

# Additivity properties for Value-at-Risk under Archimedean dependence and heavy-tailedness

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## Abstract

Mainly due to new capital adequacy standards for banking and insurance, an increased interest exists in the aggregation properties of risk measures like Value-at-Risk (VaR). We show how VaR can change from sub- to superadditivity depending on the properties of the underlying model. Mainly, the switch from a finite to an infinite mean model gives a completely different asymptotic behaviour. Our main result proves a conjecture made in Barbe et al. [3].

*Keywords:* Value-at-Risk, subadditivity, dependence structure, Archimedean copula, aggregation.

## 1 Introduction

Based on the regulatory framework for banking and insurance supervision, the financial industry has to come up with regulatory capital for several risk categories (market, credit, insurance underwriting, claims reserving, operational, ...). An overview of the underlying issues and further references can be found in McNeil et al. [17]. The quantile based Value-at-Risk (VaR) is the main risk measure used in industry to quantify regulatory capital. For instance, market risk has to be measured at a 99%-VaR with a 10-day holding period, whereas for operational risk this is 99.9% and 1 year. Hence the regulatory capital for operational risk corresponds to a one-in-1000-year event. In several papers the strengths and weaknesses of VaR as a risk measure have been discussed; in Artzner et al. [2] and

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in various subsequent papers VaR is criticized for not being subadditive; see McNeil et al. [17] for more details. However, the subadditivity axiom for risk measures is not flawless either; see for example Rootzén-Klüppelberg [27], Dhaene et al. [7], Daniélsson et al. [5] or Heyde et al. [12]. The latter paper also contains a critical discussion on robustness of risk measures and suggests for instance to replace tail conditional expectation (which is a subadditive risk measure) by the more robust tail conditional median which is simply VaR at a higher confidence level.

Though the present paper can be read context free, the main motivation comes from recent discussions around the measurement of operational risk. Current insurance regulation gives no clear guidance for quantification of operational risk. Larger international banks, complying with international banking regulation, have to come up with a one-in-1000-year VaR estimate under the so-called loss distribution approach (LDA). There is an ongoing intensive discussion between practitioners, regulators and academics concerning the appropriate way to estimate such a 99.9% quantile based capital charge. Influential papers from the regulatory side concerning several quantitative impact studies are Moscadelli [20], de Fontnouvelle et al. [10] and Dutta-Perry [8]. The Moscadelli paper relies heavily on extreme value theory (EVT) based modelling and also uses the tail conditional median as a risk measure. The Dutta-Perry paper introduces the g-and-h distribution as a parametric model for operational risk severity data. These authors claim that EVT methods may occasionally lead to unrealistically high capital charges. This is mainly due to fitted infinite mean models; see Nešlehová et al. [23], Degen et al. [6] and Böcker-Klüppelberg [4] for a discussion in the context of operational risk. Daniélsson et al. [5] discuss this issue from the point of view of multivariate regular variation.

Our paper contributes to this discussion by looking at dependent data using copula methodology. Our main results are Theorems 2.3 and 2.5 which qualitatively highlight the difficulties caused by switching from a finite to an infinite mean model. Theorem 2.3 solves an open problem posed in Barbe et al. [3].

**Organisation.** In Section 2 we define the model, state the main convergence results (see Proposition 2.2 and Corollary 2.4), and provide a regime switching picture in Theorem 2.5. These statements are based on our main theorem, Theorem 2.3, which describes the

basic behaviour of the limiting constants introduced in Proposition 2.2. In Section 3 we present general properties of asymptotic VaR behaviour. Finally, the proofs of the main results are given in Section 4.

## 2 A multivariate model and main results

Consider loss random variables  $X_1, \dots, X_n$ ,  $n \geq 2$ , defined on some probability space  $(\Omega, \mathcal{F}, P)$ . Our main focus is on the calculation of  $\text{VaR}_p(X_1 + \dots + X_n)$ , where  $p$  is a given confidence level typically close to 1. The quantile risk measure  $\text{VaR}_p(X)$  for a random variable  $X$  and  $p \in (0, 1)$  is defined as:

$$\text{VaR}_p(X) = \inf\{t \in \mathbb{R} : P(X > t) \leq 1 - p\}. \quad (2.1)$$

We do not assume independence between the loss random variables. For modelling dependence in loss data, the theory of Archimedean copulas is particularly useful; see McNeil et al. [17] for details, definitions and further references.

