

Ruin, Operational Risk and How Fast Stochastic Processes Mix

Paul Embrechts
ETH Zürich

Based on joint work with:

- Roger Kaufmann (ETH Zürich)
- Gennady Samorodnitsky (Cornell University)

Basel Committee for Banking Supervision ([3], [7])

- Basel Accord (= **Basel I**) (1988)
 - credit risk
- **Amendment** to Basel I (1996)
 - market risk
 - netting
 - derivatives, Value-at-Risk based
- **Basel II** (1998–2005/6)
 - (internal) rating models for credit risk
 - increased granularity
 - new risk category: operational risk
- Increased **collaboration** between insurance- and banking supervision: integrated risk management

Definition(s) of operational risk:

- (non)definition (early): the complement of market risk
- coming from DFA: the company specific risk, uncorrelated with capital markets, non-systematic part (frictional costs)
- current definition in use through Basel II:
Operational risk is the risk of losses resulting from inadequate or failed internal processes, people and systems or from external events.

Some examples:

- Barings, £700,— Mio
- Allied Irish (Allfirst subsidiary), US\$700,— Mio
- Bank of New York, US\$140,— Mio

Disclosed figures:

- 2001 Annual Reports, disclosure for economic capital for operational risk:
 - Deutsche Bank: € 2.5 Bio
 - JP Morgan-Chase: US\$ 6.8 Bio
- Estimated total losses 2001 in USA: US\$ 50 Bio

September 2001 BIS Quantitative Impact Study:

- credit (51%), market (23%), **operational (16%)**, other (10%)

Three Pillar concept of Basel II:

- Pillar I: Minimal Capital Requirement
- Pillar II: Supervisory Review Process
- Pillar III: Market Discipline Requirement

These apply to both credit- as well as operational risk

Pillar I (Minimal Capital Requirement) for Operational Risk

- The Basic Indicator Approach:

- $RC(OR) = \alpha GI$

- The Standardized Approach:

- $RC(OR) = \sum_{i=1}^8 \beta_i GI_i$

- The Advanced Measurement Approach:

- $RC(OR) = \sum_{i=1}^8 \sum_{k=1}^7 \gamma_{i,k} e_{i,k}$

- $RC(OR) = \sum_{i=1}^8 \rho_{i,k}$

Eight standardized business lines:

- Corporate Finance; Trading and Sales; Retail Banking; Payment and Settlement; Agency Services; Commercial Banking; Asset Management; Retail Brokerage

Seven loss types:

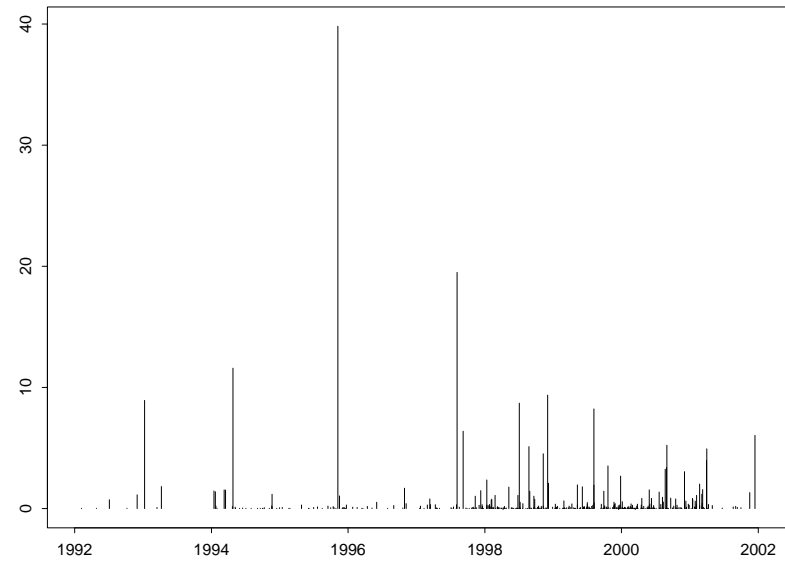
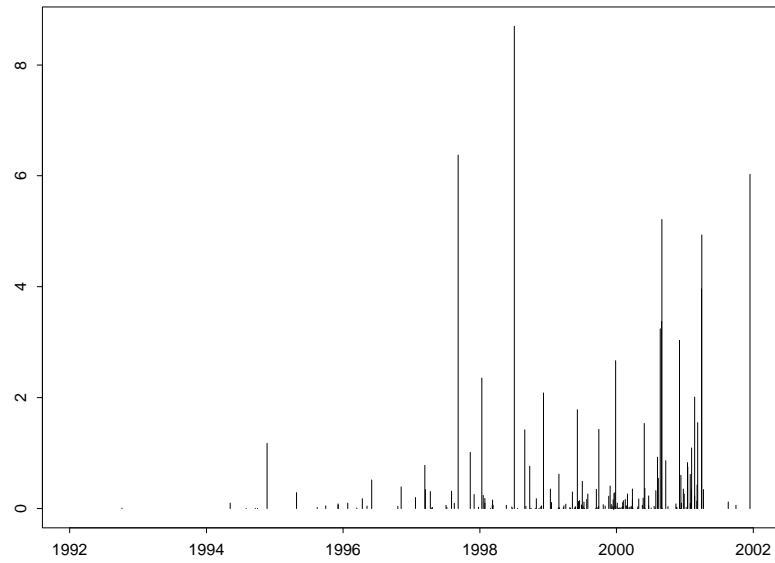
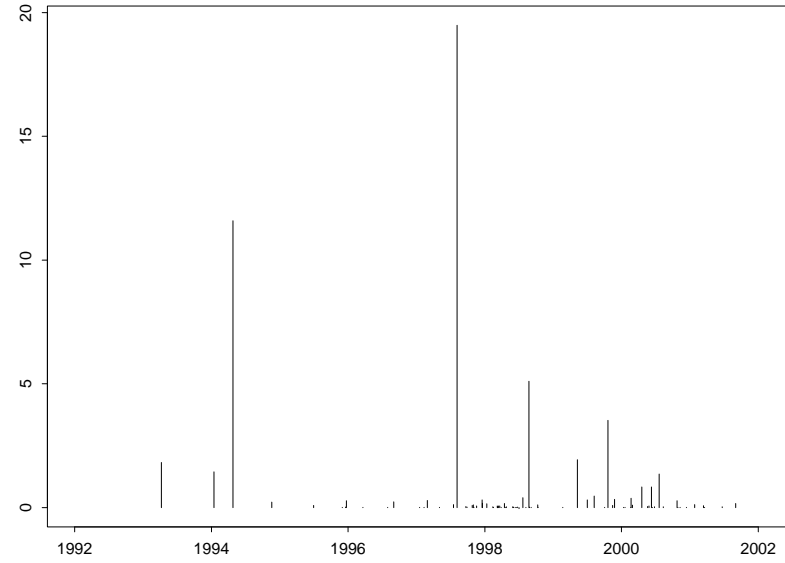
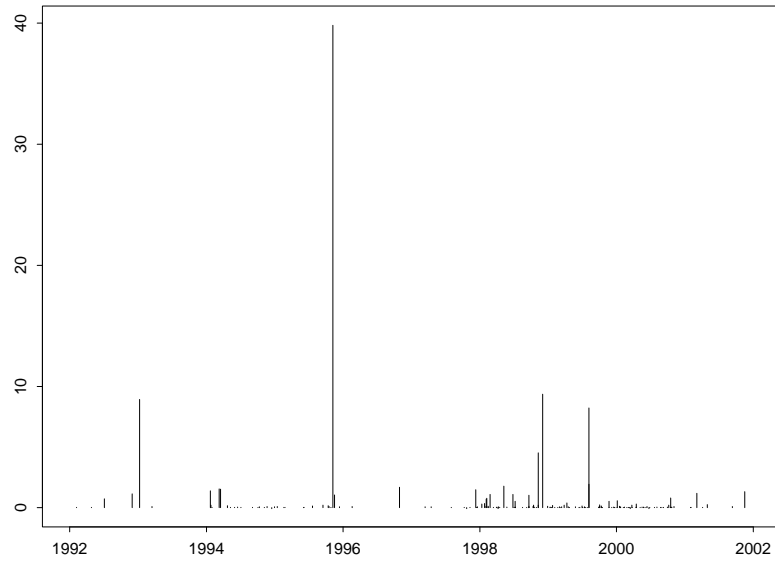
- Internal Fraud; External Fraud; Employment Practices and Workplace Safety; Clients, Products and Business Practices; Damage to Physical Assets; Business Disruption and System Failure; Execution, Delivery and Process Management

