Bounds for the sum of dependent risks having overlapping marginals

Paul Embrechts[a], Giovanni Puccetti[*,b,1]

[a] *Department of Mathematics, ETH Zurich, CH–8092 Zurich, Switzerland*
[b] *Department of Mathematics for Decisions, University of Firenze, I–50134 Firenze, Italy*

Abstract

We describe several analytical and numerical procedures to obtain bounds on the distribution function of a sum of n dependent risks having fixed overlapping marginals. As an application, we produce bounds on quantile-based risk measures for portfolios of financial/actuarial interest.

Key words: Fréchet bounds, overlapping marginals, dependent risks, mass transportation theory, copula functions, Value-at-Risk

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1. Introduction

In recent years, the problem of obtaining bounds for the distribution of a function of n dependent risks has received considerable attention. One of the main reasons for this is the New Basel Capital Accord (Basel II), under the terms of which international active banks are required to set aside capital to offset various types of risks. In fact, the new regulatory capital charges are based on Value-at-Risk (VaR), i.e. a quantile of the distribution function of the potential loss exposure of positions held by a bank. Since the aggregate exposure is typically described by a function $\psi(X)$ of a portfolio $X = (X_1, \dots, X_n)$ of n dependent loss random variables, the search for bounds on the distribution function of $\psi(X)$ becomes crucial, especially in the case $\psi = +$, the sum operator. So far, in the literature, this problem has been studied under the so-called *non-overlapping marginals* setup, i.e. only the marginal distributions $F_1, \dots, F_n$ of the risks held are known or modelled. Of course, the exact calculation of $\text{VaR}(\psi(X))$, say, needs a joint model for X; unfortunately, in applications of Quantitative Risk Management (QRM), such information is often *not* available. A two-stage

*Corresponding Author, tel.+390554796824 fax: +390554796800.
Email addresses: `embrechts@math.ethz.ch` (Paul Embrechts), `giovanni.puccetti@unifi.it` (Giovanni Puccetti)

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fitting procedure is then called for, typically leading to bounds on $\text{VaR}(\psi(X))$. Mikosch (2006) gives a critical assessment of this approach, whereas Embrechts (2009) puts the criticisms in a proper QRM perspective.

In some cases, it may be that further dependence information is available. This calls for a theory which interpolates between marginal knowledge $(F_1, \dots, F_n)$ and full joint distribution function (df) knowledge of the vector X. Examples where such intermediate dependence information may be available is to be found in the modelling of Operational Risk; see McNeil et al. (2005) for an introduction and Embrechts and Puccetti (2006a) for a more detailed modelling. Such intermediate cases we refer to as an *overlapping framework*.

To our knowledge, the only paper which deals with distributional bounds in a overlapping marginals setup is Rüschendorf (1991a), in which several theoretical results are obtained. In our paper, we describe the computational procedures which are necessary to produce numerical bounds from the results by Rüschendorf (1991a). Moreover, we apply some techniques from mass transportation theory to study some particular cases not covered in the above reference.

This paper is a further development in the analysis of distributional bounds which, within a non-overlapping marginals context, we started in Embrechts and Puccetti (2006b) (univariate homogeneous marginals), and continued in Embrechts and Puccetti (2006a) (univariate non-homogeneous marginals) and Embrechts and Puccetti (2006c) (multivariate marginals). In this paper we cover all overlapping systems of marginals which are called *regular* or *decomposable*. Future research concerning also so-called *indecomposable* systems will thus complete the panorama of risk aggregation with fixed marginals. The main motivation for writing these papers is the emergence of QRM within banking and insurance. The subprime crisis, as for instance summarized in Crouhy et al. (2008), highlights the need for a better understanding of risk estimation (both numerical as well as statistical) under incomplete model information. The resulting *model uncertainty* will no doubt become one of the major issues going forward. Whereas we use the language of banking and insurance regulation in our examples, no doubt the results obtained are more widely applicable.

A brief summary of the paper follows. In Section 2 we introduce the mathematical model used throughout the paper. In Section 3 we apply the results contained in Rüschendorf (1991a) to a three dimensional portfolio under the so-called *star-like* system of marginals. An extension of this methodology to an arbitrary number of risks is given in Section 4. In Section 5 we check the quality of the techniques introduced, while in Section 6 we study the *kissing* system of marginals using a duality theorem from mass transportation theory. Finally, in Section 7 we briefly comment on how to analyse all decomposable systems of marginals and how to produce improved distributional bounds when the above mentioned ones are not sharp. Several concrete examples are presented throughout.

2. The mathematical framework

We follow the mathematical setup described in Rüschendorf (1991a). Let $B = \Pi_{i=1}^{n} B_i$ be the product of n Borel spaces with σ-algebra $\mathcal{B} = \bigotimes_{i=1}^{n} \mathcal{B}_i$, $\mathcal{B}_i$ being the Borel σ-algebra on B_i. Define $I := \{1, \ldots, n\}$ and let $\xi \subset 2^I$, the power set of I, with $\cup_{J \in \xi} J = I$. For $J \in \xi$, let $F_J \in \mathfrak{F}(B_J)$ be a *consistent* system of probability measures on $B_J = \pi_J(B) = \Pi_{j \in J} B_j$, π_J being the natural projection from B to B_J and $\mathfrak{F}(B_J)$ denoting the set of all probability measures on B_J. Consistency of $F_J, J \in \xi$ means that $J_1, J_2 \in \xi, J_1 \cap J_2 \neq \emptyset$ implies that

$$\pi_{J_1 \cap J_2} F_{J_1} = \pi_{J_1 \cap J_2} F_{J_2}.$$

Finally, we denote by

$$\mathfrak{F}_\xi = \mathfrak{F}(F_J, J \in \xi)$$

the *Fréchet class* of all probability measures on B having marginals $F_J, J \in \xi$. Some particular choices of ξ will be relevant in what follows:

- $\xi_n = \{\{1\}, \ldots, \{n\}\}$, also called the *simple* system of marginals, which defines the Fréchet class

$$\mathfrak{F}(F_1, \ldots, F_n),$$

- $\xi_n^\star = \{\{1, j\}, j = 2, \ldots, n\}$, the *star-like* system of marginals, which defines the Fréchet class

$$\mathfrak{F}(F_{12}, F_{13}, \ldots, F_{1n}),$$

- $\xi_n^\heartsuit = \{\{j, j+1\}, j = 1, \ldots, n-1\}$, the *kissing* system of marginals, which defines the Fréchet class

$$\mathfrak{F}(F_{12}, F_{23}, \ldots, F_{n-1\,n}).$$

Note that consistency of $F_J, J \in \xi$ is a necessary condition to guarantee that F_ξ is non-empty. When ξ is *regular*, see Vorob'ev (1962) and Section 7 below, then consistency is also sufficient. The three systems of marginals listed above are examples of regular systems. When the system ξ is non-regular, the Fréchet class $\mathfrak{F}_\xi$ may happen to be empty even with consistent marginals; see Rüschendorf (1991a). When ξ is a partition of I, i.e. when all sets $J \in \xi$ are also pairwise disjoint, we speak about a *non-overlapping* system of marginals, *overlapping* otherwise.

In the following, we will consider the case $B_i = \mathbb{R}$, $B = \mathbb{R}^n$. For the sake of notational simplicity, we identify probability measures on these spaces with the corresponding distribution functions.

Consider now n $\mathbb{R}$-valued random variables $X_1, \ldots, X_n$ defined on some probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. For fixed $\xi \subset 2^I$, our aim is to calculate bounds on the df F_S of the sum $S = X_1 + \cdots + X_n$, when the distribution F of the vector $X = (X_1, \ldots, X_n)$ belongs to $\mathfrak{F}_\xi$. With the notation introduced above, and for a fixed real threshold s, we look for

$$m_\xi(s) = \inf \left\{ \int 1_{\{x_1 + \cdots + x_n < s\}} dF(x_1, \ldots, x_n), F \in \mathfrak{F}_\xi \right\}, \tag{1a}$$

$$M_\xi(s) = \sup \left\{ \int 1_{\{x_1 + \cdots + x_n \leq s\}} dF(x_1, \ldots, x_n), F \in \mathfrak{F}_\xi \right\}. \tag{1b}$$

Note that the indicator functions in (1a) and (1b) are defined differently in order to guarantee the inf and the sup are attained. With respect to the random variables $X_1, \ldots, X_n$, problems (1) translate into the bounds

$$m_\xi(s) \leq \mathbb{P}[X_1 + \cdots + X_n \leq s] \leq M_\xi(s). \tag{2}$$

Problems of this type arise especially in insurance or finance when one has to calculate bounds for the distribution of the aggregate position S deriving from a portfolio $X = (X_1, \ldots, X_n)$ of n random losses. In such contexts, the marginal information $F_1, \ldots, F_n$ about the individual risks may be available, while it is difficult to capture statistically the n-variate structure of dependence of the vector X. However, it is often the case that partial information about dependence in some subgroups J of marginals exists. This framework fits for example the case of a portfolio of operational risks as described for instance in Embrechts and Puccetti (2006a).

