Statistics* and Quantitative Risk Management

(* including computational probability)

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The Evolution of Quantitative Risk Management Tools

1938	Bond duration
1952	Markowitz mean-variance framework
1963	Sharpe's single-factor beta model
1966	Multiple-factor models
1973	Black-Scholes option-pricing model, "greeks"
1983	RAROC, risk-adjusted return
1986	Limits on exposure by duration bucket
1988	Limits on "greeks", Basel I
1992	Stress testing
1993	Value-at-Risk (VAR)
1994	RiskMetrics
1996-2000	Basel I 1/2
1997	CreditMetrics
1998-	Integration of credit and market risk
2000-	Enterprisewide risk management
2000-2008	Basel II

(Jorion 2007)

On Mathematics and Finance (1/3)

For several economics/finance problems:

- no-arbitrage theory
- pricing and hedging of derivatives (options, . . .)
- market information
- more realistic models
- . . .

mathematics provides the right tools/results:

- (semi-)martingale theory
- SDEs (Itô's Lemma), PDEs, simulation
- filtrations of sigma-algebras
- from Brownian motion to more general Lévy processes
- . . .

On Mathematics and Finance (2/3)

It is fair to say that

Thesis 1: Mathematics has had a strong influence on the development of (applied) finance

Thesis 2: Finance has given mathematics (especially stochastics, numerical analysis and operations research) several new areas of interesting and demanding research

However:

Thesis 3: Over the recent years, the two fields "Applied Finance" and "Mathematical Finance" have started to diverge perhaps mainly due to their own maturity

As a consequence: and due to events like LTCM (1998), subprime crisis (2007/8), etc. . . .

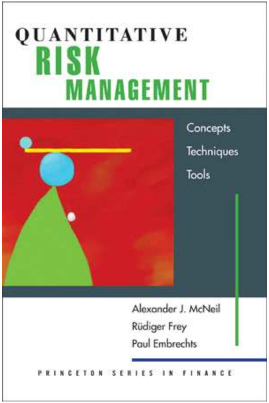
On Mathematics and Finance (3/3)

There are critical voices raised (from the press):

- Mathematicians collapse the world of financial institutions (LTCM)
- The return of the eggheads and how the eggheads cracked (LTCM)
- With their snappy name and flashy mathematical formulae, "quants" were the stars of the finance show before the credit crisis errupted (*The Economist*)
- And many more similar comments . . .

But what about Statistics and QRM?

- For this talk:
  $\{\text{Statistics}\} \cup \{\text{Computational Probability}\} \setminus \{\text{Econometrics}\}$



- QRM is an emerging field
- Fix the fundamentals
- Concentrate on applied issues
  - Interdependence and concentration of risks
  - Risk aggregation
  - The problem of scale
  - Extremes matter
  - Interdisciplinarity
- RM is as much about human judgement as about mathematical genius (*The Economist*, 17/5/07)

Let us look at some very concrete QRM issues

- The Basel Committee and Accords (I, Amendment (I 1/2), II):
  - BC established in 1974 by the Central Bank Governors of the Group of 10
  - Formulates international capital adequacy standards for financial institutions referred to as the Basel x Accords, $x \in \{\mathrm{I}, \mathrm{I}\ 1/2, \mathrm{II}\}$ so far
  - Its main aim: the avoidance of systemic risk

- Statistical quantities are hardwired into the law!
  - Value-at-Risk at confidence α and holding period d

$$\mathrm{VaR}_{\alpha,d}(X) = \inf\{x \geq 0 : \mathbb{P}(X \geq x) \geq \alpha\}$$

X: a rv denoting the (minus -) value of a position at the end of a time period $[0, d]$, $0 =$ today, $d =$ horizon

Notation: often $\mathrm{VaR}_\alpha(X)$, VaR_α, VaR . . . (↯)

Statistically speaking:

VaR is just a quantile ... ($\lightning$)

However:

- Market Risk (MR): $\alpha = 0.99$, d=10 days
- Trading desk limits (MR): $\alpha = 0.95$, d=1 day
- Credit Risk (CR): $\alpha = 0.999$, d=1 year
- Operational Risk (OR): $\alpha = 0.999$, d=1 year
- Economic Capital (EC): $\alpha = 0.9997$, d=1 year

Hence:

VaR typically is a (very) extreme quantile!