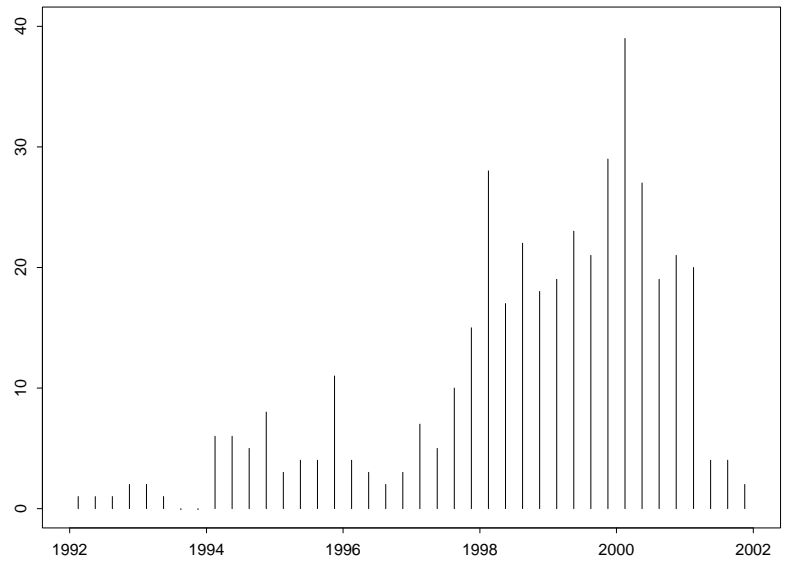
In total: 56 categories to model!

Some **critical** remarks:

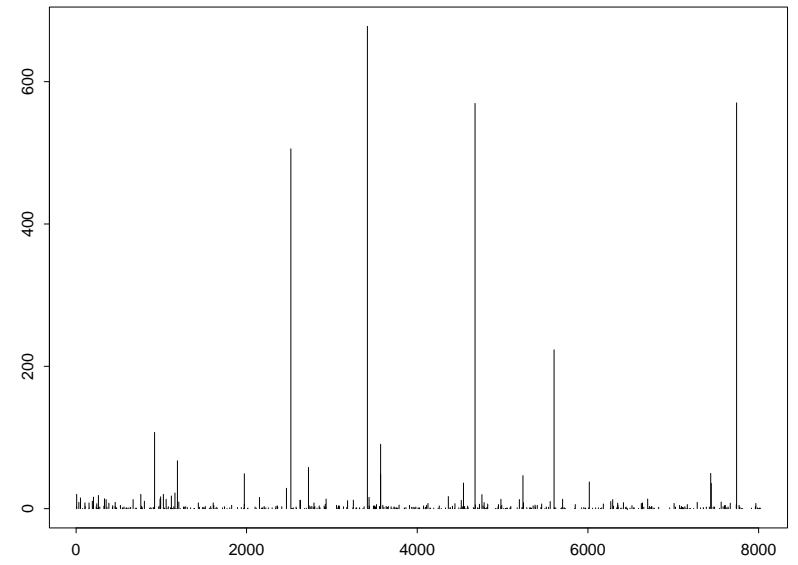
- **business risk** (though very important) is explicitly **excluded**
- **distinguish** between
 - repetitive versus non-repetitive losses
 - low frequency/high impact versus high frequency/low impact
- **lack of data**, data pooling (?), near misses (??)
- **Pillar II** very important
- for the moment: **qualitative** >> quantitative
- overall complexity (Comptroller of the Currency)

Some data ([5])

