Sensitivity of the limit shape of sample clouds from meta densities

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Why meta distributions?

Recent popularity in applications of multivariate probability theory, especially in finance

- Asymptotic independence of coordinatewise maxima as a shortcoming of the multivariate Gaussian model
- Go beyond normality by introducing stronger tail dependence while preserving normal marginals

Why meta distributions?

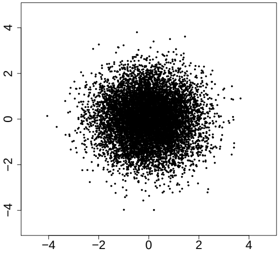
Recent popularity in applications of multivariate probability theory, especially in finance

- Asymptotic independence of coordinatewise maxima as a shortcoming of the multivariate Gaussian model
- Go beyond normality by introducing stronger tail dependence while preserving normal marginals
- A typical example:
  - Start with a multivariate Student t distribution (tail dependence and heavy tails)
  - Transform each coordinate so that the new distribution has normal marginals (light tails)
  - The new distribution is referred to as meta distribution with normal marginals based on the original t distribution (tail dependence and light tails)

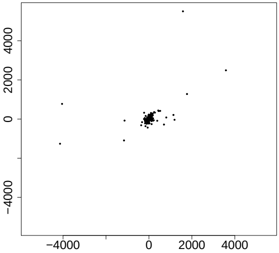
Extremes and asymptotic shape of sample clouds

- Global shape of sample clouds vs. classical EVT of coordinatewise maxima
  - Intuitive view of multivariate extremes via asymptotic behaviour of sample clouds
- The limit shape, if it exists, describes the relation between extreme observations in different directions

Examples of sample clouds

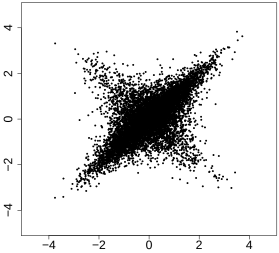


standard normal



elliptic Cauchy with dispersion matrix

$$\Sigma = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}$$



meta-Cauchy with normal marginals

Objectives & questions

- Under the assumptions of the standard set-up, we investigate stability of the shape of the limit set under changes in the original distribution which do not affect marginals (at least asymptotically)

More specifically:

- How much the original and meta distributions can be altered without affecting the asymptotic behaviour of sample clouds? ("robustness", "domains" of limit shape)
- How sensitive is the asymptotic shape of sample clouds to perturbations of the original distribution?

Preliminaries

Definition (Meta distribution)

- Random vector $\mathbf{Z}$ in $\mathbb{R}^d$ with df F and continuous marginals F_i, $i = 1, \dots, d$
- $G_1, \dots, G_d$: continuous df's on $\mathbb{R}$, strictly increasing on $\mathrm{I_i} = \{0 < \mathrm{G_i} < 1\}$
- Define transformation:

$$K(x_1, \dots, x_d) = \big(K_1(x_1), \dots, K_d(x_d)\big), \quad K_i(s) = F_i^{-1}\big(G_i(s)\big), \quad i = 1, \dots, d$$

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- The df $G = F \circ K$ is the meta distribution (with marginals G_i) based on original df F
- The coordinatewise map $K = K_1 \otimes \cdots \otimes K_d$ which maps $\mathbf{x} = (x_1, \ldots, x_d) \in \mathrm{I} = \mathrm{I_1} \times \cdots \times \mathrm{I_d}$ into the vector $\mathbf{z} = (K_1(x_1), \ldots, K_d(x_d))$ is called the meta transformation

Standard set-up (1/3)

Recall:

- A measurable function h on $(0, \infty)$ is regularly varying at ∞ with index ρ (written $h \in RV_\rho$) if for $x > 0$

$$\lim_{t \to \infty} \frac{h(tx)}{h(t)} = x^\rho$$

- A measurable function $e^{-\psi}$ on $[0, \infty)$ is a von Mises function with scale function $a(s) = 1/\psi'(s)$ if ψ is a C^2 function with a positive derivative such that

$$a'(s) \to 0 \qquad s \to \infty$$

Standard set-up (2/3)

Definition (Class $\mathcal{F}_\lambda$, $\lambda > 0$)