We assume that  $\psi : [0, 1] \rightarrow [0, \infty]$  is the generator of the Archimedean copula

$$C^\psi(x_1, \dots, x_n) = \psi^{-1} \left( \sum_{i=1}^n \psi(x_i) \right). \quad (2.2)$$

Note that in order to generate a true copula,  $\psi$  has to satisfy certain growth and regularity conditions, in particular,  $\psi$  needs to be completely monotone to generate a copula for any dimension  $n \geq 2$ ; see Kimberling [15], Nelsen [22], Joe [13] and Alink et al. [1]. The condition of complete monotonicity can be relaxed if one only requires that  $\psi$  generates a copula for a fixed dimension  $n$ ; see McNeil-Nešlehová [18].

Given that these conditions are fulfilled, the following formula always defines an  $n$ -dimensional distribution with marginals  $F_i$  and Archimedean copula  $C^\psi$

$$H(x_1, \dots, x_n) = C^\psi(F_1(x_1), \dots, F_n(x_n)) = \psi^{-1} \left( \sum_{i=1}^n \psi(F_i(x_i)) \right). \quad (2.3)$$

As has already been discussed in the literature (see for example Daniélsson et al. [5] or Nešlehová et al. [23]), the (sub- or super-)additivity properties of VaR depend on the interplay between the tail behaviour of the marginal loss random variables and their dependence structure. In the present paper, we investigate the properties of VaR in the following specific model.

**Model 2.1 (Archimedean survival copula)**

Assume that the random vector  $(X_1, \dots, X_n)$  satisfies:

- (a) All coordinates  $X_i$  have the same continuous marginal distribution function  $F$ . The tail distribution  $\bar{F} = 1 - F$  is regularly varying at infinity with index  $-\beta < 0$ , i.e.  $\bar{F}(x) = x^{-\beta}L(x)$  for some function  $L$  slowly varying at infinity.
- (b)  $(-X_1, \dots, -X_n)$  has an Archimedean copula with generator  $\psi$ , which is regularly varying at 0 with index  $-\alpha < 0$ .

For background on regular variation see Embrechts et al. [9]. Under the assumption of  $n = 2$ , the additivity properties of VaR within this model have been analyzed in Alink et al. [1] and Barbe et al. [3]. Our aim is to discuss the case  $n > 2$ . The following result yields the basics for our analysis.

**Proposition 2.2** *Under the Model 2.1 we have for all dimensions  $n \geq 2$*

$$\lim_{u \rightarrow \infty} \frac{1}{\bar{F}(u)} P \left( \sum_{i=1}^n X_i > u \right) = q_n(\alpha, \beta), \tag{2.4}$$

where the limiting constant  $q_n(\alpha, \beta) \in [n^{\beta-1/\alpha}, \infty)$  is given by the following integral representation

$$q_n(\alpha, \beta) = \int_{\mathbb{R}_+^n} 1_{\{\sum_{i=1}^n x_i^{-1} \geq 1\}} \frac{d^n}{dx_1 \cdots dx_n} \left( \sum_{i=1}^n x_i^{-\alpha\beta} \right)^{-1/\alpha} dx_1 \cdots dx_n. \tag{2.5}$$

**Proof.** Consider the transformation  $X_i \mapsto -X_i$ . Then  $-X_i$  satisfies the assumptions of Theorem 3.3 (Fréchet case) in Wüthrich [29]. Hence the existence of the limiting constant  $q_n(\alpha, \beta)$  follows from  $\lim_{u \rightarrow \infty} \bar{F}(u)/\bar{F}(un) = n^\beta$  and Theorem 3.3 in Wüthrich [29]. The integral representation (2.5) for  $q_n(\alpha, \beta)$  was then proved in Alink et al. [1], Theorem 2.2.

□

## Remarks.

- The proof of Proposition 2.2 strongly relies on the regular variation property of the marginal distribution function  $F$  and of the generator  $\psi$ ; see Alink et al. [1]. An alternative proof can be based on Proposition 5.17 of Resnick [25] or Theorem 1 of Resnick [26]. Indeed, this has recently been done in Barbe et al. [3] using the theory of multivariate regular variation: the limiting constant in (2.5) can then be expressed via the Radon exponent measure obtained in Theorem 1 of Resnick [26]. However, the analysis of the limiting constant for general dependence structures turns out to be rather difficult. Barbe et al. [3] derive some basic properties in their Proposition 2.2. In order to obtain specific asymptotic additivity properties one needs to switch to explicit dependence structures; see also Barbe et al. [3], Section 3. Whereas we restrict to Archimedean survival copulas, other approaches are possible that yield results like Proposition 2.2 under alternative dependence structures.
- The case of the two-dimensional Archimedean copula is solved in Alink et al. [1]; as we will see, the case  $n \geq 3$  is much more involved.
- Under the assumption (a) of Model 2.1, we immediately have the following results.

