance principle of EVT since the mean measures satisfy $\rho = \Gamma(\sigma)$; see above. ¶

A more direct proof runs along the lines of the proof of Theorem 3.2. Regular variation of the tails transforms the relation $\mathbf{z} = K(\mathbf{u})$ into the relation $\mathbf{w} = \Gamma(\mathbf{u})$ since $K(r\mathbf{u})/R(r) \to \Gamma(\mathbf{u})$ gives with the notation in (3.1) and (3.3):

$$\Gamma = \Lambda_{\mathbf{a}}^{1/\lambda} J^{\mu/\lambda} \Lambda_{\mathbf{b}}^{-1/\mu} = \Lambda_{\mathbf{c}} J^\gamma, \qquad \gamma = \mu/\lambda, \qquad c_i^\pm = (a_i^\pm/b_i^\pm)^{1/\lambda}. \tag{4.7}$$

The transformation Γ is homogeneous of degree γ, $\Gamma(r\mathbf{u}) = r^\gamma \Gamma(\mathbf{u})$. Hence

$$n_D(\mathbf{w}) = n_D(\Gamma(\mathbf{u})) = n_Q^\gamma(\mathbf{u}), \qquad Q = \Gamma^{-1}(D). \tag{4.8}$$

4.2 Level sets and densities

The theory for the transformation of the level sets of densities for heavy tails is similar to the theory for light tails. Both are based on the limit relation $K(r\mathbf{u})/R(r) \to \Gamma(\mathbf{u})$, see (4.7), resulting from the regular variation of functions associated with the margins. There are two differences. For heavy tails the contribution of the Jacobian is not negligible, far from it, and for heavy tails the tail of the density generator f_* together with the shape D determines the tails of the marginal densities f_i. (The margins inherit the balance conditions from the intensity $1/n_D^{\lambda+d}$, and the slowly varying component from f_*.)

If ρ has density $h(\mathbf{w}) = 1/n_D^{\lambda+d}(\mathbf{w})$ then $\sigma = \Gamma^{-1}(\rho)$ with Γ in (4.7) has density

$$k(\mathbf{u}) = \frac{\prod_i \Gamma_i'(u_i)}{n_D^{\lambda+d}(\Gamma(\mathbf{u}))} = \frac{\gamma^d \prod_i |u_i|^{\gamma-1}(c_i^- \mathbf{1}_{[u_i<0]} + c_i^+ \mathbf{1}_{[u_i \geq 0]})}{n_{\Gamma^{-1}(D)}^{\gamma\lambda+\gamma d}(\mathbf{u})} = \frac{1}{n_Q^{\mu+d}(\mathbf{u})}, \qquad c_i^\pm = \Big(\frac{a_i^\pm}{b_i^\pm}\Big)^{1/\lambda}. \tag{4.9}$$

Here $\gamma = \mu/\lambda$ and Q is an open set in $\mathbb{R}^d$, which may be unbounded (if $\gamma < 1$), which need not contain the origin (if $\gamma > 1$), and which is bounded and contains the origin for $\gamma = 1$, but need not have a continuous boundary. If $\gamma = 1$ and $c_i^- \neq c_i^+$ the intensity k jumps by a factor $c_i^-/c_i^+ \neq 1$ on crossing over from $u_i > 0$ to $u_i < 0$.

The heavy tailed meta density g with margins g_i based on the density $f \in \mathcal{F}_\lambda$ is linked to f by the symmetric relation (2.7). Write $f_*(t) = T(t)/t^d$ with $T \in \mathcal{R}_{-\lambda}$ for the density generator of f and for $S \in \mathcal{R}_{-\mu}$ define $g_*(r) = S(r)/r^d$. Let r and $t = R(r)$ be linked by $T(t) = S(r)$. Then the limit relation we want to establish reads:

$$\frac{g(r\mathbf{u})}{S(r)/r^d} = \frac{f(t\mathbf{w})}{T(t)/t^d}\frac{r^d}{t^d}\prod_i \frac{g_i(ru_i)}{f_i(tw_i)} \to \frac{\prod_i \Gamma_i'(u_i)}{n_D^{\lambda+d}(\Gamma(\mathbf{u}))} =: k(\mathbf{u}). \tag{4.10}$$

