Bounding Risk Measures for Portfolios with Known Marginal Risks

Paul Embrechts[1] Giovanni Puccetti[2]

[1]Department of Mathematics ETH Zurich, CH-8092 Zurich, Switzerland

[2]Department of Mathematics for Decisions, University of Firenze, 50134 Firenze, Italy

available at `www.math.ethz.ch/~embrechts`

The problem at hand

Consider a function $\psi : \mathbb{R}^n \to \mathbb{R}$ and a random vector

$$X := (X_1, \ldots, X_n)$$

of n one-period financial losses or insurance claims on some probability space $(\Omega, \mathfrak{A}, \mathbb{P})$.

The Value-at-Risk (quantile) at level α for the aggregate loss $\psi(X)$ can be computed once we know the joint distribution of the vector X, i.e. $F(x_1, \ldots, x_n) = \mathbb{P}[X_1 \leq x_1, \ldots, X_n \leq x_n]$.

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Unfortunately, the distribution function (df) of the random vector X is **not** completely determined by the F_i's.

There are infinitely many distributions for the vector X which are consistent with the initial choice of the marginals.

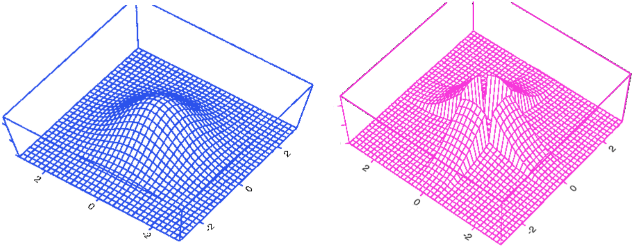

Figure: Two different bivariate dfs having $N(0,1)$-marginals and the same correlation

Which is the df giving the worst-possible Value-at-Risk (VaR) for the random variable $\psi(X)$?

History of the problem

- Makarov (1981) provided the first result for $n = 2; \psi = +$
- Frank et al. (1987) restated Makarov's result, using copulas
- Independently, Rüschendorf (1982) gave a more elegant proof of the same theorem using duality
- Williamson and Downs (1990) found the solution in the presence of partial information for non-decreasing functions ψ of **two** random variables
- Embrechts and Puccetti (2005c) provided an approximation on the real solution when no dependence information is available and $n \geq 3$.

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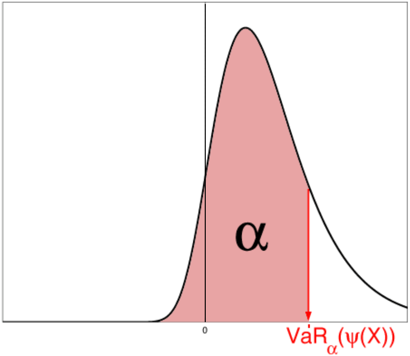
Value-at-Risk for the aggregate loss

Definition

For $\alpha \in [0,1]$, the *Value-at-Risk* at probability level α for $\psi(X)$ is its α-quantile, defined as $\mathrm{VaR}_\alpha(Y) := G^{-1}(\alpha)$, where G is the df of $\psi(X)$.



Searching for the worst-possible VaR means looking for

$$m_\psi(s) := \inf\{\mathbb{P}[\psi(X) < s] : X_i \backsim F_i, i = 1, \dots, n\}.$$

Indeed, according to the definition of VaR, we have

$$\mathrm{VaR}_\alpha(\psi(X)) \leq m_\psi^{-1}(\alpha), \alpha \in [0, 1].$$

The distribution of $\psi(X)$ van be uniquely defined through the marginal dfs and their interdependence,
which can be modeled by the concept of **copula**.

Definition

A *copula* is any n-dimensional df restricted to $[0,1]^n$ having standard uniform marginals.

Given a copula C and a set of n marginals $F_1, \ldots, F_n$ one can always define a df F on $\mathbb{R}^n$ having these marginals by

$$F(x_1, \ldots, x_n) := C(F_1(x_1), \ldots, F_n(x_n)). \tag{1}$$

Sklar's theorem states conversely that we can always find a copula C coupling the marginals of a fixed df F trough (1).

