Risk Aggregation and Model Uncertainty

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Outline

Regulation

Three regulatory documents

R1: BCBS-Consultative Document, May 2012,
Fundamental Review of the Trading Book (⇐ Basel 3.5)

R2: United States Senate, March 15, 2013,
JPMorgan Chase Whale Trades: A case History of Derivatives Risks and Abuses

R3: UK House of Lords/House of Commons, June 12, 2013,
Changing banking for good, Volumes I and II

- (In total, more than 1000 pages!)

Regulation

And some statements:

From R1: Page 41, Question 8:
"What are the likely constraints with moving from VaR to ES, including any challenges in delivering robust backtesting, and how might these be best overcome?"

From R2: Page 13, and further detailed in the report:
"End of Q1, 2012, the RWAs were down from 20 to 13 Bio USD, and this based on three VaR-model changes. The change in VaR methodology effectively masked the significant changes in the portfolio."

Regulation

And finally:

From R3: Volume II, page 119. A quote from a former employee of HBOS:
«We actually got an external advisor [to asses how frequently a particular event might happen] and they came out with one in 100,000 years and we said «no», and I think we submitted one in 10,000 years. But that was a year and a half before it happened. It doesn't mean to say it was wrong: it was just unfortunate that the 10,000th year was so near.»

Basel 3.5

From R1, Page 41, Question 8:

"What are the likely constraints with moving from VaR to ES, including any challenges in delivering robust backtesting, and how might these be best overcome?"

Definition

$\mathrm{VaR}_\alpha(X)$, for $\alpha \in (0,1)$,

$$\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}$$

Definition

$\mathrm{ES}_\alpha(X)$, for $\alpha \in (0,1)$, if $\mathbb{E}[X] < \infty$,

$$\mathrm{ES}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_\delta(X)\,d\delta \underset{(F\,cont.)}{=} \mathbb{E}\left[X \mid X > \mathrm{VaR}_\alpha(X)\right]$$

VaR Bounds

Consider:

- One-period risk positions $X_1, \ldots, X_d$ with known distribution functions (dfs) F_i, $i = 1, \ldots, d$
- Portfolio position $X_d^+ = X_1 + \cdots + X_d$
- $\mathrm{VaR}_\alpha(X_i)$, $i = 1, \ldots, d$, the marginal VaR's at the common confidence level $\alpha \in (0, 1)$

Calculate $\mathrm{VaR}_\alpha(X_d^+)$

Problem:

- We need a *joint* model for the random vector $\mathbf{X} = (X_1, \ldots, X_d)'$

VaR Bounds

- **X** elliptical

$$\mathrm{VaR}_\alpha(X_d^+) \leq \sum_{1=1}^{d} \mathrm{VaR}_\alpha(X_i)$$

Examples: multivariate Gaussian, multivariate Student t

- **X** comonotone i.e. there exist increasing functions ψ_i, i=1,...,d and a random variable Z so that

$$X_i = \psi_i(Z)$$

then

$$\mathrm{VaR}_\alpha(X_d^+) = \sum_{i=1}^{d} \mathrm{VaR}_\alpha(X_i)$$

i.e. VaR_α (like ES_α) is comonotonically additive

- «Diversification» often uses:

$$(1-\delta)\sum_{i=1}^{d} \mathrm{VaR}_\alpha(X_i),\ 0 < \delta < 1$$

VaR Bounds

The Fréchet (unconstrained) problem

$$\underline{\mathrm{VaR}}_\alpha(X_d^+) = \inf_F \{\mathrm{VaR}_\alpha(X_1^F + \cdots + X_d^F) : X_i \overset{d}{\sim} F_i, i = 1, \ldots, d\}$$

$$\overline{\mathrm{VaR}}_\alpha(X_d^+) = \sup_F \{\mathrm{VaR}_\alpha(X_1^F + \cdots + X_d^F) : X_i \overset{d}{\sim} F_i, i = 1, \ldots, d\}$$

Equivalently, for $\mathcal{C}_d$ the space of all d-copulas

$$\underline{\mathrm{VaR}}_\alpha(X_d^+) = \mathop{inf}_{C \in \mathcal{C}_d} \{\mathrm{VaR}_\alpha(X_1^C + \cdots + X_d^C) : \; X_i \overset{d}{\sim} F_i, \; i = 1, \ldots, d\}$$

$$\overline{\mathrm{VaR}}_\alpha(X_d^+) = \mathop{sup}_{C \in \mathcal{C}_d} \{\mathrm{VaR}_\alpha(X_1^C + \cdots + X_d^C) : \; X_i \overset{d}{\sim} F_i, \; i = 1, \ldots, d\}$$