So assume $\mathbf{Z}$ has density $f \in \mathcal{F}_\lambda$. From Theorem 4.1, the meta density g lies in the domain of attraction of the homogeneous measure $\sigma \in \mathcal{H}_\mu$. The relation $\rho = \Gamma(\sigma)$ yields the density k of σ in (4.9). For $\mu < \lambda$, the derivatives Γ_i' are negative powers. The intensity k of the limiting Poisson point process becomes infinite along all coordinate planes. We shall show that even in this case the limit relation (4.10) holds in $\mathbf{L}^1$ on the complement of centered balls and uniformly on compact sets of $\mathbb{R}^d \setminus \{\mathbf{0}\}$ in the sense that $g(s_n\mathbf{u}_n)/g_*(s_n) \to k(\mathbf{u}) \in [0,\infty]$ holds when $\mathbf{u}_n \to \mathbf{u} \neq \mathbf{0}$ and $s_n \to \infty$. First we show that the margins f_i are well-behaved.

Proposition 4.2. *Let $Y = \eta\mathbf{Z}$ for a non-trivial linear functional η with df F_0. Then Y has a continuous density f_0 which is asymptotic to $\lambda(1-F_0(t))/t$ for $t \to \infty$.*

Proof Think of η as the vertical coordinate and write $\mathbf{z} = (\mathbf{x}, y)$, where $\mathbf{x}$ denotes the horizontal part of the vector. Assume $f = f_*(n_D)$. Then

$$f_0(y) = \int f_*(n_D(\mathbf{x}, y))d\mathbf{x} = y^{d-1}\int f_*(yn_D(\mathbf{u}, 1))d\mathbf{u} = y^{d-1}J(y), \qquad y > 0,$$

by homogeneity of the gauge function n_D. The function $y \mapsto J(y)$ is decreasing since $y \mapsto f_*(ya)$ is for $a \geq 0$. The function $A(t) = (1 - F_0(t))/t^{d-1}$ has derivative $-f_0(t)/t^{d-1} - (d-1)(1 - F_0(t))/t^d$. Since $1 - F_0 \in \mathcal{R}_{-\lambda}$ by the arguments in the previous section, the second term varies regularly, and hence so does its integral $I(t) \sim ((d-1)/(d-1+\lambda))A(t)$. The function $A(t) + I(t) \in \mathcal{R}_{-(\lambda+d-1)}$ has a monotone derivative $-f_0(t)/t^{d-1}$, and $f_0(t)/t^{d-1}$ then varies regularly by the Monotone Density Theorem in [6] (where the case of slow variation has to be excluded!). Regular variation of f_0 gives the desired asymptotic equality. ¶

Proposition 4.3. *Let the df F have density f in $\mathcal{F}_\lambda$. The function f satisfies (2.3) pointwise on $\mathbb{R}^d \setminus \{\mathbf{0}\}$ and in $\mathbf{L}^1$ on the complement of centered balls ϵB, $\epsilon > 0$. The margins F_i satisfy (4.1), where $T(t) \sim t^d f_*(t)$ lies in $\mathcal{R}_{-\lambda}$, and $a_i^\pm$ are positive constants depending on D and λ.*

Proof Regular variation of f_* follows by writing this relation out for $(st, \mathbf{w})$ and $(t, s\mathbf{w})$ and using the homogeneity of n_D. This implies regular variation of T. The $2d$ constants $a_i^\pm$ in (4.2) are positive since D contains the origin. Let $\mathbf{Z}$ have df F. Then $\mathbf{W} = \mathbf{Z}/t$ has density $g_t(\mathbf{w}) = t^d f(t\mathbf{w})$ and $g_t \sim T(t)h$ by (2.3) implies $F_i(-st) = \mathbb{P}\{W_i < -s\} \sim T(t)\rho\{w_i \leq -s\} \sim a_i^- T(ts)$ for $t \to \infty$. This gives (4.1). ¶

Theorem 4.4. *Let $f \in \mathcal{F}_\lambda$. Let $S \in \mathcal{R}_{-\mu}$ for some $\mu > 0$ and let $g_1, \dots, g_d$ be continuous positive densities such that*

$$g_i(-s) \sim b_i^- \mu S(s)/s, \qquad g_i(s) \sim b_i^+ \mu S(s)/s, \qquad s \to \infty,\ i = 1, \dots, d$$

with $b_i^\pm > 0$. The meta density g with margins g_i based on f is continuous. Set $g_*(s) = S(s)/s^d$. Then $g(s\mathbf{u})/g_*(s) \to k(\mathbf{u}) := 1/n_Q^{\mu+d}(\mathbf{u})$. Here Q is an open set. Convergence holds uniformly on compact sets which are disjoint from the coordinate planes and in $\mathbf{L}^1$ on the complement of centered balls.