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- **independent** marginals are merged by the

$$\Pi : [0,1]^n \to [0,1]; \Pi(u_1, \ldots, u_n) := \prod_{i=1}^{n} u_i$$

- **comonotonic** marginals are merged by the so-called *upper Fréchet bound*

$$M : [0,1]^n \to [0,1]; M(u_1, \ldots, u_n) := \min\{u_1, \ldots, u_n\}$$

- **countermonotonic** marginals are merged by the so-called *lower Fréchet bound*

$$W : [0,1]^n \to [0,1]; W(u_1, \ldots, u_n) := \left[\sum_{i=1}^{n} u_i - n + 1 \right]^+$$

Any copula C lies between the lower and upper Fréchet bounds:

$$W \leq C \leq M.$$

Dependence information

By Sklar's theorem, our problem can be equivalently expressed as

$$m_\psi(s) = \inf \left\{ \mathbb{P}_C \left[\psi(X) < s \right] : C \in \mathfrak{C}_n \right\},$$

where $\mathfrak{C}_n$ denotes the set of all n-dimensional copulas.

Putting a lower bound on the copula C of the portfolio can be interpreted as having partial information regarding the dependence structure of our portfolio of risks.
If this is the case, the problem reduces to

$$m_{C_L,\psi}(s) := \inf \left\{ \mathbb{P}_C \left[\psi(X) < s \right] : C \geq C_L \right\}.$$

If $C_L = W$, then come back to our original problem.

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Main Result with Dependence information

**When a lower copula-bound on the portfolio copula C is assumed and $n = 2$
the problem at hand is fully solved.**

This result goes back to Williamson and Downs (1990) for any function ψ which is continuous and non-decreasing in each place. Embrechts et al. (2003) state the same theorem also for $n \geq 3$ but, unfortunately, their proof contains a gap: the bound is correct but its sharpness is not proved for $n \geq 3$.

Define

$$\tau_{C,\psi}(F_1, \ldots, F_n)(s) := \sup_{x_1,\ldots,x_{n-1}\in\mathbb{R}} C(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), F_n^-(\widehat{\psi_{x_{-n}}}(s))).$$

Theorem 1 (bound for general functionals ψ) Let $X = (X_1, \ldots, X_n)$ be a random vector on $\mathbb{R}^n$ ($n > 1$) having marginal dfs $F_1, \ldots, F_n$ and copula C. Assume that there exists a copula C_L such that $C \geq C_L$. If $\psi : \mathbb{R}^n \to \mathbb{R}$ is non-decreasing in each coordinate, then, for every $\alpha \in [0, 1]$, we have

$$\mathrm{VaR}_\alpha(\psi(X)) \leq \tau_{C_L,\psi}(F_1, \ldots, F_n)^{-1}(\alpha). \tag{2}$$

Theorem 2 (sharpness of the bound) Assume ψ is also continuous and $n = 2$. Define the function $C_t : [0, 1]^2 \to [0, 1]$ as follows:

$$C_t(u) := \begin{cases} \max\{t, C_L(u)\} & \text{if } u = (u_1, u_2) \in [t, 1]^2, \\ \min\{u_1, u_2\} & \text{otherwise,} \end{cases}$$

where $t = \tau_{C_L,\psi}(F_1, F_2)^{-1}(\alpha)$. Then C_t is a copula and it attains bound (2), i.e. under C_t we have

$$\mathrm{VaR}_\alpha(\psi(X)) = t.$$

Important Remark on the Theorems

- A priori assumptions such as $C \geq \Pi$ may lead to a critical undervaluation of the portfolio risk since the componentwise ordering in the class $\mathfrak{C}^2$ is not complete.

Therefore, in the following we will restrict to the case in which we do not assume any information on the copula of the portfolio, i.e. $C_L = W$.

Main Result without information on dependence

Consider then

$$C_L = W.$$

Though the *standard* bound stated in Theorem 1 still holds in arbitrary dimension, but when

$$n \geq 3$$

it may fail to be sharp.

