Quantitative Models for Operational Risk

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Contents

A. The New Accord (Basel II)

- **1988**: Basel Accord (Basel I): minimal capital requirements against credit risk, one standardised approach, Cooke ratio
- **1996**: Amendment to Basel I: market risk, internal models, netting...VaR is born
- **1999**: First Consultative Paper on the New Accord (Basel II)
- **to date**: Several Consultative Papers on the New Basel Capital Accord (www.bis.org/bcbs/)
- **2007+**: full implementation of Basel II

Basel II: What is new?

- **Rationale** for the New Accord: More flexibility and risk sensitivity
- **Structure** of the New Accord: Three-pillar framework:
  1. Pillar 1: minimal capital requirements (risk measurement)
  2. Pillar 2: supervisory review of capital adequacy
  3. Pillar 3: public disclosure

- Two options for the measurement of credit risk:
  - Standard approach
  - Internal rating based approach (IRB)
- Pillar 1 sets out the minimum capital requirements (Cooke Ratio, McDonough Ratio):

$$\frac{\text{total amount of capital}}{\text{risk-weighted assets}} \geq 8\%$$

- MRC (minimum regulatory capital) $\stackrel{\text{def}}{=}$ 8% of risk-weighted assets
- Explicit treatment of operational risk

Operational Risk:

> The risk of losses resulting from inadequate or failed internal processes, people and systems, or external events.

Remark: This definition includes legal risk, but excludes strategic and reputational risk.

Note: Solvency 2

- **Notation**: C_{OP}: capital charge for operational risk
- **Target**: $C_{\text{OP}} \approx 12\%$ of minimum risk capital (down from initial 20%)
- **Estimated total losses** in the US (2001): $50b
- **Not uncommon that** $C_{\text{OP}} > C_{\text{MR}}$
- **Some examples**
  - 1977: Credit Suisse Chiasso-affair
  - 1995: Nick Leeson/Barings Bank, £1.3b
  - 2001: September 11
  - 2001: Enron (largest US bankruptcy so far)
  - 2002: Allied Irish, £450m

B. Risk measurement methods for OP risks

Pillar 1 regulatory minimal capital requirements for operational risk:

Three distinct approaches:

1. Basic Indicator Approach
2. Standardised Approach
3. Advanced Measurement Approach (AMA)

Basic Indicator Approach (BIA)

- Capital charge:

$$\boxed{C_{\mathrm{OP}}^{\mathrm{BIA}} = \alpha \times GI}$$

- $C_{\mathrm{OP}}^{\mathrm{BIA}}$: capital charge under the Basic Indicator Approach
- GI: average annual gross income over the previous three years
- $\alpha = 15\%$ (set by the Committee based on CISs)

Standardised Approach (SA)

- Similar to the BIA, but on the level of each business line:

$$C_{\mathrm{OP}}^{\mathrm{SA}} = \sum_{i=1}^{8} \beta_i \times GI_i$$

$\beta_i \in [12\%, 18\%]$, $i = 1, 2, \ldots, 8$ and 3-year averaging

- 8 business lines:

Corporate finance (18%)	Payment & Settlement (18%)
Trading & sales (18%)	Agency Services (15%)
Retail banking (12%)	Asset management (12%)
Commercial banking(15%)	Retail brokerage (12%)

Advanced Measurement Approach (AMA)

- Allows banks to use their internally generated risk estimates
- Preconditions: Bank must meet qualitative and quantitative standards before being allowed to use the AMA
- Risk mitigation via insurance possible ($\leq 20\%$ of $C_{\text{OP}}^{\text{SA}}$)
- Incorporation of risk diversification benefits allowed
- "Given the continuing evolution of analytical approaches for operational risk, the Committee is not specifying the approach or distributional assumptions used to generate the operational risk measures for regulatory capital purposes."
- Example:

Loss distribution approach

Internal Measurement Approach

- Capital charge (similar to Basel II model for Credit Risk):

$$\boxed{C_{\mathrm{OP}}^{\mathrm{IMA}} = \sum_{i=1}^{8} \sum_{k=1}^{7} \gamma_{ik}\, e_{ik}} \qquad \text{(first attempt)}$$

e_{ik}: expected loss for business line i, risk type k
γ_{ik}: scaling factor

- 7 loss types:
  - Internal fraud
  - External fraud
  - Employment practices and workplace safety
  - Clients, products & business practices
  - Damage to physical assets
  - Business disruption and system failures
  - Execution, delivery & process management