The class $\mathcal{F}_\lambda$, $\lambda > 0$, consists of all positive continuous densities f on $\mathbb{R}^d$ which are asymptotic to a function of the form $f_*(n_D(\mathbf{z}))$, where

- $f_*(r) \in RV_{-(\lambda+d)}$ is a continuous decreasing function on $[0, \infty)$
- n_D is the gauge function of the set D; i.e.

$$\{n_D < 1\} = D \qquad n_D(r\mathbf{z}) = r n_D(\mathbf{z}) \qquad r > 0,\ \mathbf{z} \in \mathbb{R}^d$$

- The set D is bounded, open, star-shaped ($\mathbf{z} \in D \ \Rightarrow\ t\mathbf{z} \in D$ for $0 \le t < 1$), contains the origin and has a continuous boundary

Standard set-up (3/3)

Definition (standard set-up)

In the standard set-up, the meta density g is based on the original density f and has marginals which are all equal to g_0, where

- $f \in \mathcal{F}_\lambda$ for some $\lambda > 0$
- g_0 is continuous, positive, symmetric, and asymptotic to a von Mises function $e^{-\psi}$ with $\psi \in RV_\theta$ for $\theta > 0$

Remark: conditions on g_0 are satisfied for normal, Laplace, Weibull densities and densities of the form $g_0(s) \sim as^b e^{-ps^\theta}$, $s \to \infty$, $a, p, \theta > 0$

Convergence of sample clouds (1/3)

- An ***n*-point sample cloud** is the point process consisting of the first n points of a sequence of independent observations from a given distribution, after proper scaling:

$$N_n = \{\mathbf{Z}_1/a_n, \ldots, \mathbf{Z}_n/a_n\}$$

- $N_n(A)$ counts the number of the points of the sample cloud that fall into the set A

Convergence of sample clouds (2/3)

- For density $f \in \mathcal{F}_\lambda$,

$$\frac{f(r_n \mathbf{w}_n)}{f_*(r_n)} \to 1/n_D(\mathbf{w})^{\lambda+d} =: h(\mathbf{w}), \qquad \mathbf{w}_n \to \mathbf{w} \neq \mathbf{0},\ r_n \to \infty,$$

uniformly and in $\mathbf{L}^1$ on the complement of centered balls

- For independent observations $\mathbf{Z}_1, \mathbf{Z}_2, \ldots$ from f, the sample clouds N_n converge in distribution to the Poisson point process with intensity h weakly on the complement of centered balls under a suitable choice of scaling constants a_n

Convergence of sample clouds (3/3)

For a compact set E in $\mathbb{R}^d$, the sample clouds N_n converge onto set E if

- $\mathbb{P}\{N_n(U^c) > 0\} \to 0$ for open sets U containing E
- $\mathbb{P}\{N_n(\mathbf{p} + \epsilon B) > m\} \to 1$ for $m \geq 1,\ \epsilon > 0,\ \mathbf{p} \in E$

This set E is called a **limit set**

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This set E is called a **limit set**

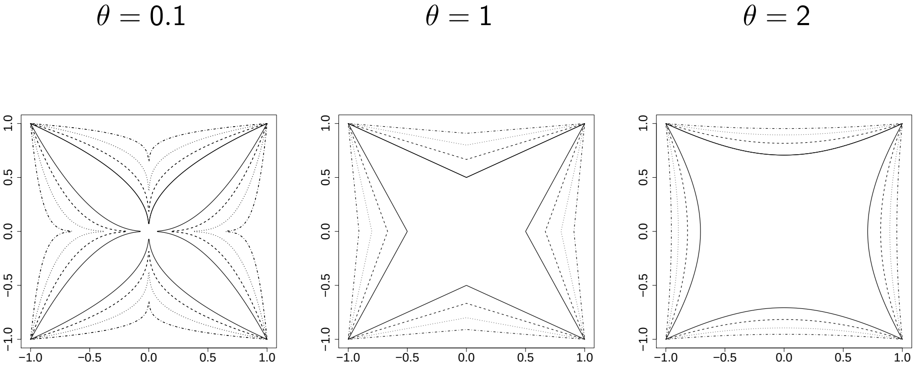
Theorem

- Let f, g and $g_0 \sim e^{-\psi}$ satisfy assumptions of the standard set-up
- For sequence of i.i.d. observations $\mathbf{X}_1, \mathbf{X}_2, \ldots$ from meta density g, sample clouds $M_n = \{\mathbf{X}_1/r_n, \ldots, \mathbf{X}_n/r_n\}$ converge onto $E = E_{\lambda,\theta}$ where