The risk measure most often used in finance/insurance to evaluate the risk of a loss portfolio X is Value-at-Risk (VaR) at some probability level $\alpha \in (0, 1)$, which is simply the α-quantile $F_S^{-1}(\alpha)$ of the distribution of S. Since banks are obliged to calculate a VaR-based capital charge to offset S, equations (1) become useful since they provide, by (numerical) inversion, bounds on VaR, i.e.

$$M_\xi^{-1}(\alpha) \leq \mathrm{VaR}_\alpha(S) := F_S^{-1}(\alpha) \leq m_\xi^{-1}(\alpha). \tag{3}$$

Having mainly financial/actuarial applications in mind, in the following we will often compute the bounds (3) on VaR, instead of the equivalent bounds (2) on probabilities. For an extensive treatment of Fréchet problems in insurance and finance, we refer the reader to Section 6.2 in McNeil et al. (2005). Finally note that $\xi = \{I\}$ defines a *trivial* system in which $\mathfrak{F}_I$ has only one element and therefore $m_\xi = M_\xi$.

3. Bounds for the case $n = 3$

When $n = 2$, ξ can always be considered to be trivial ($\xi = \{\{1,2\}\}$) or to be simple ($\xi = \{\{1\},\{2\}\}$) . In this latter case, solutions to problems (1) have been given in Rüschendorf (1982); see also Embrechts and Puccetti (2006b) for further details. We hence start our analysis with the case $n = 3$ and the system $\xi_3^\star = \{\{1,2\},\{1,3\}\}$, which is equivalent to the system $\xi_3^\heartsuit$ up to a relabeling of the marginals.

Proposition 7 in Rüschendorf (1991a) states that bounds on F_S for the system $\xi_3^\star$ can be obtained by integrating the corresponding bounds for the Fréchet class $\mathfrak{F}(F_{2|x_1}, F_{3|x_1})$, defined by the two univariate marginals obtained by conditioning F_{12} and F_{13} on the value x_1 of the common marginal F_1. The calculation of bounds in an overlapping system is thus reduced to the calculation of bounds in a non-overlapping system. We reformulate the above cited proposition within our specific framework.

Theorem 1 (Rüschendorf (1991a))**.** *Define $F_{2|x_1}$ as the conditional distribution of F_{12} given that X_1 takes the value x_1. Analogously for $F_{3|x_1}$. We then have that*

$$m_{\xi_3^\star}(s) = \int m_{23|x_1}(s - x_1)\, dF_1(x_1), \tag{4a}$$

$$M_{\xi_3^\star}(s) = \int M_{23|x_1}(s - x_1)\, dF_1(x_1), \tag{4b}$$

where

$$m_{23|x_1}(s - x_1) = \inf\left\{\int 1_{\{x_2+x_3<s-x_1\}} dF(x_2,x_3), F \in \mathfrak{F}(F_{2|x_1}, F_{3|x_1})\right\},$$
$$M_{23|x_1}(s - x_1) = \sup\left\{\int 1_{\{x_2+x_3\leq s-x_1\}} dF(x_2,x_3), F \in \mathfrak{F}(F_{2|x_1}, F_{3|x_1})\right\}.$$

Theorem (1) is very useful, since $m_{23|x_1}(s-x_1)$ and $M_{23|x_1}(s-x_1)$ are known; see for example Proposition 1 in Rüschendorf (1982):

$$m_{23|x_1}(s - x_1) = \sup_{x\in\mathbb{R}}\{F^-_{2|x_1}(x) + F_{3|x_1}(s - x_1 - x) - 1\}, \tag{5a}$$

$$M_{23|x_1}(s - x_1) = \inf_{x\in\mathbb{R}}\{F^-_{2|x_1}(x) + F_{3|x_1}(s - x_1 - x)\}, \tag{5b}$$

where $F^-_{2|x_1}(x)$ denotes the left-hand limit of $F_{2|x_1}$ in x.

From a theoretical viewpoint, equations (4) and (5) are the solution to problems (1) for the system $\xi_3^\star$. Computationally, things are different because the pointwise calculation of $m_{23|x_1}(s - x_1)$ and $M_{23|x_1}(s - x_1)$ in (5) may be very complicated depending on the conditional distributions $F_{2|x_1}$ and $F_{3|x_1}$. Embrechts and Puccetti (2006a), Section 3.1, show that under certain conditions, the computational complexity reduces considerably. This is in particular so if the conditional densities $F'_{2|x_1}$ and $F'_{3|x_1}$ are monotone in the tail. A further issue is that we want to compute $m_{23|x_1}(s-x_1)$ and $M_{23|x_1}(s-x_1)$ also when $(s-x_1)$ approaches zero, say. As a consequence, monotonicity of the densities of the conditional distributions should be required on the entire domains, and this last requirement is seldomly fulfilled. However, we can adapt the technique used in the above cited paper to obtain at least bounds on $m_{\xi_3^\star}$ and $M_{\xi_3^\star}$.

In order to illustrate our approach, we take the case in which the marginals are homogeneous, $F_{12} = F_{13}$, non-negative, $F_{12}(0,0) = 0$, with $F_{2|x_1} = F_{3|x_1} = G_{x_1}$, for all $x_1 \geq 0$. We also assume that G_{x_1} is continuous with density G'_{x_1}. Under the above assumed hypotheses, we can write (5) as

$$m_{23|x_1}(s-x_1) = \sup_{0\leq x\leq s-x_1} \{G_{x_1}(x) + G_{x_1}(s-x_1-x) - 1\}, \tag{6a}$$

$$M_{23|x_1}(s-x_1) = \inf_{0\leq x\leq s-x_1} \{G_{x_1}(x) + G_{x_1}(s-x_1-x)\}. \tag{6b}$$

Instead of solving (6) numerically, we calculate the corresponding objective functions at some points being candidates to be optimizers. In (6a) we choose $x = 0$ (or $x = s - x_1$) as a point at the boundary of the domain and $x = (s-x_1)/2$ as a zero-derivative point. In (6b) we choose the same candidates, the objective function being (up to a constant) the same. Formally, we calculate the bounds $\hat{m}_{23|x_1}$ and $\hat{M}_{23|x_1}$ defined as:

$$m_{23|x_1}(s-x_1) \geq \hat{m}_{23|x_1}(s-x_1) = \max\{2G_{x_1}((s-x_1)/2) - 1, G_{x_1}(s-x_1) - 1, 0\}, \tag{7a}$$

$$M_{23|x_1}(s-x_1) \leq \hat{M}_{23|x_1}(s-x_1) = \min\{2G_{x_1}((s-x_1)/2), G_{x_1}(s-x_1), 1\}. \tag{7b}$$

We also define $\hat{m}_{\xi_3^\star}(s)$ and $\hat{M}_{\xi_3^\star}(s)$ as the corresponding bounds obtained by substituting $\hat{m}_{23|x_1}(s-x_1)$ and $\hat{M}_{23|x_1}(s-x_1)$ in (4). It is easy to verify that, for every threshold s, we have

$$\hat{m}_{\xi_3^\star}(s) \leq m_{\xi_3^\star}(s) \quad \text{and} \quad M_{\xi_3^\star}(s) \leq \hat{M}_{\xi_3^\star}(s). \tag{8}$$

Consequently, the range $[\hat{m}_{\xi_3^\star}(s), \hat{M}_{\xi_3^\star}(s)]$ contains the solution to (4), i.e.

$$\hat{m}_{\xi_3^\star}(s) \leq \mathbb{P}[X_1 + X_2 + X_3 \leq s] \leq \hat{M}_{\xi_3^\star}(s), \tag{9}$$

for all $(X_1, X_2, X_3) \in \mathfrak{F}(F_{12}, F_{13})$, or equivalently

$$\hat{M}^{-1}_{\xi_3^\star}(\alpha) \leq F_S^{-1}(\alpha) \leq \hat{m}^{-1}_{\xi_3^\star}(\alpha). \tag{10}$$

If G'_{x_1} is decreasing on its entire domain, and this for all $x_1 \geq 0$, we deduce from Embrechts and Puccetti (2006a) that the equations (8) hold with equality and hence the bounds (9) and (10) are sharp. We will remark in the following examples when this case occurs.

As an application relevant for insurance and finance, we choose each random variable X_i to be of Pareto type with tail parameter $\theta > 0$, i.e.

$$F_i(x) = \mathbb{P}[X_i \leq x] = 1 - (1 + x)^{-\theta},\ x \geq 0,\ i = 1, \ldots n. \tag{11}$$

In order to generate the bivariate distributions F_{12} and F_{23} from these Pareto marginals we use the concept of *copula*; this is not necessary but analytically as well as computationally convenient. A copula C is a bivariate distribution on $[0, 1]^2$ with uniform marginals. Given a copula C and two univariate marginals F_1, F_2, one can always define a distribution F_{12} on $\mathbb{R}^2$ having these marginals by

$$F_{12}(x_1, x_2) = C(F_1(x_1), F_2(x_2)),\ x_1, x_2 \in \mathbb{R}. \tag{12}$$

Sklar's theorem (see for instance Nelsen (2006)) states conversely that we can always find a copula C coupling the marginals of a fixed joint distribution F_{12} trough the above expression (12). Any copula C satisfies the *Fréchet bounds*, i.e. for all $u_1, u_2 \in [0, 1]$:

$$\max\{u_1 + u_2 - 1, 0\} \leq C(u_1, u_2) \leq \min\{u_1, u_2\}.$$

In the following, we denote the upper Fréchet bound by $M(u_1, u_2)$. M is the so-called *comonotonic* copula which describes perfect *positive* dependence between two coupled marginals; see Dhaene et al. (2002) for a detailed discussion of the concept of comonotonicity. The lower Fréchet bound $W(u_1, u_2)$ is called the *countermonotonic* copula and describes perfect *negative* dependence between F_1 and F_2. Another fundamental copula is the *product* copula $\Pi(u_1, u_2) = u_1 u_2$, which represents *independence* between the coupled marginals. For any additional details on the concept of copula and its applications, we refer the reader to Nelsen (2006) and Chapter 5 in McNeil et al. (2005).

