Aggregating Risk Capital:
Lessons Learned From Basel II

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Basel II: the basics

- late nineties: more refined (granular/risk sensitive) measurement of credit risk
- A new risk class: operational risk
- overall risk capital before/after equal
- three pillar approach

- McDonough (before Cooke) Ratio:

$$\frac{\text{total amount of capital}}{\text{risk weighted assets}} \geq 8\%$$

  - regulatory capital: Tiers 1, 2 and 3
  - risk weighted assets: that's where the Pillar 1 action is

- several quantitative impact studies

- start: 2007+?

- $X_1, \dots, X_d$ $\quad d$ 1-period risks, $\mathbf{X} = (X_1, \dots, X_d)$

- $\Psi\,(X_1, \dots, X_d)$ $\quad$ a financial "instrument" / "position"

  - $\Psi(\mathbf{x}) = s_d = \sum_{k=1}^{d} x_k$

  - $\Psi(\mathbf{x}) = m_d = \max\{x_1, \dots, x_d\}$

  - $\Psi(\mathbf{x}) = \sum_{k=1}^{d} (x_k - c_k)^+$ $\quad$ excess–of–loss

  - $\Psi(\mathbf{x}) = \left(\sum_{k=1}^{d} x_k - c\right)^+$ $\quad$ stop–loss

  - $\Psi(\mathbf{x}) = m_d I_{\{s_d > q_\alpha\}}$ $\quad$ digital trigger

  - ... (credit derivatives)

- a risk or pricing measure $\mathcal{R}$
  - $\mathcal{R}(Y) = \text{VaR}_\alpha(Y)$ Value–at–Risk
  - $\mathcal{R}(Y) = E\,(Y \mid Y > \text{VaR}_\alpha(Y))$ Expected Shortfall (warning)
  - $\mathcal{R}(Y) = E\left((Y-\mu)^k\right)$ moments
  - $\mathcal{R}(Y) = F_Y$, the distribution function
  - $\ldots$

- aim, calculate $\mathcal{R}(\Psi(\mathbf{X}))$ given
  - some information on $X_1, \ldots, X_d$ $(F_1, \ldots, F_d)$
  - for a specific $\mathcal{R}$

- Questions: is this a well–posed problem?
  is this relevant for practice?
  give examples!

We start with an example:

Operational Risk: The risk of losses resulting from inadequate or failed internal processes, people and systems, or external events. Included is legal risk, excluded are strategic/business risk and reputational risk

Examples: many!
Barings, ...

Recent QISs:

- Basel Committee QIS (Moscadelli 2004, 40 000+ observations)

- Federal Reserve Bank of Boston Loss–Date Collection Exercise (Dutta and Perry 2006, 50 000+ observations)

Similar issues (as in Basel II): Solvency 2 for Insurance

The OpRisk pillar 1 issue: Loss Distribution Approach (LDA)

- Data matrix $\mathcal{X}^t = \left\{X_{ij}^t : i = 1, \ldots, 8\,\text{BLs}, \quad j = 1, \ldots, 7\,\text{RTs}\right\}$ for year t, $t = 1, \ldots, T$ (only few years so far)