Recall from Sklar's Theorem: $F = C(F_1, \ldots, F_d)$

VaR Bounds

d=2

The sharp bounds $\overline{\text{VaR}}_\alpha(X_2^+)$ and $\underline{\text{VaR}}_\alpha(X_2^+)$ are known for *any* type of marginal distributions F_1, F_2. Analytic formulas are given in Makarov (1981) and Rüschendorf (1982) ▸ here

d $\geq$ 2

1. Homogeneous case
   - $\overline{\text{VaR}}_\alpha(X_d^+)$: A dual bound technique provides an analytic solution under specific conditions on F (mostly satisfied in practice); see Puccetti and Rüschendorf (2013)
   - $\underline{\text{VaR}}_\alpha(X_d^+)$: Results obtained, based on the concept of complete mixability by Bernard, Jiang and Wang (2013)
2. Heterogeneous case
   - Rearrangement Algorithm of Embrechts, Puccetti, Rüschendorf (2013) yields a powerful computational tool for the calculation of $\overline{\text{VaR}}_\alpha(X_d^+)$ and $\underline{\text{VaR}}_\alpha(X_d^+)$, and possibly $d \geq 1000$

Model Uncertainty

Two important measures

Measure 1 Superadditivity ratio

$$\triangle_{\alpha,d}(X_d^+) = \frac{\overline{\mathrm{VaR}}_\alpha(X_d^+)}{\sum_{i=1}^d \mathrm{VaR}_\alpha(X_i)}$$

Measure 2 Ratio between worst-ES and worst-VaR

$$\mathcal{B}_{\alpha,d}(X_d^+) = \frac{\overline{\mathrm{ES}}_\alpha(X_d^+)}{\overline{\mathrm{VaR}}_\alpha(X_d^+)} = \frac{\sum_{i=1}^d \mathrm{ES}_\alpha(X_i)}{\overline{\mathrm{VaR}}_\alpha(X_d^+)}$$

Model Uncertainty

Superadditivity ratio: Some examples

- Short tailed risks
  1. LogNormal(2,1)-distributed risks $\Rightarrow \triangle_{0.999,d}(X_d^+) \approx 1.4$
  2. Gamma(3,1)-distributed risks $\Rightarrow \triangle_{0.999,d}(X_d^+) \approx 1.1$
- Heavy tailed risks
  - Pareto(2)-distributed risks $\Rightarrow \triangle_{0.999,d}(X_d^+) \approx 2$ (even $\theta \geq 2$)

In QRM applications often Pareto(θ) with $\theta \in [0.5, 5]$:

[0.5,1] catastrophe insurance

[3,5] market return data

$\theta \geq 0.5$ for operational risk

Model Uncertainty

Value-at-Risk

1. Always exists
2. Only frequency
3. Non-coherent risk measure (non-subadditive)
   - Heavy tailed
   - Very skew
   - Special dependencies
4. Backtesting straightforward
5. Estimation (EVT)
6. Model Uncertainty

Expected Shortfall

1. Needs second moment
2. Frequency and severity
3. Coherent risk measure (diversification)
4. Backtesting an issue (elicitability!)
5. Estimation (EVT)
6. Model Uncertainty

Model Uncertainty

VaR versus ES

- For all $\alpha \in (0,1) \Rightarrow ES_\alpha(X) \geq VaR_\alpha(X)$
- For $X \sim N(\mu, \sigma^2)$

$$\lim_{\alpha \to 1} \frac{\mathrm{ES}_\alpha(X)}{\mathrm{VaR}_\alpha(X)} = 1$$

- For $P(X > x) = x^{-1/\xi} L(x),\ 0 < \xi < 1,\ L$ slowly varying,

$$\lim_{\alpha \to 1} \frac{\mathrm{ES}_\alpha(X)}{\mathrm{VaR}_\alpha(X)} = \frac{1}{1-\xi}$$