In the following, the bivariate distributions $F_{12} = F_{13}$ (by way of example) will be generated by coupling two Pareto marginals via two different families of copulas:

(i) The *(bivariate) Pareto* copula with parameter $\gamma > 0$,

$$C_\gamma^{Pa}(u, v) = ((1 - u)^{-1/\gamma} + (1 - v)^{-1/\gamma} - 1)^{-\gamma} + u + v - 1,$$

as defined in Hutchinson and Lai (1990). The *survival* copula of C_γ^{Pa} is the function

$$\hat{C}_\gamma^{Pa}(u, v) = u + v + -1 + C_\gamma^{Pa}(1 - u, 1 - v) = (u^{-1/\gamma} + v^{-1/\gamma} - 1)^{-\gamma},$$

which is known in the literature to be of Clayton type. See Section 2.6 and Example 2.14 in Nelsen (2006) for the definition of survival copula and for further details on the parameterization used here. Under the Pareto copula, the joint distribution function F_{12} is given by

$$F_{12}(x_1, x_2) = 1 + ((1 + x_1)^{\theta/\gamma} + (1 + x_2)^{\theta/\gamma} - 1)^{-\gamma} - (1 + x_1)^{-\theta} - (1 + x_2)^{-\theta}. \tag{13}$$

The Pareto copula has a density everywhere $c_\gamma^{Pa}(u, v) = \frac{\partial^2}{\partial u \partial v} C_\gamma^{Pa}(u, v)$ and interpolates between comonotonicity ($C_0^{Pa} = M$) and independence ($C_{+\infty}^{Pa} = \Pi$). Elementary calculations give for the conditional distribution:

$$\begin{aligned} G_{x_1}(x) = F_{2|x_1}(x) &= \int_0^x c_\gamma^{Pa}\left(F_1(x_1), F_2(x_2)\right) F_2'(x_2) dx_2 \\ &= 1 - (1 + x_1)^{\theta(1/\gamma+1)} \left((1 + x)^{(\theta/\gamma)} + (1 + x_1)^{(\theta/\gamma)} - 1\right)^{-\gamma-1}. \end{aligned} \tag{14}$$

(ii) The *Frank* copula with parameter $\delta \in \mathbb{R} \setminus \{0\}$,

$$C_\delta^F(u, v) = -\frac{1}{\delta} \ln\left(1 + \frac{(e^{-\delta u} - 1)(e^{-\delta v} - 1)}{e^{-\delta} - 1}\right). \tag{15}$$

The Frank family is continuous with density $c_\delta^F(u, v)$ and comprehensive, i.e. includes W (when $\delta \to -\infty$), Π ($\delta \to 0$) and M ($\delta \to +\infty$). Under the Frank copula, the joint distribution function F_{12} is given by

$$F_{12}(x_1, x_2) = -\frac{1}{\delta} \ln\left(1 + \frac{(e^{-\delta(1-(1+x_1)^{-\theta})} - 1)(e^{-\delta(1-(1+x_2)^{-\theta})} - 1)}{e^{-\delta} - 1}\right). \tag{16}$$

Straightforward calculations yield:

$$G_{x_1}(x) = \frac{e^{\delta(1+x_1)^{-\theta}}(e^\delta - 1)}{e^{\delta(1+x_1)^{-\theta}} - e^\delta} \left[\left(1 + e^{\delta((1+x_1)^{-\theta}+(1+x)^{-\theta}-1)} - e^{\delta(1+x)^{-\theta}} - e^{\delta(1+x_1)^{-\theta}}\right)^{-1} - \left(1 - e^\delta\right)^{-1}\right].$$

The motivation for choosing these copula families is mainly pedagogical in order to obtain different conditional distributions $F_{2|x_1}$ and $F_{3|x_1}$ which can be written in closed form. In this latter case, a numerical computation of (5) is possible using the techniques described in Embrechts and Puccetti (2006a), but it may be cumbersome to compute numerically the two integrals in (4) if $m_{23|x_1}$ and $M_{23|x_1}$ are not expressed in closed analytic form.

In Figure 1, we provide the bounds $\hat{m}_{\xi_3^\star}^{-1}(\alpha)$ and $\hat{M}_{\xi_3^\star}^{-1}(\alpha)$ on the VaR for the sum of three Pareto-distributed risks under the bivariate Pareto (left) and Frank (right) dependence scenarios introduced above. In these

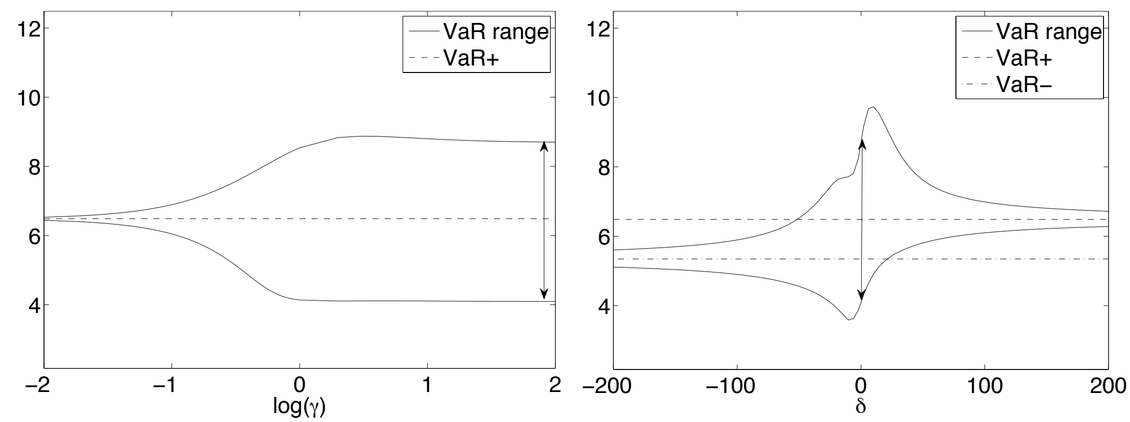

Figure 1: VaR ranges $[\hat{M}^{-1}_{\xi_3^\star}(0.90), \hat{m}^{-1}_{\xi_3^\star}(0.90)]$ for the sum of three Pareto(2)-distributed risks coupled by a bivariate Pareto (left) and Frank (right) copula. The ranges are plotted against the dependence parameters $\log(\gamma)$ (left) for the Pareto and δ (right) for the Frank copula. The total y-axis ranges correspond to the VaR ranges under the non-overlapping marginal system ξ_3 with the same univariate marginals. Limit values VaR^+ and VaR^- are also displayed. The arrows indicate the VaR ranges given by the independence copula Π.

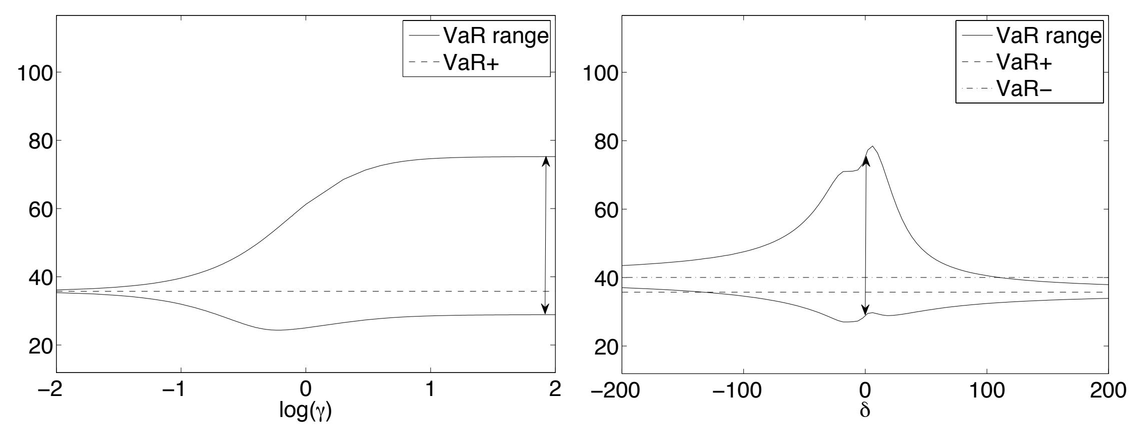

Figure 2: The same as Figure 1 for three Pareto(0.9) distributed risks.

pictures, the Pareto tail parameter is set to $\theta = 2$ (finite marginal mean, infinite marginal variance), while the probability level at which the quantile range is computed is $\alpha = 0.90$. Both plots show how the bounds on Value-at-Risk vary over the parameter $\log(\gamma)$ (respectively δ) chosen for the bivariate Pareto (respectively Frank) dependence.