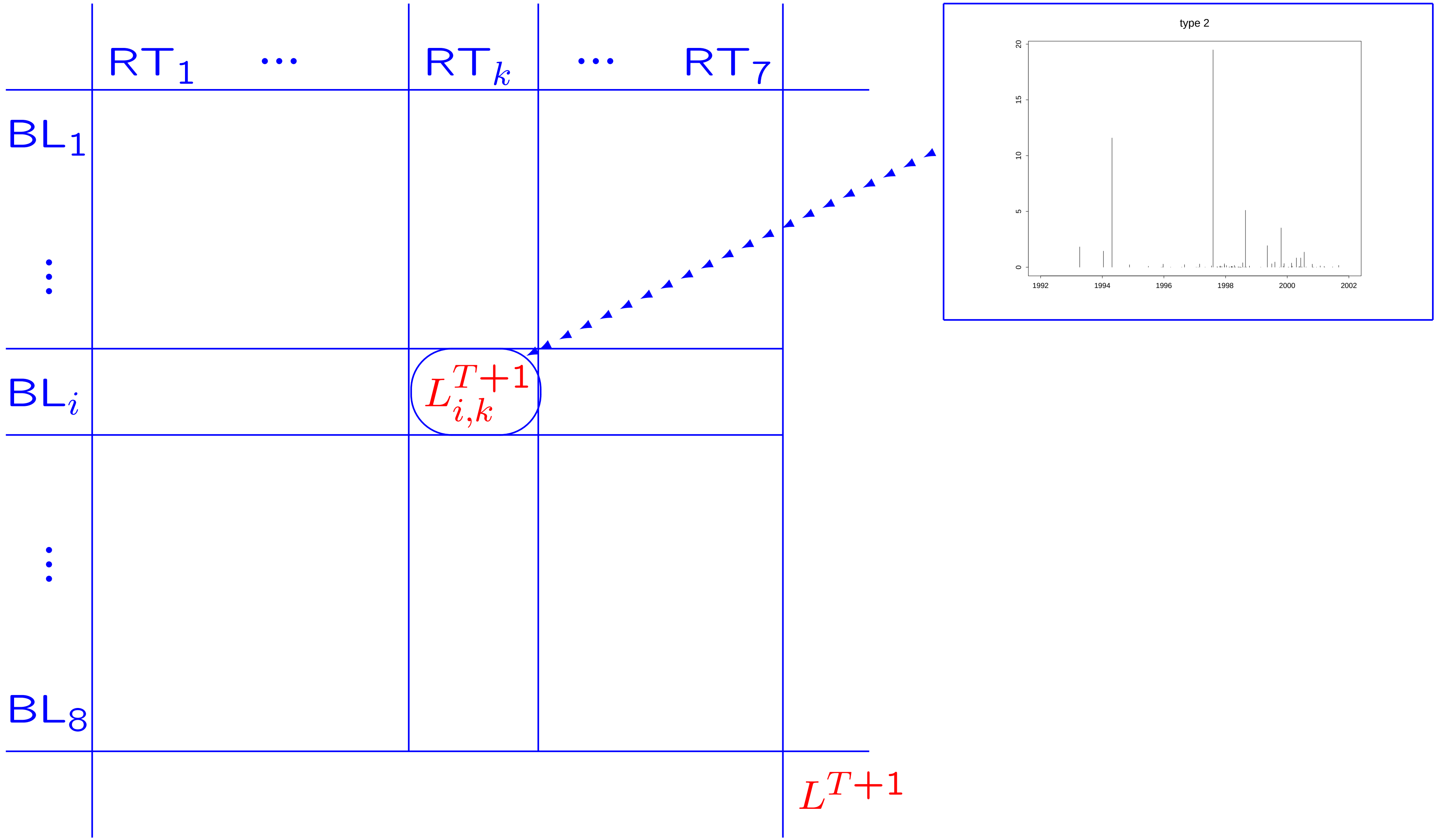
- Pillar 1 in LDA based on $\text{VaR}_{99.9\%}^{1\text{ year}}$, i.e. a 1 in 1000 year event!

- Various approaches are possible/allowed (Basel II spirit)

- Take care of: loss frequency, severity, lower truncation, and use internal/external/expert data

- insurance (up to 20%) is allowed

Loss Distribution Approach (LDA)



A complicated stochastic structure

$$L^{T+1} = \sum_{i=1}^{8} \sum_{k=1}^{7} L_{i,k}^{T+1}$$

$$L_{i,k}^{T+1} = \sum_{\ell=1}^{N_{i,k}^{T+1}} X_{i,k}^{\ell}$$

$X_{i,k}^{\ell}$: loss severities

$N_{i,k}^{T+1}$: loss frequencies

As an example, aggregate BL–wise, yielding risk capital estimates:

$$\mathrm{VaR}(1), \ldots, \mathrm{VaR}(8)$$

Calculate (as indicated in Basel **II**):

$$\mathrm{VaR}(+) = \mathrm{VaR}(1) + \cdots + \mathrm{VaR}(8)$$

Use diversification/corrrelation arguments to find the "real" capital charge

$$\mathrm{VaR} < \mathrm{VaR}(+)$$

Question: how reliable is the above procedure?