$$E_{\lambda,\theta} = \{\mathbf{u} \in \mathbb{R}^d \mid |u_1|^\theta + \cdots + |u_d|^\theta + \lambda \geq (\lambda + d)\|\mathbf{u}\|_\infty^\theta\},$$

and r_n satisfy $\psi(r_n) \sim \log n$

Examples of limit sets for meta densities



Legend: $\lambda = 1$ (solid), $\lambda = 2$ (dashed), $\lambda = 4$ (dotted), $\lambda = 10$ (dotdash)

Additional definitions and conventions

- All univariate dfs are assumed to be continuous and strictly increasing
- Heavy-tailed dfs F^* and F^{**} have the same asymptotics if
  - the marginals are tail asymptotic
  - the sample clouds converge to the same Poisson point process
  - scaling constants c_n satisfy $1 - F_0(c_n) \sim 1/n$
- Light-tailed dfs G^* and G^{**} have the same asymptotics if
  - the marginals are tail asymptotic
  - the scaled sample clouds converge onto the same compact set E^*
  - scaling constants b_n satisfy $-\log(1 - G_0(b_n)) \sim \log n$

Results

Simplifying assumption for the rest of the talk:

marginal densities of F are all equal to a positive continuous symmetric density f_0

$$\Downarrow$$

the meta transformation K has equal components:

$$K : \mathbf{x} \mapsto \mathbf{z} = (K_0(x_1), \ldots, K_0(x_d))$$

$$K_0 = F_0^{-1} \circ G_0 \qquad K_0(-t) = -K_0(t)$$

Motivating questions

Let F^* be a df with marginal densities $f_0 \;\Rightarrow$
$G^* = F^* \circ K$ is the meta distribution based on F^* with marginals g_0

Qn.1 If the scaled sample clouds from G^* and from G converge onto the same set E, do the scaled sample clouds from F^* converge to the same point process N as those from F?

Qn.2 If the scaled sample clouds from F^* and from F converge to the same point process N, do the scaled sample clouds from G^* converge onto the same set E as those from G?

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Answer:

- For coordinatewise maxima and their exponent measures: "Yes"
- For sample clouds and their limit shape: "No" $\Rightarrow$ examples to follow ...

Basic ideas

- fix the marginals f_0 and g_0 (this determines meta tranformation K)
- vary the copula
- check the limit behaviour of the sample clouds (imposing condition that both converge)

That is, we look for dfs F^* and G^* with the properties:

- F^* has marginal densities f_0
- G^* is the meta distribution based on F^* with marginal densities g_0
- the scaled sample clouds from F^* converge to a point process N^*
- the scaled sample clouds from G^* converge onto a compact set E^*
- either $E^* = E_{\lambda,\theta}$ or N^* has mean measure $\rho^* = \rho$ with intensity h

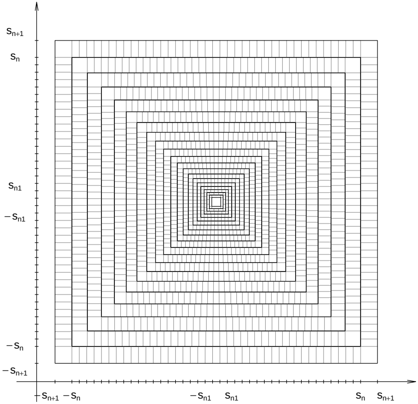
Constructions are based on two procedures: block partitions and mixtures

Block partitions - Overview

- a partition into coordinate blocks
- if blocks are small $\Rightarrow$ asymptotics of a distribution do not change if the distribution is replaced by one which gives the same mass to each block
- effect of meta transformation K:
  - block partitions are mapped into block partitions
  - mass of the blocks is preserved
  - size and shape of the blocks may change drastically
- allow to gain insight in the relation between asymptotic behaviour of measures dF^* and dG^*