In the no-information scenario, it is convenient to express our problem using a duality result given in Rüschendorf (1982):

$$m_\psi(s) = \inf\{\mathbb{P}[\psi(X) < s] : X_i \backsim F_i, i = 1, \ldots, n\}$$
$$= 1 - \inf\left\{ \sum_{i=1}^n \int f_i dF_i : f_i \in L^1(F_i), i \in N \text{ s.t.} \right.$$
$$\left. \sum_{i=1}^n f_i(x_i) \geq 1_{[s,+\infty)}(\psi(x)) \text{ for all } x \in \mathbb{R}^n \right\}.$$

Some remarks on the dual problem

- The dual optimization problem seems to be very difficult to solve;
- Explicit results are known only for uniformly or binomially distributed risks;
- Unfortunately, the solution in the case of the sum of uniform marginals does not work in the general case.

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Dual bounds

We use the dual problem to provide a bound which is better (i.e. $\geq$) than the *standard* one.

Theorem 3 (dual bound) Let F be a non-negative, continuous df. If $F_i = F, i = 1, \ldots, n$, then for every $s \geq 0$,

$$m_+(s) \geq 1 - n \inf_{r \in [0, s/n)} \frac{\int_r^{s-(n-1)r} (1 - F(x))dx}{s - nr}.$$

- For $n = 2$ this theorem gives the sharp bound already stated
- This *dual* bound is strictly larger than the standard bound for most dfs and thresholds s of interest
- The theorem can be easily extended to consider non-homogeneous portfolios, i.e. different marginal distributions, see Embrechts and Puccetti (2005b).

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Application 1: Finding the real solution (homogeneous portfolios)

Under the assumptions of Theorem 3 (**homogeneous portfolios**), it is easy to show that, for s large enough, the standard bound reduces to

$$\tau_{W,+}(F,\dots,F)(s) = [nF(s/n) - n + 1]^{+}.$$

Moreover, the dual bound can be easily calculated numerically also for huge portfolios ($n = 100000$) by finding the zero-derivative points of a real-valued function.

How can we compare the quality of the dual bound with respect to the standard bound?

We define the two dfs $\underline{F}_N, \overline{F}_N$ by

$$\underline{F}_N(x) := \frac{1}{N} \sum_{i=1}^{N} 1_{[q_r,+\infty)}(x),$$
$$\overline{F}_N(x) := \frac{1}{N} \sum_{i=0}^{N-1} 1_{[q_r,+\infty)}(x),$$

the jump points $q_0, \ldots, q_N$ being the quantiles of F defined by

$$q_0 := \inf \operatorname{supp}(F), q_N := \sup \operatorname{supp}(F) \text{ and}$$
$$q_r := F^{-1}(r/N), \ r = 1, \ldots, N-1.$$

It is straightforward that

$$\underline{F}_N \leq F \leq \overline{F}_N,$$

from which it follows that

$$\underline{m}_+(s) \leq m_+(s) \leq \overline{m}_+(s),$$

where $\underline{m}_+(s)$ and $\overline{m}_+(s)$ are naturally defined as:

$$\underline{m}_+(s) := \inf\left\{\mathbb{P}\left[\sum_{i=1}^{n} X_i < t\right] : X_i \backsim \underline{F}_N, i = 1, \ldots, n\right\},$$

$$\overline{m}_+(s) := \inf\left\{\mathbb{P}\left[\sum_{i=1}^{n} X_i < t\right] : X_i \backsim \overline{F}_N, i = 1, \ldots, n\right\}.$$