If $\mu > \lambda$, convergence holds uniformly on compact sets in $\mathbb{R}^d \setminus \{\mathbf{0}\}$. If $\mu \neq \lambda$ then $k_{s_n}(\mathbf{u}_n) \to k(\mathbf{u})$ for $s_n \to \infty$ and $\mathbf{u}_n \to \mathbf{u} \neq \mathbf{0}$, where the limit is infinite if $\mu < \lambda$ and $\mathbf{u}$ lies on a coordinate plane. If $\mu = \lambda$ and the balance in each coordinate is preserved, $b_i^-/b_i^+ = a_i^-/a_i^+$ for $i = 1, \dots, d$, then Γ in (4.7) is a linear map and $Q = c\Gamma^{-1}(D)$ with $c > 0$. If $\mu = \lambda$ and the balance condition for the index i is violated there is a jump discontinuity over the corresponding coordinate plane by a factor $\neq 1$.

Proof We may and shall assume that T and S are strictly decreasing, continuous and map $(0, \infty)$ onto itself. Write $S = T \circ R$. Then $R \in \mathcal{R}_\gamma$ with $\gamma = \mu/\lambda$ is continuous and strictly increasing. Let $t_n = R(s_n) \to \infty$. Let $\mathbf{u}_n \to \mathbf{u}$ and set $\mathbf{w}_n = K(s_n\mathbf{u}_n)/t_n$. Then in (4.10) the factor $f(t_n\mathbf{w}_n)/(T(t_n)/t_n^d)$ tends to $h(\Gamma(\mathbf{u}))$ uniformly on compact sets in $\mathbb{R}^d \setminus \{\mathbf{0}\}$ by Proposition (4.3) since $f_*(t) \sim T(t)/t^d$ and $\mathbf{w}_n = K(r_n\mathbf{u}_n)/R(r_n) \to \Gamma(\mathbf{u})$, see (4.7). Convergence of the product on the right is less obvious. Let $i \in \{1, \ldots, d\}$. Write $Q_n = g_i(s_nu_n)/f_i(t_nw_n)$. Claim:

$$u_n \to u \neq 0 \quad \Rightarrow \quad Q_ns_n/t_n \sim \mu w_n/\lambda u_n \to \gamma\Gamma_i(u)/u,$$

and the left side converges to zero for $u_n \to 0$ provided $\gamma = \mu/\lambda > 1$. The asymptotic relations between the tails of the density and the df give the asymptotic equality provided $s_nu_n \to \infty$ since $G_i(s_nu_n) = F_i(t_nw_n)$. Then $w_n = K_n(s_nu_n)/t_n \to \Gamma_i(u)$ gives the limit for $u \neq 0$. Now assume $\gamma > 1$, We have to prove $Q_ns_n/t_n \to 0$ for $u_n \to 0$. First assume $s_nu_n \to \infty$. Then $K_i(s) \sim c_i^+R(s)$ and Potter's bounds, see [6], give $w_n = K_i(s_nu_n)/R(s_n) \leq 2c_i^+u_n^{(1+\gamma)/2}$ for $s_nu_n \geq M_0$. A similar bound holds for $s_nu_n \leq -M_0$. It is possible that $u_n = 0$ and $w_n \neq 0$ if $K_i(0) \neq 0$. But K_i is a homeomorphism. Hence if $|s_nu_n| \leq M_0$ then A_n is bounded and $R \in \mathcal{R}_\gamma$ with $\gamma > 1$ implies $s_n/t_n = s_n/R(s_n) \to 0$ for $s_n \to \infty$ and hence in this case also $Q_ns_n/t_n \to 0$. By symmetry, a similar result holds for $\mu < \lambda$, with the limit value ∞. $\mathbf{L}^1$ convergence on the complement of centered balls follows from the almost sure convergence of the densities and the weak convergence of the measures by Fatou's Lemma as in the proof of Scheffé's Theorem. ¶

The change in the intensity on decreasing the parameter λ is dramatic. The spikes of the new level sets may perhaps be interpreted as an indication of extra asymptotic independence for lighter tails. Student densities with spherical level sets tend to complete independence of the coordinates as the exponent λ goes to ∞, and the Student distribution converges to the Gaussian distribution.