C. Loss Distribution Approach (LDA)

- For each business line/loss type cell (i, k) one models

$L_{i,k}^{T+1}$: OP risk loss for business line i, loss type k over the future (one year, say) period $[T, T+1]$

$$L_{i,k}^{T+1} = \sum_{\ell=1}^{N_{i,k}^{T+1}} X_{i,k}^{\ell} \quad \text{(next period's loss for cell } (i,k))$$

Note that $X_{i,k}^{\ell}$ is truncated from below

Remark:

Look at the structure of the loss random variable L^{T+1}

$$L^{T+1} = \sum_{i=1}^{8} \sum_{k=1}^{7} L_{i,k}^{T+1} \quad \text{(next period's total loss)}$$

$$= \sum_{i=1}^{8} \sum_{k=1}^{7} \sum_{\ell=1}^{N_{i,k}^{T+1}} X_{i,k}^{\ell}$$

A methodological pause I

$$L = \sum_{k=1}^{N} X_k \quad \text{(compound rv)}$$

where (X_k) are the severities and N the frequency

Models for X_k:

- gamma, lognormal, Pareto (≥ 0, skew)

Models for N:

- binomial (individual model)
- Poisson(λ) (limit model)
- negative binomial (randomize λ as a gamma rv)

Loss Distribution Approach continued

Choose:

- **Period** T
- **Distribution** of $L_{i,k}^{T+1}$ for each cell i, k
- **Interdependence** between cells
- **Confidence level** $\alpha \in (0, 1)$, $\alpha \approx 1$
- **Risk measure** g_α

Capital charge for:

- **Each cell**: $C_{i,k}^{T+1,\text{OR}} = g_\alpha(L_{i,k}^{T+1})$
- **Total OR loss**: $C^{T+1,\text{OR}}$ based on $C_{i,k}^{T+1,\text{OR}}$

Basel II proposal

- **Period**: one year
- **Distribution**: should be based on
  - internal data/models
  - external data
  - expert opinion
- **Confidence level**: $\alpha = 99.9\%$, for economic capital purposes even $\alpha = 99.95\%$ or $\alpha = 99.97\%$
- **Risk measure**: VaR_α
- **Total capital charge**:

$$C^{T+1,\mathrm{OR}} = \sum_{i,k} \mathrm{VaR}_\alpha(L_{i,k}^{T+1})$$

  - possible reduction due to correlation effects

Basel II proposal: Some issues

- **Very high confidence level**:
  - lack of data, difficult (if not impossible) in-sample estimation
  - high variability/uncertainty
  - robustness, scaling
- **Distribution of $L_{i,k}^{T+1}$**:
  - extreme value theory necessarily enters
  - credibility theory (combination of internal data, expert opinion and external data)
  - non-stationarity, dependence, inhomogeneity, contamination ...
- **Choice of VaR as a risk measure**:
  - VaR is not subadditive
  - other risk measures exist, but require finite mean
- **"Correlation effects"**:
  - dynamic dependence models between loss processes
  - multivariate extreme value theory, copulas ...

Summary

- Marginal VaR calculations

$$\mathrm{VaR}_\alpha^1, \dots, \mathrm{VaR}_\alpha^l$$

- Global VaR estimate

$$\mathrm{VaR}_\alpha^+ = \mathrm{VaR}_\alpha^1 + \cdots + \mathrm{VaR}_\alpha^l$$

- Reduction because of “correlation effects”

$$\mathrm{VaR}_\alpha < \mathrm{VaR}_\alpha^+$$

- Further possibilities: insurance, pooling, ...

Subadditivity

A risk measure g_α is called subadditive if

$$g_\alpha(X+Y) \leq g_\alpha(X) + g_\alpha(Y)$$

VaR_α is in general not subadditive:

- skewness
- special dependence
- very heavy-tailed losses

VaR_α is subadditive for:

- elliptical distributions

Skewness

- 100 iid loans: 2%-coupon, 100 face value, 1% default probability (period: 1 year):

$$X_i = \begin{cases} -2 & \text{with probability } 99\% \\ 100 & \text{with probability } 1\% \quad (\text{loss}) \end{cases}$$

- Two portfolios $L_1 = \sum_{i=1}^{100} X_i$, $L_2 = 100X_1$
- $\underbrace{\mathrm{VaR}_{95\%}(L_1)}_{\mathrm{VaR}_{95\%}\left(\sum_{i=1}^{100} X_i\right)} > \underbrace{\mathrm{VaR}_{95\%}(100X_1)}_{\sum_{i=1}^{100} \mathrm{VaR}_{95\%}(X_i)}$ (!)