There are several interesting points to remark about these plots. First of all, note that the bounds obtained under the Pareto family of copulas are sharp, since, in this latter case, the conditional distribution G_{x_1} has a monotone density on its entire domain and then the conditions specified in Remark 3.2 in Embrechts and Puccetti (2006a) apply. When γ tends to 0, the bivariate Pareto copula C_γ^{Pa} tends to the comonotonic copula M, and the distribution $F_{12} = F_{13}$ hence goes to $M(F_1, F_2)$. It is elementary to show that the class $\mathfrak{F}(M(F_1, F_2), M(F_1, F_3))$ has only one element, which is the 3-variate comonotonic distribution $M(F_1, F_2, F_3)$, defined by the 3-dimensional copula $M(u, v, z) = \min\{u, v, z\}$. As a consequence, the range for VaR reduces to a single value, i.e.

$$\hat{M}^{-1}_{\xi_3^\star}(\alpha) = \hat{m}^{-1}_{\xi_3^\star}(\alpha) = \mathrm{VaR}^+.$$

In general, VaR^+ is simply the sum of marginal VaRs, i.e. $\mathrm{VaR}^+ = F_1^{-1}(\alpha) + F_2^{-1}(\alpha) + F_3^{-1}(\alpha)$; see Proposition 6.15 in McNeil et al. (2005).

An analogous behavior characterizes the Frank dependence when δ goes to $+\infty$ (the VaR range again reduces to VaR^+). Moreover, since the Frank family is comprehensive, a similar case occurs when δ goes to $-\infty$, indeed $\mathfrak{F}(W(F_1, F_2), W(F_1, F_3))$ is a singleton. We call the corresponding limit VaR^-. Note that in this latter case, the subvectors (X_1, X_2) and (X_2, X_3) are countermonotonic, while the subvector (X_1, X_3) is comonotonic. It is well known that a 3-dimensional countermonotonic distribution does not exist; see Nelsen (2006, p.47). In both plots, the ranges for VaR under independence are highlighted. Note that independence may not be the most dangerous (=highest upper bound) case. Moreover, for Frank dependencies, $\mathrm{Var}^+ > \mathrm{VaR}^-$ holds. The strange behavior of VaR bounds around $\delta = 0$ under the Frank dependence scenario is probably due to the fact that the Frank copula C_δ^F rapidly changes in the neighborhood of $\delta = 0$. In Figure 2 we provide the same bounds as in Figure 1, except for the fact that the Pareto tail parameter is set to $\theta = 0.9$. This makes a big difference since, with such a choice, the marginals F_i enter the *infinite-mean world*, where many pitfalls regarding risk aggregation have been established; see Nešlehová et al. (2006), Embrechts et al. (2009) and Ibragimov and Walden (2008). For instance, with $\theta = 0.9$ and under the Frank copula model we get that $\mathrm{VaR}^- > \mathrm{VaR}^+$.

Finally note that in the plots displayed in Figures 1 and 2, the total y-axis range corresponds to the bounds for Value-at-Risk within the same portfolios of risks, but when the marginal system chosen is the simple $\xi_3 = \{\{1\},\{2\},\{3\}\}$. Since under this last marginal system only the univariate marginals F_i of the portfolio are fixed, the VaR bound does not depend upon the dependence parameters. Moreover, the overlapping VaR range is always narrower than the non-overlapping range on the y-axis due to the fact that

$$\mathfrak{F}(F_{12}, F_{23}) \subset \mathfrak{F}(F_1, F_2, F_3),$$

i.e. the Fréchet class of risks gets smaller when switching from a non-overlapping to an overlapping system of marginals. The same applies to Figures 3- 5 below.

4. Bounds for the case $n > 3$, star-like system of marginals

The techniques described in Section 3 can be easily extended to consider the star-like system of marginals $\xi_n^\star$ for an arbitrary number n of marginals. Again, the calculation of bounds in the case of $\xi_n^\star$ is reduced to the calculation of bounds in a non-overlapping system. The following theorem is essentially Proposition 8 in Rüschendorf (1991a).

Theorem 2 (Rüschendorf (1991a)). *Define $F_{i|x_1}$ as the distribution of $X_i|X_1 = x_1$, and this for all $i = 2, \dots, n$. Then we have that*

$$m_{\xi_n^\star}(s) = \int m_{2,\dots,n|x_1}(s - x_1)\, dF_1(x_1), \tag{17a}$$

$$M_{\xi_n^\star}(s) = \int M_{2,\dots,n|x_1}(s - x_1)\, dF_1(x_1), \tag{17b}$$

where

$$m_{2,\dots,n|x_1}(s - x_1) = \inf\left\{\int 1_{\{\sum_{i=2}^n x_i < s - x_1\}} dF(x_2, \dots x_n), F \in \mathfrak{F}(F_{2|x_1}, \dots, F_{n|x_1})\right\}, \tag{18a}$$

$$M_{2,\dots,n|x_1}(s - x_1) = \sup\left\{\int 1_{\{\sum_{i=2}^n x_i \le s - x_1\}} dF(x_2, \dots x_n), F \in \mathfrak{F}(F_{2|x_1}, \dots, F_{n|x_1})\right\}. \tag{18b}$$

The main difference between the case $n = 3$ (Theorem 1) and the case $n > 3$ (Theorem 2) is that the solutions to problems (18) are *not* known in the literature. Therefore, we need to find bounds similar to those in Section 3.

Under the simplifying assumptions and notation of Section 3, the bounds (6) can be extended to an arbitrary number n of random variables; see for example equations (12) and (13) in Denuit et al. (1999). We can then write that

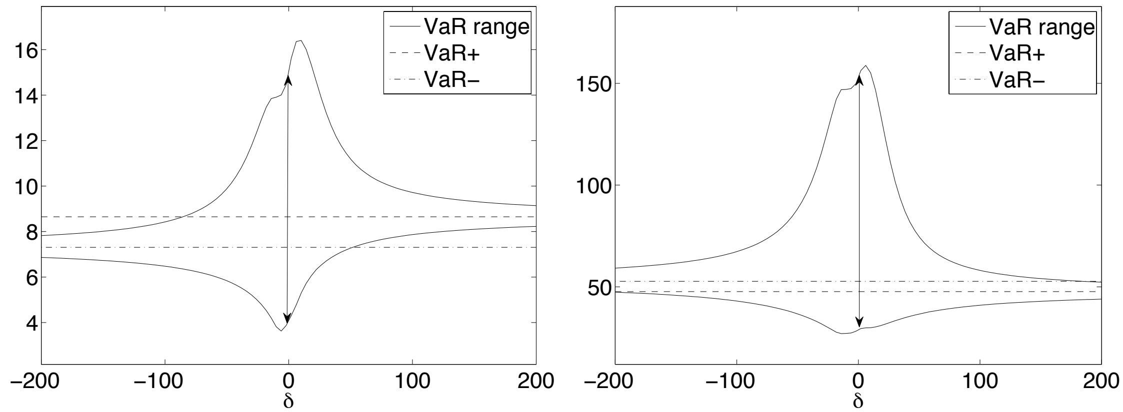

Figure 3: VaR ranges $[\hat{M}^{-1}_{\xi_4^\star}(0.90), \hat{m}^{-1}_{\xi_4^\star}(0.90)]$ for the sum of four Pareto(2)-(left) and Pareto(0.9)-(right) distributed risks coupled by a bivariate Frank copula. The ranges are plotted against the dependence parameter δ. The total y-axis ranges correspond to the VaR ranges under the non-overlapping marginal system ξ_4 with the same univariate marginals. Limit values VaR^+ and VaR^- are also displayed. The arrows indicate the VaR ranges given by the independence copula Π.

$$m_{2,\dots,n|x_1}(s-x_1) \geq \sup_{x_2,\dots,x_{n-1}\in[0,s]} \left\{ G_{x_1}(x_2) + \cdots + G_{x_1}(x_{n-1}) + G_{x_1}\left(s - \sum_{i=1}^{n-1} x_i\right) - n + 2 \right\}, \tag{19a}$$

$$M_{2,\dots,n|x_1}(s-x_1) \leq \inf_{x_2,\dots,x_{n-1}\in[0,s]} \left\{ G_{x_1}(x_2) + \cdots + G_{x_1}(x_{n-1}) + G_{x_1}\left(s - \sum_{i=1}^{n-1} x_i\right) \right\}, \tag{19b}$$

Analogously to the case $n = 3$, we define the bounds $\hat{m}_{2,\dots,n|x_1}$ and $\hat{M}_{2,\dots,n|x_1}$ as

$$m_{2,\dots,n|x_1}(s-x_1) \geq \hat{m}_{2,\dots,n|x_1}(s-x_1) = \max\left\{(n-1)G_{x_1}\left(\frac{s-x_1}{n-1}\right) - n + 2, G_{x_1}(s-x_1) - n + 2, 0\right\}, \tag{20a}$$

$$M_{2,\dots,n|x_1}(s-x_1) \leq \hat{M}_{2,\dots,n|x_1}(s-x_1) = \min\left\{(n-1)G_{x_1}\left(\frac{s-x_1}{n-1}\right), G_{x_1}(s-x_1), 1\right\}. \tag{20b}$$

Defining $\hat{m}_{\xi_n^\star}(s)$ and $\hat{M}_{\xi_n^\star}(s)$ as the corresponding bounds obtained by substituting (20) in (17), it is easy to verify that, for all thresholds s, we have

$$\hat{m}_{\xi_n^\star}(s) \leq \mathbb{P}[X_1 + \cdots + X_n \leq s] \leq \hat{M}_{\xi_n^\star}(s), \tag{21}$$

for all $F \in \mathfrak{F}(F_{12}, \dots, F_{1n})$, or equivalently

$$\hat{M}^{-1}_{\xi_n^\star}(\alpha) \leq F_S^{-1}(\alpha) \leq \hat{m}^{-1}_{\xi_n^\star}(\alpha).$$

As an application, we consider for the F_{1i}'s the same set of bivariate marginals defined in Section 3. In Figure 3, we provide bounds on the VaR for the sum of $n = 4$ Pareto-distributed risks under the bivariate

Frank dependence scenario and the tail parameter of the Pareto df set to $\theta = 2$ (left) and $\theta = 0.9$ (right). The quantile level is $\alpha = 0.90$. Both plots show how the bounds on Value-at-Risk vary as a function of the Frank parameter δ. Apart from sharpness of the bounds, which does not hold in these cases, similar comments as made for Figure 2 apply, in particular the total y-axis range corresponds to the bounds for Value-at-Risk within the same portfolios of risks, but when the marginal system chosen is the simple $\xi_4 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$. Finally note that the computational time needed for the calculation of the bounds (20) does not change considerably with n. As noted in Section 3, matters may change considerably if the portfolio X is assumed to be non-homogeneous.