5 Heavy tails to light tails

The transformation from heavy to light tails gives new and unexpected results. If we import the copula of the Student t density $f = f_*(n_D)$ with $f_*(r) = c/(\lambda + r^2)^{(\lambda+d)/2} \sim c/r^{\lambda+d}$ and D a centered ellipsoid into a density with standard Gaussian margins, the resulting density is continuous and its level sets have an asymptotic shape D_λ, whose boundary is given by a quadratic expression. In Figure 2 this shape is clearly visible in a sample cloud of a ten thousand points. The shape depends only on the parameter λ. All other information is lost in the transformation from heavy to light tails. All $f \in \mathcal{F}_\lambda$ yield the same shape D_λ. We restrict ourselves here to citing the corresponding theorem from [2], where the proof may be found and a discussion.

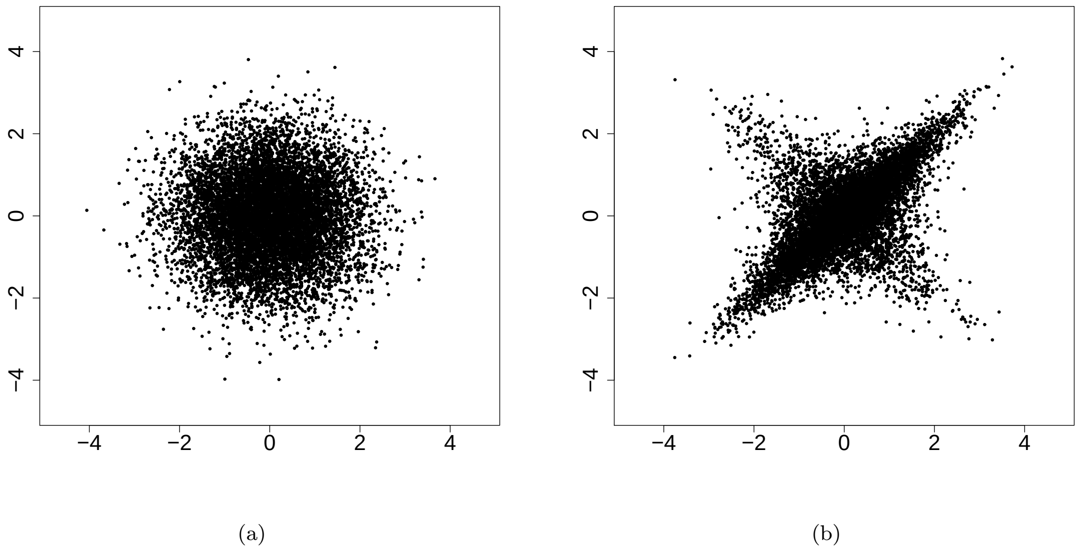


Figure 2: Bivariate sample clouds of 10,000 points from (a) the standard normal distribution, and (b) the meta-Cauchy distribution with standard normal margins based on the centered Cauchy density with level sets shaped like the ellipse $5x^2 + 6xy + 5y^2 = 1$.

Theorem 5.1. *Suppose f is a density on $\mathbb{R}^d$ in $\mathcal{F}_\lambda$ for some $\lambda > 0$, and g_0 is a continuous, positive, symmetric density on $\mathbb{R}$ asymptotic to a von Mises function $e^{-\psi}$ with $\psi \in RV_\theta$ for some $\theta > 0$. Let $\psi(r_n) = \log n$, and let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be independent observations from the meta density g based on f with equal margins g_0. Set $c(r) = g(r, \ldots, r)$. Then the level sets $\{g > c(r)\}$, scaled by r, converge to the limit set $D_{\lambda,\theta} = \{\chi < \lambda\}$ where*

$$\chi(\mathbf{u}) = (\lambda + d)\|\mathbf{u}\|_\infty^\theta - (|u_1|^\theta + \cdots + |u_d|^\theta). \tag{5.1}$$

The sample clouds $N_n = \{\mathbf{X}_1/r_n, \ldots, \mathbf{X}_n/r_n\}$ from g converge onto the closure of $D_{\lambda,\theta}$.

The shape of the limit set $D_{\lambda,\theta} \in \mathcal{D}$ varies continuously in λ. For fixed θ the shape $D_{\lambda,\theta}$ reflects the change in the copula as the tail parameter λ varies over $(0, \infty)$ for $f \in \mathcal{F}_\lambda$ with the shape D of the level sets fixed. The good behaviour of the function $\lambda \mapsto D_{\lambda,\theta}$ unfortunately is unstable. One can alter the bivariate circle symmetric Cauchy density without affecting the limit measure $\rho \in \mathcal{H}_1$ so that the sample clouds from the meta density with Gaussian margins based on the perturbed density converge onto the diagonal cross $E_\times$, the union of the two diagonals of the square $[-1,1]^2$. See [3] for details.