Here are some of the main issues:

- reliability of VaR(i) estimates (statistical issue)
- which arguments lead to VaR(+) reduction? (understanding correlation/diversification)
- are there practically relevant cases where

$$\text{VaR} > \text{VaR}(+) \quad ?$$

  (non–coherence of VaR)
- what about different risk measures

A related question: aggregating risk capital at the level of the bank

• MR:	$\mathrm{VaR}_{\alpha_1}^{d_1}$	$d_1 = 10$ days	$\alpha_1 = 99\%$
• CR:	$\mathrm{VaR}_{\alpha_2}^{d_2}$	$d_2 = 1$ year	$\alpha_2 = 99.9\%$
• OR:	$\mathrm{VaR}_{\alpha_3}^{d_3}$	$d_3 = 1$ year	$\alpha_3 = 99.9\%$
• EC:	$\mathrm{VaR}_{\alpha_4}^{d_4}$	$d_4 = 1$ year	$\alpha_4 = 99.97\%$

Calculate $\mathrm{VaR}_{\alpha}^{d}$ (Total) for given confidence level α and holding period d and link to EC

Problems to solve:

- scale $\text{VaR}_{\alpha_i}^{d_i}$ to a common $\text{VaR}_{\alpha}^{d}(i)$ for risk class $i = 1, 2, 3$

- calculate $\text{VaR}_{\alpha}^{d}(1)+\text{VaR}_{\alpha}^{d}(2)+\text{VaR}_{\alpha}^{d}(3)$

- discuss "diversification reduction"

Back to the concrete case of Operational Risk

- How to estimate a capital charge VaR(i) for BL i?
  - Using EVT ($\alpha = 99.9\%!$) as in Moscadelli, 2004?
  - Using the g–and–h distribution (r.v.)

    $$Y = A\frac{e^{gZ} - 1}{g}\, e^{\frac{1}{2}hZ^2} + B\,, \quad Z \sim N(0,1)$$

    as in Dutta and Perry, 2006?
  - Other methods?

- Key drivers towards choice (five “dimensions” in Dutta and Perry):
  - good fit to the available data
  - realistic wrt capital charge
  - well specified wrt data properties (e.g. losses are positive)
  - flexible wrt possible shape of loss densities
  - simple: “As simple as possible but not simpler”

- The VaR $< \sum_{i=1}^{8}$ VaR(i) issue is related to the coherence/non–coherence (in particular sub/superadditivity) of VaR:

$$\text{VaR}(X+Y) \leq \text{versus} \geq \text{VaR}(X) + \text{VaR}(Y)$$

$\geq$ may happen, even in practice

$=$ holds for comonotonic risks

$X = \Psi_1(W)\,,\quad Y = \Psi_2(W)\,,\quad \Psi_i$ increasing, $\quad W$ a rv

also yields maximal correlation between X and Y

- An important result: if $(X_1, \ldots, X_d)$ are multivariate normal then VaR is subadditive (coherent), even more generally this result holds for all multivariate elliptical vectors (e.g. t, logistic, hyperbolic, ...)

$$\mathbf{X} = \boldsymbol{\mu} + \sqrt{V}\mathbf{Z}\,, \quad \mathbf{Z} \sim N_d(\mathbf{0}, \boldsymbol{\Sigma}) \text{ independent of } V > 0$$

Why does this hold: look at the normal case

– $X \sim N\left(\mu, \sigma^2\right) \Rightarrow \text{VaR}_\alpha(X) = \mu + \sigma\Phi^{-1}(\alpha)$ (easy!)
VaR is driven by standard deviation σ and hence coherent (Cauchy–Schwartz Inequality)

– $(X_1, X_2) \sim N_2(\boldsymbol{\mu}, \boldsymbol{\sigma}, \rho)$, then $X_1 + X_2 \sim N\left(\mu_1 + \mu_2, \sigma_{1,2}^2\right)$
where $\sigma_{1,2}^2 = \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2$, hence

$$
\begin{aligned}
\text{VaR}_\alpha(X_1 + X_2) &= \mu_1 + \mu_2 + \sigma_{1,2}\Phi^{-1}(\alpha) \\
&= \mu_1 + \mu_2 + \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}\,\Phi^{-1}(\alpha) \\
(\rho \leq +1) \quad &\leq \mu_1 + \mu_2 + \sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2}\,\Phi^{-1}(\alpha) \\
&= \mu_1 + \mu_2 + (\sigma_1 + \sigma_2)\,\Phi^{-1}(\alpha) \\
&= \text{VaR}_\alpha(X_1) + \text{VaR}_\alpha(X_2)
\end{aligned}
$$

standard deviation drives subadditivity!