5. Quality of the bounds

In this section we illustrate the quality of the estimates of the sharp bounds $m_{\xi_n^\star}$ and $M_{\xi_n^\star}$ (see (17)) provided by the bounds $\hat{m}_{\xi_n^\star}$ and $\hat{M}_{\xi_n^\star}$ defined in Section 4 (see (21)). To this end, we define the univariate discrete distribution F_Q as

$$F_Q(x) := \frac{1}{N} \sum_{i=1}^{N} 1_{[q_i, +\infty)}(x),$$

where $Q = \{q_1, \ldots, q_N\}$ denotes the set of N jump points of F_Q. Hence, F_Q gives mass $1/N$ to each q_i. Then, we define the bivariate distribution $H_Q = C_\delta^F(F_Q, F_Q)$, where C_δ^F is a Frank copula as defined in (15). Hence H_Q is a discrete distribution taking values in $Q \times Q$, with generalized density h_Q given by

$$h_Q(q_i, q_j) = H_Q(q_i, q_j) - H_Q(q_{i-1}, q_j) - H_Q(q_i, q_{j-1}) + H_Q(q_{i-1}, q_{j-1}),\ i, j = 2, \ldots, N,$$
$$h_Q(q_i, q_1) = h_Q(q_1, q_i) = H_Q(q_i, q_1) - H_Q(q_{i-1}, q_1),\ i = 2, \ldots, N,\ h_Q(q_1, q_1) = H_Q(q_1, q_1).$$

We now consider the star-like marginal system ξ_n^* with fixed homogeneous bivariate marginals $F_{1i} = F_{12}$ as defined in (16) under the Frank dependence scenario. Let us define the sets

$$\overline{Q}_N = \{t_0, \ldots, t_{N-1}\} \quad \text{and} \quad \underline{Q}_N = \{t_1, \ldots, t_N\},$$

where $t_0, \ldots, t_N$ are the quantiles of a Pareto(θ) distribution, i.e. $t_r = (1 - r/N)^{-1/\theta} - 1, r = 0, \ldots, N-1$ and $t_N = +\infty$. The two discrete distributions $\underline{F}_{12} = H_{\underline{Q}_N}$ and $\overline{F}_{12} = H_{\overline{Q}_N}$ satisfy

$$\underline{F}_{12} \leq F_{12} \leq \overline{F}_{12},$$

from which it follows that, for every real s,

$$\underline{m}_{\xi_n^*}(s) \leq m_{\xi_n^*}(s) \leq \overline{m}_{\xi_n^*}(s). \tag{22}$$

Hence $\underline{m}_{\xi_n^*}(s)$ and, respectively, $\overline{m}_{\xi_n^*}(s)$ are naturally defined as

$$\underline{m}_{\xi_n^*}(s) := \inf\left\{\int 1_{\{x_1+\cdots+x_n<s\}} dF(x_1,\ldots,x_n), F \in \mathfrak{F}(F_{12},F_{13},\ldots,F_{1n})\right\} \text{ with } F_{1i} = \underline{F}_{12}, i = 2,\ldots,n,$$

$$\overline{m}_{\xi_n^*}(s) := \inf\left\{\int 1_{\{x_1+\cdots+x_n<s\}} dF(x_1,\ldots,x_n), F \in \mathfrak{F}(F_{12},F_{13},\ldots,F_{1n})\right\} \text{ with } F_{1i} = \overline{F}_{12}, i = 2,\ldots,n.$$

Due to (22), a numerical range for the sharp bound $m_{\xi_n^\star}$ can be found by solving two discrete linear problems (LPs). In fact, given that $\underline{F}_{12}$ is a discrete df, $\underline{m}_{\xi_n^*}(s)$ is the solution of the following LP:

$$\underline{m}_{\xi_n^*}(s) = \min_{p_{j_1,\ldots,j_n}} \sum_{j_1=1}^{N} \cdots \sum_{j_n=1}^{N} p_{j_1,j_2,\ldots,j_n} 1_{\{\sum_{i=1}^n t_{j_i}<s\}} \text{ subject to}$$

$$\begin{cases} \sum_{j_3=1}^N \sum_{j_4=1}^N \cdots \sum_{j_n=1}^N p_{j_1,\ldots,j_n} = h_{\underline{Q}}(t_{j_1},t_{j_2}) & j_1,j_2 = 1,\ldots,N, \\ \sum_{j_2=1}^N \sum_{j_4=1}^N \cdots \sum_{j_n=1}^N p_{j_1,\ldots,j_n} = h_{\underline{Q}}(t_{j_1},t_{j_3}) & j_1,j_3 = 1,\ldots,N, \\ \cdots & \\ \sum_{j_2=1}^N \sum_{j_3=1}^N \cdots \sum_{j_{n-1}=1}^N p_{j_1,\ldots,j_n} = h_{\underline{Q}}(t_{j_1},t_{j_n}) & j_1,j_n = 1,\ldots,N, \\ 0 \le p_{j_1,\ldots,j_n} \le 1 & j_i = 1,\ldots,N, \\ & i = 1,\ldots,n. \end{cases} \tag{23}$$

The function $\overline{m}_{\xi_n^*}(s)$ is the solution of an analogous LP. Since for N tending to infinity the distributions $\overline{F}_{12}$ and $\underline{F}_{12}$ converge to the original distribution F_{12}, calculating $m_{\xi_n^*}(s)$ with any given level of accuracy is a matter of solving (23) and the corresponding LP for $\overline{m}_{\xi_n^*}(s)$ with N large enough. As we will see below, this is a non-trivial task.

Similar considerations are valid if one wants to obtain a numerical range for the sharp bound $M_{\xi_n^\star}$. Defining

$$\underline{M}_{\xi_n^*}(s) := \sup\left\{\int 1_{\{x_1+\cdots+x_n\le s\}} dF(x_1,\ldots,x_n), F \in \mathfrak{F}(F_{12},F_{13},\ldots,F_{1n})\right\}, \text{ with } F_{1i} = \underline{F}_{12}, i = 1,\ldots,n,$$

$$\overline{M}_{\xi_n^*}(s) := \sup\left\{\int 1_{\{x_1+\cdots+x_n\le s\}} dF(x_1,\ldots,x_n), F \in \mathfrak{F}(F_{12},F_{13},\ldots,F_{1n})\right\}, \text{ with } F_{1i} = \overline{F}_{12}, i = 1,\ldots,n,$$

we have that, for every s,

$$\underline{M}_{\xi_n^*}(s) \le M_{\xi_n^*}(s) \le \overline{M}_{\xi_n^*}(s). \tag{24}$$

Also $\underline{M}_{\xi_n^*}(s)$ and $\overline{M}_{\xi_n^*}(s)$ are the solution of two LPs analogous to (23).

5.1. Plots of the best-possible bounds

The dimension of the two LPs giving $\underline{m}_{\xi_n^*}(s)$ (see (23)) and $\overline{m}_{\xi_n^*}(s)$ is N^n rows (variables) per $(n-1)N^2$ columns (constraints). As a consequence, the computational time and the memory needed to solve the two

LPs increase exponentially in N. Exploiting the symmetry of the problem with homogeneous marginals, it is possible to reduce the dimensionality of the two LPs. As an example for $n = 3$, each LP can be reduced to having $N^2(N+1)/2$ rows per N^2 columns, instead of N^3 per $2N^2$. For $n = 4$ the dimensionality is $N^2(N^2+3N+2)/6$ per N^2, instead of N^4 per $3N^2$. However, these numbers remain problematic since accurate approximations for $m_{\xi_n^*}(s)$ and $M_{\xi_n^*}(s)$ can generally be obtained for $N > 100$. Finally note that a numerical solution will truncate $\underline{F}_{12}$ at a certain finite value. We automatically sets this upper limit so that (22) and (24) are maintained.

In Figure 4 (left), we plot the lower bound $\hat{m}_{\xi_n^*}(s)$ for a portfolio of $n = 3$ Pareto(2)-distributed risks under the star-like marginal system described above. The parameter of the Frank copula is set to $\delta = 1$. The quality of the bound $\hat{m}_{\xi_3^*}(s)$ is indicated by the fact that it always falls within the range $[\underline{m}_{\xi_3^*}(s), \overline{m}_{\xi_3^*}(s)]$, which we plot at some thresholds of interest choosing $N = 150$. In the same figure, we provide also the lower bound on $\mathbb{P}[X_1 + X_2 + X_3 < s]$ for the same portfolio of risks, but when the marginal system chosen is the simple $\xi_3 = \{\{1\},\{2\},\{3\}\}$. Similar to Figures 1 and 2, the bound obtained under the overlapping model is better than the corresponding one obtained in the non-overlapping case. In Figure 4(right) we plot the upper bound $\hat{M}_{\xi_3^*}(s)$ with analogous probability ranges. Again, $\hat{M}_{\xi_3^*}(s)$ turns out to be very accurate and improves the corresponding bound in the non-overlapping marginal system.