But:

What to do with it?

Minimal Capital Adequacy: the Cook Ratio

The Quants (us!)

The Accountants, Management, Board (them!)

$$\frac{\text{Regulatory Capital}}{\text{Risk Weighted Positions}} = 8\% \qquad (1)$$

Basel I, I 1/2: CR, MR (crude)

Basel II: CR, MR, OR (fine)

Important remark

Larger international banks use internal models, hence opening the door for non-trivial mathematics and statistics

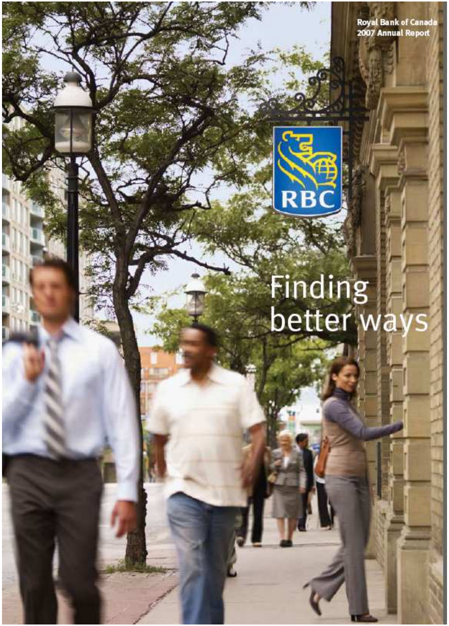
An example from the denominator for MR at day t:

$$RC_{\mathrm{IM}}^{t}(MR) = \max\left\{\mathrm{VaR}_{0.99,\,10}^{t}\,,\ \frac{k}{60}\sum_{i=1}^{60}\mathrm{VaR}_{0.99,\,10}^{t-i+1}\right\} + RC_{SR}^{t} \qquad (2)$$

where: RC = Risk Capital
IM = Internal Model
$k \in [3,5]$ Stress Factor
MR = Market Risk
SR = Specific Risk

Remarks:

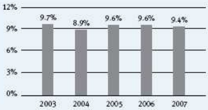
- All the numbers are statistical estimates
- k depends on statistical backtesting and the quality of the statistical methodology used
- A detailed explanation of (2) fills a whole course!
- The underlying rv X typically (and also dynamically) depends on several hundred (or more) factors / time series


Royal Bank of Canada
2007 Annual Report
RBC
Finding
better ways

Regulatory capital and capital ratios (1)		Table 38
(C$ millions, except percentage amounts)	2007	2006
Tier 1 capital		
Common equity (2)	$ 22,272	$ 21,065
Non-cumulative preferred shares	2,344	1,345
Trust capital securities	3,494	3,222
Other non-controlling interest in subsidiaries	25	28
Goodwill	(4,752)	(4,182)
	23,383	21,478
Tier 2 capital		
Permanent subordinated debentures (3)	779	839
Non-permanent subordinated debentures (3)	5,473	6,313
General allowances	1,221	1,223
Trust capital securities (excess over 15% Tier 1)	–	249
Trust subordinated notes	1,027	–
Accumulated net unrealized gain on available-for-sale equity securities (4)	105	–
	8,605	8,624
Other deductions from capital		
Investment in insurance subsidiaries	(2,912)	(2,795)
Other	(505)	(643)
Total capital	$ 28,571	$ 26,664
Capital ratios		
Tier 1 capital to risk-adjusted assets	9.4%	9.6%
Total capital to risk-adjusted assets	11.5%	11.9%
Assets-to-capital multiple	19.9X	19.7X

(1) As defined in the guidelines issued by the OSFI.
(2) This amount is Shareholders' equity less preferred shares of $2,050 million and other items not included in regulatory capital of $117 million.
(3) Subordinated debentures that are within five years of maturity are subject to straight-line amortization to zero during their remaining term and, accordingly, are included above at their amortized value.
(4) As prescribed by the OSFI, certain components of Accumulated other comprehensive income (AOCI) are included in the determination of regulatory capital. Accumulated net foreign currency translation adjustments are included in Tier 1 capital in common equity. Net unrealized fair value losses on available-for-sale (AFS) equities are deducted in the determination of Tier 1 capital while net unrealized fair value gains on AFS equities are included in Tier 2 capital.

