Risk Aggregation under Dependence Uncertainty
Challenges in Theory and Practice

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Outline

1. Framework
2. VaR and ES Bounds
3. Asymptotic Equivalence
4. Challenges
5. References
Fundamental problem in Finance/Insurance

- Risk factors: $\mathbf{X} = (X_1, \ldots, X_d)$
- Model assumption: $X_i \sim F_i$, $F_i$ known, $i = 1, \ldots, d$
- A financial position $\Psi(\mathbf{X})$
- A risk measure/pricing function: $\rho(\Psi(\mathbf{X}))$

Calculate $\rho(\Psi(\mathbf{X}))$
Calculating $\rho(\Psi(X))$

Example:

- $\Psi(X) = \sum_{i=1}^{d} X_i$
- $\rho = \text{VaR}_p$ or $\rho = \text{ES}_p$

Challenge:

- We need a *joint* model for the random vector $X$
- Joint models are hard to get by

We will focus on the above special choices of $\Psi$ and $\rho$.  

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VaR and ES

\( \text{VaR}_p(X) \)

For \( p \in (0, 1) \),

\[
\text{VaR}_p(X) = F_X^{-1}(p) = \inf\{x \in \mathbb{R} : F_X(x) \geq p\}
\]

\( \text{ES}_p(X) \)

For \( p \in (0, 1) \),

\[
\text{ES}_p(X) = \frac{1}{1 - p} \int_p^1 \text{VaR}_q(X) dq = \mathbb{E}[X|X > \text{VaR}_p(X)]
\]
A related quantity **Left-tail-ES**: 

\[ \text{LES}_p(X) = \frac{1}{p} \int_0^p \text{VaR}_q(X) dq = -\text{ES}_{1-p}(-X) \]
Fréchet problem

Denote

\[ S_d = S_d(F_1, \ldots, F_d) = \left\{ \sum_{i=1}^{d} X_i : X_i \sim F_i, \ i = 1, \ldots, d \right\} \]

- Every element in \( S_d \) is a possible risk position.
- Determination of \( S_d \): very challenging.
  - Think about \( S_2(U[0, 1], U[0, 1]) \) ... open question!
Worst- and best-values of VaR and ES

The Fréchet (unconstrained) problems for $\text{VaR}_p$

$$\overline{\text{VaR}}_p(S_d) = \sup \{ \text{VaR}_p(S) : S \in S_d(F_1, \ldots, F_d) \},$$

$$\underline{\text{VaR}}_p(S_d) = \inf \{ \text{VaR}_p(S) : S \in S_d(F_1, \ldots, F_d) \}. $$

Same notation for $\text{ES}_p$ and $\text{LES}_p$. 
Worst- and best-values of VaR and ES

- ES is subadditive:
  \[ \overline{ES}_p(S_d) = \sum_{i=1}^{d} ES_p(X_i). \]

Similarly \( \underline{ES}_p(S_d) = \sum_{i=1}^{d} LES_p(X_i). \)

- \( \overline{VaR}_p(S_d), \overline{VaR}_p(S_d) \) and \( ES_p(S_d) \): generally open questions

**Challenge for \( ES_p(S_d) \)**

To calculate \( ES_p(S_d) \) one naturally seeks a safest risk in \( S_d \).
Common understanding of the most dangerous scenario:

- Comonotonicity - well accepted notion

Understanding concerning the safest scenario:

- $d = 2$: counter-monotonicity
- $d \geq 3$: question mark! (?!)
  - Calls for notions of extremal negative dependence.
ES respects **convex order**: the natural order of risk preference.

**Convex order**

We write $X \leq_{cx} Y$ if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all convex functions $f$ such that the two expectations exist.

**Finding $\text{ES}_p(S_d)$**

Search for a **smallest element** in $S_d$ with respect to convex order, if it exists.
VaR does not respect convex order: more tricky

- Good news: the questions for $\overline{\text{VaR}}_p(S_d)$, $\underline{\text{VaR}}_p(S_d)$ and $\text{ES}_p(S_d)$ are mathematically similar.