*Independent case.* If  $X_1, \dots, X_n$  are i.i.d. then

$$\lim_{u \rightarrow \infty} \frac{1}{\overline{F}(u)} P \left( \sum_{i=1}^n X_i > u \right) = n. \quad (2.6)$$

This property is referred to as subexponentiality of  $\overline{F}$  and holds for a wider class of distribution functions; see Corollary 1.3.2 in Embrechts et al. [9]. We would also like to note that numerical approximations based on (2.4) and (2.6) are often bad. Indeed, in a slightly different context Mikosch-Nagaev [19] show that rates of convergence for tail approximation of compound sums can be arbitrarily slow and hence may numerically be very poor.

*Comonotonic case.* If  $X_1, \dots, X_n$  are comonotone then

$$\lim_{u \rightarrow \infty} \frac{1}{\overline{F}(u)} P \left( \sum_{i=1}^n X_i > u \right) = n^\beta. \quad (2.7)$$

It turns out that  $q_n(\alpha, \beta)$  lies between these two extreme cases; see Lemma 3.1 below. Moreover, note that the case  $\beta = 1$  plays a special role, since then the limiting constants of the independent and the comonotonic case are the same.

□

The main difficulty is to determine how the limiting constant  $q_n(\alpha, \beta)$  behaves as a function of  $\alpha$  and  $\beta$ , respectively. For  $n = 2$ , the representation for the constant  $q_2(\alpha, \beta)$  yields that for every  $\alpha > 0$ ,  $q_2(\alpha, \beta)$  is strictly increasing in  $\beta$ ; see Theorem 2.5 in Alink et al. [1]. Moreover, we have:

- (a) For  $\beta > 1$ ,  $q_2(\alpha, \beta)$  is strictly increasing in  $\alpha$ .
- (b) For  $\beta = 1$ ,  $q_2(\alpha, \beta) = 2$ .
- (c) For  $\beta < 1$ ,  $q_2(\alpha, \beta)$  is strictly decreasing in  $\alpha$ .

Our main theorem extends the above result to the case  $n \geq 3$ . It is proved in Section 4 and reads as follows:

**Theorem 2.3** *Take  $n \geq 2$  in the Model 2.1 setup.*

- (a) *For  $\beta > 1$ ,  $q_n(\alpha, \beta)$  is increasing in  $\alpha$ .*
- (b) *For  $\beta = 1$ ,  $q_n(\alpha, \beta) = n$ .*
- (c) *For  $\beta < 1$ ,  $q_n(\alpha, \beta)$  is decreasing in  $\alpha$ .*

**Remark.** The decreasing/increasing property of the limiting constant is crucial for obtaining asymptotic superadditivity or subadditivity for VaR. Figure 1 gives the graph  $\alpha \mapsto q_3(\beta, \alpha)$  for different  $\beta$ 's in dimension  $n = 3$ . Note that we cannot prove strict monotonicity for  $n \geq 3$  though we believe this to hold. Theorem 2.3 proves the conjecture made by Barbe et al. [3] in Remark 4.2, where it was stated that “a proof for the monotonicity property remains elusive”.

□

An immediate consequence of Proposition 2.2 and regular variation of the marginals is the following asymptotic VaR behaviour.

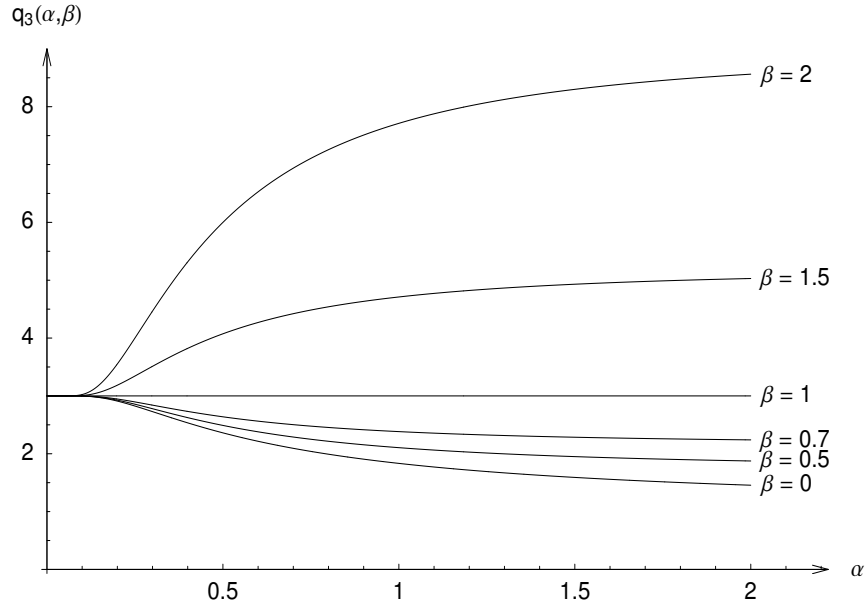


Figure 1: Function  $\alpha \mapsto q_3(\beta, \alpha)$  for different  $\beta \geq 0$

**Corollary 2.4** *Under the Model 2.1 and the assumption  $n \geq 2$ ,*

$$\lim_{p \rightarrow 1} \frac{\text{VaR}_p \left( \sum_{i=1}^n X_i \right)}{\text{VaR}_p (X_1)} = q_n(\alpha, \beta)^{1/\beta}, \quad (2.8)$$

where  $q_n(\alpha, \beta)$  is defined in (2.5).