Tier 1 capital ratio

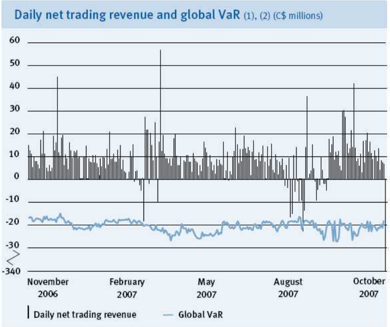


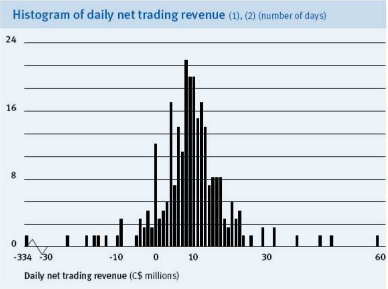
As at October 31, 2007, the Tier 1 capital ratio was 9.4% and the Total capital ratio was 11.5%.

The Tier 1 capital ratio was down 20 bps from a year ago. The decrease was largely due to business growth, including acquisitions, which resulted in an increase in RAA and a higher goodwill deduction from capital. The impact of our common share repurchases under our normal course issuer bid also contributed to the decrease. These factors were partially offset by strong generation of capital from earnings and the issuance of preferred shares.

The Total capital ratio was down 40 bps from a year ago due to growth in RAA and the redemption of subordinated debentures. These factors were partially offset by the issuance of trust subordinated notes.

As at October 31, 2007, our assets-to-capital multiple was 19.9 compared to 19.7 a year ago. Our assets-to-capital multiple





(1) Trading revenue on a taxable equivalent basis excluding revenue related to consolidated VIEs.
(2) The $357 million writedown on the valuation of U.S. subprime RMBS and CDOs of ABS was included on October 31, 2007.

So far for the global picture, now to some concrete research themes:

- an axiomatic theory of risk measures and their estimation
- backtesting risk measure performance
- rare event estimation and (M)EVT
- a statistical theory of stress scenarios
- combining internal, external and expert opinion data (Bayes!)
- scaling of risk measures, e.g. $\mathrm{VaR}_{\alpha_1, T_1} \rightarrow \mathrm{VaR}_{\alpha_2, T_2}$
- risk aggregation, e.g. $\mathrm{VaR}^{MR}_{\alpha_1, T_1} + \mathrm{VaR}^{CR}_{\alpha_2, T_2} + \mathrm{VaR}^{OR}_{\alpha_3, T_3}$ (+?)
- understanding diversification and concentration of risk
- robust estimation of dependence
- high-dimensional covariance matrix estimation
- Fréchet-space problems
- ...

Some recent regulatory developments:

- Incremental Risk Charge (MR, 99.9%, one year)
- Consultative Document on Fair Value (November 2008)
  - Strengthening of Pillar 2
  - Challenging of valuation models
  - Understand underlying assumptions and their appropriateness
  - Understand the model's theoretical soundness and mathematical integrity
  - Sensitivity analysis under stress conditions
  - Backtesting, etc ...