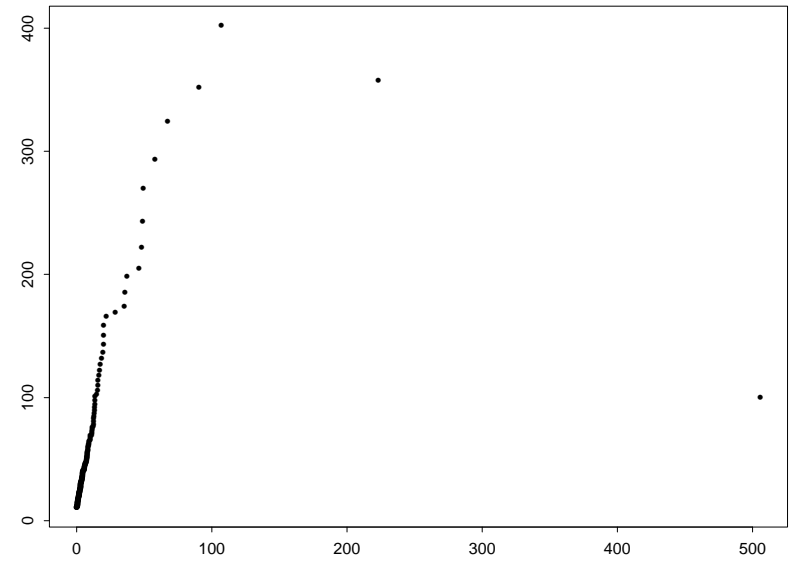
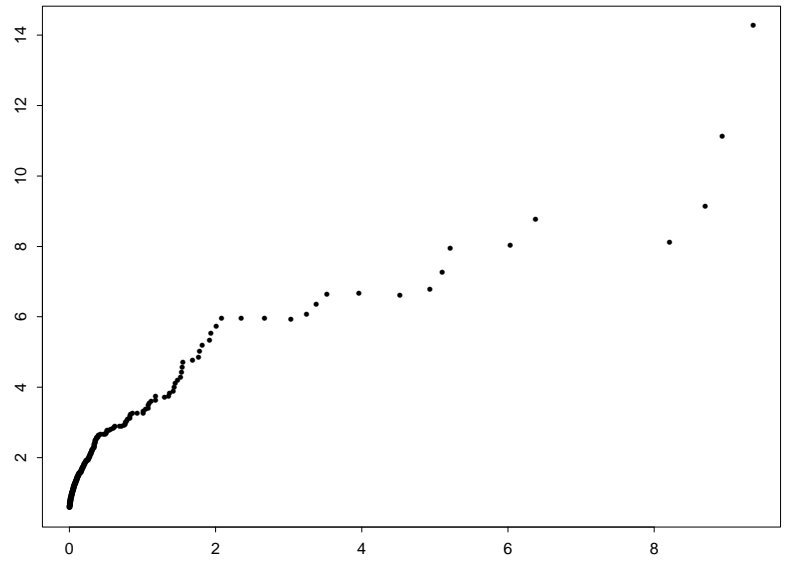