Block partitions - Construction

- Partitions of $\mathbb{R}^d$ into bounded Borel sets B_n, where B_n are coordinate blocks
- Start with an increasing sequence of cubes: $s_n C = [-s_n, s_n]^d$ with $0 < s_1 < s_2 < \cdots, \; s_n \to \infty$
- Subdivide ring $R_n = s_{n+1} C \setminus s_n C$ into blocks by a symmetric partition of interval $[-s_{n+1}, s_{n+1}]$ with division points $\pm s_{nj}$, $j = 1, \ldots, m_n$, with



$$-s_{n+1} < -s_n < \cdots < -s_{n1} < s_{n1} < \cdots < s_{nm_n} = s_n < s_{nm_n+1} = s_{n+1}$$

Definition

A partition of $\mathbb{R}^d$ into Borel sets A_n is regular if:

(1) The sets A_n are bounded and have positive volume $|A_n| > 0$

(2) Every compact set is covered by a finite number of sets A_n

(3) The sets A_n are relatively small: There exist points $\mathbf{p}_n \in A_n$ with norm $\|\mathbf{p}_n\| = r_n > 0$ such that for any $\epsilon > 0$

$$A_n \subset \mathbf{p}_n + \epsilon r_n B \qquad n \geq n_\epsilon$$

Remark: the block partition is regular if and only if

$$s_{n+1} \sim s_n \text{ and } \Delta_n / s_n \to 0$$

where $\Delta_n = \max\{s_{n1}, s_{n2} - s_{n1}, \ldots, s_{nm_n} - s_{nm_n - 1}\}$

Block partitions - Domains (1/2)

$\Rightarrow$ Simple answer to the question: If f or g are replaced by a discrete distribution, how far apart are the atoms allowed to be so that the asymptotic behaviour of the sample clouds is retained?

Block partitions - Domains (2/2)

Theorem

- $A_1, A_2, \ldots$ block partition in $\mathbf{x}$-space
- $B_1 = K(A_1), B_2 = K(A_2), \ldots$ corr. block partition in $\mathbf{z}$-space
- $\tilde{\mu}$ and $\tilde{\pi}$ prob. measures in $\mathbf{x}$-space and $\mathbf{z}$-space, resp., linked via $\tilde{\pi} = K(\tilde{\mu})$, so that $\tilde{\pi}(A_n) = \tilde{\mu}(B_n)$ for all n

If (A_n) and (B_n) are both regular and one of the asymptotic equalities holds

$$\tilde{\pi}(A_n) \sim \int_{A_n} f(\mathbf{z})d\mathbf{z} \quad \Longleftrightarrow \quad \tilde{\mu}(B_n) \sim \int_{B_n} g(\mathbf{x})d\mathbf{x}$$

then:

- sample clouds from $\tilde{\pi}$ scaled by c_n converge to the Poisson point process with intensity h
- sample clouds from $\tilde{\mu}$ scaled by b_n converge onto the set $E = E_{\lambda,\theta}$

Block partitions - Sensitivity

Due to non-linearity of the meta transformation K, regularity of one block partition does not imply regularity of the other block partition

$$\Downarrow$$

distinguish two cases:

1. (A_n) in **x**-space is regular, but (B_n) is not
2. (B_n) in **z**-space is regular, but (A_n) is not

$$\Downarrow$$

What are the implications?

Block partitions - Sensitivity - Case 1

Theorem

- Assume the standard set-up
- Let $\tilde{\rho}$ be an excess measure on $\mathbb{R}^d \setminus \{\mathbf{0}\}$ with marginal densities $\lambda/|t|^{\lambda+1}$, $\lambda > 0$

One may choose $\tilde{F}$ such that

- marginals are tail asymptotic to F_0
- sample clouds converge to point process $\tilde{N}$ with mean measure $\tilde{\rho}$
- meta dfs $\tilde{G} = \tilde{F} \circ K$ and G have the same asymptotics

