Bounds on Value-at-Risk

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The problem

Consider a function $\psi : \mathbb{R}^n \to \mathbb{R}$ and a random vector

$$X := (X_1, \ldots, X_n).$$

Fix marginals dfs $F_i = \mathbb{P}[X_i \leq x], i = 1, \ldots, n$.

X is a portfolio of one-period financial **losses** or insurance **claims**.

The distribution function of the random variable $\psi(X)$ is not completely determined by the F_i's.

Which is the df giving the worst-possible Value-at-Risk (VaR) for the random variable $\psi(X)$?

History of the problem

- Makarov (1981) provided the first result for $n = 2, \psi = +$.
- Frank, Nelsen, and Schweizer (1987) restated Makarov's result, using copulas.
- Independently, Rüschendorf (1982) gave a more elegant proof of the same theorem using duality.
- Williamson and Downs (1990) introduced the use of dependence information.
- Embrechts, Höing, and Juri (2003) gave the most general theorem, stating sharpness of the bound in the presence of information for non-decreasing functions ψ.

The latter paper however contains a gap in the main proof. We revisit the proof and correct the statement of their main result.

Definitions and Preliminaries

Definition 1. *For* $\alpha \in [0,1]$, *the* Value-at-Risk at probability level α *of a random variable* S *is its* α*-quantile, defined as*

$$\mathrm{VaR}_\alpha(S) := \inf\{x \in \mathbb{R} : G(x) \geq \alpha\}.$$

where G *is the df of* S.

Searching for the worst-possible VaR means looking for

$$m_\psi(s) := \inf\{\mathbb{P}[\psi(X) < s] : X_i \backsim F_i, i = 1, \ldots, n\}. \quad (1)$$

Indeed, according to Definition 1, we have

$$\mathrm{VaR}_\alpha(\psi(X)) \leq m_\psi^{-1}(\alpha), \alpha \in [0,1]. \quad (2)$$

Of course, every quantile of $\psi(X)$ can be computed once we know

$$F(x_1, \dots, x_n) = \mathbb{P}[X_1 \leq x_1, \dots, X_n \leq x_n].$$

The latter is determined by the **fixed marginal dfs** and the **copula** of the portfolio.

Definition 2. *A* copula *is any n-dimensional df restricted to $[0,1]^n$ having standard uniform marginals.*

Given a copula C and a set of n marginals $F_1, \dots, F_n$ one can always define a df F on $\mathbb{R}^n$ having these marginals by

$$F(x_1, \dots, x_n) := C(F_1(x_1), \dots, F_n(x_n)). \tag{3}$$

Sklar's theorem states conversely that we can always find a copula C coupling the marginals of a fixed df F through (3).

For example, if the marginals are merged by the copula

$$\Pi : [0,1]^n \to [0,1]; \Pi(u) := \prod_{i=1}^{n} u_i,$$

then the vector X has independent components, and we can calculate the VaR under independence.

Any copula C lies between the lower and upper Fréchet bounds $W, M : [0,1]^n \to [0,1]$;

$$W(u_1, u_2, \ldots, u_n) := (\sum_{i=1}^{n} u_i - n + 1)^+,$$

$$M(u_1, u_2, \ldots, u_n) := \min\{u_1, \ldots, u_n\}, \text{ namely}$$

$$W \leq C \leq M.$$

Denote by μ_C the probability measure on $\mathbb{R}^n$ corresponding to a copula C, and define:

$$\sigma_{C,\psi}(F_1, \ldots, F_n)(s) := \mu_C[\psi(X) < s],$$
$$\tau_{C,\psi}(F_1, \ldots, F_n)(s) :=$$
$$\sup_{x_1,\ldots,x_{n-1} \in \mathbb{R}} C(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), F_n^-(\widehat{\psi_{x_{-n}}}(s))),$$

where $\widehat{\psi_{x_{-n}}}(s) := \sup\{x_n \in \mathbb{R} : \psi(x_{-n}, x_n) < s\}$ for fixed $x_{-n} \in \mathbb{R}^{n-1}$.

Hence we can write our problem as

$$\boldsymbol{m_\psi(s) = \inf\{\sigma_{C,\psi}(F_1, \ldots, F_n)(s) : C \in \mathfrak{C}_n\}},$$

where $\mathfrak{C}_n$ denotes the set of all n-dimensional copulas.

Dependence information

Putting a lower bound on the copula C of the portfolio can be interpreted as having partial information regarding its dependence structure.

$$\begin{aligned} m_{C_L,\psi}(s) &:= \inf\{\sigma_{C,\psi}(F_1,\dots,F_n)(s) : C \geq C_L\} \\ &= \inf\{\mathbb{P}[\psi(X) < s] : X_i \backsim F_i, i = 1,\dots,n, \\ &\quad F \geq C_L(F_1,\dots,F_n)\}. \end{aligned}$$

If $C_L = W$,

we are completely ignorant about the dependence structure of the random vector X.