Given that $\underline{F}_N$ is a (possibly defective) discrete df, $\underline{m}_+(s)$ is the solution of the following LP:

$$\underline{m}_+(s) = \min_{p_{j_1,\dots,j_n}} \sum_{j_1=1}^{N} \cdots \sum_{j_n=1}^{N} p_{j_1,j_2,\dots,j_n} 1_{(-\infty,t)}\left(\sum_{i=1}^{n} q_{j_i}\right) \text{ subject to}$$

$$\begin{cases} \sum_{j_2=1}^N \sum_{j_3=1}^N \cdots \sum_{j_n=1}^N p_{j_1,\dots,j_n} = \frac{1}{N} & j_1 = 1,\dots,N, \\ \sum_{j_1=1}^N \sum_{j_3=1}^N \cdots \sum_{j_n=1}^N p_{j_1,\dots,j_n} = \frac{1}{N} & j_2 = 1,\dots,N, \\ \dots, & \\ \sum_{j_1=1}^N \sum_{j_2=1}^N \cdots \sum_{j_{n-1}=1}^N p_{j_1,\dots,j_n} = \frac{1}{N} & j_n = 1,\dots,N, \\ 0 \le p_{j_1,\dots,j_n} \le 1 & j_i = 1,\dots,N, \\ & i = 1,\dots,n. \end{cases}$$

The function $\overline{m}_+(s)$ is the solution of an analogous LP.

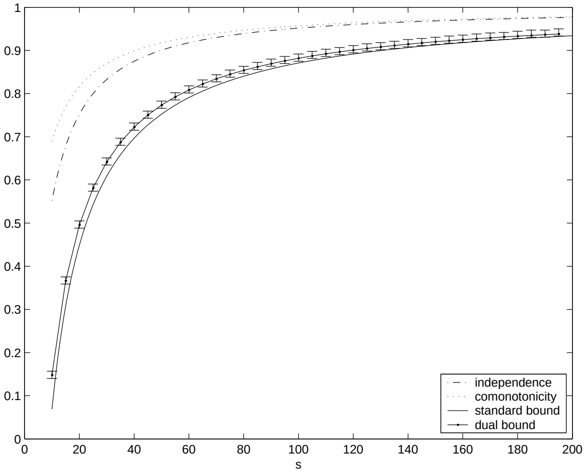

Figure: Range for $\mathbb{P}[X_1 + X_2 + X_3 < s]$ for a Pareto(1.5,1)-portfolio

Some remarks on this plot

- The ranges for the true solutions have been calculated solving the two LPs with $N = 180$ and using ILOG CPLEX® C Callable Libraries (a powerful tool).
- Switching to $n = 5$ drastically lowers the quality of approximation to $N < 50$.
- The worst VaR does not occur under the comonotonicity assumptions, i.e. VaR is not a coherent measure of risk.

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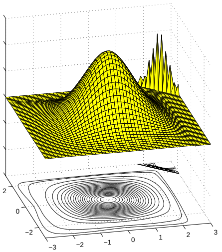
Non-coherence of VaR

$$\mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2) < \mathrm{VaR}_\alpha\,(X_1 + X_2)$$

- X_1, X_2 independent but very skew
- X_1, X_2 independent but very heavy-tailed
- $X_1, X_2 \backsim N(0,1)$ but special dependence, see picture below.



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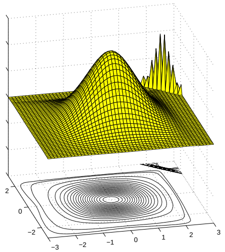

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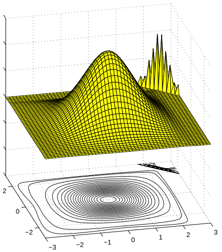

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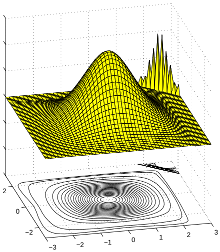

Bounds on Value-at-Risk

	$\mathrm{VaR}_\alpha(\sum_{i=1}^3 X_i)$, *exact*		$\mathrm{VaR}_\alpha(\sum_{i=1}^3 X_i)$, *upper bound*	
α	independence	comonoton.	dual	standard
0.90	7.54	8.85	14.44	15.38
0.95	9.71	12.73	19.50	20.63
0.99	16.06	25.16	35.31	37.03
0.999	29.78	53.99	69.98	73.81

Table: Range for VaR for a Log-Normal(-0.2,1)-portfolio.

