Bounds on Value-at-Risk

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Abstract The problem of finding the worst-possible Value-at-Risk (VaR) for a non-decreasing function of a vector of n dependent risks is solved when $n = 2$ or a lower bound on the copula of the portfolio is provided. In this paper we correct the statement and the proof of this result, given in Embrechts, Höing, and Juri (2003). The problem gets much more complicated in arbitrary dimensions when no information on the structure of dependence of the random vector is available. In this case we provide a bound on the VaR for the sum of risks which we prove to be better than the one generally used in literature.

Key words Value-at-Risk – copulas – dependent risks – dependency bounds – Fréchet bounds

1 Introduction

Consider a n-valued real function ψ and a random vector $X := (X_1, \ldots, X_n)$. In this paper we study the problem of finding the best-possible lower bound on the distribution function (df) of $\psi(X)$ when the marginal distributions of the individual risks X_i are given and the structure of dependence of X is partially or completely unknown. Equivalently stated, we search for the worst-possible Value-at-Risk (VaR) for the random variable $\psi(X)$. This problem has a long history. Makarov (1981), in response to a question formulated by A.N. Kolmogorov, provided the first result for $n = 2$ and $\psi = +$, the sum operator. Some years later Frank, Nelsen, and Schweizer (1987) restated Makarov's result, using the well-known formulation of the problem based on copulas. Independently from this *geometric* approach, Rüschendorf (1982) gave a much more elegant proof of the same theorem using a dual result proved for a more general purpose. The *dual* approach of Rüschendorf was related to a much earlier issue, dating back to 1871: the so-called *Monge-Kantorovich mass-transportation problem*; in particular he solved a special case of its *Kantorovich version*. A complete analysis of this kind of problems is given in Rachev and Rüschendorf (1998). The use of dependence information to tighten the bound on the df of a two-dimensional portfolio firstly appeared in Williamson and Downs (1990), where sharpness was proved for non-decreasing functionals. Denuit, Genest, and Marceau (1999) extended the bound for the sum to arbitrary dimensions and provided some applications. Finally Embrechts, Höing, and Juri (2003) gave the most general theorem till now, stating sharpness of the bound in the presence of information for a larger class of functions ψ. The latter paper however contains a gap in the main proof; in our paper, we revisit the proof and correct the statement of their main result. While the problem can be considered fully solved if a lower bound on the copula of the vector X is available, the search for the worst-possible VaR is still open in the no-information scenario for $n > 2$, even for the case of the sum. In this case, a bound for the df of the sum of risks can be obtained by the previously cited theorems, but it fails to be sharp whenever $n > 2$. Exploiting the dual result of Rüschendorf we give a better bound which, though not proved to be sharp, improves considerably the previous estimate of the VaR of a sum for identically distributed risks. Some applications of our result are provided to prove the usefulness of the new estimate for actuarial/financial applications. A full solution of the general problem seems still out of reach.

1.1 Notation

We first fix some notation. The inverse of a non-decreasing function $\psi : \mathbb{R} \to \mathbb{R}$ is the function $\psi^{-1} : \mathbb{R} \to \overline{\mathbb{R}}$;

$$\psi^{-1}(y) := \inf\{x \in \mathbb{R} | \psi(x) \geq y\}.$$

Given a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we write

$$x_{-i} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

to indicate the $(n-1)$-valued vector obtained from x by deleting the i-th component. The indicator function of the set $B \subset \mathbb{R}$ is the function $1_B : \mathbb{R} \to \mathbb{R}$;

$$1_B(b) := \begin{cases} 1 & \text{if } b \in B; \\ 0 & \text{otherwise.} \end{cases}$$

Finally, for k (possibly identical) real numbers $s_1, \dots, s_k$, we define $U_{\{s_1,\dots,s_k\}} : \mathbb{R} \to \mathbb{R}$;

$$U_{\{s_1,\dots,s_k\}}(s) := \frac{1}{k} \sum_{i=1}^{k} 1_{[s_i,+\infty)}(s)$$

the probability measure uniformly distributed on $\{s_1, \dots, s_k\}$.

2 Definitions and preliminaries

In this section we introduce the main mathematical problem and recall some well-known concepts about copulas.

2.1 Copulas as dependence structures

Let $X_1, \dots, X_n$ be n real-valued random variables on some probability space $(\Omega, \mathfrak{A}, \mathbb{P})$, with given dfs $F_i(x) = \mathbb{P}[X_i \leq x], i = 1, \dots, n$. The random vector $X := (X_1, \dots, X_n)$ can be seen as a portfolio of one-period financial or insurance risks. For some function $\psi : \mathbb{R}^n \to \mathbb{R}$, we consider the problem of bounding the VaR for the financial position $\psi(X)$, over the class of possible dfs for X having fixed marginals.

Definition 2.1. *For $\alpha \in [0,1]$, the* Value-at-Risk at probability level α *of a random variable S is its α-quantile, defined as*

$$\mathrm{VaR}_\alpha(S) := G^{-1}(\alpha)$$

where G is the df of S.

Instead of operating directly with the VaR, we can equivalently search for

$$m_{\psi}(s) := \inf\{\mathbb{P}[\psi(X) < s] : X_i \backsim F_i, i = 1, \ldots, n\}. \tag{2.1}$$

Indeed, according to Definition 2.1, we have

$$\mathrm{VaR}_{\alpha}(\psi(X)) \leq m_{\psi}^{-1}(\alpha), \alpha \in [0, 1]. \tag{2.2}$$

Of course every quantile of $\psi(X)$ can be computed once the df $F(x_1, \ldots, x_n) = \mathbb{P}[X_1 \leq x_1, \ldots, X_n \leq x_n]$ is known. The latter is uniquely defined through the marginal dfs and their interdependence. The tool for modelling these dependencies is offered by the concept of copula.

Definition 2.2. *A* copula *is any function $C : [0, 1]^n \to [0, 1]$ which has the following three properties:*

(i) *C is non-decreasing in each argument.*

(ii) *$C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$ for all $u_i \in [0, 1], i = 1, \ldots, n$.*

(iii) *C is* n-increasing, *i.e. for all $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in [0, 1]^n$ with $a_i \leq b_i, i = 1, \ldots, n$ we have*

$$\sum_{j_1=1}^{2} \cdots \sum_{j_n=1}^{2} (-1)^{j_1+\cdots+j_n} C(u_{1j_1}, \ldots, u_{nj_n}) \geq 0,$$

where $u_{i1} = a_i, u_{i2} = b_i, i = 1, \ldots, n$.

It is equivalent to say that a copula is a n-dimensional df restricted to $[0, 1]^n$ having standard uniform marginals. It easily follows that given a copula C and a set of n univariate marginals $F_1, \ldots, F_n$ one can always define a df F on $\mathbb{R}^n$ having these marginals by

$$F(x_1, \ldots, x_n) := C(F_1(x_1), \ldots, F_n(x_n)). \tag{2.3}$$

Sklar's theorem (see Theorem 1 in Sklar (1973)) states conversely that we can always find a copula C coupling the marginals of a fixed df F trough (2.3). For continuous marginal dfs, this copula is unique. In our set-up it is convenient to identify the df F of X with the copula C merging the given marginals into the df $C(F_1(x_1), \ldots, F_n(x_n))$. Denote by μ_C the corresponding probability measure on $\mathbb{R}^n$ and define:

$$\begin{aligned} \sigma_{C,\psi}(F_1, \ldots, F_n)(s) : &= \int_{\{\psi < s\}} dC(F_1(x_1), \ldots, F_n(x_n)) \\ &= \mu_C[\psi(X) < s], \end{aligned} \tag{2.4}$$

$$\begin{aligned} \tau_{C,\psi}(F_1, \ldots, F_n)(s) : &= \sup_{x_1, \ldots, x_{n-1} \in \mathbb{R}} C(F_1(x_1), \ldots, \\ &\qquad F_{n-1}(x_{n-1}), F_n^{-}(\psi^{\wedge}_{x_{-n}}(s))), \end{aligned} \tag{2.5}$$

where $\psi^{\wedge}_{x_{-n}}(s) := \sup\{x_n \in \mathbb{R} : \psi(x_{-n}, x_n) < s\}$ for fixed $x_{-n} \in \mathbb{R}^{n-1}$.