- Replace X by X_d^+ under specific assumptions on the joint df of $\mathbf{X}$

Model Uncertainty

VaR versus ES

- Worst-dependence scenarios:

$$\overline{\text{VaR}}_\alpha(X_d^+) = \sup_F \{\text{VaR}_\alpha(X_1^F + \cdots + X_d^F) : X_i \overset{d}{\sim} F_i, 1 \le i \le d\}$$

$$\overline{\text{ES}}_\alpha(X_d^+) = \sup_F \{\text{ES}_\alpha(X_1^F + \cdots + X_d^F) : X_i \overset{d}{\sim} F_i, 1 \le i \le d\} \underset{(co.)}{=} \sum_{i=1}^d \text{ES}_\alpha(X_i)$$

- Asymptotic equivalence for large dimensions of the risk portfolio, under some general conditions:

$$\lim_{d\to\infty} \frac{\overline{\text{ES}}_\alpha(X_d^+)}{\overline{\text{VaR}}_\alpha(X_d^+)} = 1$$

Model Uncertainty

Recall from R1, Page 41, Question 8
"... *robust backtesting* ..."

Backtesting

- A new notion for comparing risk measure forecasts: elicitability; see Gneiting (2011)
- **Theorem** (Gneiting (2011)) Under general conditions,
  1. VaR is elicitable
  2. ES is not elicitable

Model Uncertainty

- A new risk measure: expectiles
  For $0 < \tau < 1$ and $X \in L^2$

  $$e_\tau(X) = \arg\min_{x \in \mathbb{R}} \mathbb{E}[\tau \max(X - x, 0)^2 + (1-\tau)\max(x - X, 0)^2]$$

  $e_\tau(x)$ is the unique solution x of the equation

  $$\tau \mathbb{E}[(X - x)^+] = (1-\tau)\mathbb{E}[(x - X)^+]$$

- Risk measure

  $$\rho_{e_\tau}(X) = e_\tau(X) - \mathbb{E}(X)$$

  has the following properties:

  1. homogeneous and law-invariant
  2. subadditive for $1/2 < \tau < 1$, superadditive for $0 < \tau \leq 1/2$
  3. elicitable
  4. not comonotone additive for $1/2 < \tau < 1$

  see Ziegel (2013), Delbaen (2013)

Model Uncertainty

- Link to Keating's Omega Ratio (as alternative to Sharpe Ratio):

$$\Omega(X) = \frac{\mathbb{E}[(X - x)]^+]}{\mathbb{E}[(x - X)^+]}$$

see Rémillard (2013), Section 4.4

- Under specific EVT-based conditions, backtesting of ES is possible, see McNeil et al. (2005), p. 163; using the Pickands-Balkema-de Haan Theorem and the Generalized Pareto Distribution within the Peaks-Over-Threshold (POT) approach.

Model uncertainty

Robustness

- A precise definition matters!
- Some preliminary results:
  1. Kou et al. (2013): "Coherent risk measures are not robust", propose Median Shortfall (VaR-like)
  2. Stahl (2009): "Use stress testing based on mixture models . . . contamination"
  3. Cont et al. (2010): "Our results illustrate in particular, that using recently proposed risk measures such as CVaR/Expected Shortfall leads to a less robust risk measurement procedure than Value-at-Risk."

Model uncertainty

Robustness

4. Cambou, Filipovic (2013): "ES is robust, and VaR is non-robust based on the notion of ϕ-divergence"
5. Krätschmer et al. (2013): "We argue here that Hampel's classical notion of qualitative robustness is not suitable for risk measurement and we propose and analyze a refined notion of robustness that applies to tail-dependent law-invariant convex risk measures on Orlicz spaces." (Introduce an index of qualitative robustness)

Fazit: Much more work in needed!

Application: Operational Risk

Definition

Operational risk is the risk of losses resulting from inadequate or failed internal processes, people and systems, or external events.

Remark: This definition includes legal risk but excludes reputational and strategic risk.

Some examples:

- 1995: Nick Leeson/Barings Bank
- 2001: September 11
- 2001: Enron
- 2012: Libor
- 2013: Abacus 2700–AC1

Application: Operational Risk

The LDA Operational risk capital calculation under Basel II

The ingredients:

- Risk measure VaR_α
- Holding period: 1 year
- Confidence level: 99.9%, $\alpha = 0.999$
- The data 8x7 matrix; 8 Business lines, 7 Loss types
- Often: aggregate column-wise$\Rightarrow$ $\mathrm{VaR}_\alpha(1), \dots, \mathrm{VaR}_\alpha(8)$

  Aggregate: $\sum_{i=1}^{8} \mathrm{VaR}_\alpha(i) = \mathrm{VaR}_\alpha^+$
- Discussion!