Overall Key Words:

- Valuation Uncertainty (Basel Committee)
- Model Uncertainty (RiskLab, ETH Zurich)

I. A Fréchet-type problem

d one-period risks:

$$rvs\ X_i : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}, \quad i = 1, \ldots, d$$

a financial position in $\mathbf{X} = (X_1, \ldots, X_d)^T$:

$$\Psi(\mathbf{X}) \text{ where } \Psi : \mathbb{R}^d \to \mathbb{R} \text{ measurable}$$

a risk measure $\mathcal{R}$:

$$\mathcal{R} : \mathcal{C} \to \mathbb{R}, \quad \mathcal{C} \subset L^\circ(\Omega, \mathcal{F}, \mathbb{P}) \text{ a cone, } \mathbf{X} \in \mathcal{C}^d$$

Assume:

$$X_i \sim F_i \,(\text{or } \hat{F}_i) \quad i = 1, \ldots, d \qquad (\mathcal{A})$$

some idea of dependence

Task: Calculate $\mathcal{R}\big(\Psi(\mathbf{X})\big)$ under $(\mathcal{A})$ (3)

In general (3) is not well-defined (one, no or ∞-many solutions), hence in the latter case calculate so-called Fréchet bounds:

$$\mathcal{R}_{\text{inf}} \leq \mathcal{R}\big(\Psi(\mathbf{X})\big) \leq \mathcal{R}_{\text{sup}}$$

where

$$\begin{aligned} \mathcal{R}_{\text{inf}} &= \inf\big\{\mathcal{R}\big(\Psi(\mathbf{X})\big) \text{ under } (\mathcal{A})\big\} \\ \mathcal{R}_{\text{sup}} &= \sup\big\{\mathcal{R}\big(\Psi(\mathbf{X})\big) \text{ under } (\mathcal{A})\big\} \end{aligned}$$

Prove sharpness of these bounds and work out numerically

Remark:

Replace in $(\mathcal{A})$ knowledge of $\{F_i : i = 1, \ldots, d\}$ by knowledge of overlapping or non-overlapping sub-vectors $\{\mathbf{F}_j : j = 1, \ldots, \ell\}$

For instance $d = 3$:

Scenario 1: $(\mathbf{F}_1 = F_1,\ \mathbf{F}_2 = F_2,\ \mathbf{F}_3 = F_3)$

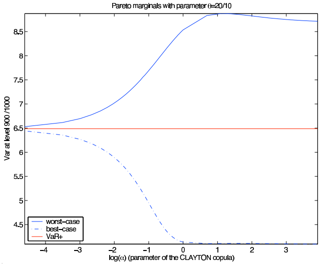
Scenario 2: $(\mathbf{F}_1 = F_{12},\ \mathbf{F}_2 = F_3)$ $\qquad + \text{ dependence}$

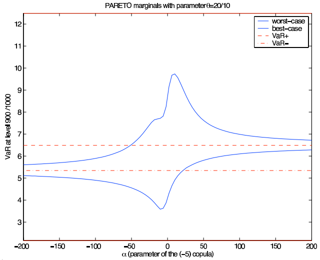
Scenario 3: $(\mathbf{F}_1 = F_{12},\ \mathbf{F}_2 = F_{23})$

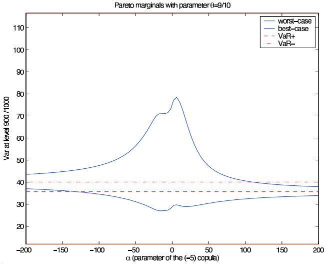
Theorem (Rüschendorf (1991))

$$\inf_{\mathcal{F}(F_{12},F_{23})} \mathbb{P}(X_1+X_2+X_3 < s) = \int \inf_{\mathcal{F}(F_{12|x_2},F_{23|x_2})} \mathbb{P}(X_1+X_3 < s-x_2)dF_2(x_2)$$

Examples: Scenario 3







II. Operational Risk

Basel II Definition

The risk of loss resulting from inadequate or failed internal processes, people and systems or from external events. This definition includes legal risk, but excludes strategic and reputational risk.