By the above discussion, problem (2.1) can be equivalently expressed as

$$m_{\psi}(s) = \inf\{\sigma_{C,\psi}(F_1, \ldots, F_n)(s) : C \in \mathfrak{C}_n\} \tag{2.6}$$

where $\mathfrak{C}_n$ denotes the set of all n-dimensional copulas.

2.2 Dependency information

If we don't have the perfect knowledge of the copula C coupling the fixed marginal dfs of the portfolio X, the quantiles of $\psi(X)$ cannot be determined exactly and problem (2.6) arises. However, it can be the case that partial information regarding C is known.

Given two copulas C_1 and C_2, we say that $C_1 \geq$ (resp. $\leq$) C_2 if and only if $C_1(u) \geq$ (resp. $\leq$) $C_2(u)$ for all $u \in [0,1]^n$. Using the properties of a copula it can be easily shown that any copula C lies between the so-called lower and upper Fréchet bounds $W, M : [0,1]^n \to [0,1]$;

$$W(u_1, u_2, \ldots, u_n) := (\sum_{i=1}^{n} u_i - n + 1)^+,$$
$$M(u_1, u_2, \ldots, u_n) := \min\{u_1, \ldots, u_n\},$$

namely

$$W \leq C \leq M.$$

A third copula of interest is the product copula $\Pi : [0,1]^n \to [0,1]; \Pi(u) := \prod_{i=1}^{n} u_i$ which represents independence among coupled random variables.

The copula of a df F contains all the dependency information of F, hence putting a lower bound on the copula C of the portfolio can be interpreted as having partial information regarding its dependence structure. For instance, assuming that $C \geq M$ directly characterizes the risks of our portfolio as *comonotonic*, i.e. as being increasing functions of a common random variable. See Dhaene, Denuit, Goovaerts, Kaas, and Vyncle (2001) for more details on comonotonicity. Moreover, assuming that $C \geq \Pi$ identifies the risks as *positive lower orthant dependent* (PLOD).

If we assume that a lower bound on C is known, we can reduce our search to

$$\begin{aligned} m_{C_L,\psi}(s) &:= \inf\{\sigma_{C,\psi}(F_1, \ldots, F_n)(s) : C \geq C_L\} \\ &= \inf\{\mathbb{P}[\psi(X) < t] : X_i \backsim F_i, i = 1, \ldots, n, F \geq C_L(F_1, \ldots, F_n)\}. \end{aligned} \tag{2.7}$$

Note that $m_{W,\psi}(t) = m_\psi(t)$: saying that $C \geq W$ corresponds to the situation in which we are completely ignorant about the dependence structure of the random vector X. Obviously, $m_{C_L,\psi}(t) \geq m_\psi(t)$ but we warn the reader that the last inequality is often strict even for a non-decreasing function ψ. Due to the fact that $\geq$ is not a complete ordering on $\mathfrak{C}_n$, letting $C \geq C_L$ is not necessarily a prudent assumption. In fact, for any $C_L \neq W$, we get rid of all copulas which are not comparable to C_L with respect to $\geq$. By doing so we possibly exclude the riskiest copula, i.e. the one possibly solving (2.6). Roughly speaking, if we limit our attention to copulas greater than a given one, let say Π, we are not in general restricting to riskier portfolios.

Finally note that, contrary to M, W is not a copula for $n > 2$: this fact will play a fundamental role in the next sections.

3 Main result with partial information

When partial information on the copula of a vector X is known, it is easy to find a general lower bound on $\sigma_{C,\psi}(F_1, \ldots, F_n)(s)$.

Theorem 3.1. *Let $X = (X_1, \dots, X_n)$ be a random vector on $\mathbb{R}^n$ $(n > 1)$ having marginal distribution functions $F_1, \dots, F_n$ and copula C. Assume that there exists a copula C_L such that $C \geq C_L$. If $\psi : \mathbb{R}^n \to \mathbb{R}$ is non-decreasing in each coordinate, then for every real s we have*

$$\sigma_{C,\psi}(F_1, \dots, F_n)(s) \geq \tau_{C_L,\psi}(F_1, \dots, F_n)(s). \tag{3.1}$$

Proof. First observe that for arbitrary $x \in \mathbb{R}^n$, the uniform continuity of a copula C implies that

$$\begin{aligned}
&\mu_C[X_1 \leq x_1, \dots, X_{n-1} \leq x_{n-1}, X_n < x_n] \\
&= \mu_C[\cup_{k\in\mathbb{N}}\{X_1 \leq x_1, \dots, X_{n-1} \leq x_{n-1}, X_n \leq x_n - \frac{1}{k}\}] \\
&= \lim_{k\to\infty} \mu_C[X_1 \leq x_1, \dots, X_{n-1} \leq x_{n-1}, X_n \leq x_n - \frac{1}{k}] \\
&= \lim_{k\to\infty} C(F_1(u_1), \dots, F_{n-1}(x_{n-1}), F_n(u_n - \frac{1}{k})) \\
&= C(F_1(u_1), \dots, F_{n-1}(x_{n-1}), \lim_{k\to\infty} F_n(u_n - \frac{1}{k})) \\
&= C(F_1(u_1), \dots, F_{n-1}(x_{n-1}), F_n^-(u_n)).
\end{aligned}$$

Now fix $(\bar{x}_1, \dots, \bar{x}_{n-1}) \in \mathbb{R}^{n-1}$ and assume that $\bar{x}_n := \psi^{\wedge}_{\bar{x}_{-n}}(s)$ is finite. Then

$$\{X_1 \leq \bar{x}_1, \dots, X_{n-1} \leq \bar{x}_{n-1}, X_n < \bar{x}_n\} \subset \{\psi(X) < s\}$$

and hence

$$\begin{aligned}
\mu_C[\psi(X) < s] \geq& \mu_C[X_1 \leq \bar{x}_1, \dots X_{n-1} \leq \bar{x}_{n-1}, X_n < \bar{x}_n] \\
=& C(F_1(\bar{x}_1), \dots, F_{n-1}(\bar{x}_{n-1}), F_n^-(\bar{x}_n)) \\
\geq& C_L(F_1(\bar{x}_1), \dots, F_{n-1}(\bar{x}_{n-1}), F_n^-(\psi^{\wedge}_{\bar{x}_{-n}}(s))).
\end{aligned}$$

If $\bar{x}_n = +\infty$, then $\psi(\bar{x}_{-n}, x_n) < s$ for all $x_n \in \mathbb{R}$, and hence

$$\begin{aligned}
\mu_C[\psi(X) < s] \geq& \mu_C[X_1 \leq \bar{x}_1, \dots, X_{n-1} \leq \bar{x}_{n-1}, X_n \in \mathbb{R}] \\
=& C(F_1(\bar{x}_1), \dots, F_{n-1}(\bar{x}_{n-1}), 1) \\
\geq& C_L(F_1(\bar{x}_1), \dots, F_{n-1}(\bar{x}_{n-1}), F_n^-(+\infty)).
\end{aligned}$$

Analogously, if $\bar{x}_n = -\infty$ then $\psi(\bar{x}_{-n}, x_n) \geq s$ for all $x_n \in \mathbb{R}$, so that

$$\begin{aligned}
\mu_C[\psi(X) < s] \geq 0 &= C_L(F_1(\bar{x}_1), \dots, F_{n-1}(\bar{x}_{n-1}), 0) \\
&= C_L(F_1(\bar{x}_1), \dots, F_{n-1}(\bar{x}_{n-1}), F_n^-(-\infty)).
\end{aligned}$$

The theorem follows by taking the supremum over all $(\bar{x}_1, \dots, \bar{x}_{n-1}) \in \mathbb{R}^{n-1}$. □

We now prove that, if a non-trivial lower bound on C is assumed, then there will always be a copula attaining bound (3.1), i.e. that bound cannot be tightened.

Theorem 3.2. *In the hypotheses of Theorem 3.1 assume ψ is also right-continuous in its last argument. Define the copula $C_t : [0,1]^n \to [0,1]$;*

$$C_t(u) := \begin{cases} \max\{t, C_L(u)\} & \text{if } u = (u_1, \dots, u_n) \in [t,1]^n; \\ \min\{u_1, \dots, u_n\}, & \text{otherwise,} \end{cases}$$

where $t = \tau_{C_L,\psi}(F_1, \dots, F_n)(s)$. Then this copula attains bound (3.1)*, i.e.*

$$\sigma_{C_t,\psi}(F_1, \dots, F_n)(s) = t. \tag{3.2}$$

Proof. We prove the theorem in two steps; first we show that the above definition makes sense.