Combining Corollary 2.4 and Theorem 2.3 leads to our final result whose proof is found in Section 4: it shows that in Model 2.1, VaR is superadditive or subadditive according to the value of  $\beta$  with respect to 1.

**Theorem 2.5** *For  $n \geq 2$  and assuming Model 2.1 we have that:*

(a) *For all  $\beta > 1$  and  $\alpha > 0$  there exists  $p_0 > 0$  such that for all  $p_0 < p < 1$*

$$\text{VaR}_p \left( \sum_{i=1}^n X_i \right) < \sum_{i=1}^n \text{VaR}_p (X_i). \quad (2.9)$$

(b) *For all  $\beta < 1$  and  $\alpha > 0$  there exists  $p_1 > 0$  such that for all  $p_1 < p < 1$*

$$\text{VaR}_p \left( \sum_{i=1}^n X_i \right) > \sum_{i=1}^n \text{VaR}_p (X_i). \quad (2.10)$$

Theorem 2.5 gives the basic qualitative picture for the asymptotic VaR behaviour. It clearly highlights the switch from subadditivity to superadditivity when moving from a finite mean model ( $\beta > 1$ ) to an infinite mean model ( $\beta < 1$ ). The crucial property for these results to hold is whether the limiting constant  $q_n(\alpha, \beta)$  is an increasing or decreasing function in  $\alpha$ . Note that this happens regardless the explicit choice of the Archimedean dependence structure as long as the conditions of Model 2.1 (b) hold. In other words, within Model 2.1, asymptotic super- or subadditivity of VaR is not driven by the dependence structure but entirely by the marginals. This is a special property of Model 2.1 and it would be wrong to think that this feature holds in greater generality.

### 3 Some properties of $q_n(\alpha, \beta)$ for $n \geq 3$

It follows from Corollary 2.4 that the asymptotic behaviour of VaR is determined by the limiting constant  $q_n(\alpha, \beta)$ . This section is devoted to the discussion of additional properties of the latter constant; in particular, we address its monotonicity as a function of  $\beta$  and its limiting behaviour in  $\alpha$  and  $\beta$ , respectively. Our results complement those of Barbe et al. [3].

Following Section 2.3 in Barbe et al. [3], we first reformulate the problem in polar coordinates. Define  $\mathbf{x} = (x_1, \dots, x_n)'$  and

$$g_\alpha(\mathbf{x}) = \frac{d^n}{dx_1 \cdots dx_n} \left( \sum_{i=1}^n x_i^\alpha \right)^{-1/\alpha} = \prod_{i=0}^{n-1} \left( \frac{1}{\alpha} + i \right) \left( \sum_{i=1}^n x_i^\alpha \right)^{-1/\alpha - n} \prod_{i=1}^n \alpha x_i^{\alpha-1}. \quad (3.1)$$

Use the transformation  $x_i \mapsto x_i^{-1/\beta}$  to obtain

$$q_n(\alpha, \beta) = \int_{\mathbb{R}_+^n} 1_{\{\sum_{i=1}^n x_i^{1/\beta} \geq 1\}} g_\alpha(\mathbf{x}) d\mathbf{x}. \quad (3.2)$$

The function  $g_\alpha(\cdot)$  is well-known and corresponds to the class of logistic distributions; consider for instance the densities  $h_{m,c}(\mathbf{w})$  in Kotz-Nadarajah [16] with  $\psi_{k,c} = 1$ .

On  $\mathbb{R}_+^n$  we now introduce the polar coordinate transformation

$$\mathbf{x} \mapsto \left( |\mathbf{x}|, \frac{\mathbf{x}}{|\mathbf{x}|} \right), \quad \text{with} \quad |\mathbf{x}| = \sum_{i=1}^n x_i.$$



Using this transformation we rewrite (3.2) as follows

$$\begin{aligned} q_n(\alpha, \beta) &= \int_{[0, \infty) \times \mathcal{N}_+} 1_{\left\{ \left( \sum_{i=1}^n w_i^{1/\beta} \right)^{-\beta} \leq r \right\}} r^{-2} g_\alpha(\mathbf{w}) dr d\mathbf{w} \\ &= \int_{\mathcal{N}_+} \left( \sum_{i=1}^n w_i^{1/\beta} \right)^\beta g_\alpha(\mathbf{w}) d\mathbf{w}, \end{aligned} \quad (3.3)$$

where  $\mathcal{N}_+ = \{\mathbf{w} \in \mathbb{R}_+^n : |\mathbf{w}| = 1\}$  is the unit simplex. This is exactly the integral representation (20) in Barbe et al. [3]. The following lemmas contain some results which are essentially known. We add short proofs where necessary.