Under what circumstances is VaR superadditive ($>$)

– very skew distributions

– very heavy–tailed distributions

– special dependence but possible nice marginals

Are these examples relevant for practice?

- for MR perhaps not

- for CR slightly for very skew positions

- for OR highly relevant

  OpRisk data are:
  - heavy–tailed (power laws, Pareto)
  - skew
  - we have no clear view on interdependence

So "There is an issue!"

Here are the (counter)examples in short:

- Skewness
  100 iid loans: 2% yearly coupon, 100 face value, 1% yearly default, no recoverable, P&L X_i (use easy binomial calculation):

$$\text{VaR}_{95\%}\left(\sum_{i=1}^{100} X_i\right) > \text{VaR}_{95\%}(100X_1) = \sum_{i=1}^{100} \text{VaR}_{95\%}(X_i)$$

- Heavy–tailedness
  X_1, X_2 independent, $P(X_i > x) = x^{-\delta}L(x)$ with $\delta < 1$ (infinite mean!) and L slowly varying, then for α large enough

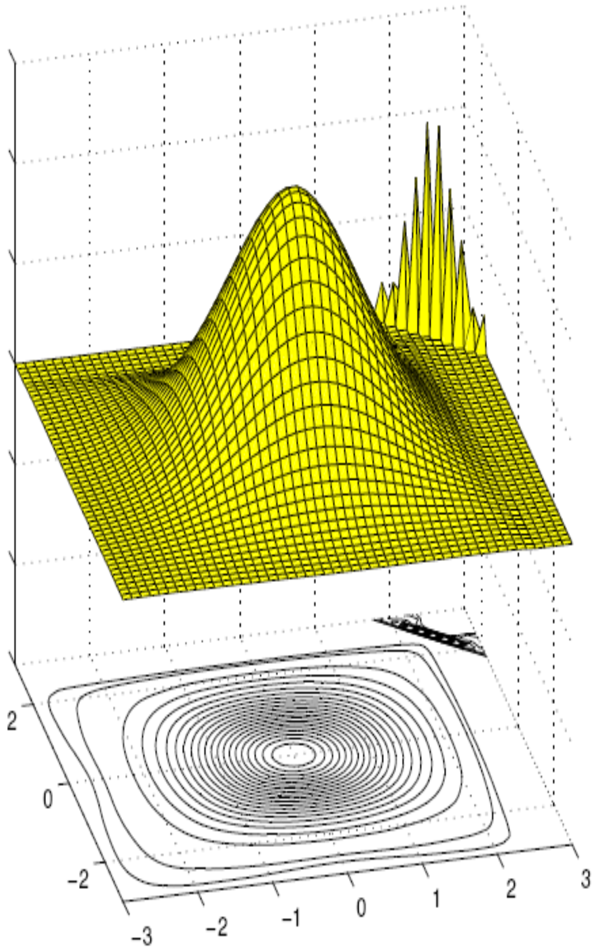
$$\text{VaR}_{\alpha}(X_1 + X_2) > \text{VaR}_{\alpha}(X_1) + \text{VaR}_{\alpha}(X_2)$$

- Special dependence

  given marginal risks $X_1 \sim F_1$, $X_2 \sim F_2$, then one can always construct a joint model F with marginals F_1, F_2 so that under F and for any $0 < \alpha < 1$:

$$\mathrm{VaR}_\alpha(X_1 + X_2) > \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2)$$

  use the notion of copula $F = C(F_1, F_2)$, in particular one can take $F_1 = F_2 = N(0,1)$ and construct the worst dependence structure with respect to VaR



“A risk manager’s worst nightmare”

Going beyond VaR, does that help?