Lemma 3.1. *The function C_t is a copula.*

Proof. Properties (i) and (ii) of Definition 2.2 are trivially satisfied by C_t. It therefore remains to show that C_t is n-increasing on its domain.
If we take $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n)$ arbitrary vectors in $[0,1]^n$, with $a_i \leq b_i, i \in N := \{1, \dots, n\}$, we have to show that

$$\sum_{j_1=1}^{2} \cdots \sum_{j_n=1}^{2} (-1)^{j_1+\cdots+j_n} C_t(u_{1j_1}, \dots, u_{nj_n}) \geq 0, \tag{3.3}$$

where $u_{i1} = a_i, u_{i2} = b_i$ for all $i \in N$. If $F_i(a_i) \geq t$ for all $i \in N$, then $C_t(u) = C_L(u)$ for every $u \in \prod_{i=1}^{n}[a_i, b_i]$ and (3.3) follows from n-increasingness of C_L. Note that we can always find a permutation $\sigma : N \to N$ and an $m \in N$ such that

$$F_{\sigma(1)}(a_{\sigma(1)}) \leq \cdots \leq F_{\sigma(m)}(a_{\sigma(m)}) < t \leq F_{\sigma(m+1)}(a_{\sigma(m+1)}) \leq \dots \\ \leq F_{\sigma(n)}(a_{\sigma(n)}).$$

Changing the order of summation, (3.3) can be rewritten as

$$\sum_{j_{\sigma(1)}=1}^{2} \cdots \sum_{j_{\sigma(n)}=1}^{2} (-1)^{j_1+\cdots+j_n} C_t(u_{1j_1}, \dots, u_{nj_n}) \geq 0.$$

In the following we denote

$$C_t(u_{[\sigma(1)j_{\sigma(1)}]}, \dots, u_{[\sigma(n)j_{\sigma(n)}]}) := C_t(u_{1j_1}, \dots, u_{nj_n}),$$

for $j_i = 1, 2, i \in N$. Observe that

$$C_t(u_{[\sigma(1)1]}, u_{[\sigma(2)j_{\sigma(2)}]}, \dots, u_{[\sigma(n)j_{\sigma(n)}]}) = F_{\sigma(1)}(a_{\sigma(1)}),$$

for $j_{\sigma(i)} = 1, 2, i = 2, \dots, n$. Hence

$$\sum_{j_{\sigma(2)}=1}^{2} \cdots \sum_{j_{\sigma(n)}=1}^{2} (-1)^{1+j_{\sigma(2)}+\cdots+j_{\sigma(n)}} C_t(u_{[\sigma(1)1]}, u_{[\sigma(2)j_{\sigma(2)}]}, \dots, \\ u_{[\sigma(n)j_{\sigma(n)}]}) = 0,$$

the last expression being the sum of an even number of terms, all equal in absolute value but with alternate signs. Analogously we have that

$$C_t(u_{[\sigma(1)2]}, u_{[\sigma(2)1]}, u_{[\sigma(3)j_{\sigma(3)}]} \dots, u_{[\sigma(n)j_{\sigma}(n)]}) \\ = \min\{F_{\sigma(1)}(b_{\sigma(1)}), F_{\sigma(2)}(a_{\sigma(2)})\}$$

for $j_{\sigma(i)} = 1, 2, i = 3, \dots, n$, and again

$$\sum_{j_{\sigma(3)}=1}^{2} \cdots \sum_{j_{\sigma(n)}=1}^{2} (-1)^{1+j_{\sigma(3)}+\cdots+j_{\sigma(n)}} \\ \times C_t(u_{[\sigma(1)2]}, u_{[\sigma(2)1]}, u_{[\sigma(3)j_{\sigma(3)}]}, \dots, u_{[\sigma(n)j_{\sigma(n)}]}) = 0.$$

For $k = 1, \dots, m$ we can show that

$$C_t(u_{[\sigma(1)2]}, \dots, u_{[\sigma(k-1)2]}, u_{[\sigma(k)1]}, u_{[\sigma(k+1)j_{\sigma(k+1)}]}, \dots, u_{[\sigma(n)j_{\sigma(n)}]}) = \min\{F_{\sigma(k)}(a_{\sigma(k)}), \min_{1\le i\le k-1} F_{\sigma(i)}(b_{\sigma(i)})\}$$

for all $j_{\sigma(i)} = 1, 2, i = k+1, \dots, n$, and again

$$\sum_{j_{\sigma(k+1)}=1}^{2} \cdots \sum_{j_{\sigma(n)}=1}^{2} (-1)^{1+j_{\sigma(k+1)}+\cdots+j_{\sigma(n)}} \times C_t(u_{[\sigma(1)2]}, \dots, u_{[\sigma(k-1)2]}, u_{[\sigma(k)1]}, u_{[\sigma(k+1)j_{\sigma(k+1)}]}, \dots, u_{[\sigma(n)j_{\sigma(n)}]}) = 0. \quad (3.4)$$

By (3.4), (3.3) reduces to

$$\sum_{j_{\sigma(m+1)}=1}^{2} \cdots \sum_{j_{\sigma(n)}=1}^{2} (-1)^{j_{\sigma(m+1)}+\cdots+j_{\sigma(n)}} \times C_t(u_{[\sigma(1)2]}, \dots, u_{[\sigma(m)2]}, u_{[\sigma(m+1)j_{\sigma(m+1)}]}, \dots, u_{[\sigma(n)j_{\sigma(n)}]}). \quad (3.5)$$

If there exists $i \in \{1, \dots, m\}$ so that $F_{\sigma(i)}(b_{\sigma(i)}) < t$ then, as before, (3.5) is zero because $F_{\sigma(i)}(a_{\sigma(i)}) \ge t, i = m+1, \dots, n$. If instead $F_{\sigma(i)}(b_{\sigma(i)}) \ge t$, for all $i = 1, \dots, m$, then $C_t = C_L$ on the terms of summation in (3.5) and hence

$$\begin{aligned}\sum_{j_{\sigma(m+1)}=1}^{2} \cdots \sum_{j_{\sigma(n)}=1}^{2} & (-1)^{j_{\sigma(m+1)}+\cdots+j_{\sigma(n)}} \times C_t(u_{[\sigma(1)2]}, \dots, u_{[\sigma(m)2]}, u_{[\sigma(m+1)j_{\sigma(m+1)}]}, \dots, u_{[\sigma(n)j_{\sigma(n)}]}) \\ &= \mu_{C_t}[U_{\sigma(1)} \le b_{\sigma(1)}, \dots, U_{\sigma(m)} \le b_{\sigma(m)}, U_{\sigma(m+1)} \in [a_{\sigma(m+1)}, b_{\sigma(m+1)}], \dots, \\ &\qquad U_{\sigma(n)} \in [a_{\sigma(n)}, b_{\sigma(n)}]] \ge 0,\end{aligned}$$

where $(U_1, \dots, U_n) \backsim C_t$ on $[0,1]^n$. The Lemma follows from arbitrariness of a and b. □

We now turn to the proof of Theorem 3.2. First note that, since $\min\{x_1, \dots, x_n\}$ yields the upper Fréchet bound, $C_t \ge C_L$. Hence by Theorem 3.1 we have

$$\sigma_{C_t,\psi}(F_1, \dots, F_n)(s) \ge t \quad (3.6)$$

and it remains to prove the converse inequality. Consider the set

$$B_s := \{x \in \mathbb{R}^n : \psi(x) < s\}.$$

If t=1, (3.6) leads to $\sigma_{C,\psi}(F_1, \dots, F_n)(s) = 1$ for every copula $C \ge C_L$. Consider $t \in [0,1)$ and assume that B_s is non-empty (otherwise $\sigma_{C_t,\psi}(F_1, \dots, F_n)(s) = 0 = t$). For an arbitrary $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n) \in B_s$, $\psi(\tilde{x}) < s$ and

$$\widehat{\psi_{\tilde{x}_{-n}}}(s) = \sup\{x_n \in \mathbb{R} : \psi(\tilde{x}_{-n}, x_n) < s\} \ge \tilde{x}_n.$$