Special dependence

- Given rvs $X_1, \ldots, X_n$ with marginal dfs $F_1, \ldots, F_n$, then one can always find a copula C so that for the joint model

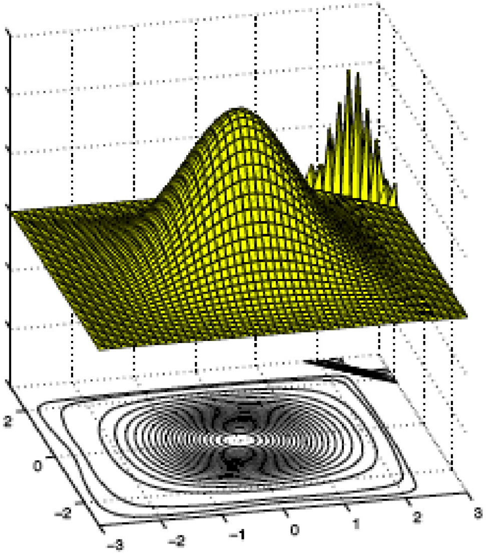
$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

  VaR_α is superadditive:

$$\mathrm{VaR}_\alpha\left(\sum_{k=1}^{n} X_k\right) > \sum_{k=1}^{n} \mathrm{VaR}_\alpha(X_k)$$

- In particular, take the "nice" case

$$F_1 = \cdots = F_n = N(0,1)$$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3

Very heavy-tailedness

- Take X_1, X_2 independent with $\mathrm{P}(X_i > x) = x^{-1/2}$, $x \geq 1$ then for $x > 2$

$$\mathrm{P}(X_1 + X_2 > x) = \frac{2\sqrt{x-1}}{x} > \mathrm{P}(2X > x)$$

so that

$$\mathrm{VaR}_\alpha(X_1 + X_2) > \mathrm{VaR}_\alpha(2X_1) = \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2)$$

- Similar result holds for

$$\mathrm{P}(X_i > x) = x^{-1/\xi} L(x),$$

with $\xi > 1$, L slowly varying
- For $\xi < 1$, we obtain subadditivity!

WHY?

Several reasons:

- (Marcinkiewicz-Zygmund) Strong Law of Large Numbers
- Argument based on stable distributions
- Main reason however comes from functional analysis

> *In the spaces $\mathcal{L}^p$, $0 < p < 1$, there exist no convex open sets other than the empty set and $\mathcal{L}^p$ itself.*

  Hence as a consequence 0 is the only continuous linear functional on $\mathcal{L}^p$; this is in violent contrast to $\mathcal{L}^p$, $p \geq 1$

- Discussion:
  - no reasonable risk measures exist
  - diversification goes the wrong way

Definition

An $\mathbb{R}^d$-valued random vector $\mathbf{X}$ is said to be regularly varying if there exists a sequence (a_n), $0 < a_n \uparrow \infty$, $\mu \neq 0$ Radon measure on $\mathcal{B}\left(\overline{\mathbb{R}}^d \backslash \{0\}\right)$ with $\mu(\overline{\mathbb{R}}^d \backslash \mathbb{R}) = 0$, so that for $n \to \infty$,

$$nP(a_n^{-1}\mathbf{X} \in \cdot) \to \mu(\cdot) \quad \text{on } \mathcal{B}\left(\overline{\mathbb{R}}^d \backslash \{0\}\right).$$

Note that:

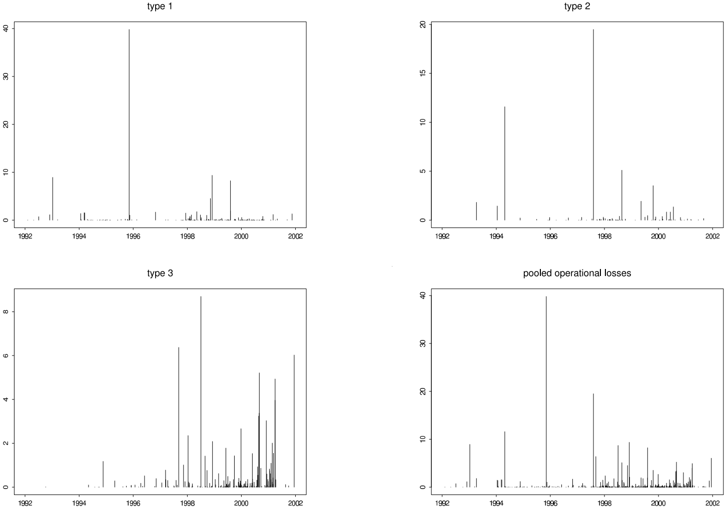
- $(a_n) \in RV_\xi$ for some $\xi > 0$
- $\mu(uB) = u^{-1/\xi}\mu(B)$ for $B \in \mathcal{B}\left(\overline{\mathbb{R}}^d \backslash \{0\}\right)$

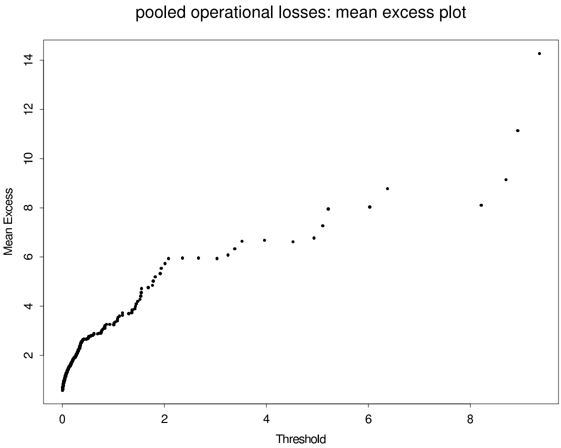
Theorem (several versions – Samorodnitsky)

If $(X_1, X_2)' \in RV_{-1/\xi}$, $\xi < 1$, *then for* α *sufficiently close to* 1, VaR_α *is subadditive.*

Is this relevant for Operational Risk?

Some data





- $\mathrm{P}(L > x) \sim x^{-1/\xi} L(x)$

- Stylized facts about OP risk losses:
  - Loss amounts show extremes
  - Loss occurence times are irregularly spaced in time (reporting bias, economic cycles, regulation, management interactions, structural changes, . . .)
  - Non-stationarity (frequency(!), severity(?))
- Large losses are of main concern
- Repetitive versus non-repetitive losses
- **However**: severity is of key importance

A methodological pause II

- severity models need to go beyond the classical models (binomial, homogeneous Poisson, negative binomial: $\longrightarrow$ stochastic processes)
- as stochastic processes:
  - Poisson(λt), $\lambda > 0$ deterministic (1)
  - Poisson($\lambda(t)$), $\lambda(t)$ deterministic
    non-homogeneous Poisson, via time change $\longrightarrow$ (1)
  - Poisson($\Lambda(t)$), $\Lambda(t)$ stochastic process
    - double stochastic (or Cox-) process
    - basic model for credit risk
- industry example: (NB,LN)
- desert island model: (Poisson, Pareto)

Analysis of the Basel II data

- $\mathrm{P}(L_i > x) = x^{-1/\xi_i} L_i(x)$

Business line	$\widehat{\xi_i}$
Corporate finance	1.19 (*)
Trading & sales	1.17
Retail banking	1.01
Commercial banking	1.39 (*)
Payment & settlement	1.23
Agency services	1.22 (*)
Asset management	0.85
Retail brokerage	0.98
	* means significant at 95% level

$\widehat{\xi_i} > 1$: infinite mean

- **Remark**: different picture at level of individual banks

Some issues regarding infinite mean models

- Reason for $\xi > 1$?
- Potentially:
  - wrong analysis
  - EVT conditions not fulfilled
  - contamination, mixtures
- We concentrate on the latter:
  Two examples:
  - Contamination above a high threshold
  - Mixture models
- Main aim: show through examples how certain data-structures can lead to infinite mean models

Contamination above a high threshold

Example (1)

Consider the model

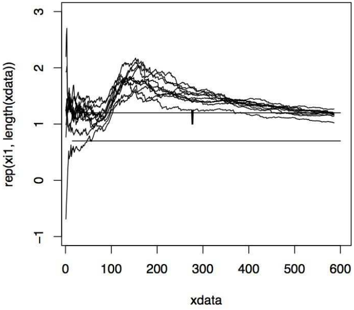
$$F_X(x) = \begin{cases} 1 - \left(1 + \frac{\xi_1 x}{\beta_1}\right)^{-1/\xi_1} & \text{if} \quad x \le v, \\ 1 - \left(1 + \frac{\xi_2 (x - v^*)}{\beta_2}\right)^{-1/\xi_2} & \text{if} \quad x > v, \end{cases}$$

where $0 < \xi_1 < \xi_2$ and $\beta_1, \beta_2 > 0$.