In Figure 5(left) we plot the bound $\hat{m}_{\xi_n^*}(s)$ for a portfolio of $n = 4$ risks with the same Pareto parameter and Frank parameter $\delta = 2$. Contrary to the case $n = 3$, the bound $\hat{m}_{\xi_4^*}(s)$ fails to fall within the range $[\underline{m}_{\xi_4^*}(s), \overline{m}_{\xi_4^*}(s)]$ for certain values of s. At this point, we recall that the bound $\hat{m}_{\xi_n^*}(s)$ obtained under the Frank family of copulas may fail to be sharp, as remarked in Section 4. However, $\hat{m}_{\xi_n^*}(s)$ still improves the corresponding bound in a non-overlapping marginal system with the same marginals. For the case $d = 4$, we were able to calculate numerical ranges only with $N = 60$. As a consequence, numerical ranges in Figure 5 are broader than those in Figure 4.

For $n = 5$ accurate numerical ranges for the sharp bounds $m_{\xi_n^\star}$ and $M_{\xi_n^\star}$ are out of reach. This however clearly shows the need for bounds $\hat{m}_{\xi_n^\star}$ and $\hat{M}_{\xi_n^\star}$ when $n > 5$. Moreover, we note that the computational time of a single range $[\underline{m}_{\xi_n^*}(s), \overline{m}_{\xi_n^*}(s)]$, with the above values of N, is approximately 20 minutes on a MacBook laptop (2.4 GHz Intel Core 2 Duo, 2GB RAM). Instead of this, the entire curves $\hat{m}_{\xi_n^\star}(s)$ and $\hat{M}_{\xi_n^\star}(s)$ are obtained within a second, and this independently from the number n of variables under study. To illustrate this property, in Table 1 we give upper VaR limits $\hat{m}^{-1}_{\xi_n^\star}(\alpha)$ for Frank-Pareto portfolios of increasing dimensions. As quantile levels, we take $\alpha = 0.99$ and $\alpha = 0.999$; these are relevant within a risk management context. Accurate numerical VaR ranges in high dimensions cannot be obtained, yet the overlapping bounds consider-

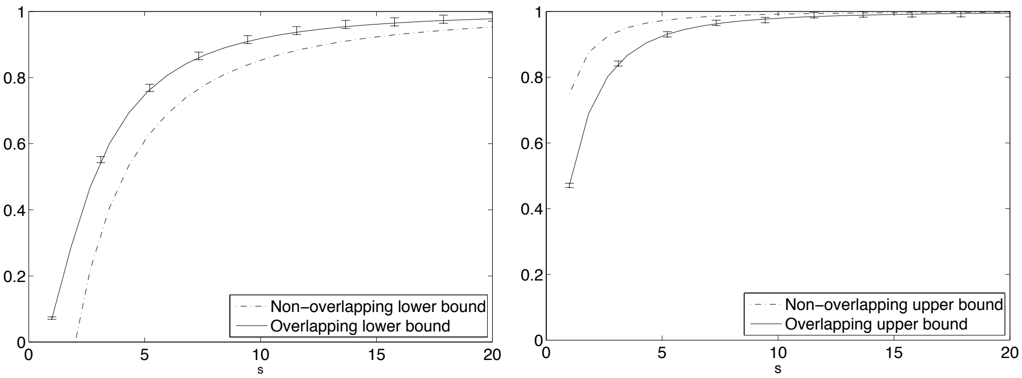

Figure 4: Lower bound $\hat{m}_{\xi_3^\star}(s)$ (left) and upper bound $\hat{M}_{\xi_3^\star}(s)$ for the sum of three Pareto(2)-distributed risks coupled by a bivariate Frank copula with parameter $\delta = 1$. Numerical ranges for the sharp bounds $m_{\xi_3^\star}(s)$ (left) and $M_{\xi_3^\star}(s)$ (right), and lower (left) an upper (right) bounds on $\mathbb{P}[X_1 + X_2 + X_3 < s]$ in the non-overlapping marginal system ξ_3 with the same marginals are also displayed.

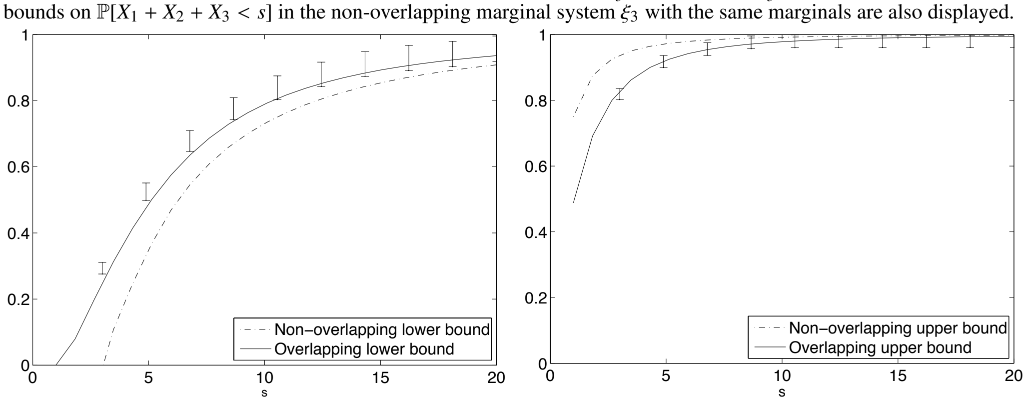

Figure 5: Lower bound $\hat{m}_{\xi_4^\star}(s)$ (left) and upper bound $\hat{M}_{\xi_4^\star}(s)$ for the sum of four Pareto(2)-distributed risks coupled by a bivariate Frank copula with parameter $\delta = 2$. Numerical ranges for the sharp bounds $m_{\xi_4^\star}(s)$ (left) and $M_{\xi_4^\star}(s)$ (right), and lower (left) an upper (right) bounds on $\mathbb{P}[X_1 + X_2 + X_3 + X_4 < s]$ in the non-overlapping marginal system ξ_4 with the same marginals are also displayed.

ably improve the corresponding bounds in a non-overlapping setting. This fact has important consequences for applications. Switching from a non-overlapping simple system to a overlapping star-like marginal system means having more information about the dependence structure of the portfolio $(X_1, \ldots, X_n)$ of risks held. Under this extra information, less capital is needed to offset the underlying portfolio risk.

5.2. Optimal dependence structure

It is interesting to study the structure of dependence of the vector $(X_1^*, X_2^*, X_3^*) \in \mathfrak{F}(F_{12}, F_{13})$ attaining the bound $m_{\xi_3^\star}(s)$ under the Frank-Pareto homogeneous portfolio of risks assumed above. For the threshold $s = 5$, Figure 6 shows the density (left) and the contour plot (right) for the conditional distribution of

	α = **0.99**		α = **0.999**	
n	*overlapping*	*non-overlapping*	*overlapping*	*non-overlapping*
3	29.98	46.70	95.17	156.98
4	51.82	70.75	167.24	248.98
5	78.46	98.44	253.83	348.55
6	108.99	129.36	352.62	458.76
7	143.03	178.20	463.35	578.66
8	180.12	218.27	584.19	707.54
9	220.14	261.00	712.03	844.81
10	262.83	306.27	850.30	990.00

Table 1: Upper bounds on Value-at-Risk for the sum of n Pareto(2)-distributed risks within the *overlapping* star-like $\xi_n^\star$ and the *non-overlapping* marginal system ξ_n. Under the star-like system, the marginals are coupled by a Frank copula with parameter $\delta = 1$.

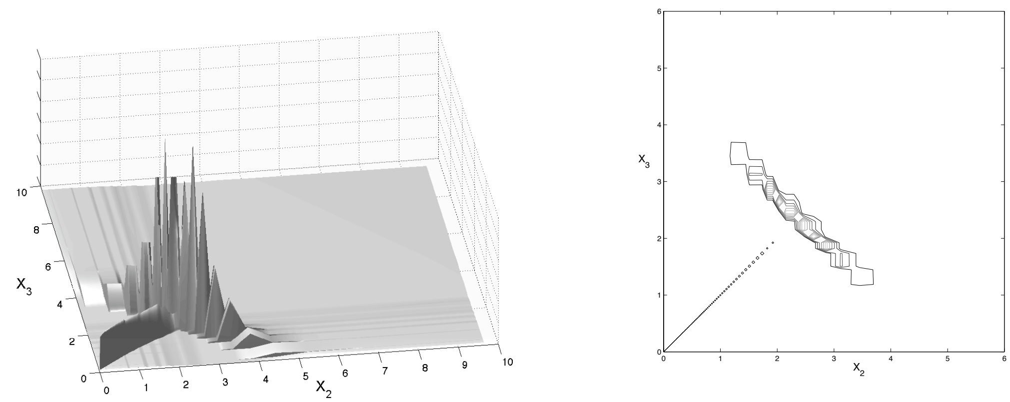

Figure 6: Density and contour plot for the distribution of (X_2^*, X_3^*) conditioned to $X_1^* = 0.40$, when the bound $m_{\xi_3^\star}(5)$ is attained.