If $\widehat{\psi_{\tilde{x}_{-n}}}(s) = \tilde{x}_n$, then $\psi(\tilde{x}_{-n}, \tilde{x}_n + \epsilon) \ge s$ for all $\epsilon > 0$, and by right-continuity of ψ in its last argument, we obtain

$$\psi(\tilde{x}) = \lim_{\epsilon \to 0} \psi(\tilde{x}_{-n}, \tilde{x}_n + \epsilon) \ge s;$$

this contradicts the fact that $\tilde{x} \in B_s$. Hence $\psi_{\widehat{\tilde{x}_{-n}}}(s) > \tilde{x}_n$ and

$$F_n^-(\psi_{\widehat{\tilde{x}_{-n}}}(s)) = \mu_{C_t}[X_n < \psi_{\widehat{\tilde{x}_{-n}}}(s)] \geq \mu_{C_t}[X_n \leq \tilde{x}_n] = F_n(\tilde{x}_n),$$

which leads to

$$\begin{aligned} C_L(F_1(\tilde{x}_1), \ldots, F_n(\tilde{x}_n)) &\leq C_L(F_1(\tilde{x}_1), \ldots, F_n^-(\psi_{\widehat{\tilde{x}_{-n}}}(s)) \\ &\leq \sup_{x_1, \ldots, x_{n-1} \in \mathbb{R}} C_L(F_1(x_1), \ldots, F_n^-(\psi_{\widehat{x_{-n}}}(s))) = t. \end{aligned} \tag{3.7}$$

From the definition of C_t, (3.7) implies that, for any $x \in B_s$,

$$C_t(F_1(x_1), \ldots, F_n(x_n)) = \min\{t, F_1(x_1), \ldots, F_n(x_n)\}. \tag{3.8}$$

Note that for $t = 0$, we have $C_0(F_1(x_1), \ldots, F_n(x_n)) = 0$, for all $x \in B_s$ and hence

$$\sigma_{C_0,\psi}(F_1, \ldots, F_n)(s) = 0.$$

We can then restrict to $t \in (0, 1)$. Define now $x^o = (x_1^o, \ldots, x_n^o)$ by

$$\begin{aligned} x_i^o &:= \sup\{x_i : F_i(x_i) < t\}, i = 1, \ldots, n-1, \\ x_n^o &:= \psi_{\widehat{x_{-n}^o}}(s) = \sup\{x_n \in \mathbb{R} : \psi(x_{-n}^o, x_n) < s\}. \end{aligned}$$

Note that x_i^o is finite for $i = 1, \ldots, n-1$ because the F_i's are (non-defective) distribution functions on $\mathbb{R}$ and

$$x_i < x_i^o \Rightarrow F_i(x_i) < t, \text{for all } i = 1, \ldots, n-1. \tag{3.9}$$

Moreover, right-continuity of the marginals implies that

$$x_i \geq x_i^o \Rightarrow F_i(x_i) \geq t, \text{for all } i = 1, \ldots, n-1. \tag{3.10}$$

This claim also holds for $i = n$. Indeed, suppose there exists $x_n' \geq x_n^o = \psi_{\widehat{x_{-n}^o}}(s)$ such that $F_n(x_n') < t$, and fix an arbitrary vector $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. If $x_i < x_i^o$ for some $i = 1, \ldots, n-1$ then $F_i(x_i) < t$ and

$$\begin{aligned} C_L(F_1(x_1), \ldots, F_i(x_i), \ldots, F_n^-(\psi_{\widehat{x_{-n}}})) &\leq C_L(1, \ldots, 1, F_i(x_i), 1, \ldots, 1) \\ &\leq F_i(x_i) < t. \end{aligned}$$

If instead $x_i \geq x_i^o$ for all $i = 1, \ldots, n-1$, then $\psi_{\widehat{x_{-n}}} \leq \psi_{\widehat{x_{-n}^o}} \leq x_n'$ and

$$\begin{aligned} C_L(F_1(x_1), \ldots, F_n^-(\psi_{\widehat{x_{-n}}})) &\leq C_L(1, \ldots, 1, F_n^-(x_n')) \\ &\leq F_n^-(x_n') < t. \end{aligned} \tag{3.11}$$

Hence we have

$$t = \sup_{x_1, \ldots, x_{n-1} \in \mathbb{R}} C_L(F_1(x_1), \ldots, F_n^-(\psi_{\widehat{x_{-n}}})) < t,$$

which is a contradiction and thus we can extend (3.10) to

$$x_i \geq x_i^o \Rightarrow F_i(x_i) \geq t, \text{for all } i = 1, \ldots, n. \tag{3.12}$$

Combining (3.8), (3.9) and (3.12) we obtain, for all $x \in B_s$,

$$\begin{aligned} &C_t(F_1(x_1), \ldots, F_n(x_n)) \\ &\quad = \begin{cases} \min\{t, F_1(x_1), \ldots, F_{i-1}(x_{i-1}), F_{i+1}(x_{i+1}), \ldots, F_n(x_n)\} & \text{if } x_i \geq x_i^o \\ & \text{for some } i \in \{1, \ldots, n-1\}; \\ \min\{t, F_1(x_1), \ldots, F_{n-1}(x_{n-1})\} & \text{if } x_n \geq x_n^o; \\ \min\{F_1(x_1), \ldots, F_n(x_n)\} & \text{if } x_i < x_i^o, i = 1, \ldots, n. \end{cases} \end{aligned}$$

Now recall that

$$\sigma_{C_t,\psi}(F_1, \ldots, F_n) = \mu_{C_t}[B_s]$$

and consider the following covering of B_s

$$B_s \subseteq \cup_{i=1}^n I_i \cup T$$

where

$$\begin{aligned} I_i &:= \{x \in \mathbb{R}^n : x_i > x_i^o\} \cap B_s, i = 1, \ldots, n, \\ T &:= \prod_{i=1}^n (-\infty, x_i^o]. \end{aligned}$$

Hence

$$\mu_{C_t}[B_s] \leq \sum_{i=1}^n \mu_{C_t}[I_i] + \mu_{C_t}[T] = \mu_{C_t}[T]$$

for C_t is constant along the i-th dimension on I_i. If $x_n^o = +\infty$ then $(x_{-n}^o, x_n) \in B_s$ for all real x_n, and hence

$$\begin{aligned} \mu_{C_t}[T] &= \mu_{C_t}[X_1 \leq x_1^o, \ldots, X_{n-1} \leq x_{n-1}^o] \\ &= \lim_{x_n \to +\infty} \mu_{C_t}[X_1 \leq x_1^o, \ldots, X_{n-1} \leq x_{n-1}^o, X_n \leq x_n] \\ &= \lim_{x_n \to +\infty} C_t(F_1(x_1^o), \ldots, F_{n-1}(x_{n-1}^o), F_n(x_n)) \leq t. \end{aligned}$$

If instead x_n^o is finite, observe that $\psi(x^o) = s$ by right-continuity of ψ in the last argument, so it is sufficient to show that

$$\mu_{C_t}[T \setminus \{x^o\}] = \mu_{C_t}[\cup_{i=1}^n \{X_{-i} \leq x_{-i}^o, X_i < x_i^o\}] \leq t. \tag{3.13}$$

For $P \subset \{1, \ldots, n\}$ define

$$A_P := \{X_i < x_i^o \text{ for } i \in P, X_i \leq x_i^o \text{ for } i \in \{1, \ldots, n\} \setminus P\}.$$

From elementary probability we have

$$\begin{aligned} &\mu_{C_t}[\cup_{i=1}^n \{X_{-i} \leq x_{-i}^o, X_i < x_i^o\}] = \mu_{C_t}[\cup_{i=1}^n A_{\{i\}}] \\ &= \sum_{1 \leq i_1 \leq n}^n \mu_{C_t}[A_{\{i_1\}}] - \sum_{1 \leq i_1 \leq i_2 \leq n} \mu_{C_t}[A_{\{i_1,i_2\}}] \\ &+ \sum_{1 \leq i_1 \leq i_2 \leq i_3 \leq n} \mu_{C_t}[A_{\{i_1,i_2,i_3\}}] - \cdots + (-1)^{n+1} \mu_{C_t}[A_{\{1,\ldots,n\}}]. \end{aligned} \tag{3.14}$$

