Aggregating Risk Capital, with an Application to Operational Risk

2007 version

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The present talk is mainly based on the following papers:

JMVA Embrechts, P. and G. Puccetti (2006). Bounds for functions of multivariate risks. *J. Mult. Analysis*, *97*(2), 526–547.

F&S Embrechts, P. and G. Puccetti (2006b). Bounds for functions of dependent risks. *Finance Stoch. 10*(3), 341–352.

GRIR Embrechts, P. and G. Puccetti (2006c). Aggregating risk capital, with an application to operational risk. *Geneva Risk. Insur. Rev.*, *31*(2), 71–90.

The problem at hand

On some probability space $(\Omega, \mathfrak{A}, \mathbb{P})$, consider a random vector

$$X := (X_1, \dots, X_n)$$

of n one-period financial losses or insurance claims,

and fix its marginal dfs $F_1, \dots, F_n$.

The **joint** df of the random vector X
is **not** completely determined by the F_i's.

There are infinitely many distributions for the vector X which are consistent with the initial choice of the marginals.

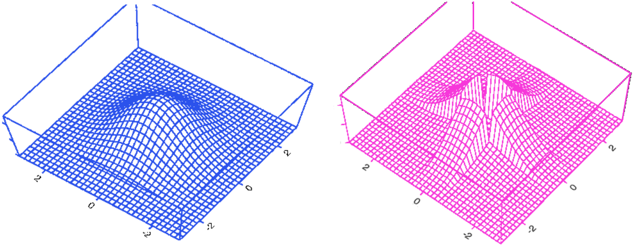
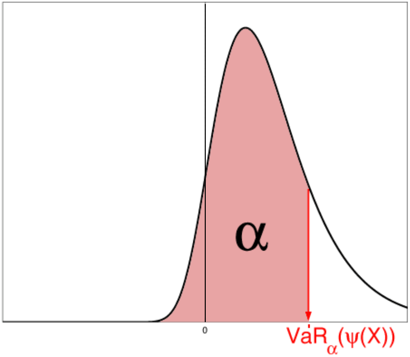


Figure: Two bivariate dfs having $N(0,1)$-marginals and the same correlation

Given an aggregating function $\psi : \mathbb{R}^n \to \mathbb{R}$, which is the df giving the **worst-possible Value-at-Risk (VaR)** for $\psi(X)$?

Value-at-Risk for the aggregate loss $\psi(X)$

For $\alpha \in [0,1]$, the **Value-at-Risk** at probability level α for $\psi(X)$ is its α-quantile, defined as $\mathrm{VaR}_\alpha(Y) := G^{-1}(\alpha)$, where G is the df of $\psi(X)$.



Searching for the worst-possible VaR means looking for

$$m_\psi(s) := \inf\{\mathbb{P}[\psi(X) < s] : X_i \backsim F_i, i = 1, \ldots, n\}, s \in \mathbb{R}.$$

Indeed, according to the definition of VaR, we have

$$\mathrm{VaR}_\alpha(\psi(X)) \le m_\psi^{-1}(\alpha), \alpha \in [0, 1].$$

The distribution of $\psi(X)$ is uniquely defined through the marginal dfs **and** their interdependence, which can be modeled by the concept of **copula**.

A **copula** is any n-dimensional df restricted to $[0,1]^n$ having standard uniform marginals.

Given a copula C and a set of n marginals $F_1, \ldots, F_n$ one can always define a df F on $\mathbb{R}^n$ having these marginals by

$$F(x_1, \ldots, x_n) := C(F_1(x_1), \ldots, F_n(x_n)). \tag{1}$$

Sklar's theorem states conversely that we can always find a copula C coupling the marginals of a fixed df F trough (1).