Finding $\overline{\text{VaR}}_p(S_d)$

Search for a smallest element in $S_d(\hat{F}_1, \ldots, \hat{F}_d)$ with respect to convex order, where $\hat{F}_i$ is the $p$-tail-conditional distribution of $F_i$.

- $\overline{\text{VaR}}_p(S_d)$ is symmetric to $\underline{\text{VaR}}_p(S_d)$. 
Summary of existing results

\[ d = 2: \]
- fully solved analytically

\[ d \geq 3: \]
- Homogeneous model \((F_1 = \cdots = F_d)\)
  - \(\text{ES}_p(S_d)\) solved analytically for decreasing densities, e.g. Pareto, Exponential
  - \(\text{VaR}_p(S_d)\) solved analytically for tail-decreasing densities, e.g. Pareto, Gamma, Log-normal
- Inhomogeneous model
  - Few analytical results: current research
- Numerical methods available: Rearrangement Algorithm
$d = 2$, Makarov (1981) and Rüschendorf (1982)

For any $p \in (0, 1)$,

$$\overline{\text{VaR}}_p(S_2) = \inf_{x \in [0, 1-p]} \{ F_1^{-1}(p + x) + F_2^{-1}(1 - x) \},$$

and

$$\underline{\text{VaR}}_p(S_2) = \sup_{x \in [0, p]} \{ F_1^{-1}(x) + F_2^{-1}(p - x) \}.$$  

- A large outcome is coupled with a small outcome.
Sharp VaR bounds (Wang, Peng and Yang, 2013)

Suppose that the density function of $F$ is decreasing on $[b, \infty)$ for some $b \in \mathbb{R}$. Then, for $p \in [F(b), 1)$, and $X \sim F$,

$$\overline{\text{VaR}}_p(S_d) = d \mathbb{E} \left[ X \middle| X \in \left[ F^{-1}(p + (d - 1)c), F^{-1}(1 - c) \right] \right],$$

where $c$ is the smallest number in $[0, \frac{1}{d}(1 - p)]$ such that

$$\int_{p + (d - 1)c}^{1-c} F^{-1}(t) \, dt \geq \frac{1 - p - dc}{d} ((d - 1)F^{-1}(p + (d - 1)c) + F^{-1}(1 - c)).$$

Red part clearly has an ES-type form.

- $c = 0$: $\overline{\text{VaR}}_p(S_d) = \overline{\text{ES}}_p(S_d)$. 

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Sharp VaR bounds II

Suppose that the density function of $F$ is decreasing on its support. Then for $p \in (0, 1)$ and $X \sim F$,

$$\text{VaR}_p(S_d) = \max\{(d - 1)F^{-1}(0) + F^{-1}(p), d\mathbb{E}[X|X \leq F^{-1}(p)]\}.$$ 

Red part has an LES form.
ES bounds - homogeneous model

Sharp ES bounds (Bernard, Jiang and Wang, 2014)

Suppose that the density function of $F$ is decreasing on its support. Then for $p \in (1 - dc, 1)$, $q = (1 - p)/d$ and $X \sim F$,

$$\text{ES}_p(S_d) = \frac{1}{q} \int_0^q \left( (d - 1)F^{-1}((d - 1)t) + F^{-1}(1 - t) \right) dt,$$

$$= (d - 1)^2 \text{LES}_{(d-1)q}(X) + \text{ES}_{1-q}(X),$$

where $c$ is the smallest number in $[0, \frac{1}{d}]$ such that

$$\int_{(d-1)c}^{1-c} F^{-1}(t) dt \geq \frac{1-dc}{d}((d - 1)F^{-1}((d - 1)c) + F^{-1}(1 - c)).$$

- One large outcome is coupled with $d - 1$ small outcomes.
The homogeneous VaR and ES bounds are based on the notion of complete mixability:

**Complete mixability, Wang and Wang (2011)**

A distribution function $F$ on $\mathbb{R}$ is called $d$-completely mixable ($d$-CM) if there exist $d$ random variables $X_1, \ldots, X_d \sim F$ such that

$$
\mathbb{P}(X_1 + \cdots + X_d = dk) = 1,
$$

for some $k \in \mathbb{R}$.