Application: Operational Risk

Sharp bounds on the VaR and ES for the sum of d **Pareto(2)** distributed rvs for $\alpha = 0.999$; VaR_α^+ corresponds to the comonotonic case

Pareto(2)	**d=8**	**d=56**
$\underline{\mathrm{VaR}}_\alpha$	31	53
$\underline{\mathrm{ES}}_\alpha$	145	472
VaR_α^+	245	1715
$\overline{\mathrm{VaR}}_\alpha$	465	3454
$\overline{\mathrm{ES}}_\alpha$	498	3486

Application: Operational Risk

Sharp bounds on the VaR and ES for the sum of d=8 Pareto(θ)-distributed rvs for $\alpha = 0.999$; VaR_α^+ corresponds to the comonotonic case, different θ values

θ	θ=1.5	θ=2	θ=3	θ=5	θ=10
$\underline{\text{VaR}}_\alpha$	99	31	9	3	1
$\underline{\text{ES}}_\alpha$	793	145	26	6	2
VaR_α^+	792	245	72	24	8
$\overline{\text{VaR}}_\alpha$	1897	465	110	31.65	9.72
$\overline{\text{ES}}_\alpha$	2392	498	112	31.81	9.73

Conclusion

C1 Q8 and Basel 3.5: a short question with many ramifications. No clear answer so far.

C2 On ES or Var? ES!

C3 Concerning MU and VaR bounds:

1. Find sharp couplings
2. Are they realistic in practice?
3. Impose extra dependence assumptions
4. Add statistical uncertainty

C4 Many more examples needed

C5 Other approaches: Robust Optimization

THANK YOU !

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Makarov and Rüschendorf

For d=2 sharp tail bounds for any $s \in \mathbb{R}$ are:

$$\sup\{P(X_1 + X_2 \geq s) : X_i \sim F_i\} = \inf_{x \in \mathbb{R}} \{\overline{F}_1(x-) + \overline{F}_2(s-x)\}$$

where $\overline{F}_i(x) = 1 - F_i(x) = P(X_1 > x)$ and $\overline{F}_1(x-) = P(X_1 \geq x)$

Theorem (Sharpness of dual bounds)

Under several conditions (involving a below), we obtain the sharp bound for α sufficiently close to 1,

$$\overline{VaR}_\alpha(X_d^+) = D^{-1}(1-\alpha)$$

where b=s-(d-1)a

$$D(s) = \inf_{t<s/d} \frac{d\int_t^{s-(d-1)t} \overline{F}(x)dx}{(s-dt)} = \frac{d\int_a^b \overline{F}(x)dx}{(b-a)}$$

Definition (Complete mixability)

A distribution function F on $\mathbb{R}$ is called d-completely mixable (d-CM) if there exist d-random variables $X_1, \ldots, X_d$ identically distributed as F such that

$$P(X_1 + \cdots + X_d = dk) = 1, \quad (1)$$

for some $k \in \mathbb{R}$. Any such k is called a center of F and any vector $(X_1, \ldots, X_d)'$ satisfying (1) with $X_i \sim F$, $1 \leq i \leq d$, is called a d-complete mix. F is completely mixable (CM) if it is d-CM for all d $\in \mathbb{N}$.

Some examples of CM distributions: Normal, Student t, Cauchy distribution.

Theorem (Puccetti, Wang and Wang (2013))

Fix $m \in \mathbb{N}$, and let $F_1, \ldots, F_m$ be marginal dfs. Assume for $1 \le j \le m$, F_j has finite mean and a continuous density, strictly positive on $[F_j^{-1}(\alpha), F_j^{-1}(1))$. Suppose that for $(X_i)_i$, $F_{X_i} \in \{F_1, \ldots, F_m\}$, and $ES_\alpha(X_i) > 0$. Then

$$\lim_{d \to \infty} \frac{\overline{ES}_\alpha(X_d^+)}{\overline{VaR}_\alpha(X_d^+)} = 1$$

◂ Theorem