Main result with partial information

When a lower bound on the copula of our portfolio is assumed,

the problem at hand is fully solved.

Theorem 1. *Let $X = (X_1, \dots, X_n)$ be a random vector on $\mathbb{R}^n$ $(n > 1)$ having marginal distribution functions $F_1, \dots, F_n$ and copula C. Assume that there exists a copula C_L such that $C \geq C_L$. If $\psi : \mathbb{R}^n \to \mathbb{R}$ is non-decreasing in each coordinate, then for every real s we have*

$$\sigma_{C,\psi}(F_1, \dots, F_n)(s) \geq \tau_{C_L,\psi}(F_1, \dots, F_n)(s). \tag{4}$$

Theorem 2. *Assume ψ is also right-continuous in its last argument. Define the copula $C_t : [0,1]^n \to [0,1]$;*

$$C_t(u) := \begin{cases} \max\{t, C_L(u)\} & \textit{if } u = (u_1, \dots, u_n) \in [t,1]^n; \\ \min\{u_1, \dots, u_n\}, & \textit{otherwise,} \end{cases}$$

where $t = \tau_{C_L,\psi}(F_1, \dots, F_n)(s)$. Then

$$\sigma_{C_t,\psi}(F_1, \dots, F_n)(s) = t. \tag{5}$$

Some remarks on Theorem 1 and Theorem 2

- The proof of Theorem 2 provided in our paper is long and laborious, but the elegant proof of the same theorem given in Embrechts, Höing, and Juri (2003) contains a gap.
- Moreover, the correct statement of the theorem requires the definition of the operator σ given in our paper.
- For applications of Theorem 2, including how to calculate numerically the bound $\tau_{C_L,\psi}(F_1,\ldots,F_n)(s)$, Embrechts, Höing, and Juri (2003) is an excellent source.

Main result without information on dependence

Consider now

$$C_L = W.$$

The fact that the function W is not a copula for $n \geq 3$ causes problems.

Though the *standard* bound holds in arbitrary dimensions, when

$$n > 2 \text{ and } C_L = W,$$

it may fail to be sharp.

Mutually exclusive risks

Consider a portfolio of **mutually exclusive risks**, i.e. risks that can be positive at most one at a time.

In this specific case, of actuarial interest, we have that

the lower Fréchet bound W is a proper df

and the *standard* bound is then sharp for arbitrary n, even in the no-info scenario.

A dual formulation

In the no-info scenario, it is convenient to express our problem (1) by a duality result given in Rüschendorf (1982):

$$\inf\{\mathbb{P}[\psi(X) < s] : X_i \sim F_i, i = 1, \ldots, n\} =$$

$$1 - \inf\{\sum_{i=1}^{n} \int f_i dF_i : f_i \text{ are bounded functions on } \mathbb{R} \text{ s.t.}$$

$$\sum_{i=1}^{n} f_i(x_i) \geq 1_{[s,+\infty)}(\psi(x_1, \ldots, x_n))$$

$$\text{for all } x_i \in \mathbb{R}, i = 1, \ldots, n\}.$$

Some remarks on the dual problem

The dual optimization problem is very difficult to solve.

Explicit results are known only for uniformly or binomially distributed rvs.

Unfortunately, the solution in the case of the sum of uniform marginals does not work in the general case, where

it may depend upon the marginals chosen.

This is much in contrast to the case of the copula C_t, which gave the solution for all choices of $F_1, \dots, F_n$.

Main Result

We use the dual to provide a bound which is better (i.e. $\geq$) than the *standard* one.

Theorem 3. *Let F be a continuous df with non-negative support. If $F_i = F, i = 1, \ldots, n$, then for every $s \geq 0$*

$$m_+(s) \geq 1 - n \inf_{r \in [0, s/n)} \frac{\int_r^{s-(n-1)r} 1 - F(x)dx}{s - nr}. \tag{6}$$

Remark 1. (i) For $n = 2$, this theorem gives the sharp bound already stated.

(ii) This *dual* bound is strictly greater than the standard bound for most dfs and thresholds of interest.

(iii) For $n > 2$, this bound can be easily calculated numerically.

The assumptions under which Theorem 3 is valid are considerable with respect to the setting of the previous sections,

but still consistent with most dfs F and thresholds s of actuarial/financial interest.

Under such assumptions, it is easy to show that, for s large enough, the *standard* bound reduces to

$$\boldsymbol{\tau_{W,+}(F,\ldots,F)(s) = [nF(s/n) - n + 1]^+.}$$

Applications

How can we compare the quality of the *dual* bound with respect to the *standard* bound generally used in the literature?