6 Light tails to heavy tails

Assume the level sets of a light-tailed density g may be scaled to converge to a set $D \in \mathcal{D}$, or more generally, assume the sample clouds from the light-tailed distribution dG converge onto the closure of D. Turn to the meta distribution dF with heavy-tailed margins. Can one describe the tails of dF asymptotically by a homogeneous measure ρ? Is there a limiting point process N for the sample clouds?

The limit shape of the level sets of the light-tailed meta densities in the previous section gives no information on the asymptotic shape of the original heavy-tailed density. In the transition from heavy to light tails, information about the limit shape is blurred to such an extent that one cannot go back from the asymptotic shape of the level sets for light-tailed density to the asymptotic shape for heavy-tailed one. Yet we do have some results. As in EVT all that matters is the shape of D in the vertices of the circumscribed coordinate box.

6.1 Asymptotic independence

Theorem 6.1. *Let* $\mathbf{X}$ *have a positive continuous density* g*. Suppose there exist* $c_n > 0$ *and* $0 < r_n \to \infty$ *such that* $c_{n+1}/c_n \to 0$*,* $r_{n+1}/r_n \to 1$ *and* $\{g > c_n\}/r_n \to D \in \mathcal{D}$*. Let* $F_1, \dots, F_d$ *be strictly increasing continuous dfs such that*

$$F_i(-t) \sim a_i^- T(t), \qquad 1 - F_i(t) \sim a_i^+ T(t) \qquad t \to \infty$$

for positive constants $a_i^\pm$ *and* $T \in \mathcal{R}_{-\lambda}$*,* $\lambda > 0$*. Let* $\mathbf{Z}_1, \mathbf{Z}_2, \dots$ *be independent observations from the meta df* F *with margins* F_i *based on the density* g*. Suppose* D *is convex with a* C^1 *boundary (in each boundary point there is a unique tangent plane). Choose* $a_n > 0$ *such that* $nT(a_n) \to 1$*. Then the sample clouds* *converge:*

$$N_n = \{\mathbf{Z}_1/a_n, \ldots, \mathbf{Z}_n/a_n\} \Rightarrow N \quad \text{weakly on } \epsilon B^c, \qquad \epsilon > 0.$$

The limit N is a Poisson point process with mean measure ρ which lives on the 2d *halfaxes. It is determined by*

$$\rho\{x_i < -t\} = a_i^-/t^\lambda, \qquad \rho\{x_i > t\} = a_i^+/t^\lambda, \qquad t > 0.$$

Proof The condition on the shape of D implies asymptotic independence of all coordinates, both positive and negative. See [10] and [5]. ¶

The charm of the condition in the theorem above is that it is geometric. Any two linear combinations of the coordinates $X = \xi\mathbf{X}$ and $Y = \eta\mathbf{X}$ are asymptotically independent provided the linear functionals ξ and η are linearly independent.

6.2 Asymptotic dependence and homothetic densities

We now turn to bivariate distributions. For asymptotic independence it suffices that the limit set $\mathrm{cl}(D)$ of the sample clouds does not contain the coordinatewise supremum of the points in D: $\sup D \notin \mathrm{cl}(D)$. Such sets D are called *blunt*. This condition ensures that for large sample clouds the maximal horizontal and the maximal vertical coordinate come from sample points in disjoint subsets. See [5] for details. Now suppose D is the triangle with vertices $(1, 1), (-1, 0), (0, -1)$. This set certainly is not blunt. Yet there exist continuous positive densities g with light tails and convex level sets which, properly scaled, converge to D such that the vector $\mathbf{X}$ with density g has asymptotically independent components. (The level sets are triangles tD with a tip of size $\sqrt{t}$ cut off to blunt them. See [5] for details.) For asymptotic dependence we need strong conditions. So assume $g \sim g_*(n_D)$ where g_* is a von Mises function. We focus on the positive quadrant. So one could restrict D to the positive quadrant or assume that D and g are invariant under sign changes, of assume that the behaviour of D outside the positive quadrant is harmless. We shall do the latter.