**Lemma 3.1** *For all  $n \geq 2$  and  $\alpha > 0$  the following properties hold:*

- (a)  $g_\alpha(\cdot)/n$  is a probability density on  $\mathcal{N}_+$ ;
- (b)  $q_n(\alpha, \beta)$  is strictly increasing in  $\beta$ ;
- (c)  $q_n(\alpha, 1) = n$ , and
- (d)  $\min\{n^\beta, n\} \leq q_n(\alpha, \beta) \leq \max\{n^\beta, n\}$ .

**Proof.**

- (a) See for instance Kotz-Nadarajah [16], formula (3.7).
- (b) This was proved in Proposition 2.2 of Barbe et al. [3].
- (c) This was proved in Proposition 4.1 of Barbe et al. [3].
- (d) From (b) and (c) it is clear that  $q_n(\alpha, 1)$  is an upper bound on  $q_n(\alpha, \beta)$  for  $\beta < 1$  and that  $q_n(\alpha, 1)$  is a lower bound on  $q_n(\alpha, \beta)$  for  $\beta > 1$ . The maximum and minimum of  $n \left( \sum_{i=1}^n w_i^{1/\beta} \right)^\beta$  on  $\mathcal{N}_+$  are given by  $\max(n^\beta, n)$  and  $\min(n^\beta, n)$ , respectively. Hence, since  $g_\alpha(\cdot)/n$  is a probability density on  $\mathcal{N}_+$ , the proof of (d) is straightforward. □

**Lemma 3.2** *For all  $n \geq 2$  and  $\alpha > 0$  we have*

$$\lim_{\beta \rightarrow 0} q_n(\alpha, \beta) = \int_{\mathcal{N}_+} \max\{w_1, \dots, w_n\} g_\alpha(\mathbf{w}) d\mathbf{w} = \sum_{k=1}^n \binom{n}{k} (-1)^{k-1} k^{-1/\alpha}, \quad (3.4)$$

and

$$\lim_{\beta \rightarrow \infty} \frac{q_n(\alpha, \beta)}{n^\beta} = \int_{\mathcal{N}_+} \prod_{i=1}^n w_i^{1/n} g_\alpha(\mathbf{w}) d\mathbf{w}. \quad (3.5)$$

**Proof.** Claim (3.5) as well as the first equality in (3.4) follow from Proposition 2.2 in Barbe et al. [3]. Define  $|\mathbf{w}|_\infty = \max\{w_1, \dots, w_n\}$ . Hence it remains to be shown that

$$\int_{\mathcal{N}_+} |\mathbf{w}|_\infty g_\alpha(\mathbf{w}) d\mathbf{w} = \sum_{k=1}^n \binom{n}{k} (-1)^{k-1} k^{-1/\alpha}.$$

For this, note that

$$\begin{aligned} \int_{\mathcal{N}_+} |\mathbf{w}|_\infty g_\alpha(\mathbf{w}) d\mathbf{w} &= \int_{[0, \infty) \times \mathcal{N}_+} 1_{\{1/|\mathbf{w}|_\infty \leq r\}} r^{-2} g_\alpha(\mathbf{w}) dr d\mathbf{w} \\ &= \int_{\mathbb{R}_+^n} 1_{\{|\mathbf{x}|_\infty \geq 1\}} g_\alpha(\mathbf{x}) d\mathbf{x} = \int_{B_\infty^c(1)} g_\alpha(\mathbf{x}) d\mathbf{x}, \end{aligned}$$

where  $B_\infty^c(1) = \{\mathbf{x} \in \mathbb{R}_+^n : |\mathbf{x}|_\infty \geq 1\}$ . Choose  $\mathbf{a} \in \mathbb{R}_+^n$  and define the set  $A_n = [a_1, \infty) \times \dots \times [a_n, \infty) \subset \mathbb{R}_+^n$ . Then we have

$$\int_{A_n} g_\alpha(\mathbf{x}) d\mathbf{x} = \left( \sum_{i=1}^n a_i^\alpha \right)^{-1/\alpha},$$

completing the proof. □

**Lemma 3.3** *For all  $n \geq 2$  and  $\beta > 0$  we have*

$$\lim_{\alpha \rightarrow \infty} q_n(\alpha, \beta) = n^\beta, \tag{3.6}$$

and

$$\lim_{\alpha \rightarrow 0} q_n(\alpha, \beta) = n. \tag{3.7}$$

**Proof.** Claim (3.6) is proved in Proposition 4.1 of Barbe et al. [3]. The proof of (3.7) is more involved; Barbe et al. [3] give a hint in their Section 3.1. We decided to give the explicit arguments since one needs to distinguish the two cases  $\beta \leq 1$  and  $\beta > 1$ .