We define the two dfs $\underline{F}_N, \overline{F}_N$ by

$$\underline{F}_N(x) := \frac{1}{N}\sum_{i=1}^{N} 1_{[q_r,+\infty)}(x),$$

$$\overline{F}_N(x) := \frac{1}{N}\sum_{i=0}^{N-1} 1_{[q_r,+\infty)}(x),$$

the jump points $q_0, \ldots, q_N$ being the quantiles of F defined by $q_0 := \inf \text{supp}(F)$, $q_N := \sup \text{supp}(F)$ and $q_r := F^{-1}(\frac{r}{N}), r = 1, \ldots, N-1$.

It is straightforward that

$$\underline{F}_N \leq F \leq \overline{F}_N,$$

from which it follows that

$$\underline{m}_+(s) \leq m_+(s) \leq \overline{m}_+(s), \tag{7}$$

where $\underline{m}_+(s)$ and $\overline{m}_+(s)$ are naturally defined as:

$$\underline{m}_+(s) := \inf\{\mathbb{P}[\sum_{i=1}^{n} X_i < t] : X_i \backsim \underline{F}_N, i = 1, \dots, n\}$$

$$\overline{m}_+(s) := \inf\{\mathbb{P}[\sum_{i=1}^{n} X_i < t] : X_i \backsim \overline{F}_N, i = 1, \dots, n\}.$$

Given that $\underline{F}_N$ is a (possibly defective) discrete df, $\underline{m}_+(s)$ is the solution of the following LP:

$$\underline{m}_+(s) = \min_{p_{j_1,\dots,j_n}} \sum_{j_1=1}^{N} \cdots \sum_{j_n=1}^{N} p_{j_1,j_2,\dots,j_n} 1_{(-\infty,t)}(\sum_{i=1}^{n} q_{j_i}) \text{ subject to}$$

$$\begin{cases} \sum_{j_2=1}^{N} \sum_{j_3=1}^{N} \cdots \sum_{j_n=1}^{N} p_{j_1,\dots,j_n} & = \frac{1}{N} \quad j_1 = 1,\dots,N, \\ \sum_{j_1=1}^{N} \sum_{j_3=1}^{N} \cdots \sum_{j_n=1}^{N} p_{j_1,\dots,j_n} & = \frac{1}{N} \quad j_2 = 1,\dots,N, \\ \quad \dots, & \\ \sum_{j_2=1}^{N} \sum_{j_3=1}^{N} \cdots \sum_{j_{n-1}=1}^{N} p_{j_1,\dots,j_n} & = \frac{1}{N} \quad j_n = 1,\dots,N, \\ 0 \leq p_{j_1,\dots,j_n} \leq 1 & \qquad j_i = 1,\dots,N, \\ & \qquad i = 1,\dots,n. \end{cases}$$