- **independent** marginals ⇔ *independence copula*

$$\Pi : [0,1]^n \to [0,1]; \Pi(u_1, \ldots, u_n) := \prod_{i=1}^{n} u_i$$

- **comonotonic** marginals ⇔ *upper Fréchet bound*

$$M : [0,1]^n \to [0,1]; M(u_1, \ldots, u_n) := \min\{u_1, \ldots, u_n\}$$

- **countermonotonic** marginals ⇔ *lower Fréchet bound* (which is a copula only for $n = 2$)

$$W : [0,1]^n \to [0,1]; W(u_1, \ldots, u_n) := \left[\sum_{i=1}^{n} u_i - n + 1\right]^+$$

$$W \leq C \leq M, \text{ for any copula } C$$

Dependence information

By Sklar's theorem, our problem can be equivalently expressed as

$$m_\psi(s) = \inf\left\{\mathbb{P}_C\left[\psi(X) < s\right] : C \in \mathfrak{C}_n\right\},$$

where $\mathfrak{C}_n$ denotes the set of all n-dimensional copulas.

If we have partial information regarding the dependence structure of our portfolio of risks, we put a lower bound C_L on the copula C.

$$m_{C_L,\psi}(s) := \inf\left\{\mathbb{P}_C\left[\psi(X) < s\right] : C \geq C_L\right\}.$$

If $C_L = W$, then we come back to $m_\psi(s)$

Main Result with Dependence information

When:

- a lower copula-bound C_L on the portfolio copula C is assumed (also $C_L = W$ for $n \geq 3$ is used throughout)
- $n = 2$
- ψ is continuous and non-decreasing in each argument

the problem at hand is fully solved ($m_{C_L,\psi}(s)$ is found) by Williamson and Downs (1990).

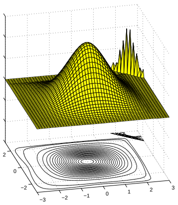
Note that Embrechts et al. (2003) state a lower bound for $m_{C_L,\psi}(s)$ also for $n \geq 3$ but this bound is not sharp .

Theorem 1 (STANDARD BOUND for the sum)

Let $X = (X_1, \ldots, X_n)$ be a random vector on $\mathbb{R}^n$ $(n > 1)$ having marginal dfs $F_1, \ldots, F_n$ and copula C. Assume that there exists a copula C_L such that $C \geq C_L$. Then we have

$$\mathrm{VaR}_\alpha(X_1 + \cdots + X_n) \leq \tau_{C_L,+}(F_1, \ldots, F_n)^{-1}(\alpha), \alpha \in [0,1], \text{ where} \tag{2}$$

$$m_{C_L,+}(s) \geq \tau_{C_L,+}(F_1, \ldots, F_n)(s) := \sup_{x_1,\ldots,x_{n-1} \in \mathbb{R}} C_L(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), F_n^-(s - \sum_{i=1}^{n-1} x_i)).$$



If $n = 2$, there exists a copula C_α (see picture with $C_L = W$) under which (2) holds with $=$.

Important Remark on the Theorem

The **standard bound** (stated in Theorem 1)
still holds in arbitrary dimensions but when

$$n \geq 3$$

it fails to be sharp.

A priori assumptions such as $C_L = \Pi$ may lead to a critical **undervaluation** of the portfolio risk
(the componentwise ordering $\geq$ defined in $\mathfrak{C}_n$ is not complete).

Therefore, in the following we will restrict to the case in which we do not assume any information on the copula of the portfolio, i.e.
$C_L = W$.

In the no-information scenario

$$C_L = W,$$

it is convenient to express our problem using a duality result given in Rüschendorf (1982):

$$\begin{aligned} m_\psi(s) &= \inf\{\mathbb{P}[\psi(X) < s] : X_i \backsim F_i, i = 1, \ldots, n\} \\ &= 1 - \inf\Bigg\{ \sum_{i=1}^{n} \int f_i dF_i : f_i \in L^1(F_i), i \in N \text{ s.t.} \\ &\qquad \sum_{i=1}^{n} f_i(x_i) \geq 1_{[s,+\infty)}(\psi(x)) \text{ for all } x \in \mathbb{R}^n \Bigg\}. \end{aligned}$$

Some remarks on the dual problem

- The dual optimization problem seems to be very difficult to solve;
- Explicit results are known only for uniformly or binomially distributed risks;
- Unfortunately, the solution in the case of the sum of uniform marginals does not work in the general case.