Observe that for every non-empty P

$$A_P = \cup_{k\in\mathbb{N}} \cap_{i=1}^n \{X_i \le x_i^o - \frac{1}{k}1_P(i)\},$$

hence

$$\mu_{C_t}[A_P] = \lim_{k\to\infty} C_t(F_1(x_1^o - \frac{1}{k}1_P(1)), \dots, F_n(x_n^o - \frac{1}{k}1_P(n))).$$

If $P \cap \{1, \dots, n-1\} \neq \varnothing$ then $F_i(x_i^o - \frac{1}{k}) < t$ for some $i \in \{1, \dots, n-1\}$ and all integers k; if instead $n \in P$ then $(x_{-n}^o, x_n^o - \frac{1}{k}) \in B_s$ for all integers k by definition of x_n^o. These facts lead to

$$\begin{aligned}\mu_{C_t}[A_P] &= \lim_{k\to\infty} \min\{t, F_1(x_1^o - \frac{1}{k}1_P(1), \dots, F_n(x_n^o - \frac{1}{k}1_P(n))\} \\ &= \min\{t, \min_{i\in P} F_i^-(x_i^o)\}.\end{aligned}$$

We assume, without loss of generality, that $F_1^-(x_1^o) \le \dots \le F_{n-1}^-(x_{n-1}^o)$. Noting that (3.11) implies that $F_n^-(x_n^o) \ge t > F_{n-1}^-(x_{n-1}^o)$, we can calculate (3.14):

$$\begin{aligned}&\mu_{C_t}[T \setminus \{u^o\}] \\ &\quad = \sum_{i=1}^{n-1} F_i^-(x_i^o) + \min\{t, F_n^-(x_n^o)\} \\ &\quad - \binom{n-1}{1} F_1^-(x_1^o) - \binom{n-2}{1} F_2^-(x_2^o) - \dots - \binom{1}{1} F_{n-1}^-(x_{n-1}^o) \\ &\quad + \binom{n-1}{2} F_1^-(x_1^o) + \binom{n-2}{2} F_2^-(x_2^o) + \dots + \binom{1}{1} F_{n-2}^-(x_{n-2}^o) \\ &\quad - \dots + (-1)^{n+1} F_1^-(x_1^o).\end{aligned}$$

Rearranging all the terms we obtain

$$\begin{aligned}&\mu_{C_t}[T \setminus \{u^o\}] \\ &= F_1^-(x_1^o)[\binom{n-1}{0} - \binom{n-1}{1} + \binom{n-1}{2} - \dots + (-1)^{n-1}\binom{n-1}{n-1}] \\ &+ F_2^-(x_2^o)[\binom{n-2}{0} - \binom{n-2}{1} + \binom{n-2}{2} - \dots + (-1)^{n-2}\binom{n-2}{n-2}] \\ &+ \dots \\ &+ F_{n-1}^-(x_{n-1}^o)[\binom{1}{0} - \binom{1}{1}] + t.\end{aligned} \tag{3.15}$$

Recall that $\sum_{i=0}^n (-1)^i \binom{n}{i} = 0$ for all integer n, hence (3.15) simplifies to

$$\mu_{C_t}[T \setminus \{u^o\}] = t,$$

which completes the proof. □

Remark 3.1. There are several points worth noting regarding this theorem.

(i) For $n = 2$, $C_L = W$ and $\psi(x) = x_1 + x_2$ we get Proposition 1 in Rüschendorf (1982) and in equivalent form Theorem 1 in Makarov (1981) and Theorem 3.2 in Frank, Nelsen, and Schweizer (1987). In these papers, as well as in Embrechts, Höing, and Juri (2003), a sharp upper bound on the df of $\psi(X)$ is also given.

(ii) The theorem cannot be strengthened to read

$$\mu_{C_t}[\psi(X) \le t] = \sup_{x_1,\dots,x_{n-1}\in\mathbb{R}} C(F_1(x_1),\dots,F_{n-1}(x_{n-1}),F_n(\psi^{\wedge}_{x_{-n}}(s))),$$

for $\sigma_{C,\psi}(F_1,\dots,F_n)(s+) := \mu_C[\psi(X) \le s]$ may have no minimum over the set $\mathfrak{C}_n$. Hence, contrary to Embrechts, Höing, and Juri (2003, page 151), (2.4) is the correct way of defining the operator σ, if one wants to state Theorem (3.2) correctly. See Nelsen (1999, page 187) for more details in the case of the sum of risks.

(iii) Note that C_t is not the unique copula attaining the bound t for we can always change it on $[0,t]^n$ by substituting for the upper Fréchet bound any other copula $C \ge C_L$.

(iv) The last part of the proof (from (3.13) on) is necessary only if all $F_i's$ are discontinuous at x^o. Indeed, if there exists $F_i, i = 1,\dots,n-1$ which is continuous at x_i^o, then $F_i(x_i^o) = t$ and

$$\mu_{C_t}[T] = \mu_{C_t}[X_1 \le x_1^o,\dots,X_n \le x_n^o] \le \mu_{C_t}[X_i \le x_i^o] = F_i(x_i^o) = t.$$

If instead it is F_n to be continuous at x_n^o, then $\mu_{C_t}[X_n = x_n^o] = 0$ and

$$\begin{aligned}\mu_{C_t}[T] &= \mu_{C_t}[X_1 \le x_1^o,\dots,X_n \le x_n^o]\\ &= \mu_{C_t}[X_1 \le x_1^o,\dots,X_{n-1} \le x_{n-1}^o, X_n < x_n^o]\\ &= \lim_{x_n \to x_n^o} \min\{t, F_1(x_1^o),\dots,F_{n-1}(x_{n-1}^o),F_n^-(x_n)\} \le t.\end{aligned}$$

To this extent our theorem generalizes Theorem 3 in Williamson and Downs (1990), where the case of multiple discontinuities was excluded. Note however that the theorem in Williamson and Downs (1990) holds for dfs on $\overline{\mathbb{R}}$ (defective dfs) also.

(v) The hypothesis of right-continuity in the last argument of the function ψ is necessary to prove sharpness of the bound. Take for instance $n = 2, X_1 \backsim F_1 = U_{\{0,\frac{1}{2},1\}}, X_2 \backsim F_2 = U_{\{0,1,1\}}$ and $\psi(x_1,x_2) = 1_{\{x_1\ge 1,x_2>1\}}$. Note that

$$\psi(\cdot,x_2) = \begin{cases} 0 & \text{if } x_1 < 1;\\ 1_{\{x_2>1\}} & \text{otherwise,}\end{cases}$$

is not right-continuous. We have

$$\psi^{\wedge}_{x_1}(1) = \begin{cases} +\infty & \text{if } x_1 < 1;\\ 1 & \text{otherwise,}\end{cases}$$

and then $\tau_{\psi,W}(F_1, F_2)(1) = \sup_{x_1\in\mathbb{R}}[F_1(x_1) + F_2^-(\widehat{\psi_{x_1}}(1)) - 1]^+ = \frac{2}{3}$. By Theorem 3.2 we should have that

$$\mu_{\frac{2}{3}}[\psi(X_1, X_2) < 1] = \frac{2}{3},$$

but this is impossible because it is evident that $\psi(X_1, X_2) = 0 < 1$ $\mathbb{P}$-a.s. for every probability measure $\mathbb{P}$ having F_1 and F_2 as marginals. The reader can verify that the theorem works with $\hat{\psi}(x_1, x_2) = 1_{\{x_1>1,x_2\geq 1\}}$. Finally, one can easily check that defining $\tau_{W,\psi}(F_1, F_2)(1) := \sup_{x_1\in\mathbb{R}}[F_1(x_1) + F_2(\widehat{\psi_{x_1}}(1)) - 1]^+$ in the above example does not lead to a more general result.

The proof of Theorem 3.2 provided in this paper is long and laborious, but the elegant proof of the same theorem given in Embrechts, Höing, and Juri (2003) contains a gap. First of all note that, as we said in Remark 3.1, (ii), the correct statement of the theorem requires the definition of the operator σ as given in (2.4). Embrechts, Höing, and Juri (2003), stated in our notation, affirm that

> In particular μ_{C_t} assigns mass t to any set $[0, u_1] \times \cdots \times [0, u_n]$ such that $C_L(u_1, \ldots, u_n) = t$, whence $\mu_{C_t}[\{C_L \leq t\}] = t \ldots$

From this the theorem follows easily. This claim is not correct, since, even in the simplest case of two uniformly distributed risks and $C_L = W$, we have that $\mu_{C_t}[\{W \leq t\}] = 1$ for every $t \in [0, 1]$. Hence the correct statement is

> In particular μ_{C_t} assigns mass t to any set $[0, u_1] \times \cdots \times [0, u_n]$ such that $C_L(u_1, \ldots, u_n) = t$, whence $\mu_{C_t}[\{C_L \leq t\}] \geq t \ldots$

This does not yield the theorem and a new proof is then required.