- Equivalently, $S_d(F, \ldots, F)$ contains a constant.
Complete mixability

- Some examples of $d$-CM distributions for all $d \geq 2$: Normal, Student t, Cauchy, Uniform.

- Most relevant result: $F$ has a monotone density on a finite interval with a mean condition (depends on $d$) is $d$-CM.
  - Examples: (truncated) Pareto, Gamma, Log-normal.

- Inhomogeneous version called joint mixability.

- A full characterization of these classes is at the moment widely open.
Numerical calculation


- A fast numerical procedure
- Based on the CM-idea
- Discretization of relevant quantile regions
- $d$ possibly large
- Applicable to $\overline{VaR}_p$, $VaR_p$ and $ES_p$
Asymptotic equivalence

Consider the case $d \to \infty$. What would happen to $\overline{\text{VaR}}_p(S_d)$?

- Clearly always $\overline{\text{VaR}}_p(S_d) \leq \overline{\text{ES}}_p(S_d)$.
- Recall that $\overline{\text{VaR}}_p(S_d)$ has an ES-type part.

Under some weak conditions,

$$
\lim_{d \to \infty} \frac{\overline{\text{ES}}_p(S_d)}{\overline{\text{VaR}}_p(S_d)} = 1.
$$

This was shown first for homogeneous models and then extended to general inhomogeneous models.
Asymptotic equivalence - homogeneous model

**Theorem 1**

In the homogeneous model, $F_1 = F_2 = \cdots = F$, for $p \in (0, 1)$ and $X \sim F$, we have that

$$\lim_{d \to \infty} \frac{1}{d} \text{VaR}_p(S_d) = \text{ES}_p(X).$$

- Similar limits hold for a large class of risk measures
Asymptotic equivalence - worst-cases

**Theorem 2 (Embrechts, Wang and Wang, 2014)**

Suppose the continuous distributions $F_i$, $i \in \mathbb{N}$ satisfy that for $X_i \sim F_i$ and some $p \in (0, 1)$,

(i) $\mathbb{E}[|X_i - \mathbb{E}[X_i]|^k]$ is uniformly bounded for some $k > 1$;

(ii) $\liminf_{d \to \infty} \frac{1}{d} \sum_{i=1}^{d} \text{ES}_p(X_i) > 0$.

Then as $d \to \infty$,

$$\frac{\overline{\text{ES}}_p(S_d)}{\text{VaR}_p(S_d)} = 1 + O(d^{1/k-1}).$$
Asymptotic equivalence - best-cases

Similar results hold for $\text{VaR}_p$ and $\text{ES}_p$: assume (i) and

$$(iii) \quad \liminf_{d \to \infty} \frac{1}{d} \sum_{i=1}^{d} \text{LES}_p(X_i) > 0,$$

then

$$\lim_{d \to \infty} \frac{\text{VaR}_p(S_d)}{\text{LES}_p(S_d)} = 1,$$

$$\lim_{d \to \infty} \frac{\text{ES}_p(S_d)}{\sum_{i=1}^{d} \mathbb{E}[X_i]} = 1,$$

and

$$\frac{\text{VaR}_p(S_d)}{\text{ES}_p(S_d)} \approx \frac{\sum_{i=1}^{d} \text{LES}_p(X_i)}{\sum_{i=1}^{d} \mathbb{E}[X_i]} \leq 1, \quad d \to \infty.$$
Example: Pareto(2) risks

Bounds on VaR and ES for the sum of $d$ Pareto(2) distributed rvs for $p = 0.999$; $\text{VaR}_p^+$ corresponds to the comonotonic case.