For  $\beta \leq 1$  we know that  $q_n(\alpha, \beta) \leq q_n(\alpha, 1) = n$ ; see Lemma 3.1. Moreover, from Lemmas 3.1 and 3.2 we know that for all  $\beta > 0$  and  $\alpha > 0$  we have

$$q_n(\alpha, \beta) \geq \sum_{k=1}^n \binom{n}{k} (-1)^{k-1} k^{-1/\alpha}.$$

This implies that

$$\liminf_{\alpha \rightarrow 0} q_n(\alpha, \beta) \geq \lim_{\alpha \rightarrow 0} \sum_{k=1}^n \binom{n}{k} (-1)^{k-1} k^{-1/\alpha} = n.$$

Hence for  $\beta \leq 1$  we have

$$\lim_{\alpha \rightarrow 0} q_n(\alpha, \beta) = n \lim_{\alpha \rightarrow 0} \int_{\mathcal{N}_+} \left( \sum_{i=1}^n w_i^{1/\beta} \right)^\beta g_\alpha(\mathbf{w})/n \, d\mathbf{w} = n. \quad (3.8)$$

Observe that  $g_\alpha(\mathbf{w})/n$  is a probability density on  $\mathcal{N}_+$  for all  $\alpha > 0$  and that the right-hand side of (3.8) is independent of  $\beta \leq 1$ . Moreover,  $\left( \sum_{i=1}^n w_i^{1/\beta} \right)^\beta$  takes its maxima in the corners of  $\mathcal{N}_+$  for  $\beta < 1$  (i.e. in  $\mathcal{C} = \{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}$ ) and

$$\left( \sum_{i=1}^n w_i^{1/\beta} \right)^\beta = 1 \quad \text{for } \mathbf{w} \in \mathcal{C}.$$

This immediately implies that  $g_\alpha(\mathbf{w})/n$  converges to a point measure on  $\mathcal{C}$  for  $\alpha \rightarrow 0$ , giving weight  $1/n$  to each corner of  $\mathcal{N}_+$  using a symmetry argument; this corresponds to the independent case. Having a uniform distribution in the corners of  $\mathcal{N}_+$  also proves the claim for  $\beta > 1$ . This completes the proof. □

## 4 Proofs of the results from Section 2

In order to prove our main result, Theorem 2.3, we need to introduce supermodular ordering and present some related results.

### 4.1 Stochastic Ordering

**Definition 4.1** *A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is supermodular if for all  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$*

$$f(\mathbf{x} \wedge \mathbf{y}) + f(\mathbf{x} \vee \mathbf{y}) \geq f(\mathbf{x}) + f(\mathbf{y}). \quad (4.1)$$

Here  $\wedge$  and  $\vee$  denote the componentwise minimum and maximum, respectively. In statistical physics, condition (4.1) is often referred to as the FKG lattice condition; see Newman [24] and Fortuin et al. [11].

From Kemperman [14], it follows that a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is supermodular if and only if  $f(\dots, x_i, \dots, x_j, \dots)$  is supermodular in all pairs  $(x_i, x_j)$ ; see also Wei-Hu [28].

Theorem 2.5 in Müller [21] implies that, if  $f$  is twice differentiable, then it is supermodular if and only if

$$\frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x}) \geq 0 \quad \text{for all } \mathbf{x} = (x_1, \dots, x_n)' \in \mathbb{R}^n \text{ and } 1 \leq i < j \leq n. \quad (4.2)$$

**Definition 4.2** A random vector  $\mathbf{X} = (X_1, \dots, X_n)$  is said to be smaller than a random vector  $\mathbf{Y} = (Y_1, \dots, Y_n)$  in supermodular ordering if for all supermodular functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  we have

$$Ef(\mathbf{X}) \leq Ef(\mathbf{Y}).$$

Write  $\mathbf{X} \leq_{sm} \mathbf{Y}$  and  $F_{\mathbf{X}} \leq_{sm} F_{\mathbf{Y}}$  (where  $F_{\mathbf{X}}$  and  $F_{\mathbf{Y}}$  denote the distribution functions of  $\mathbf{X}$  and  $\mathbf{Y}$ , respectively).