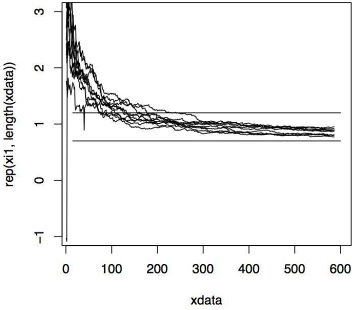
- v^* is a constant depending on the model parameters in a way that F_X is continuous
- VaR can be calculated explicitly:

$$\mathrm{VaR}_\alpha(X) = \begin{cases} \frac{1}{\xi_1}\beta_1 \left((1-\alpha)^{-\xi_1} - 1\right) & \text{if} \quad \alpha \le F_X(v), \\ v^* + \frac{1}{\xi_2}\beta_2 \left((1-\alpha)^{-\xi_2} - 1\right) & \text{if} \quad \alpha > F_X(v). \end{cases}$$

Shape plots

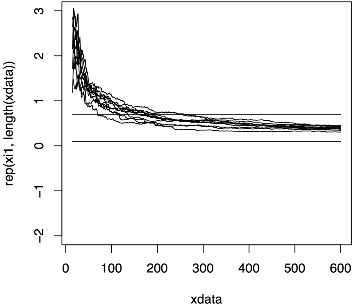


Easy case: v low



Hard case: v high

Shape plots



Careful: similar picture for v high and $\xi_1 \ll \xi_2 < 1$

Contamination above a high threshold continued

- **Easy case:** v **low**
  - Change of behavior typically visible on the mean excess plot
- **Hard case:** v **high**
  - Typically only few observations above v
  - Mean excess plot may not reveal anything
  - Classical POT analysis easily yields incorrect resuls
  - Vast overestimation of VaR possible

Mixture models

Example (2)

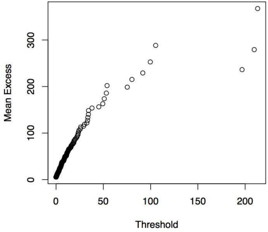
Consider

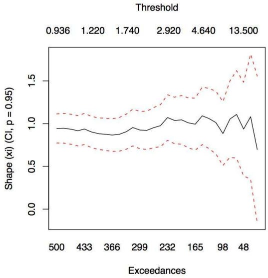
$$F_X = (1-p)F_1 + pF_2,$$

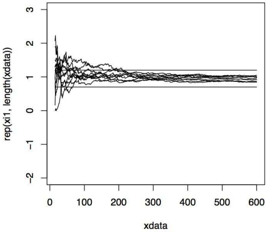
with F_i exact Pareto, i.e. $F_i(x) = 1 - x^{-1/\xi_i}$ for $x \geq 1$ and $0 < \xi_1 < \xi_2$.

- Asymptotically, the tail index of F_X is ξ_2
- VaR_α can be obtained numerically and furthermore
  - does not correspond to VaR_α of a Pareto distribution with tail-index ξ^*
  - equals VaR_{α^*} corresponding to F_2 at a level α^* lower than α

- Classical POT analysis can be very misleading:







Mixture models continued

α	$\text{VaR}_\alpha(F_X)$	$\text{VaR}_\alpha(\text{Pareto}(\xi_2))$	ξ^*
0.9	6.39	46.42	0.8
0.95	12.06	147.36	0.83
0.99	71.48	2154.43	0.93
0.999	2222.77	10^5	1.12
0.9999	10^5	$4.64 \cdot 10^6$	1.27
0.99999	$4.64 \cdot 10^6$	$2.15 \cdot 10^8$	1.33

Value-at-Risk for mixture models with $p = 0.1$, $\xi_1 = 0.7$ and $\xi_2 = 1.6$.