We use the dual problem to provide a **dual** bound which is better (i.e. $\geq$) than the *standard* one when $n \geq 3$.

Theorem 2 (DUAL BOUND, [F&S])

Let F be a non-negative, continuous df.
If $F_i = F, i = 1, \dots, n$, then for every $s \geq 0$,

$$m_+(s) \geq 1 - n \inf_{r \in [0, s/n)} \frac{\int_r^{s-(n-1)r} (1 - F(x))dx}{s - nr}.$$

- For $n = 2$ this theorem gives the sharp standard bound
- This *dual* bound is strictly better than the standard bound for most dfs and thresholds s of interest
- Extensions are stated for:
  1. non-homogeneous portfolios; see [GRIR]
  2. portfolios of vectors; see [JMVA]

Application 1: Quality of the bounds (homogeneous portfolios)

Under the assumptions of Theorem 2 (**homogeneous portfolios**), it is easy to show that, for s large enough, the **standard** bound reduces to

$$\tau_{W,+}(F,\ldots,F)(s) = [nF(s/n) - n + 1]^+.$$

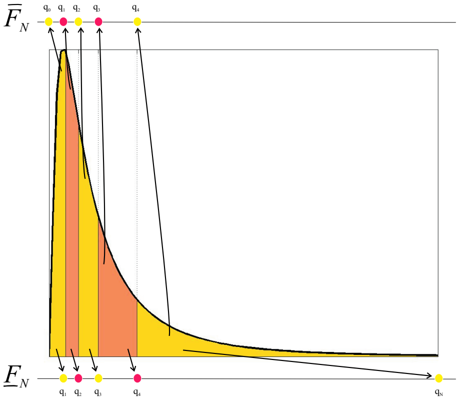
Moreover, the **dual** bound can be easily calculated numerically also for large portfolios ($n = 100000$, say).

How can we compare the quality of the dual bound with respect to the standard bound?

A df F can be bounded from above and from below by the two discrete dfs

$$\underline{F}_N(x) := \frac{1}{N}\sum_{i=1}^{N} 1_{[q_r,+\infty)}(x) \text{ and } \overline{F}_N(x) := \frac{1}{N}\sum_{i=0}^{N-1} 1_{[q_r,+\infty)}(x),$$

the jump points $q_0, \ldots, q_N$ being the quantiles of F.



It is straightforward that

$$\underline{F}_N \leq F \leq \overline{F}_N,$$

from which the following easily follows

$$\underline{m}_+(s) \leq m_+(s) \leq \overline{m}_+(s),$$

where $\underline{m}_+(s)$ and $\overline{m}_+(s)$ are naturally defined as:

$$\underline{m}_+(s) := \inf\left\{\mathbb{P}\left[\sum_{i=1}^n X_i < t\right] : X_i \backsim \underline{F}_N, i = 1, \ldots, n\right\},$$
$$\overline{m}_+(s) := \inf\left\{\mathbb{P}\left[\sum_{i=1}^n X_i < t\right] : X_i \backsim \overline{F}_N, i = 1, \ldots, n\right\}.$$

$\underline{m}_+(s)$ and $\overline{m}_+(s)$ can be found by solving two Linear Problems (LPs).