(X_2^*, X_3^*) given that X_1^* assumes the value $x_1^* = 0.40$. These figures have been obtained by solving (23) with $N = 150$. The optimal distribution is singular as it gives mass to a one-dimensional curve in the plane. To this extent, it is very similar to the optimal dependence structures found in analogous problems with non-overlapping marginals; see for instance Figure 6.2 in McNeil et al. (2005). In order to exclude such pathological dependence structures, a more appropriate problem would be to optimize over a smaller class than the full Fréchet class. A possible way to perform this task is to put lower and uppper constraint on the Fréchet class, as done for example in Embrechts et al. (2003) within a simple marginal system.

6. Bounds for the case $n > 3$, kissing system of marginals

Rüschendorf (1991a) also suggests a method to calculate (1) in the case of a kissing system of marginals $\xi_n^{\heartsuit}$ when $n > 3$. Unfortunately, as the author of the latter paper remarks, this method is hardly tractable for computational purposes. We therefore suggest a different technique to compute bounds for (1), which is based on a dual representation of these problems, which has been given in (3) and (4) in Rüschendorf (1991a); see also Rüschendorf (1991b) and Rachev and Rüschendorf (1998) for further details. Denote by $\mathcal{BM}(\mathbb{R}^2)$ the class of bounded and measurable functions $f : \mathbb{R}^2 \to \mathbb{R}$. We have that

$$\begin{aligned} m_{\xi_n^{\heartsuit}}(s) &= \inf\left\{\int 1_{\{\sum_{i=1}^n x_i < s\}} dF(x_1,\ldots,x_n), F \in \mathfrak{F}(F_{12},F_{23},\ldots,F_{n-1\,n})\right\} \\ &= 1 - \inf\Big\{\sum_{i=1}^{n-1}\int f_{i\,i+1}(x_i,x_{i+1})dF_{i\,i+1}(x_i,x_{i+1}), f_{i\,i+1} \in \mathcal{BM}(\mathbb{R}^2) \text{ s.t.} \qquad (25a) \\ &\qquad \sum_{i=1}^{n-1} f_{i\,i+1}(x_i,x_{i+1}) \geq 1_{\{\sum_{i=1}^n x_i \geq s\}}, x_1,\ldots,x_n \in \mathbb{R}\Big\}, \\ M_{\xi_n^{\heartsuit}}(s) &= \sup\left\{\int 1_{\{\sum_{i=1}^n x_i \leq s\}} dF(x_1,\ldots,x_n), F \in \mathfrak{F}(F_{12},F_{23},\ldots,F_{n-1\,n})\right\} \\ &= \inf\Big\{\sum_{i=1}^{n-1}\int f_{i\,i+1}(x_i,x_{i+1})dF_{i\,i+1}(x_i,x_{i+1}), f_{i\,i+1} \in \mathcal{BM}(\mathbb{R}^2) \text{ s.t.} \qquad (25b) \\ &\qquad \sum_{i=1}^{n-1} f_{i\,i+1}(x_i,x_{i+1}) \geq 1_{\{\sum_{i=1}^n x_i \leq s\}}, x_1,\ldots,x_n \in \mathbb{R}\Big\}. \end{aligned}$$

An exact solution of the dual problems (25) seems to be well out of reach when $n > 3$. In order to obtain bounds on $m_{\xi_n^{\heartsuit}}$ and, respectively $M_{\xi_n^{\heartsuit}}$, it is however sufficient to find a feasible choice of the functions $f_{i\,i+1}$ in (25a), respectively (25b). This method has already been used in Embrechts and Puccetti (2006a,b,c) and produces excellent bounds for the sum of risks for non-overlapping systems of marginals.

Theorem 3. *Let $(X_1,\ldots,X_n)$ be a random vector with distribution $F \in \mathfrak{F}(F_{12},F_{23},\ldots F_{n-1\,n})$. Then we have* *that*

$$m_{\xi_n^{\heartsuit}}(s) \geq 1 - \inf_{u_1,\dots,u_{n-2}\in\mathbb{R}} \left\{ \mathbb{P}\left[X_1 + \frac{X_2}{2} \geq u_1\right] + \sum_{i=2}^{n-2} \mathbb{P}\left[\frac{X_i + X_{i+1}}{2} \geq u_i\right] + \mathbb{P}\left[\frac{X_{n-1}}{2} + X_n \geq \left(s - \sum_{i=1}^{n-2} u_i\right)\right]\right\}, \tag{26a}$$

$$M_{\xi_n^{\heartsuit}}(s) \leq \inf_{u_1,\dots,u_{n-2}\in\mathbb{R}} \left\{ \mathbb{P}\left[X_1 + \frac{X_2}{2} \leq u_1\right] + \sum_{i=2}^{n-2} \mathbb{P}\left[\frac{X_i + X_{i+1}}{2} \leq u_i\right] + \mathbb{P}\left[\frac{X_{n-1}}{2} + X_n \leq \left(s - \sum_{i=1}^{n-2} u_i\right)\right]\right\}. \tag{26b}$$

Proof. Choose real numbers $u_1, \dots, u_{n-1} \in \mathbb{R}$ such that $\sum_{i=1}^{n-1} u_i = s$. Then define the functions $\hat{f}_{i\,i+1} : \mathbb{R} \to \mathbb{R}$ as

$$\begin{cases} \hat{f}_{12}(x_1, x_2) = 1_{\{x_1 + x_2/2 \geq u_1\}}, \\ \hat{f}_{i\,i+1}(x_i, x_{i+1}) = 1_{\{(x_i + x_{i+1})/2 \geq u_i\}}, i = 2, \dots, n-2, \\ \hat{f}_{n-1\,n}(x_{n-1}, x_n) = 1_{\{x_{n-1}/2 + x_n \geq u_{n-1}\}}, \end{cases}$$

We now prove that $(\hat{f}_{12}, \dots, \hat{f}_{n-1\,n})$ is a feasible choice in (25a). Since the $f_{i\,i+1}$'s are non-negative, it is sufficient to prove that $\sum_{i=1}^{n} x_i \geq s$ implies $\sum_{i=1}^{n-1} f_{i\,i+1}(x_i, x_{i+1}) \geq 1$. Suppose that $\sum_{i=1}^{n-1} f_{i\,i+1}(x_i, x_{i+1}) < 1$. By definition of the $f_{i\,i+1}$'s, this can only happen if $f_{i\,i+1} = 0$ for all $i = 1, \dots, n-1$, i.e. $x_1 + x_2/2 < u_1$,$(x_i + x_{i+1})/2 < u_i, i = 2, \dots, n-2$ and $x_{n-1}/2 + x_n < u_{n-1}$. Summing up all the latter inequalities we obtain that $\sum_{i=1}^{n-1} x_i < \sum_{i=1}^{n-1} u_i = s$. The theorem follows by checking that

$$\sum_{i=1}^{n-1} \int f_{i\,i+1}(x_i, x_{i+1}) dF_{i\,i+1} = \mathbb{P}\left[X_1 + \frac{X_2}{2} \geq u_1\right] + \sum_{i=2}^{n-2} \mathbb{P}\left[\frac{X_i + X_{i+1}}{2} \geq u_i\right] + \mathbb{P}\left[\frac{X_{n-1}}{2} + X_n \geq u_{n-1}\right], \tag{27}$$

and substituting $u_{n-1} = s - \sum_{i=1}^{n-2} x_i$. The proof for $M_{\xi_n^{\heartsuit}}$ is analogous. □

According to Theorem 3, any choice of the values $u_1, \dots u_{n-2}$ in (26a) ((26b)) produces a lower (upper) bound on $m_{\xi_n^{\heartsuit}}(s)$ ($M_{\xi_n^{\heartsuit}}(s)$). At this point, we remark that also the bounds (6) and (19), used in Sections 3 and 4, can be produced by a choice of admissible functions in the corresponding dual representation under a simple marginal system. In order to choose the u_i's in (26), we assume that the bivariate marginals are homogeneous, $F_{i\,i+1} = F_{12}, i = 2, \dots, n-1$ with a parametric form as described by (13) or (16). While Theorem 3 holds for general marginals, some simplifying assumptions are needed to avoid computational difficulties. Define the functions $w, z : \mathbb{R} \to [0, 1]$ as

$$w(u) := \mathbb{P}\left[X_1 + \frac{X_2}{2} \leq u\right], z(u) := \mathbb{P}\left[\frac{X_1 + X_2}{2} \leq u\right],$$

where (X_1, X_2) has distribution F_{12}. From (26) we can then define the bounds

$$\hat{m}_{\xi_n^{\heartsuit}}(s) = \max\left\{\sup_{u_1,\dots,u_{n-2}\geq 0}\left\{w(u_1) + \sum_{i=2}^{n-2} z(u_i) + w\left(s - \sum_{i=1}^{n-2} u_i\right) - n + 2\right\}, 0\right\}, \tag{28a}$$

$$\hat{M}_{\xi_n^{\heartsuit}}(s) = \min\left\{\inf_{u_1,\dots,u_{n-2}\geq 0}\left\{w(u_1) + \sum_{i=2}^{n-2} z(u_i) + w\left(s - \sum_{i=1}^{n-2} u_i\right)\right\}, 1\right\}. \tag{28b}$$