– $\mathrm{VaR}_\alpha(X) = \inf\{\ell \in \mathbb{R} : F_X(\ell) \geq \alpha\} = F_X^{-1}(\alpha)$ (a quantile)

– Expected Shortfall

$$\begin{aligned} ES_\alpha(X) &= \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(X)\,du \qquad (1) \\ &= E\left(X \mid X \geq \mathrm{VaR}_\alpha(X)\right) \qquad \text{(careful!)} \end{aligned}$$

is always coherent ((1)), larger than VaR and yields information on the residual loss potential above VaR

– Median Shortfall

$\mathrm{Med}_\alpha(X) = \mathrm{median}\left(X \mid X \geq \mathrm{VaR}_\alpha(X)\right)$

may yield reasonable values but is in general not coherent

Conclusion (for the Operational Risk case)

- estimating $\text{VaR}_\alpha(1), \ldots, \text{VaR}_\alpha(d)$ for $\alpha = 0.999$ is a very daunting task: no easy solution!

- VaR(+) need not yield an upper limit!
  Hence discuss diversification effect with care

- No easy short–cut via scaling (90% $\rightarrow$ 99.9%), or constrained estimation (finite mean) or putting an upper limit on the losses

- Combining data (internal, external, expert opinion) may call for credibility techniques from actuarial science

- g–and–h seems promising, be careful with EVT

Back to the calculation of

$$\mathcal{R}\left(\Psi(\mathbf{X})\right) \tag{2}$$

for a risk/pricing measure $\mathcal{R}$, financial position Ψ and risk factors $\mathbf{X} = (X_1, \ldots, X_d)$

Suppose given marginal information $X_i \sim F_i$, $i = 1, \ldots, d$ and some idea of dependence, then (2) cannot be calculated explicitly, hence approximations/bounds are called for

$$\mathcal{R}_L \leq \mathcal{R}(\Psi(\mathbf{X})) \leq \mathcal{R}_U$$

Tasks: Calculate $\mathcal{R}_L$, $\mathcal{R}_U$

Prove optimality

Some concrete examples:

- homogeneous case ($F_1 = \cdots = F_8$) (easier)

$$P\,(X_i > x) = (1+x)^{-1.5}\,, \quad x \geq 0\,, \quad i = 1,\ldots,8$$

  - comonotonic case

$$\mathrm{VaR}_{0.999}\left(\sum_{i=1}^{8} X_i\right) = \sum_{i=1}^{8} \mathrm{VaR}_{0.999}\,(X_i) = 0.79$$

  - $\mathcal{R}_U = 1.93$

- heterogeneous case ($F_1 \neq \cdots \neq F_8$) (difficult)
  $P\,(X_i > x)$ as given by Moscadelli (2004)
  - Comonotonic case $\sum_{i=1}^{8} \mathrm{VaR}_{0.999}\,(X_i) = 4.8347 \times 10^5$
  - $\mathcal{R}_U = 2.3807 \times 10^6$ (factor 5!)

Final conclusions

- Quantitative tools exist for understanding aggregation issues
- In special cases, calculation is easy
- In general difficult
- Diversification effects have to be handled with care
- Definitely more research is needed
- Basel II still has some issues to solve (2007+)