For $\tilde{a}, a \in [0, 1]$ with $\tilde{a}a < 1$ define $\mathcal{D}_{\tilde{a},a}$ as the set of all $D \in \mathcal{D}_2$ whose closure intersects the lines $x_1 = 1$ and $x_2 = 1$ only in the one point $\mathbf{e} = (1, 1)$, and whose closure does not contain the point $\inf D$. Moreover in the point $\mathbf{e}$ the set D has tangents with slope $\tilde{a}$ and $1/a$. In geometric terms this condition means that the sets $n(D - \mathbf{e})$ converge to the open sector

$$C = \{\mathbf{x} \in (-\infty, 0)^2 \mid x_1/a < x_2 < \tilde{a}x_1\}. \tag{6.1}$$

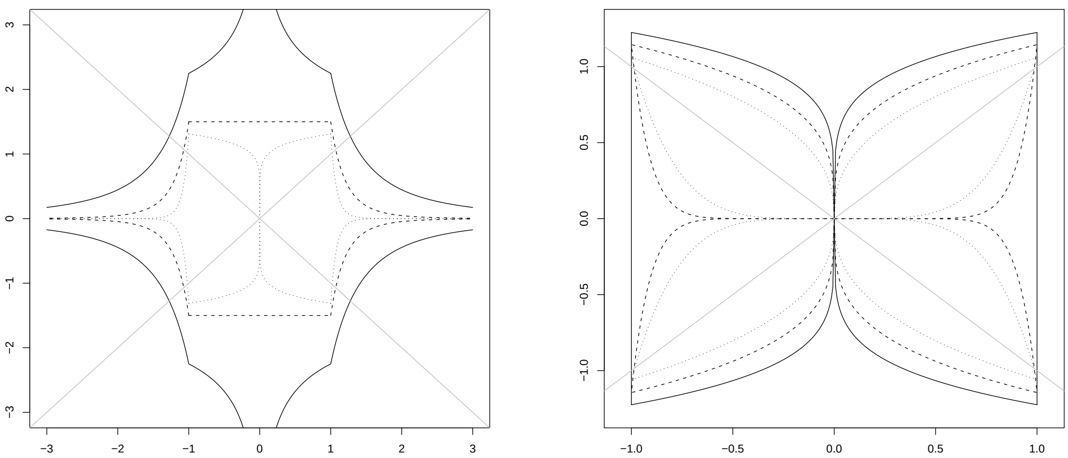
In analytic terms the functions $t \mapsto n_D(t, 1)$ and $t \mapsto n_D(1, t)$ have a left derivative in $t = 1$. We shall see that the heavy-tailed meta density then lies in the domain of a homogeneous measure ρ with a scaled *power density* on the first quadrant:

$$r(x, y) = c_0 r_0(a_0 x, b_0 y), \qquad r_0(x, y) = \begin{cases} |x|^{\tilde{\alpha}-1}/|y|^{\tilde{\beta}+1}, & |x| \le |y|, \qquad 0 < \tilde{\alpha} = \tilde{\beta} - \lambda; \\ |y|^{\alpha-1}/|x|^{\beta+1}, & |y| \le |x|, \qquad 0 < \alpha = \beta - \lambda. \end{cases} \tag{6.2}$$

Power densities are associated with exponential densities $h(u, v)$ on the plane which satisfy

$$h(u + t, v + t) = e^{-t} h(u, v), \qquad u, v \in \mathbb{R}, \qquad \{h > 1\} = C, \tag{6.3}$$

where C is an open sector in the negative quadrant.



(a) (b)

Figure 3: Level sets of the density r in (6.2) with parameters $\tilde{a} = 1/2$, $a = 1/3$ and λ. The values of λ are $\{1/2, 1, 3/2\}$ in Panel (a) and $\{2, 3, 7\}$ in Panel (b) corresponding to solid, dashed and dotted curves, respectively. The original density g in Theorem 6.2 now is assumed symmetric for sign changes. Transformations between successive level sets are described in Section 4.

Theorem 6.2. *Let $\lambda > 0$ and $0 \le a \le \tilde{a} \le 1$ with $\tilde{a}a < 1$. Let f_λ denote the standard Student t density with λ degrees of freedom. Let $g \sim g_*(n_D)$ be a continuous positive density on $\mathbb{R}^2$ where g_* is a von Mies* *function and $D \in \mathcal{D}_{\tilde{a}a}$. The meta density f with margins f_λ based on g is continuous and positive. It lies in the domain of the measure $\rho \in \mathcal{H}_\lambda$ whose margins ρ_1 and ρ_2 have density $\lambda/|t|^{\lambda+1}$ on $\mathbb{R} \setminus \{0\}$. If $\tilde{a}a = 0$ then ρ lives on the two axes. If $\tilde{a}a$ is positive then ρ lives on the two negative halfaxes and on the positive quadrant, where it has the power density r in (6.2).*