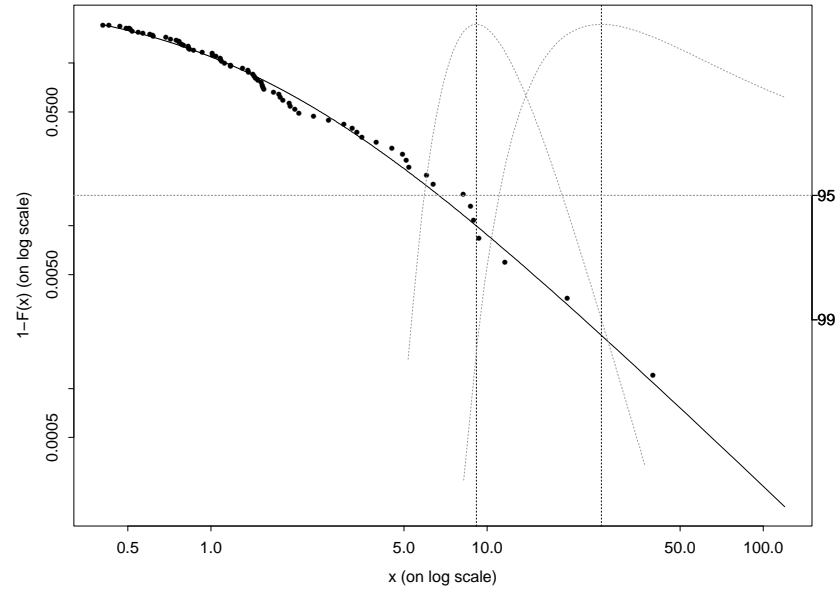


operational risk

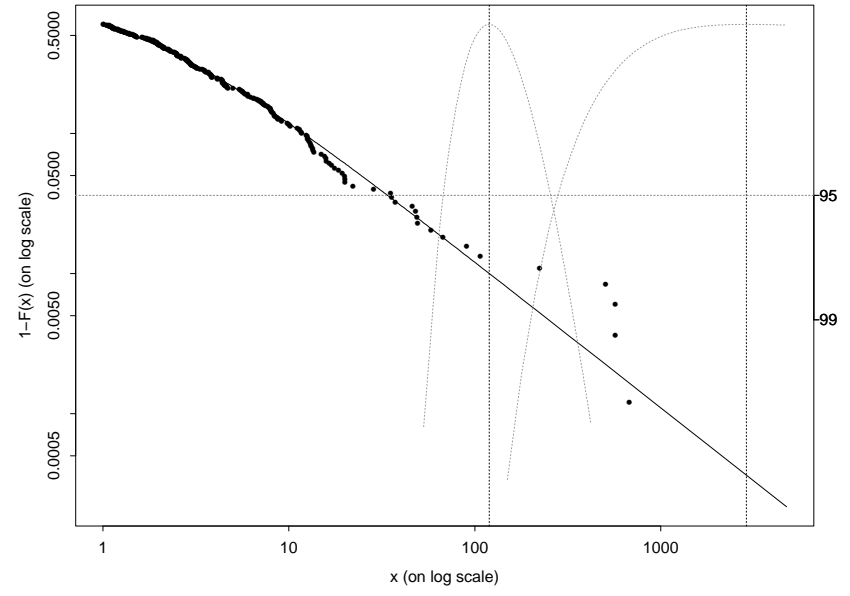


fire loss





operational risk



fire loss

A mathematical (actuarial) model:

- Operational Risk loss **database** (for each business line)

$$\{X_k^{t,i} : t = 1, \dots, T; i = 1, \dots, 7; k = 1, \dots, N^{t,i}\}$$

t (years), i (loss type), k (number of losses)

- **Truncation**

$$X_k^{t,i} = X_k^{t,i} I_{\{X_k^{t,i} > d^{t,i}\}}$$

and (random) **censoring**

- a further index indicating business line can be introduced (deleted for this talk)

Loss amounts:

$$- L_t = \sum_{i=1}^7 \sum_{k=1}^{N^{t,i}} X_k^{t,i}, \quad t = 1, \dots, T$$

$$- L_t = \sum_{i=1}^7 L_{t,i}$$

Pillar I modelling:

$$- F_{L_t} \text{ and } F_{L_{t,i}}, \quad i = 1, \dots, 7$$

- risk measurement (e.g. for L_t)

$$\text{OR-VaR}_{T+1}^{1-\alpha} = F_{L_{T+1}}^{\leftarrow}(1-\alpha), \quad \alpha \text{ (very) small}$$

$$\text{OR-CVaR}_{T+1}^{1-\alpha} = E(L_{T+1} \mid L_{T+1} > \text{OR-VaR}_{T+1}^{1-\alpha})$$

Question: Suppose we have calculated risk measures $\rho_{T+1,1-\alpha}^i$, $i = 1, \dots, 7$, for each risk category. When can we consider

$$\sum_{i=1}^7 \rho_{T+1,1-\alpha}^i$$

as a “good” risk measure for the total loss L_{T+1} ?

Answer: Ingredients

- (non-) coherence of risk measures (Artzner, Delbaen, Eber, Heath framework)
- optimization problem: given $(\rho_{T+1,1-\alpha}^i)_{i=1,\dots,7}$, what is the worst case for the overall risk for L_{T+1} ?
Solution: using copulas in [4] and references therein
- aggregation of banking risks ([1])

(Methodological) link to risk theory:

- operational risk process

$$V_{i,t} = u_i + p_i(t) - L_{t,i}, \quad t \geq 0$$

for some initial capital u_i and a premium function $p_i(t)$ satisfying

$$P(L_{t,i} - p_i(t) \rightarrow -\infty) = 1$$

- given $\epsilon > 0$, calculate $u_i(\epsilon)$ so that

$$P\left(\inf_{T \leq t \leq T+1} (u_i(\epsilon) + p_i(t) - L_{t,i}) < 0\right) \leq \epsilon \quad (1)$$

$u_i(\epsilon)$ is a **risk capital charge** (internal)

Solving for (1) is difficult:

- complicated loss process $(L_{t,i})_{t \geq 0}$
- heavy-tailed case
- finite horizon $[T, T + 1]$

Hence:

- only approach possible: Monte Carlo
- rare event simulation
- non-standard situation, see [2]!