For applications of Theorem 3.2, including how to calculate numerically the bound for every choice of $F_1, \ldots, F_n$ and C_L, see Embrechts, Höing, and Juri (2003).

4 Main result without information on dependence

Theorem 3.2 solves problem (2.7) when $n = 2$ or $C_L > W$. The fact that the function W is not a copula for $n > 2$ causes problems. The bound in (3.1) holds in arbitrary dimensions. However, when $n > 2$ and we have no information regarding the dependence structure of the portfolio (vector) X, then the bound in (3.1) may fail to be sharp. In fact, when $n > 2$ and $C_L > W$, the function C_t defined in Theorem 3.2 fails to be a copula.

4.1 Mutually exclusive risks

Actually there is an important special case when the lower Fréchet bound is a proper df and hence sharpness of the bound still holds, also in the no-information scenario. In fact, Theorem 3.7 of Joe (1997) (based on a previous result of Dall'Aglio (1972)) gives a necessary and sufficient condition for $W(F_1, \ldots, F_n)$ to be a df having marginals $F_1, \ldots, F_n$.

Theorem 4.1. *When $n > 2$, $W(F_1, \ldots, F_n)$ is a df on $\mathbb{R}^n$ if and only if*

$$\begin{aligned} &\sum_{i=1}^{n} F_i(x_i) \leq 1 \textit{ for all } x \in \mathbb{R}^n \textit{ s.t. } 0 < F_i(x_i) < 1, i = 1, \ldots, n \textit{ or} \\ &\sum_{i=1}^{n} F_i(x_i) \geq n-1 \textit{ for all } x \in \mathbb{R}^n \textit{ s.t. } 0 < F_i(x_i) < 1, i = 1, \ldots, n. \end{aligned} \tag{4.1}$$

An example of non-negative risks which satisfy condition (4.1) and have df W can be found in Dhaene and Denuit (1999). They form the class of so-called *mutually exclusive risks*, those risks that can be positive at most one at a time. In this specific case, of actuarial interest, the bound stated in Theorem 3.1 is sharp for arbitrary finite n.

4.2 Non-negative continuous and identically distributed risks

Throughout the rest of the paper we will consider $C_L = W$. In this situation, the bound (3.1) is no longer sharp if $n > 2$, and it is convenient to express (2.6) by a duality result given in Rüschendorf (1982):

$$\begin{aligned} m_\psi(s) = 1 - \inf\{\sum_{i=1}^{n} \int f_i dF_i : f_i \text{ are bounded measurable functions on } \mathbb{R} \text{ s.t.} \\ \sum_{i=1}^{n} f_i(x_i) \geq 1_{[s,+\infty)}(\psi(x_1, \ldots, x_n)) \text{ for all } x_i \in \mathbb{R}, i = 1, \ldots, n\}. \end{aligned} \tag{4.2}$$

This dual optimization problem is very difficult to solve. The only explicit results known in literature are contained again in Rüschendorf (1982) for the case of the sum of marginals being all uniformly or binomially distributed. Unfortunately, the dependence structure which solves (4.2) in the case of the sum of uniform marginals does not work in the general case, where the solution depends upon the marginals chosen. This is much in contrast to the case of the copula C_t, which satisfies (3.2) for all choices of $F_1, \ldots, F_n$. For that reason, below we restrict our attention to $\psi(x) = \sum_{i=1}^{n} x_i$ and set all marginal dfs equal to a common df F, which we assume to be non-negative and continuous. In this situation (4.2) reads as

$$\begin{aligned} m_\psi(s) = 1 - \inf\{n \int f dF : f \text{ bounded measurable function on } \mathbb{R} \text{ s.t.} \\ \sum_{i=1}^{n} f(x_i) \geq 1_{[s,+\infty)}(\sum_{i=1}^{n} x_i) \text{ for all } x_i \in [0, +\infty)^n, i = 1, \ldots, n\} \end{aligned} \tag{4.3}$$

and it is easy to show that the bound stated in (3.1), which we call *standard bound* in the following, reduces to

$$\tau_{W,+}(F, \ldots, F)(s) = [nF(s/n) - n + 1]^+ \tag{4.4}$$

for every $s \geq nx_F^*$ where $x_F^* := \inf\{x \geq 0 : F'(r) \text{ is non-increasing for all } r \geq x\}$. For example, for the numerical example given below we obtain for Pareto$(1.5, 1)$, Log-Normal$(-0.2, 1)$ and $\Gamma(3, 1)$ the values $x_F^* = 0, 2, 0.31$, respectively.

We use (4.3) to provide a bound which is better (i.e. $\geq$) than the standard one.

Theorem 4.2. *Let F be a non-negative, continuous df. If $F_i = F, i = 1, \dots, n$, then for every $s \geq 0$*

$$m_+(s) \geq 1 - n \inf_{r \in [0, s/n)} \frac{\int_r^{s-(n-1)r} 1 - F(x)dx}{s - nr}. \tag{4.5}$$

Proof. For $r \in [0, s/n]$ define $\hat{f}_r : \mathbb{R} \to \mathbb{R}$,

$$\hat{f}_r(x) := \begin{cases} 0 & \text{if } x < r; \\ \frac{x-r}{s-nr} & \text{if } r \leq x \leq s - (n-1)r; \\ 1 & \text{otherwise.} \end{cases}$$

We prove that $\hat{f}_r$ is an admissible function in (4.3). Since $\hat{f}_r$ is non-negative, it is sufficient to show that we have $\sum_{i=1}^n \hat{f}_r(x_i) \geq 1$ when $\sum_{i=1}^n x_i \geq s$. If $x_i \geq s - (n-1)r$ for some $i = 1, \dots, n$ this trivially follows, so take $x_1, \dots, x_n \in [0, s-(n-1)x]$ with $\sum_{i=1}^n x_i \geq s$. Define

$$I := \{i \leq n : x_i \geq r\}, \quad \bar{I} := \{1, \dots, n\} \setminus I$$

and observe that we have

$$\sum_{i \in I} x_i \geq s - \sum_{i \in \bar{I}} x_i \geq s - \#(\bar{I})r.$$

By definition of $\hat{f}_r$ it follows that

$$\begin{aligned} \sum_{i=1}^n \hat{f}_r(x_i) = \sum_{i \in I} \hat{f}_r(x_i) = \sum_{i \in I} \frac{x_i - r}{s - nr} &= \frac{\sum_{i \in I} x_i - (\#I)r}{s - nr} \\ &\geq \frac{s - ((\#I) + (\#\bar{I}))r}{s - nr} \geq 1. \end{aligned}$$

The theorem follows by checking that

$$\int \hat{f}_r(x) dF(x) = 1 - \frac{\int_r^{s-(n-1)r} F(x)dx}{s - nr}$$

and taking the infimum over all $r \in [0, s/n)$. □

Remark 4.1. (i) Note that

$$\lim_{r \to s/n} \left\{ 1 - n \frac{\int_r^{s-(n-1)r} 1 - F(x)dx}{s - nr} \right\} = nF(s/n) - n + 1,$$

hence it follows that (4.5) is greater or equal than the standard lower bound given in (3.1) for every threshold s at which (4.4) is valid. In Section 5 below we actually show that (4.5) is strictly greater than (3.1) in most cases.

(ii) For $n = 2$, (4.5) gives the sharp bound already stated in (3.1).

(iii) For $n > 2$, the infimum expressed in (4.5) can be easily calculated numerically by finding the zero-derivative point of its argument in the specified interval.

The assumptions under which Theorem 4.2 is valid, though considerable with respect to the setting of the previous sections, are consistent with most dfs F and thresholds s of actuarial/financial interest. In fact, such a difference in generality of results implicitly shows that the assumption of a non-trivial lower bound on the copula C of the portfolio is very strong.