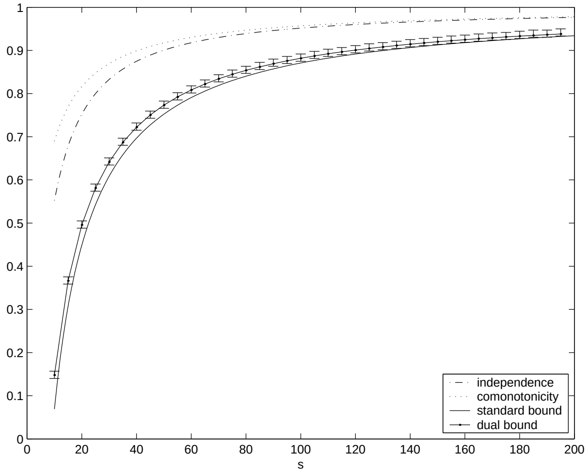


Figure: Range for $\mathbb{P}[X_1 + X_2 + X_3 < s]$ for a Pareto(1.5,1)-portfolio

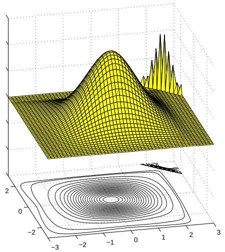
Some remarks on this plot

- Note that independence and comonotonicity curves cross.
- Numerical approximation is $O(1/N)$.
- Increasing to $n = 5$ drastically lowers the quality of approximation to $N < 50$.
- The ranges for $m_+(s)$ have been calculated solving the two LPs with $N = 180$ and using ILOG CPLEX® C Callable Libraries (a powerful tool).
- The worst VaR does not occur under the comonotonicity assumption, i.e. VaR is not a coherent measure of risk.

Non-coherence of VaR

$$\mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2) < \mathrm{VaR}_\alpha\,(X_1 + X_2)$$

- X_1, X_2 independent but very skew
- X_1, X_2 independent but very heavy-tailed
- $X_1, X_2 \backsim N(0,1)$ but special dependence (see again picture).



Bounds on Value-at-Risk

	$\mathrm{VaR}_\alpha(\sum_{i=1}^3 X_i)$, *exact*		$\mathrm{VaR}_\alpha(\sum_{i=1}^3 X_i)$, *upper bound*	
α	independence	comonoton.	dual	standard
0.90	7.54	8.85	14.44	15.38
0.95	9.71	12.73	19.50	20.63
0.99	16.06	25.16	35.31	37.03
0.999	29.78	53.99	69.98	73.81

Table: Range for VaR for a Log-Normal(-0.2,1)-portfolio.

Bounds on Value-at-Risk (large portfolios!)

	$\mathrm{VaR}_\alpha(\sum_{i=1}^{10} X_i)$		$\mathrm{VaR}_\alpha(\sum_{i=1}^{100} X_i)$		$\mathrm{VaR}_\alpha(\sum_{i=1}^{1000} X_i)$	
α	dual	standard	dual	standard	dual	standard
0.90	0.669	1.485	11.039	149.850	150.162	14998.500
0.95	1.353	2.985	22.227	229.850	301.823	29998.500
0.99	2.985	14.985	111.731	1499.850	1515.111	149998.500
0.999	68.382	149.985	1118.652	14999.850	15164.604	1499998.500

Table: Upper bounds for $\mathrm{VaR}_\alpha(\sum_{i=1}^{n} X_i)$ of three Pareto portfolios of different dimensions. Data in thousands.

Application 2: Aggregation of OpRisk losses

OpRisk The risk of losses resulting from inadequate or failed **internal processes**, **people** and **systems**, or **external events**. Included is **legal risk**, excluded are strategic/business and reputational risk.

Under the New Basel Capital Accord (**Basel II**) banks are required to set aside capital for the specific purpose of offsetting OpRisk.

Recent QISs:

- Basel Committee QIS; see Moscadelli (2004), 40.000+ obs
- Federal Reserve Bank of Boston Loss-Date Collection Exercise; see Dutta and Perry (2006), 50.000+ obs.