Back to the Basel II data:

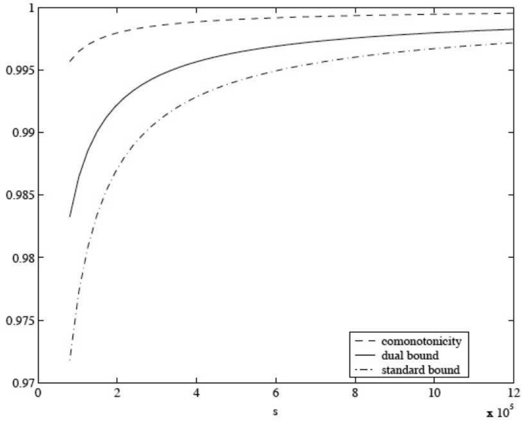


Figure: Bounds on $\mathbb{P}[\sum_{i=1}^{8} X_i < s]$ using the OR portfolio given in Moscadelli (2004), together with the comonotonic scenario.

α	comonotonic value	dual bound	standard bound
0.99	2.8924×10^4	1.4778×10^5	2.6950×10^5
0.995	6.7034×10^4	3.3922×10^5	6.1114×10^5
0.999	4.8347×10^5	2.3807×10^6	4.1685×10^6
0.9999	8.7476×10^6	4.0740×10^7	6.7936×10^7

Table: Range for $\text{VaR}_\alpha\left(\sum_{i=1}^{8} X_i\right)$ for the OR portfolio given in Moscadelli (2004).

E. One loss causes ruin problem

- based on Lorenz curve in economics
  - $20 - 80$ rule for $1/\xi = 1.4$
  - $0.1 - 95$ rule for $1/\xi = 1.01$

- for $L = L_1 + \cdots + L_d$, L_k's iid and subexponential we have that

$$P(L > x) \sim P(\max(L_1, \ldots, L_d) > x)$$
$$P(L > x) \sim dP(L_1 > x)$$

- if $L_k = \sum_{i=1}^{N_k} X_i(k)$ and some extra conditions we have that for heavy tailed loss distributions (Pareto, subexponential)

$$P(L > x) \sim cP(X(1) > x)$$

"The one-cell-dominates-all rule"

The one-cell-dominates-all rule

The basic result: *Embrechts, Goldie and Veraverbeke, ZfW, 1979*

- Suppose F is a df on $[0,\infty)$ which is infinitely divisible with Lévy measure ν, i.e.

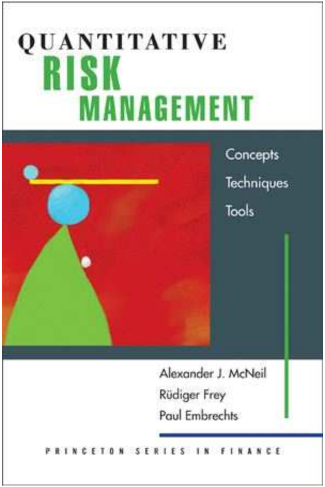
$$f(s) = \int_0^\infty e^{-sx} dF(x) = \exp\left\{-as - \int_0^\infty (1-e^{-sx})\nu(dx)\right\}$$

$a \geq 0$, ν Borel measure on $(0,\infty)$, $\int_0^1 x\nu(dx) < \infty$ and $\mu = \nu(1,\infty) < \infty$

- Then equivalent are:

(i)	$F \in \mathcal{S}$
(ii)	$\mu^{-1}\nu(1,x] \in \mathcal{S}$
(iii)	$1 - F(x) \sim \nu(x,\infty)$ as $x \to \infty$

- Link to compound Poisson dfs $F = F_1 * F_2$ where F_1 is $\mathrm{CP}(\nu)$, $\overline{F}_2(x) = o(e^{-\varepsilon x})$, $\varepsilon > 0$, $x \to \infty$.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F. References

- Embrechts, P., Klüppelberg, C., and Mikosch, T. (1997). *Modelling Extremal Events for Insurance and Finance.* Springer.
- McNeil, A.J., Frey, R., and Embrechts, P. (2005) *Quantitative Risk Management: Concepts, Techniques and Tools.* Princeton University Press.
- Moscadelli, M. (2004) The Modelling of Operational Risk: Experience with the Analysis of the Data, Collected by the Basel Committee, Banaca d'Italia, report 517-July 2004
- Nešlehová, J., Embrechts, P. and Chavez-Demoulin, V. (2006) Infinite mean models and the LDA for operational risk. Journal of Operational Risk, 1(1), 3-25.