5 Applications

In this section we show the quality of the *dual* bound expressed by (4.5) with respect to the *standard* bound stated in (4.4).

5.1 Computing numerically the worst-possible VaR

A good approximation for the real value of $m_+(s)$ can be found by solving two linear problems (LPs). We follow Williamson and Downs (1990) in defining the two dfs $\underline{F}_N, \overline{F}_N$ by

$$\underline{F}_N(x) := \frac{1}{N}\sum_{i=1}^{N} 1_{[q_r,+\infty)}(x),$$
$$\overline{F}_N(x) := \frac{1}{N}\sum_{i=0}^{N-1} 1_{[q_r,+\infty)}(x),$$

the jump points $q_0, \ldots, q_N$ being the quantiles of F defined by $q_0 := \inf \operatorname{supp}(F)$, $q_N := \sup \operatorname{supp}(F)$ and $q_r := F^{-1}(\frac{r}{N}), r = 1, \ldots, N-1$. In the applications to follow we will always take $q_0 = 0$ and $q_N = +\infty$. It is straightforward to note that

$$\underline{F}_N \leq F \leq \overline{F}_N,$$

from which it follows that for every real s

$$\sigma_{C,+}(\underline{F}_N, \ldots, \underline{F}_N)(s) \leq \sigma_{C,+}(F, \ldots, F)(s) \leq \sigma_{C,+}(\overline{F}_N, \ldots, \overline{F}_N)(s)$$

and hence

$$\underline{m}_+(s) \leq m_+(s) \leq \overline{m}_+(s), \tag{5.1}$$

where $\underline{m}_+(s)$ and $\overline{m}_+(s)$ are naturally defined as:

$$\underline{m}_+(s) := \inf\{\mathbb{P}[\sum_{i=1}^{n} X_i < t] : X_i \backsim \underline{F}_N, i = 1, \ldots, n\}$$
$$\overline{m}_+(s) := \inf\{\mathbb{P}[\sum_{i=1}^{n} X_i < t] : X_i \backsim \overline{F}_N, i = 1, \ldots, n\}.$$

Given that $\underline{F}_N$ is a (possibly defective) discrete df, $\underline{m}_+(s)$ is the solution of the following LP:

$$\underline{m}_+(s) = \min_{p_{j_1,\dots,j_n}} \sum_{j_1=1}^{N} \cdots \sum_{j_n=1}^{N} p_{j_1,j_2,\dots,j_n} 1_{(-\infty,t)}(\sum_{i=1}^{n} q_{j_i}) \text{ subject to}$$

$$\begin{cases} \sum_{j_2=1}^N \sum_{j_3=1}^N \cdots \sum_{j_n=1}^N p_{j_1,\dots,j_n} & = \frac{1}{N} \quad j_1 = 1,\dots,N, \\ \sum_{j_1=1}^N \sum_{j_3=1}^N \cdots \sum_{j_n=1}^N p_{j_1,\dots,j_n} & = \frac{1}{N} \quad j_2 = 1,\dots,N, \\ \dots, & \\ \sum_{j_2=1}^N \sum_{j_3=1}^N \cdots \sum_{j_{n-1}=1}^N p_{j_1,\dots,j_n} & = \frac{1}{N} \quad j_n = 1,\dots,N, \\ 0 \le p_{j_1,\dots,j_n} \le 1 & \quad j_i = 1,\dots,N, \\ & \quad i = 1,\dots,n. \end{cases} \tag{5.2}$$

Analogously, $\overline{m}_+(s)$ is the solution of:

$$\overline{m}_+(s) = \min_{p_{j_1,\dots,j_n}} \sum_{j_1=1}^{N} \cdots \sum_{j_n=1}^{N} p_{j_1,j_2,\dots,j_n} 1_{(-\infty,t)}(\sum_{i=1}^{n} q_{(j_i-1)}) \text{ subject to}$$

$$\begin{cases} \sum_{j_2=1}^N \sum_{j_3=1}^N \cdots \sum_{j_n=1}^N p_{j_1,\dots,j_n} & = \frac{1}{N} \quad j_1 = 1,\dots,N, \\ \sum_{j_1=1}^N \sum_{j_3=1}^N \cdots \sum_{j_n=1}^N p_{j_1,\dots,j_n} & = \frac{1}{N} \quad j_2 = 1,\dots,N, \\ \dots, & \\ \sum_{j_2=1}^N \sum_{j_3=1}^N \cdots \sum_{j_{n-1}=1}^N p_{j_1,\dots,j_n} & = \frac{1}{N} \quad j_n = 1,\dots,N, \\ 0 \le p_{j_1,\dots,j_n} \le 1 & \quad j_i = 1,\dots,N, \\ & \quad i = 1,\dots,n. \end{cases} \tag{5.3}$$

Since for N tending to infinity the dfs $\overline{F}_N$ and $\underline{F}_N$ converge to the original df F, calculating $m_+(s)$ with any given level of accuracy is a matter of solving (5.2) and (5.3) for N large enough. Unfortunately, that is not a trivial task. The dimension of the two LPs is N^n rows (variables) per nN columns (constraints) and, while the length of the interval $[\underline{m}_+(s), \overline{m}_+(s)]$ asymptotically decreases as $1/N$, the computational time and the memory needed to solve the LPs increase exponentially. Finally note that a computer solution will truncate $\underline{F}_N$ at a certain finite value to perform computations. The software used automatically sets this upper limit so that (5.1) is maintained.

5.2 Plots and tables of worst-possible VaRs

In this section we illustrate the quality of the estimate of the worst-possible $\text{VaR}_\alpha(\sum_{i=1}^n X_i)$ provided by the dual bound previously found. Some dfs of actuarial and financial interest are considered for F. In Figure 5.1, standard and dual bounds for a portfolio of three Pareto-distributed risks are given. It is relevant to note that the dual bound is strictly greater than the standard one, in accordance to Remark 4.1, (i). Most importantly, the dual value always falls within the range $[\underline{m}_+(s), \overline{m}_+(s)]$, which we plot for some thresholds of interest. This range has been calculated solving (5.2) and (5.3) with $N = 180$. The two linear problems have been solved by ILOG CPLEX® C Callable Libraries. We remark that switching to $n = 4$ drastically lowers the quality of approximation to $N < 60$. In Figure 5.1, the values of $\mu_C[X_1 + X_2 + X_3 < s]$ in case of independent ($C = \Pi$) and comonotonic ($C = M$) scenarios are also given. The fact that the worst case of VaR does not occur under the comonotonicity assumption is equivalent to non-coherence of VaR as a risk measure; see Embrechts, McNeil, and Straumann (2000) for a discussion on this.

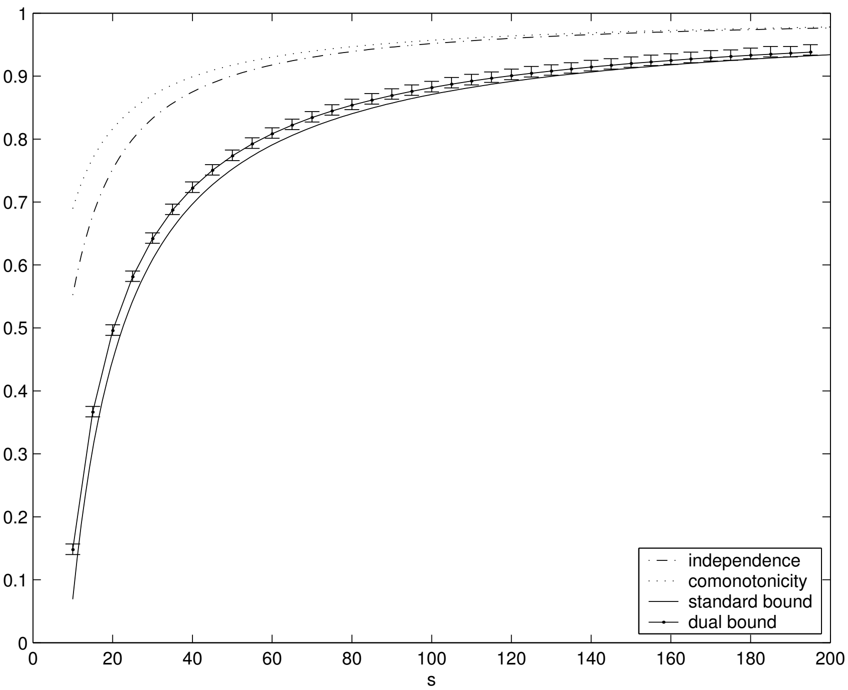

Figure 5.1: Range for $\mathbb{P}[X_1+X_2+X_3 < s]$ for a Pareto(1.5,1)-portfolio under various scenarios. Together with the independence and comonotonic situation we represent the standard and dual bound resp. given by (4.4) and (4.5). Some intervals for the true value of $m_+(s)$ are also given.