OpRisk pillar 1 issue: Loss Distribution Approach (LDA)

- Operational losses $L_{i,j}$ are separately modeled in **eight business lines** (rows) and by **seven risk types** (columns) in the 56-cell Basel matrix
- Marginal risks have **non-homogeneous** distributions
- Pillar 1 in LDA based on $\mathrm{VaR}^{1\text{ year}}_{0.999}$, i.e. a **1 in 1000 year event**
- Various approaches are possible/allowed (Basel II spirit)
- Several statistical issues related to OpRisk data: loss frequencies, severities, correlation/diversification effects, lower truncation, use of internal/external/expert data, bottom-up vs top-down
- Insurance (up to 20%) is allowed

In standard practice (see Moscadelli (2004)), the OR capital charge can be calculated aggregating risks BL-wise yielding capital estimates $\mathrm{VaR}_{1.}, \ldots, \mathrm{VaR}_{8.}$

	RT_1	$\ldots$	RT_j	$\ldots$	RT_7		
BL_1						$\rightarrow$	$\mathrm{VaR}_{1.}$
$\vdots$						$\vdots$	$\vdots$
BL_i			$L_{i,j}$			$\rightarrow$	$\mathrm{VaR}_{i.}$
$\vdots$						$\vdots$	$\vdots$
BL_8						$\rightarrow$	$\mathrm{VaR}_{8.}$

Finally calculate (as indicated in Basel II) the **comonotonic value**:

$$\mathrm{VaR}^R = \sum_{i=1}^{8} \mathrm{VaR}_{i.}$$

How reliable is this procedure?

OpRisk Standard/Dual Bounds

Problem:

Denuit et al. (1999) remark that, contrary the homogeneous case, the *standard bound* can rarely be calculated explicitly when the marginal risks are **non-homogeneous**.

Solutions contained in [GRIR]:

- easy computation of standard bounds, independently from the dimension of the portfolio, and for all continuous-marginal portfolios.
- calculation of the dual bounds, by means of a sophisticated optimization algorithm.

Application 2

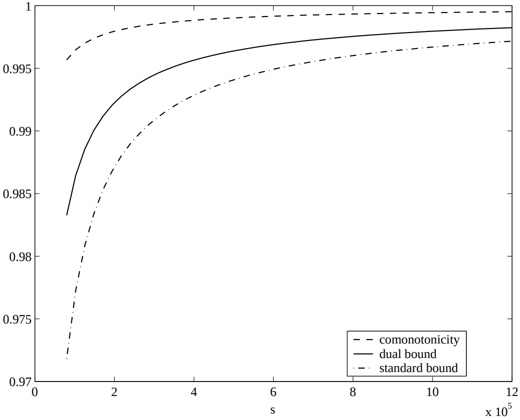


Figure: Bounds on $\mathbb{P}[\sum_{i=1}^{8} \mathrm{BL}_i < s]$ using the OR portfolio given in Moscadelli (2004), together with the comonotonic scenario.

α	Basel II value	dual bound	standard bound
0.99	2.8924×10^4	1.4778×10^5	2.6950×10^5
0.995	6.7034×10^4	3.3922×10^5	6.1114×10^5
0.999	4.8347×10^5	2.3807×10^6	4.1685×10^6
0.9999	8.7476×10^6	4.0740×10^7	6.7936×10^7

Table: Range for $\mathrm{VaR}_\alpha\left(\sum_{i=1}^{8} \mathrm{BL}_i\right)$ for the OR portfolio given in Moscadelli (2004).

- For $n = 8$ no numerical range around dual curve
- The dual bound is prudential, more realistic and economically advantageous with respect to the standard one.
- Though several studies consider the comonotonic Basel II value as over-conservative, there is no mathematical reason to drop the worst-case bounds if no dependence assumptions on the portfolio are explicitly made.