Let L denote the ray through the point $(1, b)$ with $b = (\tilde{a}/a)^{1/\lambda} \geq 1$. The level set D_λ of r containing $(1, b)$ as a boundary point is bounded by the power curves $y = bx^{\tilde{\mu}}$ above L and $y = bx^{1/\mu}$ below L with $\mu = (1-\alpha)/(1+\beta)$ and $\tilde{\mu}$ defined similarly. The constants α, β and μ are determined by a and λ:

$$\alpha = \frac{a\lambda}{1-a}, \qquad \beta = \frac{\lambda}{1-a}, \qquad \mu = \frac{\alpha-1}{\beta+1} = \frac{a\lambda-(1-a)}{\lambda+(1-a)}, \qquad b = \left(\frac{\tilde{a}}{a}\right)^{1/\lambda}.$$

The same expressions define $\tilde{\alpha}, \tilde{\beta}, \tilde{\mu}$ in terms of $\tilde{a}$ and λ. If $\tilde{a} = 1$ then $\tilde{\mu} = 1$ and the upper boundary of D_λ is the line segment from $(0,0)$ to $(1,b)$ along L, and r vanishes above L. The sets D_λ satisfy $D_{\lambda_1} \supset D_{\lambda_2}$ for $0 < \lambda_1 < \lambda_2$.

Proof The proof proceeds in three steps. (i) The limit relation (2.3) holds for g if we replace the scaling by affine normalizations. The limit function h is the density of the exponent measure σ on $\mathbb{R}^2$ associated with a max-stable limit law with Gumbel margins. (ii) By multivariate EVT, the exponent measure ρ of the meta df is the image $K(\sigma)$ under the coordinatewise exponential map

$$K : (u, v) \mapsto (x, y) = (e^{u/\lambda}, e^{v/\lambda}) \in (0, \infty)^2, \qquad (u, v) = \lambda(\log x, \log y), \tag{6.4}$$

with the coordinates scaled by a diagonal linear transformation to ensure $\rho\{y \geq 1\} = \rho\{x \geq 1\} = 1$. (iii) A computation gives the results.

Let $a(t)$ be the scale function of the von Mises function g_*. It is known that $a(t)/t \to 0$; see e.g. [9]. The set $D_0 = D - (1, 1)$, scaled by $t/a(t)$, converges to the open sector C in (6.1). Hence

$$h_t(\mathbf{w}) = \frac{g((t,t) + a(t)\mathbf{w})}{g(t,t)} \to h(\mathbf{w}), \qquad t \to \infty. \tag{6.5}$$

Here h is the exponential function in (6.3). It is continuous if $\tilde{a} < 1$, and then convergence holds uniformly on compact sets in the plane. If $\tilde{a} = 1$, it vanishes above the diagonal and convergence is uniform on compact sets disjoint from the diagonal. In both cases, convergence holds in $\mathbf{L}^1$ on halfplanes $\{(u, v) \mid c_1 u + c_2 v \geq c\}$ with $c_1, c_2 > 0$ and $c \in \mathbb{R}$. If $a = 0$ then the measure σ on $\mathbb{R}^2$ with density h is infinite on horizontal halfplanes, $\{v \geq 0\}$, and the positive coordinates of the vector (Z_1, Z_2) with density f are asymptotically independent, see [5]. So assume $0 < a \leq \tilde{a}$. Then $\sigma\{v \geq c\}$ and $\sigma\{u \geq c\}$ are finite for all $c \in \mathbb{R}$. In this case, the partial maxima converge in distribution to a max-stable limit vector with Gumbel margins if we apply the coordinatewise affine transformations $(t_n + a(t_n)u, t_n + a(t_n)v)$ for a suitable sequence $t_n \to \infty$, and σ is the exponent measure associated with this max-stable limit law. The coordinatewise maxima from f then converge in distribution to a max-stable limit vector with Fréchet margins e^{-1/t^λ} on $(0, \infty)$. The associated exponent measure ρ satisfies $\rho\{x > t\} = \rho\{y > t\} = 1/t^\lambda$. It is a positive diagonal linear transformation of the measure $\rho_1 = K(\sigma)$ with K in (6.4). The measure ρ_1 has density $\lambda^2 r_0$, where r_0 is a power function in (6.2).