From a mathematical point of view:

- heavy-tailed ruin estimation for general risk processes

Classical Cramér-Lundberg model (new notation):

- $Y(t) = \sum_{k=1}^{N(t)} Y_k$, $t \geq 0$ where
 - (Y_k) iid $\sim F_Y$, independent of $(N(t)) \sim HPOIS(\lambda)$
 - NPC: $\lambda E(Y_1) < c$
- risk process $\{u + ct - Y(t) : t \geq 0\}$
- infinite-horizon ruin probability:

$$\begin{aligned}\Psi_1(u) &= P(\inf_{t \geq 0} (u + ct - Y(t)) < 0) \\ &= P(\sup_{t \geq 0} (Y(t) - ct) > u)\end{aligned}$$

hence tail-probability of **ultimate supremum**

- NPC: $P(\lim_{t \rightarrow \infty} (Y(t) - ct) = -\infty) = 1$

In the **heavy-tailed** Cramér-Lundberg case:

$$1 - F_Y(y) \sim y^{-\beta-1} L(y) \Rightarrow \Psi_1(u) \sim cte u^{-\beta} L(u) \quad (2)$$
$$(\beta \geq 0, L \text{ s.v.}, y \rightarrow \infty) \quad (u \rightarrow \infty)$$

Question: how general does (2) hold?

Solution: given a **general** stochastic process $\{Y(t) : t \geq 0\}$ for which we have that for some $c > 0$

- $P(\lim_{t \rightarrow \infty} (Y(t) - ct) = -\infty) = 1$, and
- $\Psi_1(u) = P(\sup_{t \geq 0} (Y(t) - ct) > u) \sim u^{-\beta} L(u), \beta \geq 0$
($u \rightarrow \infty$)

Starting from $(Y(t))$ define a more general process $(Y(\Delta(t)))$ using the notion of **time change**:

- $(\Delta(t))$ is a right-continuous process, non-decreasing, Δ and Y are both defined on the same probability space (Ω, \mathcal{F}, P) and $\Delta(0) = 0$

Define a **new** ruin function:

$$\Psi_{\Delta}(u) = P(\sup_{t \geq 0} (Y(\Delta(t)) - ct) > u)$$

How sensitive is ruin as a function of Δ ?

More precisely:

Questions:

- under which (extra) conditions on Y and Δ does ruin behave similarly in both models, i.e.

$$\lim_{u \rightarrow \infty} \frac{\Psi_{\Delta}(u)}{\Psi_1(u)} = 1$$

- examples
- “link” to operational risk

References: [5] and [6]

Remark: why using time change?

- actuarial tool (Lundberg, Cramér (1930's)):
inhomogeneous Poisson \rightarrow homogeneous Poisson
- W. Doeblin (1940):
Itô's formula via time change
- Olsen's Θ -time in finance (1990's):
market data follows (geometric) BM in Θ -time
- Monroe's Theorem (1978):
every semi-martingale can be written as a time changed BM

Conclusion: very powerful tool!

Solution to our problem:

- basic assumption: $\lim_{t \rightarrow \infty} \frac{\Delta(t)}{t} = 1$, P - a.s.
- crucial: how fast does this convergence hold (**mixing**)
 $\forall \epsilon > 0 : g_\epsilon(u) = P(|\frac{\Delta(t)}{t} - 1| > \epsilon \text{ for some } t > u)$
- and define for $\epsilon > 0$ the perturbed ruin function
 $\Psi_{1,\epsilon}(u) = P(\sup_{t \geq 0} (Y(t) - c\epsilon t) > u)$

The solution very much depends on the behaviour of $g_\epsilon(u)$ and $\Psi_{1,\epsilon}(u)$ for $\epsilon > 0$.