<table>
<thead>
<tr>
<th></th>
<th>$d = 8$</th>
<th>$d = 56$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{VaR}_p$</td>
<td>31</td>
<td>53</td>
</tr>
<tr>
<td>$\text{ES}_p$</td>
<td>178</td>
<td>472</td>
</tr>
<tr>
<td>$\text{VaR}_p^+$</td>
<td>245</td>
<td>1715</td>
</tr>
<tr>
<td>$\overline{\text{VaR}}_p$</td>
<td>465</td>
<td>3454</td>
</tr>
<tr>
<td>$\overline{\text{ES}}_p$</td>
<td>498</td>
<td>3486</td>
</tr>
<tr>
<td>$\overline{\text{VaR}}_p / \text{VaR}_p^+$</td>
<td>1.898</td>
<td>2.014</td>
</tr>
<tr>
<td>$\overline{\text{ES}}_p / \text{VaR}_p$</td>
<td>1.071</td>
<td>1.009</td>
</tr>
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</table>
Example: Pareto($\theta$) risks

Bounds on the VaR and ES for the sum of $d = 8$ Pareto($\theta$)-distributed rvs for $p = 0.999$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\theta = 1.5$</th>
<th>$\theta = 2$</th>
<th>$\theta = 3$</th>
<th>$\theta = 5$</th>
<th>$\theta = 10$</th>
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<tbody>
<tr>
<td>$\overline{\text{VaR}}_p$</td>
<td>1897</td>
<td>465</td>
<td>110</td>
<td>31.65</td>
<td>9.72</td>
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<tr>
<td>$\overline{\text{ES}}_p$</td>
<td>2392</td>
<td>498</td>
<td>112</td>
<td>31.81</td>
<td>9.73</td>
</tr>
<tr>
<td>$\overline{\text{ES}}_p/\overline{\text{VaR}}_p$</td>
<td>1.261</td>
<td>1.071</td>
<td>1.018</td>
<td>1.005</td>
<td>1.001</td>
</tr>
</tbody>
</table>
Theorem 3 (Embrechts, Wang and Wang, 2014)

Take $1 > q \geq p > 0$. Suppose that the continuous distributions $F_i, i \in \mathbb{N}$, satisfy (i) and (iii), and $\lim \sup_{d \to \infty} \frac{\sum_{i=1}^{d} \mathbb{E}[X_i]}{\sum_{i=1}^{d} \mathbb{E}S_p(X_i)} < 1$, then

$$\lim \inf_{d \to \infty} \frac{\text{VaR}_q(S_d) - \text{VaR}_q(S_d)}{\text{ES}_p(S_d) - \text{ES}_p(S_d)} \geq 1.$$ 

- The uncertainty spread of VaR is generally bigger than that of ES.
- In recent Consultative Documents of the Basel Committee, VaR$_{0.99}$ is compared with ES$_{0.975}$: $p = 0.975$ and $q = 0.99$. 
Dependence-uncertainty spread

ES and VaR of $S_d = X_1 + \cdots + X_d$, where

- $X_i \sim \text{Pareto}(2 + 0.1i)$, $i = 1, \ldots, 5$;
- $X_i \sim \text{Exp}(i - 5)$, $i = 6, \ldots, 10$;
- $X_i \sim \text{Log–Normal}(0, (0.1(i - 10))^2)$, $i = 11, \ldots, 20$.

<table>
<thead>
<tr>
<th></th>
<th>$d = 5$</th>
<th>$d = 20$</th>
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<tbody>
<tr>
<td></td>
<td>best</td>
<td>worst</td>
</tr>
<tr>
<td>ES$_{0.975}$</td>
<td>22.48</td>
<td>44.88</td>
</tr>
<tr>
<td>VaR$_{0.975}$</td>
<td>9.79</td>
<td>41.46</td>
</tr>
<tr>
<td>VaR$_{0.9875}$</td>
<td>12.06</td>
<td>56.21</td>
</tr>
<tr>
<td>VaR$_{0.99}$</td>
<td>12.96</td>
<td>62.01</td>
</tr>
<tr>
<td>$\frac{ES_{0.975}}{VaR_{0.975}}$</td>
<td>1.08</td>
<td></td>
</tr>
</tbody>
</table>
Challenges

Open mathematical questions:

- Characterization of complete and joint mixability
- Characterization of $S_d$
- Find $\text{VaR}_p$ under more general settings, especially in the inhomogeneous model
- Partial dependence information and realistic scenarios
- Marginal uncertainty and statistical estimation
- Many more ...
References I


References II