Examples:

- Barings Bank (1995): \$ 1.33 bn (however ...)
- London Stock Exchange (1997): \$ 630 m
- Bank of New York (9/11/2001): \$ 242 m
- Société Générale (2008): \$7.5 bn

How to measure:

- Value-at-Risk
- 1 year
- 99.9%

} Loss Distribution Approach (LDA)

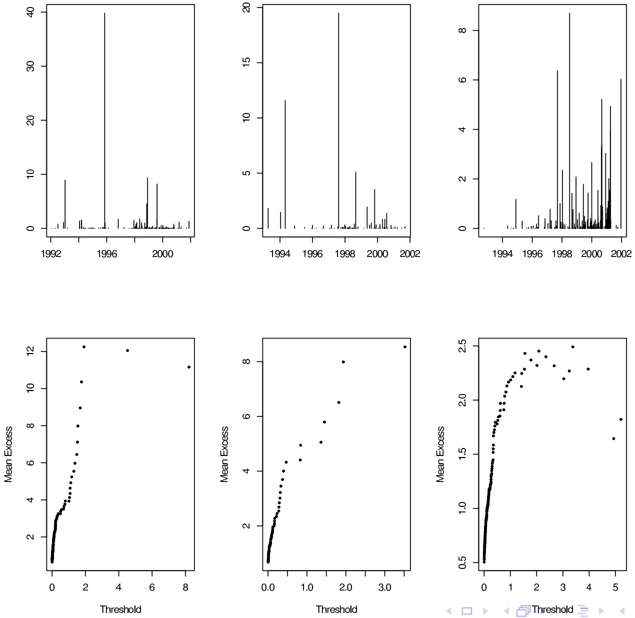
The data structure (1/2)

	RT_1	$\dots$	RT_r	$\dots$	RT_7	
BL_1						
$\vdots$						
BL_b			$L^t_{b,r}$			
$\vdots$						
BL_8						
						L^t

$$\mathfrak{X} = \left\{ X_k^{t-i,b,r} : i = 1, \dots, T;\ b = 1, \dots, 8;\ r = 1, \dots, 7;\ k = 1, \dots, N_{b,r}^{t-i} \right\}$$

$$L^t = \sum_{b=1}^{8} \sum_{r=1}^{7} L^t_{b,r} = \sum_{b=1}^{8} \sum_{r=1}^{7} \left(\sum_{k=1}^{N^t_{b,r}} X_k^{t,b,r} \right)$$

The data structure (2/2)



LDA in practice (internal data)

Step 1 Pool the data business-line wise

Step 2 Estimate $\widehat{\text{VaR}}_1, \ldots, \widehat{\text{VaR}}_8$ (99.9%, 1 year)

Step 3 Add (comonotonicity): $\widehat{\text{VaR}}_+ = \sum_{b=1}^{8} \widehat{\text{VaR}}_b$

Step 4 Use diversification argument to report

$$\text{VaR}_{reported} = (1 - \delta)\widehat{\text{VaR}}_+, \qquad 0 < \delta < 1$$

(often $\delta \in [0.1, 0.3]$)

Question: What are the statistical issues?

Step 1 Data inhomogeneity: estimation of $\widehat{\text{VaR}}_i$

Step 2 Which method to use:

(M1) EVT, POT-method

(M2) Some specific parametric model

- lognormal, loggamma
- Tukey's g-and-h

$$X = a + b\frac{e^{gZ} - 1}{g} e^{\frac{h}{2}Z^2}, \quad Z \sim \mathcal{N}(0,1)$$

Step 3 ✓

Step 4
- Justify $\delta > 0$
- Possibly $\delta < 0$: non-subadditivity of VaR!

Rare event estimation: EVT is a canonical tool!

Data: $X_1, \dots, X_n$ *iid* $\sim F$ continuous, $M_n = \max(X_1, \dots, X_n)$

Excess df: $F_u(x) = \mathbb{P}(X - u \le x | X > u), \qquad x \ge 0$

EVT basics: $\{H_\xi : \xi \in \mathbb{R}\}$ generalized extreme value dfs

$$F \in MDA(H_\xi) \Leftrightarrow \exists c_n > 0, d_n \in \mathbb{R} : \forall x \in \mathbb{R}, \lim_{n \to \infty} \mathbb{P}\Big(\frac{M_n - d_n}{c_n} \le x\Big) = H_\xi(x)$$

Basic Theorem (Pickands-Balkema-de Haan)

$$F \in MDA(H_\xi)$$

$$\Leftrightarrow$$

$$\lim_{u \to x_F} \sup_{0 \le x < x_F - u} |F_u(x) - G_{\xi,\beta(u)}(x)| = 0 \qquad (4)$$

for some measurable function β and (generalized Pareto) df $G_{\xi,\beta}$.