From Theorem 3.2 in Müller [21] it follows that the supermodular ordering for random vectors is preserved under coordinatewise increasing transformations. This immediately implies that the supermodular property  $\mathbf{X} \leq_{sm} \mathbf{Y}$  only depends on the supermodular ordering of the copulas of  $\mathbf{X}$  and  $\mathbf{Y}$ . This means that

$$\mathbf{X} \leq_{sm} \mathbf{Y} \iff C_{\mathbf{X}} \leq_{sm} C_{\mathbf{Y}}, \quad (4.3)$$

where  $C_{\mathbf{X}}$  and  $C_{\mathbf{Y}}$  are the copulas of  $\mathbf{X}$  and  $\mathbf{Y}$ , respectively. We now choose as an example a specific Archimedean copula  $\psi_{\alpha}(t) = t^{-\alpha} - 1$  for  $\alpha > 0$ . Then

$$C_{\alpha}(u_1, \dots, u_n) = \left( \sum_{i=1}^n u_i^{-\alpha} - (n-1) \right)^{-1/\alpha}$$

is the so-called Clayton copula with parameter  $\alpha$ . Observe that the generator  $\psi_{\alpha}(t)$  of the Clayton copula is regularly varying in 0 with index  $-\alpha < 0$ .

**Lemma 4.3** Assume that  $\mathbf{X}$  and  $\mathbf{Y}$  have a Clayton copula with respective parameters  $\alpha_X > 0$  and  $\alpha_Y > 0$ . Then

$$\alpha_X < \alpha_Y \implies \mathbf{X} \leq_{sm} \mathbf{Y}.$$

**Proof.** Because of (4.3), it remains to be shown that the Clayton copulas  $C_{\alpha}$  are ordered with respect to  $\alpha$ . With Theorem 3.1 in Wei-Hu [28] it suffices to prove that

$$(-1)^{k-1} \frac{\partial^k}{\partial t^k} \psi_{\alpha_X} \circ \psi_{\alpha_Y}^{-1}(t) \geq 0 \quad \text{for all } k \geq 1 \text{ and } t > 0, \quad (4.4)$$

as well as  $\psi_{\alpha_X} \circ \psi_{\alpha_Y}^{-1}(0) = 0$  and  $\psi_{\alpha_X} \circ \psi_{\alpha_Y}^{-1}(\infty) = \infty$ . However, since

$$\psi_{\alpha_X} \circ \psi_{\alpha_Y}^{-1}(t) = (1+t)^{\alpha_X/\alpha_Y} - 1,$$

(4.4) is easily verified for  $\alpha_X < \alpha_Y$ . This completes the proof. □

## 4.2 Proof of Theorem 2.3

The proof of Theorem 2.3 essentially uses a relationship between expectations (and thus characterizations of supermodular ordering) to tail probabilities, in particular, the fact that for any random variable  $Z$  and  $u \in \mathbb{R}$ ,

$$E(Z - u)_+ = \int_0^\infty P(Z > u + y) dy. \quad (4.5)$$

To simplify notation in the proofs below, we define  $s_n(\mathbf{x}) = \sum_{i=1}^n x_i$  for  $\mathbf{x} \in \mathbb{R}^n$ .

**Proof of Theorem 2.3 (b), case  $\beta = 1$ .** This was proved in Proposition 4.1 of Barbe et al. [3].

**Proof of Theorem 2.3 (a), case  $\beta > 1$ .** Because of Proposition 2.2 it suffices to show the assertion for one specific model that satisfies the assumptions of Model 2.1. We choose two random vectors  $\mathbf{X}$  and  $\mathbf{Y}$  as follows:  $-\mathbf{X}$  and  $-\mathbf{Y}$  have a Clayton copula with parameters  $0 < \alpha_X < \alpha_Y$  and identical marginals  $F$  with index of regular variation at infinity given by  $\beta > 1$ , and show that  $q_n(\alpha_X, \beta) \leq q_n(\alpha_Y, \beta)$  for  $\beta > 1$ . Proposition 2.2 implies that for this choice,

$$q_n(\alpha_X, \beta) = \lim_{u \rightarrow \infty} \frac{1}{\overline{F}(u)} P(s_n(\mathbf{X}) > u)$$

and

$$q_n(\alpha_Y, \beta) = \lim_{u \rightarrow \infty} \frac{1}{\overline{F}(u)} P(s_n(\mathbf{Y}) > u).$$