The following basic assumptions hold in most cases:

(A1) (no early ruin in the original process)

$$\lim_{\delta \searrow 0} \limsup_{u \rightarrow \infty} \frac{P(\sup_{0 \leq t \leq \delta u} (Y(t) - ct) > u)}{\Psi_1(u)} = 0$$

(A2) (a continuity assumption for ruin in the original process)

$$\lim_{\epsilon \searrow 1} \limsup_{u \rightarrow \infty} \frac{\Psi_1(u)}{\Psi_{1,\epsilon}(u)} = \lim_{\epsilon \nearrow 1} \liminf_{u \rightarrow \infty} \frac{\Psi_1(u)}{\Psi_{1,\epsilon}(u)} = 1$$

Theorem ([6])

Assume (A1) and (A2) hold, and that

$$\Psi_1(u) \sim u^{-\beta} L(u), \quad u \rightarrow \infty, \beta \geq 0.$$

If (mixing condition)

$$\forall \epsilon > 0, \delta > 0 : \lim_{u \rightarrow \infty} \frac{g_\epsilon(\delta u)}{\Psi_1(u)} = 0$$

and either

i) Δ is **continuous** with probability 1,

or

ii) $\exists a \geq 0 : Y(t) + a(t)$ is **eventually non-decreasing** with probability 1,

then

$$\lim_{u \rightarrow \infty} \frac{\Psi_\Delta(u)}{\Psi_1(u)} = 1.$$

Reformulation:

“If the mixing rate of Δ is fast enough, i.e. $\frac{\Delta(t)}{t} \rightarrow 1$ fast enough measured with respect to the original ruin probability Ψ_1 , then the ruin probability of the time-changed process Ψ_Δ is not affected by the time change.”

Further results:

- slow mixing \Rightarrow ruin is affected
- several examples (motivated by operational risk)

Example (Ingredients, details in [6])

- $\{Z_n : n \geq 0\}$ irreducible Markov chain on $\{1, \dots, K\}$, stationary distribution function (π_i)
- $\{F_j : j = 1, \dots, K\}$ holding time dfs with means (μ_i) , finite
- take (r_i) so that $\sum_{j=1}^K r_j \mu_j \pi_j = \sum_{j=1}^K \mu_j \pi_j$
- time change $\Delta(0) = 0, \frac{d\Delta(t)}{dt} = r_j$ if (Z_n) at t is in j
- **key assumption** (heavy-tailed holding times):

$$\exists \bar{F}(x) \in RV(-\gamma), \gamma > 1 \text{ and } \lim_{x \rightarrow \infty} \frac{\bar{F}_j(x)}{\bar{F}(x)} = \Theta_j \in [0, \infty)$$

Theorem:

$$\lim_{u \rightarrow \infty} \frac{g_\epsilon(u)}{u \bar{F}(u)} = \frac{1}{\epsilon^{\gamma \bar{\mu}}} \left[\sum_{j \in J_+(\epsilon)} \Theta_j \pi_j (r_j - 1 - \epsilon) (r_j - 1)^{\gamma-1} + \sum_{j \in J_-(\epsilon)} \Theta_j \pi_j (1 - r_j - \epsilon) (1 - r_j)^{\gamma-1} \right],$$

where $\epsilon > 0$ s.t. $\{j = 1, \dots, K : |r_j - 1| = \epsilon\} = \emptyset$

and $J_+(\epsilon) = \{j = 1, \dots, K : r_j > 1 + \epsilon\}$, $J_-(\epsilon) = \{j = 1, \dots, K : r_j > 1 - \epsilon\}$

Conclusion

- at the moment, **qualitative** (Pillar II) **handling** of operational risk is **more useful** than quantitative (Pillar I) modelling
- actuarial methods are useful
- more data are needed
- interesting source of mathematical problems
- challenges: choice of risk measures, aggregation of risk measures

References:

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5. P. Embrechts, R. Kaufmann, G. Samorodnitsky (2002). *Ruin theory revisited: stochastic models for operational risk*. Preprint, ETH Zürich (<http://www.math.ethz.ch/~embrechts>).
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7. www.bis.org.