The Fréchet case, $\xi > 0$ (Gnedenko):

$$F \in MDA(H_\xi) \quad \Leftrightarrow \quad \overline{F}(x) = 1 - F(x) = x^{-1/\xi} L(x)$$

L (Karamata-) slowly varying:

$$\forall t > 0 \lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1 \tag{5}$$

- Remark: Contrary to the CLT, the rate of convergence in (4) for $u \to x_F = \infty$ ($\xi > 0$) can be arbitrarily slow; it all depends on L in (5)!
- Relevance for practice (operational risk)
  - Industry discussion: EVT-POT versus g-and-h
  - Based on QISs:
    - ★ Basel committee (47 000 observations)
    - ★ Fed-Boston (53 000 observations)

- Typical (g,h)-values for OR: $g \approx 2.4$, $h \approx 0.2$

Theorem (Degen-Embrechts-Lambrigger)

For $g, h > 0$,

$$\overline{F}_{g,h}(x) = x^{-1/h} L_{g,h}(x)$$

$$L_{g,h}(x) \propto \frac{e^{\sqrt{\log x}}}{\sqrt{\log x}}$$

$$\text{rate of convergence in (4)} = O\big((\log u)^{-1/2}\big)$$

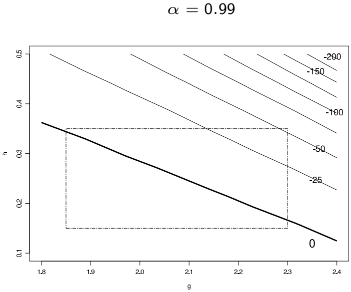
- Conclusion: in a g-and-h world ($h > 0$), statistical estimators may converge very slowly
- However: be aware of "taking models out of thin air"!

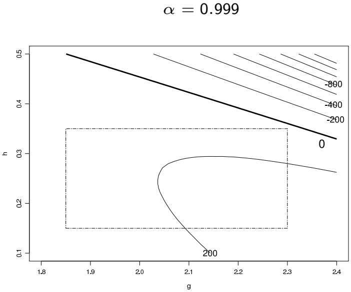
Some comments on diversification

X_1, X_2 *iid*, g-and-h, $\delta_{g,h}(\alpha) = \text{VaR}_\alpha(X_1) + \text{VaR}_\alpha(X_2) - \text{VaR}_\alpha(X_1 + X_2)$

Recall that

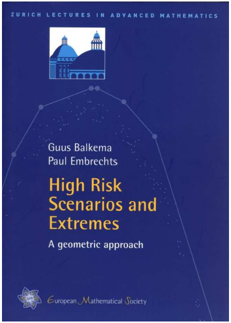
$$\delta_{g,h}(\alpha) \begin{cases} > 0 & \text{diversification potential} \\ = 0 & \text{comonotonicity} \\ < 0 & \text{non-coherence} \end{cases}$$

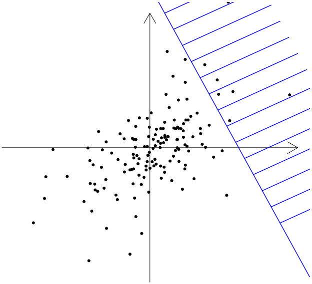




III. Multivariate Extreme Value Theory

- Recall the Rickands-Balkema-de Haan Theorem ($d = 1$)
- Question: How to generalize to $d \geq 2$?
  - componentwise approach involving multivariate regular variation, spectral decomposition and EV-copulas
  - geometric approach