For the calculation of the distribution of the sum of comononotic random variables note that in case of common df F we have:

$$\mu_M[\sum_{i=1}^{n} X_i < s] = F(s/n),$$

while the convolution is computed by iterated conditioning:

$$\mu_\Pi[\sum_{i=1}^{n} X_i < s] = \int dF(x_n)\ldots\int dF(x_2)F(s-\sum_{i=2}^{n} x_i).$$

In Figures 5.2, 5.3 we do the same for Log-Normal and Γ portfolios.

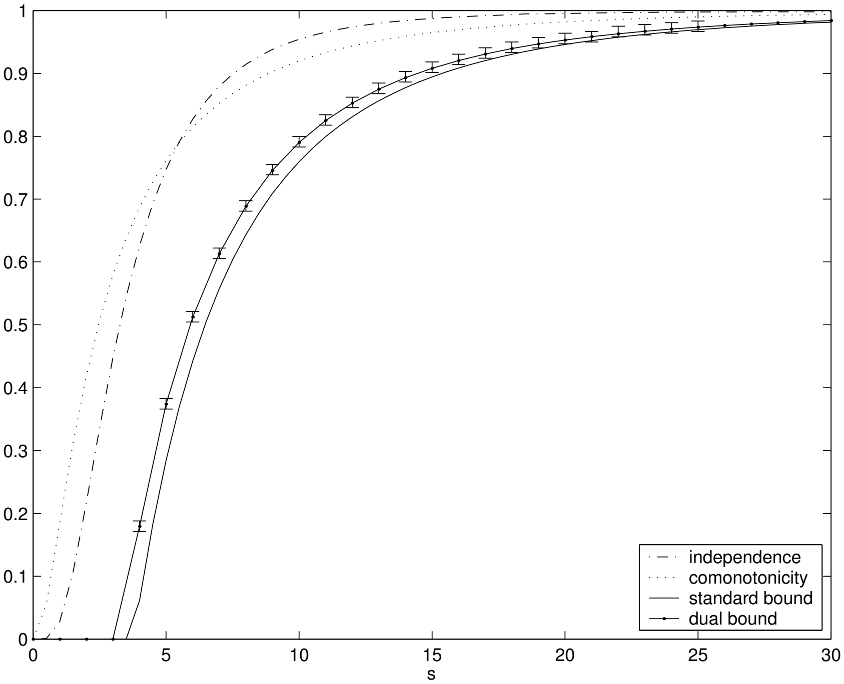

Figure 5.2: The same as Figure 1 for a Log-Normal(-0.2,1)-portfolio.

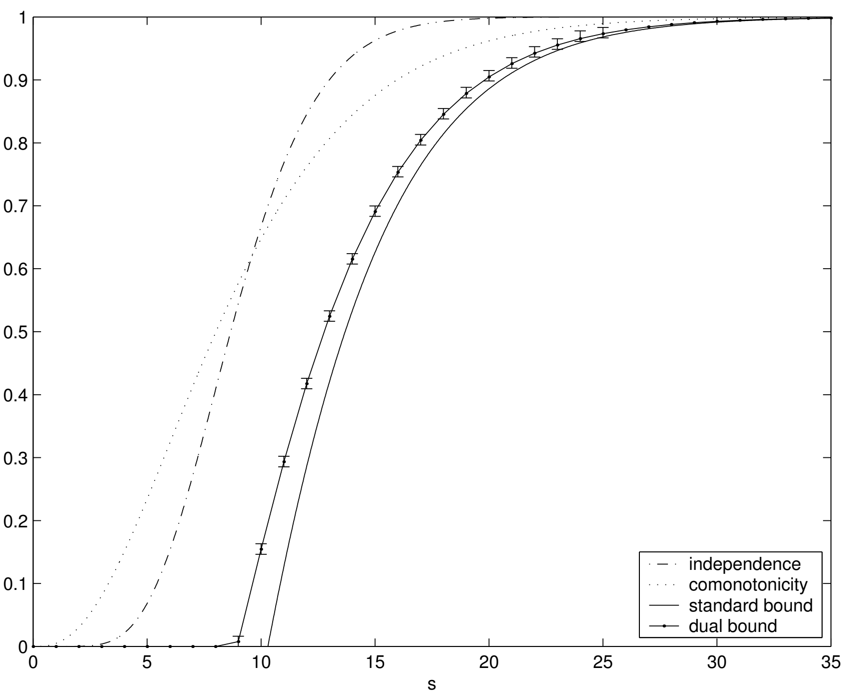

Figure 5.3: The same as Figure 1 for a $\Gamma(3,1)$-portfolio.

We finally provide some tables to emphasize the gain an end user may have in evaluating the risk of a portfolio by the the dual rather than the standard bound. In fact the lowest bound offers an evaluation of the risky position held that is prudential, more realistic and economically advantageous at the same time. The bounds on the VaR of different portfolios are numerically computed using (2.2). All values are rounded to their last digit.

	$\mathrm{VaR}_\alpha(\sum_{i=1}^3 X_i)$, *exact*		$\mathrm{VaR}_\alpha(\sum_{i=1}^3 X_i)$, *upper bound*	
α	independence	comonotonicity	dual	standard
0.90	49.71	40.50	119.06	130.50
0.95	96.80	85.50	242.61	265.50
0.99	420.76	445.50	1231.04	1345.50
0.999	871.95	4495.50	12350.90	13495.50

Table 5.1: Range for VaR for a Pareto(1.5,1)-portfolio.

	$\mathrm{VaR}_\alpha(\sum_{i=1}^3 X_i)$, *exact*		$\mathrm{VaR}_\alpha(\sum_{i=1}^3 X_i)$, *upper bound*	
α	independence	comonotonicity	dual	standard
0.90	7.54	8.85	14.44	15.38
0.95	9.71	12.73	19.50	20.63
0.99	16.06	25.16	35.31	37.03
0.999	29.78	53.99	69.98	73.81

Table 5.2: Range for VaR for a Log-Normal(-0.2,1)-portfolio.

	$\mathrm{VaR}_\alpha(\sum_{i=1}^3 X_i)$, *exact*		$\mathrm{VaR}_\alpha(\sum_{i=1}^3 X_i)$, *upper bound*	
α	independence	comonotonicity	dual	standard
0.90	13.00	15.97	19.80	20.54
0.95	14.44	18.89	22.57	23.26
0.99	17.41	25.22	28.67	29.33
0.999	21.16	33.69	36.97	37.59

Table 5.3: Range for VaR for a Γ(3,1)- portfolio.

In Table 5.4 it can be seen how the gain induced by the lowest bound increases considerably with n. A most useful fact is that the time of computation of (4.5) is not affected by the dimension of the portfolio.

	$\mathrm{VaR}_\alpha(\sum_{i=1}^{10} X_i)$		$\mathrm{VaR}_\alpha(\sum_{i=1}^{100} X_i)$		$\mathrm{VaR}_\alpha(\sum_{i=1}^{1000} X_i)$	
α	dual	standard	dual	standard	dual	standard
0.90	0.669	1.485	11.039	149.850	150.162	14998.500
0.95	1.353	2.985	22.227	229.850	301.823	29998.500
0.99	2.985	14.985	111.731	1499.850	1515.111	149998.500
0.999	68.382	149.985	1118.652	14999.850	15164.604	1499998.500

Table 5.4: Upper bounds for $\mathrm{VaR}_\alpha(\sum_{i=1}^{n} X_i)$ of three Pareto portfolios of different dimensions. Data in thousands.

6 Conclusions

The problem of finding the worst-possible VaR for a non-decreasing function of dependent risks is solved when some information on the dependence structure of the portfolio is provided or the portfolio is two-dimensional. The problem gets much more complicated in arbitrary dimensions when no information on the copula of the random vector is given. In this case we provide a new bound which we prove to be better than the standard one generally used in the literature.

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