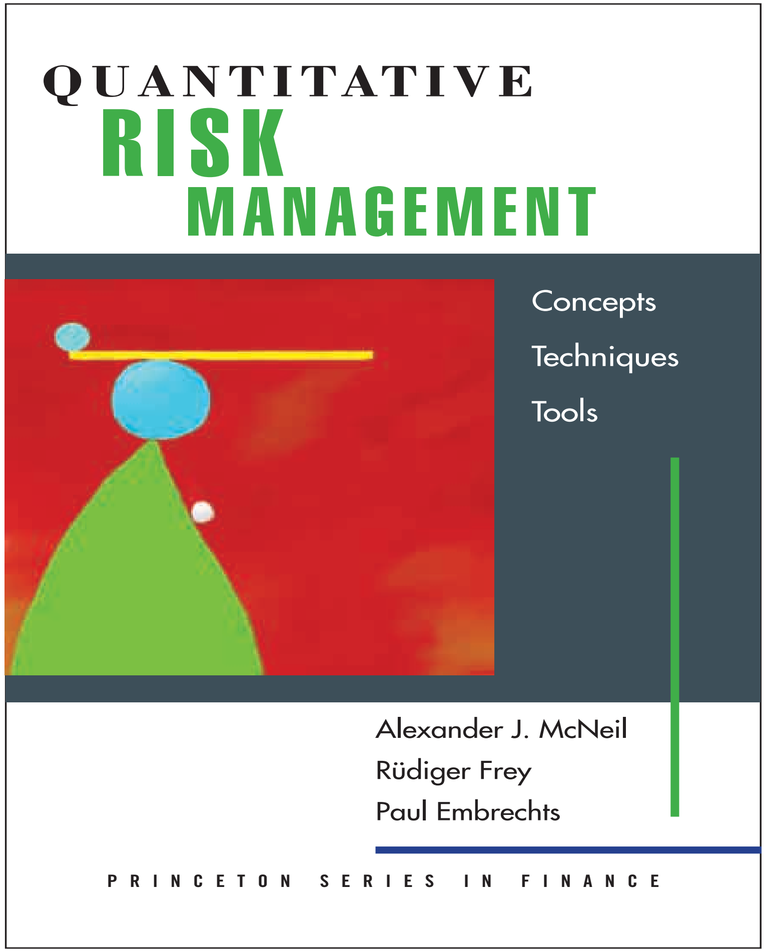
- VaR–aggregation leads to problems

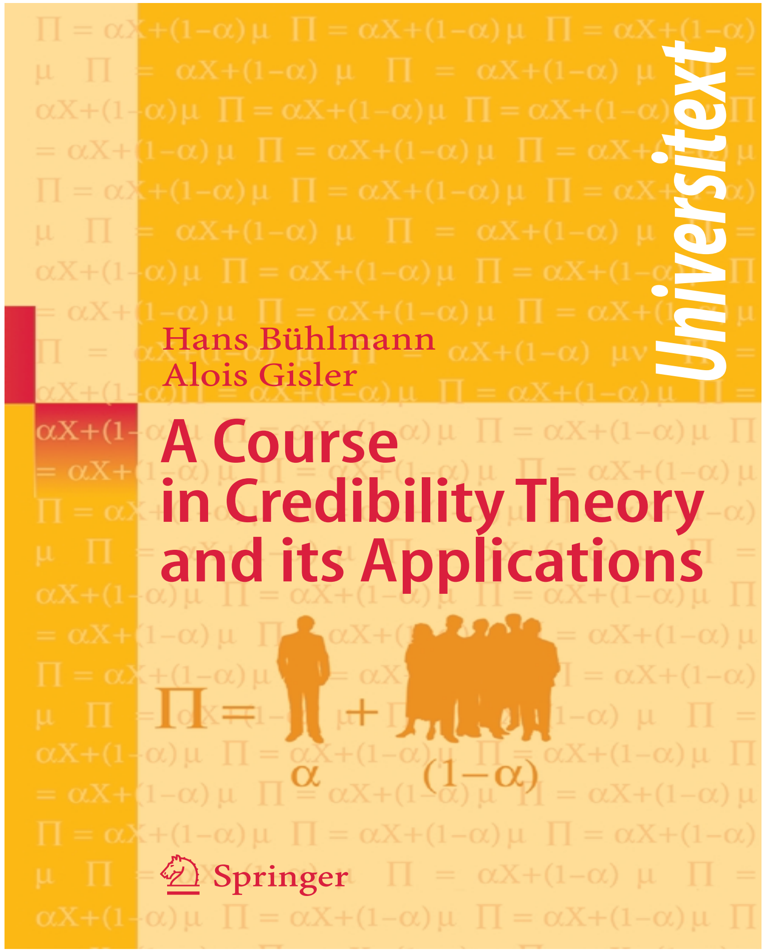
- Scaling is only understood in very specific models

- And as always

MORE WORK IS NEEDED

Further reading





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