MEVT: Geometric approach

- $\mathbf{X} = (X_1, \ldots, X_d)$
- H: a hyperspace in $\mathbb{R}^d$
- $\mathbf{X}^H$: vector with conditional df given $\{\mathbf{X} \in H\}$
- β_H: affine transformations

- Study:

$$\mathbf{W}_H = \beta_H^{-1}(\mathbf{X}^H) \xrightarrow{\mathrm{d}} \mathbf{W} \text{ for } \mathbb{P}(\mathbf{X} \in H) \to 0$$

Basic questions:

- determine all non-degenerate limits $\mathbf{W}$
- given $\mathbf{W}$, determine β_H
- characterize the domains of attraction of all possible limits

MEVT: Geometric approach

Characterization of the limit laws ($d = h + 1$, $\tau = \tau(\lambda, h)$):

$$g_0(\mathbf{u}, v) = e^{-(v+\mathbf{u}^T\mathbf{u}/2)} \qquad \mathbf{w} = (\mathbf{u}, v) \in \mathbb{R}^{h+1} \tag{6}$$

$$g_\tau(\mathbf{w}) = 1/\|\mathbf{w}\|^{d+\lambda} \qquad \mathbf{w} \neq \mathbf{0} \tag{7}$$

$$g_\tau(\mathbf{u}, v) = (-v - \mathbf{u}^T\mathbf{u}/2)_+^{\lambda-1} \qquad v < -\mathbf{u}^T\mathbf{u}/2 \tag{8}$$

Examples in the domains of attraction:

- multivariate normal distribution for (6)
- multivariate t distribution for (7)
- uniform distribution on a ball for (8)
- and distributions in a "neighbourhood" of these

Relevant research topics are:

- concrete examples
  e.g. meta distributions, skew-symmetric distributions,... (Balkema, Lysenko, Roy)

- statistical estimation of multivariate rare events
  (widely open in this context, e.g. Fougères, Soulier,...)

- stochastic simulation of such events (McLeish)

Change of paradigm:

- look at densitites rather than distribution functions; here geometry enters

- new terminology: bland data, rotund level sets, ...

IV. Two classical results from mathematics

Theorem 1

In the spaces L^p, $0 < p < 1$, there exist no convex open sets other than $\varnothing$ and L^p.

Theorem 2 (Banach-Tarski paradox)

Given any bounded subsets A, $B \subset \mathbb{R}^n$, $n \geq 3$, $int(A) \neq \varnothing$ and $int(B) \neq \varnothing$, then there exist partitions $A = A_1 \cup \ldots \cup A_k$, $B = B_1 \cup \ldots \cup B_k$ such that for all $1 \leq i \leq k$, A_i and B_i are congruent.

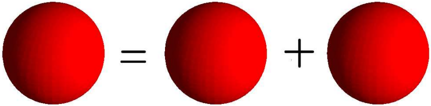
And their consequences

- (Theorem 1) On any space with infinite-mean risks there exists no non-trivial risk measure with (mild) continuity properties

  (beware: Operational Risk: joint work with Valérie Chavez-Demoulin and Johanna Nešlehová)

- (Theorem 2) Mathematics presents an idealized view of the real world; for applications, always understand the conditions

  (beware: CDOs; mark-to-market, mark-to-model, mark-to-myth!)



Conclusions

- QRM yields an exciting field of applications with numerous interesting open problems
- Applicability well beyond the financial industry
- I expect the years to come will see an increasing importance of statistics within finance in general and QRM in particular
- Key words: extremes/ rare events/ stress testing, multidimensionality, complex data structures, large data sets, dynamic/multiperiod risk measurement
- (Teaching of/ research on/ communication of) these techniques and results will be very challenging
- As a scientist: always be humble in the face of real applications

By the way, if you want to see how some of the outside world of economics views the future use of statistics, you may google:

- Super Crunchers

It is all related to the analysis of

- Large data sets
- Kryder's Law

But also google at the same time

- George Orwell, *1984*

Many Thanks!