Here are some details. Write $h(u, 0) = e^{\tilde{p}u}\mathbf{1}_{[u \leq 0]} + e^{-(1+p)u}\mathbf{1}_{[u>0]}$. Then from (6.3)

$$h(u, v) = e^{-v}h(u - v, 0) = e^{\tilde{p}u-(1+\tilde{p})v}\mathbf{1}_{[u \leq v]} + e^{pv-(1+p)u}\mathbf{1}_{[u>v]}.$$

The sector C is bounded by the lines $\tilde{p}u = (1 + \tilde{p})v$ and $pv = (1 + p)u$, which gives $1/a = 1 + 1/p$ and a similar expression for $\tilde{a}$. The measure $\rho_1 = K(\sigma)$ has density $\lambda^2 r_0$, where r_0 is the power function in (6.2) with $\alpha = p\lambda$, $\beta = (1 + p)\lambda$, $\tilde{\alpha} = \tilde{p}\lambda$ and $\tilde{\beta} = (1 + \tilde{p})\lambda$. The level set $\{r_0 > 1\}$ is bounded by two curves, $y = x^{\tilde{\mu}}$ and $x = y^\mu$ which meet at $(1, 1)$, with $\mu = (\alpha - 1)/(\beta + 1) = (a\lambda - (1 - a))/(\lambda + (1 - a))$ and a similar expression for $\tilde{\mu}$. Then $\rho_1\{y \geq 1\} = \tilde{A} := (1 - a\tilde{a})/\tilde{a}$ and $\rho_1\{x \geq 1\} = A := (1 - a\tilde{a})/a$. Now rescale by $Q = \mathrm{diag}(q, \tilde{q})$ with $q = 1/A^{1/\lambda}$ and $\tilde{q} = 1/\tilde{A}^{1/\lambda}$. Then $\rho\{x \geq 1\} = \rho\{y \geq 1\} = 1$ for $\rho = Q(\rho_1)$ and the positive diagonal maps into the ray L through the point $(1, b)$ with $b = (\tilde{a}/a)^{1/\lambda}$. The level sets of the density r of ρ are scaled copies of the set $D_\lambda \subset (0, \infty)^2$ bounded by the two curves $y = bx^{\tilde{\mu}}$ and $y = bx^{1/\mu}$ which meet at $(1, b)$. ¶

There is a discontinuity in the description of the asymptotic behaviour when a vanishes. For $a = 0$, the right tangent to D is vertical. Assume $\tilde{a} = 1/2$ and $\lambda = 2$. The exponential density h is well-defined and continuous for $a = 0$ and the limit (6.5) holds uniformly on compact sets in the plane, but the mass of vertical halfspaces is infinite since $h \equiv e^{-u}$ below the diagonal. For $a = 0$, the level set $\{r_0 > 1\}$ of the power function r_0 is bounded above by a continuous unimodal curve:

$$0 < y < x^{\tilde{\mu}}\mathbf{1}_{[0<x<1]} + x^{1/\mu}\mathbf{1}_{[1 \leq x]}, \qquad \tilde{\mu} = 1/5,\ 1/\mu = -3. \tag{6.6}$$

The ray L becomes the positive vertical axis as $a \to 0$. The normalized density r yields standard marginal densities $2/t^3$, but r vanishes uniformly on compact subsets of the positive quadrant for $a \to 1$. The information contained in r_0 and h is lost by the normalization, which pulls down the measure onto the one-dimensional axes.

7 Conclusion

The relation between the asymptotic geometric structure of multivariate densities and the copula is not as intuitive as one might hope. As observed in [8] the rate of decrease of the tails plays a crucial role.

This paper presents a systematic investigation of the relation between copula and shape of the level sets as the decay rate of the tails of the density is varied, both for light and heavy tails. The most striking results are the loss of information on shape as one passes from a heavy-tailed density to a light-tailed density, while preserving the copula, and vice versa passing from light to heavy tails - in both cases there is a reduction to a finite dimensional parametric family; and the explosive change in the asymptotic shape of the level sets of heavy-tailed densities as the tail exponent is varied.

The paper compares the asymptotic shape of level sets of two multivariate densities with the same copula but different tails. In the light-tailed case, the shape is stable. We start with a density with level sets whose asymptotic shape is a bounded open star-shaped set with a continuous boundary and which contains the origin. The asymptotic shape for the new density will have the same properties. The new shape is the image of the old shape under a coordinatewise semi-linear power transformation. In the heavy-tailed case, the change is more dramatic. Assume regular variation of the tails. A change in the slowly varying component has no effect on the shape. A change in the balance between the $2d$ marginal tails will have an effect. The new shape is still bounded and contains the origin as interior point, but the boundary is no longer continuous. As a result, the intensity also has discontinuities. If the exponent of regular variation is decreased, the new limit shape is no longer bounded and the new intensity is infinite along the coordinate planes.

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