Bounds on Value-at-Risk

	$\mathrm{VaR}_\alpha(\sum_{i=1}^{10} X_i)$		$\mathrm{VaR}_\alpha(\sum_{i=1}^{100} X_i)$		$\mathrm{VaR}_\alpha(\sum_{i=1}^{1000} X_i)$	
α	dual	standard	dual	standard	dual	standard
0.90	0.669	1.485	11.039	149.850	150.162	14998.500
0.95	1.353	2.985	22.227	229.850	301.823	29998.500
0.99	2.985	14.985	111.731	1499.850	1515.111	149998.500
0.999	68.382	149.985	1118.652	14999.850	15164.604	1499998.500

Table: Upper bounds for $\mathrm{VaR}_\alpha(\sum_{i=1}^{n} X_i)$ of three Pareto portfolios of different dimensions. Data in thousands.

Application 2: Operational Risk (non-homogeneous portfolios)

The risk management of Operational Risk (OR) under the Advanced Measurement Approach is a typical example where one has to deal with a multivariate portfolio of risks having different marginal distributions; see Moscadelli (2004).

Problems with non-homogeneous marginals

Denuit et al. (1999) remark that, contrary the homogeneous scenario, the *standard bound* can be rarely explicited analytically in practice.

Embrechts and Puccetti (2005b) shows that the computation of standard bounds can be reduced to the problem of finding numerically the root of a real-valued function, independently from the dimension of the portfolio. This result holds for all portfolio of actuarial relevance (continuous marginal distributions).

Calculating the dual bounds, contrary to the homogeneous case, calls for the use of sophisticated optimization algorithms; see Embrechts and Puccetti (2005b).

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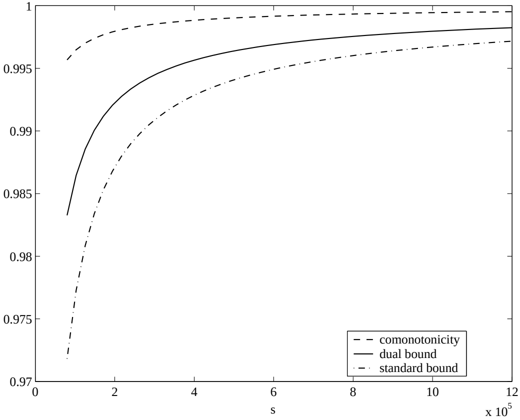

Figure: Bounds on $\mathbb{P}[\sum_{i=1}^{8} X_i < s]$ using the OR portfolio given in Moscadelli (2004), together with the comonotonic scenario.

α	comonotonic value	dual bound	standard bound
0.99	2.8924×10^4	1.4778×10^5	2.6950×10^5
0.995	6.7034×10^4	3.3922×10^5	6.1114×10^5
0.999	4.8347×10^5	2.3807×10^6	4.1685×10^6
0.9999	8.7476×10^6	4.0740×10^7	6.7936×10^7

Table: Range for $\mathrm{VaR}_\alpha\left(\sum_{i=1}^{8} X_i\right)$ for the OR portfolio given in Moscadelli (2004).

Remark on the VaR table

- With respect to the standard one, the dual bound offers an evaluation of the risky position held that is prudential, more realistic and economically advantageous at the same time.
- Though Frachot et al. (2004) among others consider even the comonotonic OR scenario as over-conservative, there is no mathematical reason to drop the worst-case bounds if one uses VaR to evaluate the risk of the position held and no dependence assumptions on the portfolio is explicitly made.

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Conclusions

The worst-possible VaR for a non-decreasing function of dependent risks can be calculated when the portfolio is **two-dimensional**.

When dealing with more than two risks, the problem gets much more complicated and we provide a new bound which we prove to be better than the standard one generally used in the literature.

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Extensions

- Other portfolio functions ψ;
- Multivariate marginals; see Embrechts and Puccetti (2005);
- Other risk measures; see Embrechts et al. (2005).
- For a textbook treatment, see

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