Recall that $\mathrm{VaR}^R = \sum_{i=1}^{8} \mathrm{VaR}_{i.}$
The OR capital charge could be calculated aggregating risks also RT-wise, yielding the capital estimate $\mathrm{VaR}^C = \sum_{j=1}^{7} \mathrm{VaR}_{.j}$

	RT_1	$\dots$	RT_j	$\dots$	RT_7		
BL_1						$\rightarrow$	$\mathrm{VaR}_{1.}$
$\vdots$						$\vdots$	$\vdots$
BL_i			$L_{i,j}$			$\rightarrow$	$\mathrm{VaR}_{i.}$
$\vdots$						$\vdots$	$\vdots$
BL_8						$\rightarrow$	$\mathrm{VaR}_{8.}$
	$\downarrow$	$\dots$	$\downarrow$	$\dots$	$\downarrow$		
	$\mathrm{VaR}_{.1}$	$\dots$	$\mathrm{VaR}_{.j}$	$\dots$	$\mathrm{VaR}_{.7}$		

- $\mathrm{VaR}^C \neq \mathrm{VaR}^R$?
- Which factors does $\Delta := \mathrm{VaR}^C - \mathrm{VaR}^R$ depend upon?

A *toy model* approach

- Simplified 2×3 Basel matrix
- $L_{i,j} \sim$ Pareto(4) for all $i = 1, 2, j = 1, 2, 3$
- Dependence among the $L_{i,j}$ is modeled by a six-dimensional **Gumbel copula** with parameter θ
- the Gumbel copula is symmetric and has Gumbel projections, losses are identically distributed

$$\downarrow$$

The only asymmetry in OR aggregation is caused by the fact that the Basel matrix is not square

Conclusions from the toy model approach

θ		VaR^C	VaR^R	Δ
1.00	independence	18.3135	14.4560	3.8575
1.25	dependence	23.7057	22.6	1.1057
1.50	higher dependence	25.4817	24.8	0.6817
$+\infty$	comonotonicity	27.7404	27.7404	0

- Large differences between the two VaR-aggregation methods.
- Starting from a maximum in the independence set-up, Δ becomes smaller as the strength of dependence increases.
- Under the comonotonic assumption, $\Delta = 0$ due to VaR additivity.

Conclusions

- The worst-possible VaR for a non-decreasing function of dependent risks can be calculated when the portfolio is **two-dimensional**.
- When dealing with more than two risks, the problem gets much more complicated and we provide a **dual** bound which we prove to be better than the standard one generally used in the literature.
- OpRisk VaR-aggregation leads to problems and diversification effects have to be handled with care.

Extensions (more research is needed!)

- Exact VaR bounds when $n > 2$, calculation of bounds for other portfolio functions ψ
- Basel II has some issues to solve (2008+)
- Problems of scaling when fixing marginal dfs (Market + Credit + Op Risk).
- For a textbook treatment, see

Visit the book zone: `www.ma.hw.ac.uk/~mcneil/book/index.html`

For Further Reading I

Denuit, M., C. Genest, and É. Marceau (1999). Stochastic bounds on sums of dependent risks. *Insurance Math. Econom. 25*(1), 85–104.

Dutta, K. and J. Perry (2006). A tale of tails: an empirical analysis of loss distribution models for estimating operational risk capital *Working Papers*, Federal Reserve Bank of Boston

Embrechts, P., A. Höing, and A. Juri (2003). Using copulae to bound the Value-at-Risk for functions of dependent risks. *Finance Stoch. 7*(2), 145–167.

Embrechts, P., A. Höing, and G. Puccetti (2005). Worst VaR scenarios *Insurance Math. Econom.*, *37*(1), 115–134.

Moscadelli, M. (2004). The modelling of operational risk: experience with the analysis of the data collected by the Basel Committee. Preprint, Banca d'Italia.

Rüschendorf, L. (1982). Random variables with maximum sums. *Adv. in Appl. Probab. 14*(3), 623–632.

Williamson, R. C. and T. Downs (1990). Probabilistic arithmetic. I. Numerical methods for calculating convolutions and dependency bounds. *Internat. J. Approx. Reason. 4*(2), 89–158.