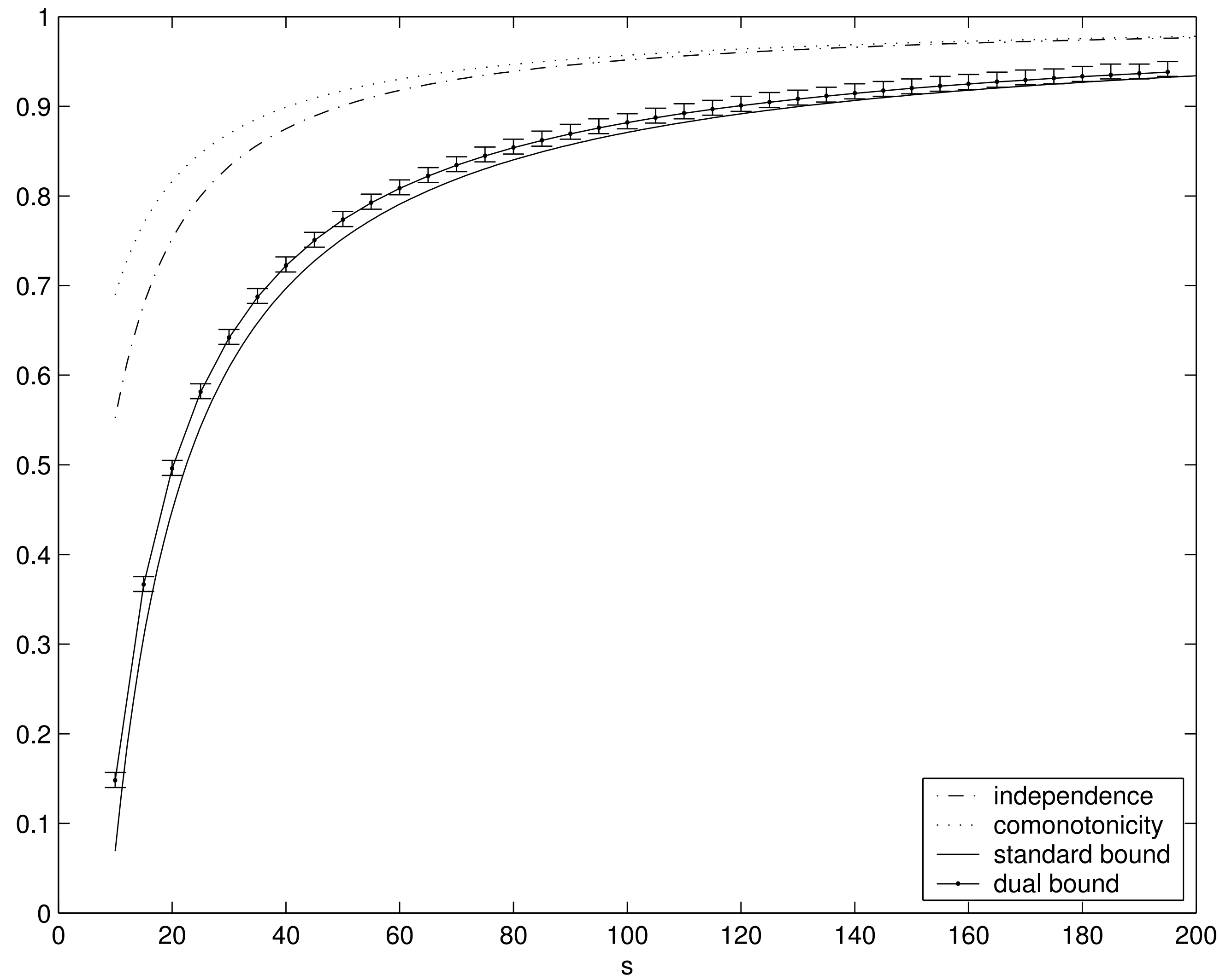

Figure 1: Range for $\mathbb{P}[X_1 + X_2 + X_3 < s]$ for a Pareto(1.5,1)-portfolio.

Some remarks on this plot

- The ranges for the true solutions have been calculated solving the two LPs with $N = 180$ and by using ILOG CPLEX® C Callable Libraries (a powerful tool).
- Switching to $n = 4$ drastically lowers the quality of approximation to $N < 60$.
- The worst VaR does not occur under the comonotonicity assumption, i.e. VaR is not a coherent measure of risk.

Non-coherence of VaR for $\mathcal{N}(0,1)$-marginals

Let assume X_1, X_2 are $\mathcal{N}(0,1)$-distributed. Since the normal distribution is symmetric, we can set

$$X_2 = -X_1$$

and obtain $\mathbb{P}[X_1 + X_2 = 0] = 1$, i.e. $m_+(0) = 0$.

The copula yielding the worst-possible VaR for the sum is then the **countermonotonic** copula W, under which X_2 is a.s. a non-increasing function of X_1.

Note that assuming comonotonicity between the marginals gives

$$\sigma_{M,+}(\mathcal{N}(0,1), \mathcal{N}(0,1))(0) = 1/2 > 0.$$

	$\mathrm{VaR}_\alpha(\sum_{i=1}^3 X_i)$, *exact*		$\mathrm{VaR}_\alpha(\sum_{i=1}^3 X_i)$, *upper bound*	
α	independence	comonotonicity	dual	standard
0.90	7.54	8.85	14.44	15.38
0.95	9.71	12.73	19.50	20.63
0.99	16.06	25.16	35.31	37.03
0.999	29.78	53.99	69.98	73.81

Table 1: Range for VaR for a Log-Normal(-0.2,1)-portfolio.

	$\mathrm{VaR}_\alpha(\sum_{i=1}^{10} X_i)$		$\mathrm{VaR}_\alpha(\sum_{i=1}^{100} X_i)$		$\mathrm{VaR}_\alpha(\sum_{i=1}^{1000} X_i)$	
α	dual	standard	dual	standard	dual	standard
0.90	0.669	1.485	11.039	149.850	150.162	14998.500
0.95	1.353	2.985	22.227	229.850	301.823	29998.500
0.99	2.985	14.985	111.731	1499.850	1515.111	149998.500
0.999	68.382	149.985	1118.652	14999.850	15164.604	1499998.500

Table 2: Upper bounds for $\mathrm{VaR}_\alpha(\sum_{i=1}^{n} X_i)$ of three Pareto(1.5,1) portfolios of different dimensions. Data in thousands.

Conclusions

The worst-possible VaR for a non-decreasing function of dependent risks can be calculated when:

- some information on the copula of the portfolio is provided
- the portfolio is two-dimensional

When no dependency information is given, the problem gets much more complicated and we provide a new bound which we prove to be better than the standard one generally used in the literature.

Acknowledgements

The authors would like to thank Ludger Rüschendorf and Andrea Höing for helpful discussions.

The second author would like to thank the Department of Informatics, University of Pisa, for use of licensed software.

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