The functions $\hat{m}_{\xi_n^{\heartsuit}}$ and $\hat{M}_{\xi_n^{\heartsuit}}$ represent bounds on (25) in the sense that, for all threshold s, we have

$$\hat{m}_{\xi_n^{\heartsuit}}(s) \leq \mathbb{P}[X_1 + \cdots + X_n < s] \leq \hat{M}_{\xi_n^{\heartsuit}}(s)$$

for all $(X_1, \dots, X_n) \in \mathfrak{F}(F_{12}, F_{23}, \dots, F_{n-1\, n})$, or equivalently

$$\hat{M}^{-1}_{\xi_n^{\heartsuit}}(\alpha) \leq F_S^{-1}(\alpha) \leq \hat{m}^{-1}_{\xi_n^{\heartsuit}}(\alpha). \tag{29}$$

Again, a good choice of the u_i's can be based on the search for local optimizers of (28). A set of u_i's at which the partial derivatives of the objective functions in (28) are zero is given by the following set of equations:

$$\begin{cases} w'\left(\frac{s-(n-3)u_2}{2}\right) = z'(u_2), \\ u_2 = \cdots = u_{n-1}, \\ u_1 = \frac{s-(n-3)u_2}{2}. \end{cases} \tag{30}$$

The first equation has to be solved numerically. Under the model (13) for the bivariate distribution F_{12}, it reads as:

$$\int_0^{\frac{s-(n-3)u_2}{2}} G'_{x_1}(s-(n-3)u_2 - 2x_1)F'_1(x_1)dx_1 = \int_0^{2u_2} G'_{x_1}(2u_2 - x_1)F'_1(x_1)dx_1, \tag{31}$$

where G'_{x_1} and, respectively, F'_1 are the densities of the conditional distribution defined in (14) and, respectively, of the Pareto distribution in (11). Together with the solution of (30), we look for boundary solutions $u_i = 0, i \in \mathcal{G}$ for all possible choice of $\mathcal{G} \in 2^{\{1,\dots,n-2\}}$. For instance, by setting $u_1, \dots, u_{n-2} = 0$, we find the bound $\max\{w(s)-n+2, 0\}$. By canceling a different set of u_i's instead, one has to solve a system of equations analogous to (30) but with the remaining u_i's as variables. Finally, one chooses the set of $u_i' s$ giving the larger bound in (28a) and the smaller bound in (28b), and this for each value of s. Note that the n-dimensional problems (25) are reduced to finding the root of a single equation, a task which is manageable by standard mathematical software. This also means that going to larger dimensions does not increase computational complexity, provided that the portfolio X under study is homogeneous.

In Figure 7 we provide the bounds $\hat{m}^{-1}_{\xi_5^{\heartsuit}}$ and $\hat{M}^{-1}_{\xi_5^{\heartsuit}}$ on the VaR for the sum of five risks under the bivariate Pareto dependence scenario with dependence parameter $\gamma = 1$ and marginal tail parameter $\theta = 2$ (left) and $\theta = 0.9$ (right). Both plots show how the bounds on $F_S^{-1}(\alpha)$ vary over the quantile level α. The value $\mathrm{VaR}^+(\alpha)$, obtained under comonotonicity of the five risks, is also displayed. Figure 7 displays bounds on VaR for a fixed value of γ since, contrary to the equivalent bounds on VaR for the star-like marginal systems studied above, these bounds are much less sensitive to a change in the dependence parameter γ. This is a consequence of the fact that it is not possible to condition on a single marginal as done for the star-like marginal system in Sections 3 and 4. Moreover, the bounds (26) do not contain as a particular case the comonotonic case arising for example from the choice of $\gamma = \infty$. One can obtain this case for the kissing system of marginals only by taking piecewise linear functions as admissible choices in the dual formulations (25). At this point heavy computational issues arise; see Section 7 below. From a computational viewpoint, the kissing marginal system turns out to be much more onerous than the star-like system. For instance, when $n > 3$, bounds in the case of a kissing marginal system cannot be tested accurately as done in Section 5 for the star-like marginal system, since it is not possible to operate an analogous reduction of dimensionality of the LPs to be solved. Useful numerical ranges for sharp bounds then remains out of reach due to heavy computational issues. However, also in these cases, the VaR range (29) improves the corresponding one in the non-overlapping system with the same marginals; see Figure 7.

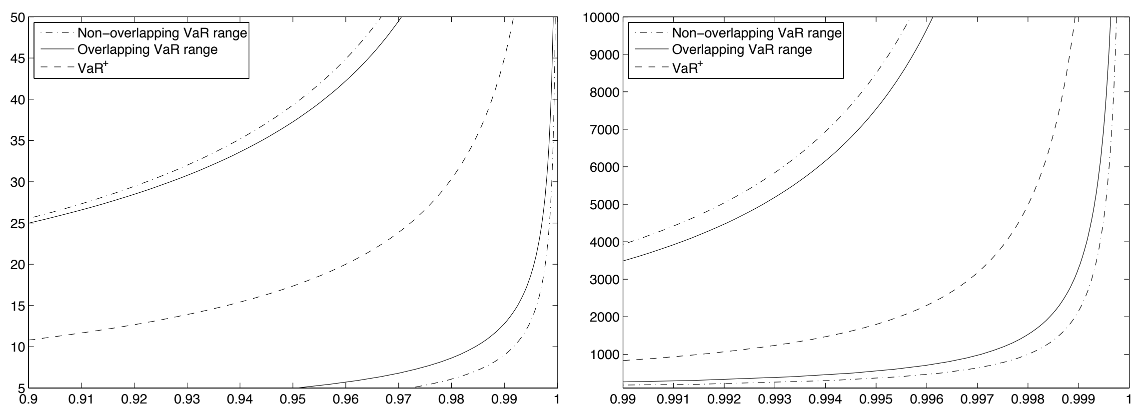

Figure 7: VaR ranges $[\hat{M}^{-1}_{\xi_5^{\heartsuit}}(\alpha), \hat{m}^{-1}_{\xi_5^{\heartsuit}}(\alpha)]$ for the sum of five Pareto(2)-(left) and Pareto(0.9)-(right) distributed risks coupled by a bivariate Pareto copula with dependence parameter $\gamma = 1$. VaR ranges in the non-overlapping marginal system ξ_5 with the same univariate marginals, and $\mathrm{VaR}^+(\alpha)$ are also displayed.

7. Dual bounds and final comments

The bounds on $m_{\xi_n^{\heartsuit}}$ and $M_{\xi_n^{\heartsuit}}$ stated in Theorem 3 have been produced by choosing a particular set of admissible indicator functions in the corresponding dual formulation of the problems, as stated in (25). As we have already remarked, these bounds are not sharp but can be improved if one chooses a more sophisticated set of admissible functions. This technique has been used for instance in Embrechts and Puccetti (2006b) in the definition of so-called *dual bounds*. Though it is theoretically possible to give dual bounds also in the context of overlapping marginals, this technique is here very difficult to apply because of heavy computational issues. Moreover, when dealing with multivariate marginals, the dual function has to be tailored to the specific marginals chosen (see Embrechts and Puccetti (2006c)) and often produces only a poor improvement of the bounds.

The importance given in this paper to the three systems of marginals introduced in Section 2 is justified by the fact that every regular system of marginals can be reduced to the study of these ones. Regularity is a consequence of the absence of *cycles*, e.g. the system $\xi = \{\{1,2\},\{2,3\},\{3,1\}\}$ is *non-regular*. The procedure to reduce a regular system to a sequence of simple, star-like and kissing systems of marginals is described in the proof of Theorem 4 in Rüschendorf (1991a). It would be much more effective to produce a bound using the dual representation given in (25) with specific indicator functions as dual admissible functions. For instance, for the system $\hat{\xi} = \{345, 234, 267, 12, 18\}$ one could choose the admissible functions

$$\begin{cases} \hat{f}_{345}(x_3,x_4,x_5) = 1_{\{x_3/2+x_4/2+x_5 \geq u_1\}}, \\ \hat{f}_{234}(x_2,x_3,x_4) = 1_{\{x_2/3+x_3/2+x_4/2 \geq u_2\}}, \\ \hat{f}_{267}(x_2,x_6,x_7) = 1_{\{x_2/3+x_6+x_7 \geq u_3\}}, \\ \hat{f}_{12}(x_1,x_2) = 1_{\{x_1/2+x_2/3 \geq u_4\}}, \\ \hat{f}_{18}(x_1,x_8) = 1_{\{x_1/2+x_8 \geq u_5\}}, \end{cases}$$

with $\sum_{i=1}^5 u_i = s$, in order to obtain the bound

$$m_{\hat{\xi}}(s) \geq 1 - \inf_{u_1,\dots,u_4 \in \mathbb{R}} \left\{ \mathbb{P}\left[\frac{X_3+X_4}{2} + X_5 \geq u_1\right] + \mathbb{P}\left[\frac{X_2}{3} + \frac{X_3+X_4}{2} \geq u_2\right] \right.$$
$$\left. + \mathbb{P}\left[\frac{X_2}{3} + X_6 + X_7 \geq u_3\right] + \mathbb{P}\left[\frac{X_1}{2} + \frac{X_2}{3} \geq u_4\right] + \mathbb{P}\left[\frac{X_1}{2} + X_8 \geq s - \sum_{i=1}^4 u_i\right] \right\}.$$

We remark again that all the theorems stated in this paper can be analogously stated choosing arbitrary non-negative marginal distributions. The computational complexity for obtaining the corresponding bounds in general depends upon the number of non-identical marginals. In order to find a good choice of the u_i's in these non-homogeneous cases, a multivariate local search algorithm can be useful; see Section 4.4 in Embrechts and Puccetti (2006a) and references therein.

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