The function  $\mathbf{x} \in \mathbb{R}^n \mapsto s_n(\mathbf{x})$  is convex and supermodular; by preservation of convexity, so is the function  $\mathbf{x} \mapsto (s_n(\mathbf{x}) - u)_+$ . It then follows by Lemma 4.3 and (4.5) that for any positive  $u$  (see also Theorem 3.1 in Müller [21])

$$\int_0^\infty P(s_n(\mathbf{X}) > u + y) dy \leq \int_0^\infty P(s_n(\mathbf{Y}) > u + y) dy. \quad (4.6)$$

Dividing both sides of this inequality by  $\bar{F}(u)$ , the left-hand side becomes

$$\int_0^\infty \frac{P(s_n(\mathbf{X}) > u + y)}{\bar{F}(u + y)} \frac{\bar{F}(u + y)}{\bar{F}(u)} dy,$$

which, by Proposition 2.2 and Karamata's theorem (see Theorem A3.6 in Embrechts et al. [9]), converges to  $q_n(\alpha_X, \beta)/(\beta - 1)$  as  $u$  tends to infinity. Consequently, (4.6) implies  $q_n(\alpha_X, \beta) \leq q_n(\alpha_Y, \beta)$ . □

**Proof of Theorem 2.3 (c), case  $\beta < 1$ .** We use the same (explicit) model and the same notations as in the proof of part (a), but this time we choose identical marginals with  $\beta < 1$ . Hence, the marginal distributions have an infinite mean, which means that the previous proof breaks down for the lack of integrability of the function  $y \mapsto \bar{F}(u + y)$ . The trick is then to use truncation. For this purpose, note that for any real number  $z$  and positive real numbers  $u, v$

$$z \wedge (u + v) = z \wedge u + ((z \wedge (u + v)) - u)_+.$$

Consequently, using (4.5), we have for any random variable  $Z$

$$\begin{aligned} E(Z \wedge (u + v)) &= E(Z \wedge u) + \int_0^\infty P(Z \wedge (u + v) > u + y) dy \\ &= E(Z \wedge u) + \int_0^v P(Z > u + y) dy. \end{aligned} \quad (4.7)$$

The function  $\mathbf{x} \in \mathbb{R}^n \mapsto -(s_n(\mathbf{x}) \wedge (u + v))$  is convex and supermodular for any positive real numbers  $u, v$ . Hence it then follows by Lemma 4.3 and (4.7) that for any positive  $u, v$

$$\begin{aligned} 0 &\leq E(s_n(\mathbf{X}) \wedge (u + v)) - E(s_n(\mathbf{Y}) \wedge (u + v)) \\ &= E(s_n(\mathbf{X}) \wedge u) - E(s_n(\mathbf{Y}) \wedge u) \\ &\quad + \int_0^v P(s_n(\mathbf{X}) > u + y) dy - \int_0^v P(s_n(\mathbf{Y}) > u + y) dy. \end{aligned} \quad (4.8)$$

Choose  $\varepsilon > 0$ . By Proposition 2.2 we then have that for all sufficiently large  $u$  and for any positive  $v$  the difference of the integrals in (4.8) is at most

$$(1 + \varepsilon) q_n(\alpha_X, \beta) \int_0^v \bar{F}(u + y) dy - (1 - \varepsilon) q_n(\alpha_Y, \beta) \int_0^v \bar{F}(u + y) dy.$$

As  $v$  tends to infinity, the integral  $\int_0^v \bar{F}(u + y) dy$  tends to infinity, and therefore (4.8) forces  $(1 + \varepsilon) q_n(\alpha_X, \beta) - (1 - \varepsilon) q_n(\alpha_Y, \beta)$  to be nonnegative. The result then follows because  $\varepsilon$  was arbitrary. □

### 4.3 Proof of Theorem 2.5

**Proof of Theorem 2.5.** For  $\beta > 1$  and  $\alpha > 0$  we know from the proofs of Proposition 4.1 in Barbe et al. [3], Lemma 3.3 and Theorem 2.3 that  $q_n(\alpha, \beta) < n^\beta$ . Hence choose  $\varepsilon \in (0, n^\beta/q_n(\alpha, \beta) - 1)$ . From Corollary 2.4 we know that there exists  $p_0 = p_0(\alpha, \beta) > 0$  such that for all  $p > p_0$  we have

$$\begin{aligned} \text{VaR}_p(s_n(\mathbf{X})) &\leq ((1 + \varepsilon)q_n(\alpha, \beta))^{1/\beta} \text{VaR}_p(X_1) \\ &< n \text{VaR}_p(X_1) = \sum_{i=1}^n \text{VaR}_p(X_i). \end{aligned}$$

The claim for  $\beta < 1$  follows analogously from the fact that  $q_n(\alpha, \beta) > n^\beta$ ; see proofs of Proposition 4.1 in Barbe et al. [3], Lemma 3.3 and Theorem 2.3. This finishes the proof.  $\square$

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