Block partitions - Sensitivity - Case 1

(A_n) in $\mathbf{x}$-space is regular, (B_n) is not

- (A_n) is based on a sequence of cubes $[-s_n, s_n]^d$ with $s_{n+1} \sim s_n$ ($\Rightarrow$ (A_n) is regular)
- It is possible that $t_n << t_{n+1}$ ($\Rightarrow$ (B_n) with $B_n = K(A_n)$ is non-regular)
- Choose $s_{nm_n-1} = s_{n-1}$; define sets:
  - $U = \bigcup_n [t_{n-1}, t_{n+1}]^d$
  - Note: $U/t_n \to (0, \infty)^d$ for $t_n \to \infty$ if $t_n << t_{n+1}$
  - Let U_δ be the image of U in orthant Q_δ for $\delta \in \{-1, 1\}^d$

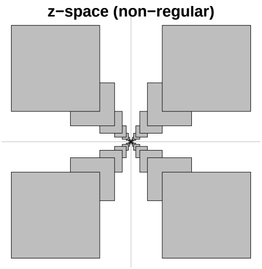
Sketch of construction

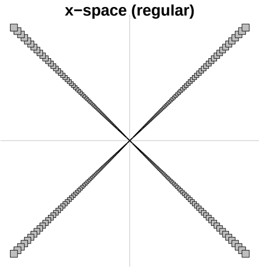
- Choose density $\hat{f}$ such that sample clouds converge to $\tilde{N}$ with mean measure $\tilde{\rho}$
- Let $\tilde{f}$ agree with $\hat{f}$ on sets U_δ and with f elsewhere $\Rightarrow$ $\tilde{f}$ and $\hat{f}$ differ only on asymptotically negligible set
- Alter $\tilde{f}$ on a bounded set to make it a probability density

$\Rightarrow$ sample clouds from $\tilde{F}$ converge to $\tilde{N}$

- In corr. partition (A_n) on **x**-space, the measure is changed only on "tiny" blocks $[s_{n-1}, s_{n+1}]^d$ (with $s_{n-1} \sim s_{n+1}$) and their reflections

$\Rightarrow$ sample clouds from $\tilde{G} = \tilde{F} \circ K$ converge onto $E_{\lambda,\theta}$





Block partitions - Sensitivity - Case 2

Theorem

- Assume the standard set-up

There exists a df $\tilde{F}$ such that

- original df F and $\tilde{F}$ have the same asymptotics
- scaled sample clouds from the corresponding meta distribution $\tilde{G}$ converge onto the diagonal cross E_{00}

$$E_{00} = \{r\delta \mid 0 \le r \le 1,\ \delta \in \{-1,1\}^d\}$$

Block partitions - Sensitivity - Case 2

(B_n) in **z**-space is regular, (A_n) is not

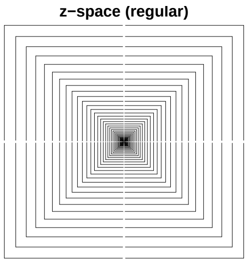
- Convenient to define partition division points in terms of upper quantiles with $p_n = e^{-\sqrt{n}}$ and $m_n = n$ $(j = 1, \dots, n)$:

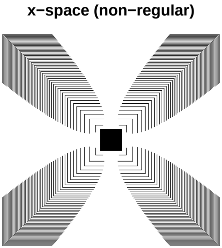
$$1 - F_0(t_n) = 1 - G_0(s_n) = p_n,\ 1 - F_0(t_{nj}) = 1 - G_0(s_{nj}) = np_n/j$$

- Then (B_n) is regular $(t_{n1}/t_n \to 0)$, but (A_n) is not $(s_{n1} \sim s_n)$

Sketch of proof

- Construct a density $\tilde{f}$ such that $\tilde{f}$ and f agree on every block which does not intersect a coordinate plane (except for block containing the origin)
- $t_{n1}/t_n \to 0 \ \Rightarrow\ \tilde{f}$ and f agree outside a vanishing neighborhood around the coordinate planes
- $s_{n1} \sim s_n \ \Rightarrow\ \tilde{g}$ and g only agree on a vanishing neighborhood around the diagonals, and $\tilde{g}(r_n\mathbf{w}_n) = 0$ eventually for $r_n \to \infty$ and $\mathbf{w}_n \to \mathbf{w}$ with $\mathbf{w}$ not on a diagonal ray





Mixtures - Overview

- Let prob. measure $d\tilde{F}$ agree outside a bounded set with a mixture $d(F + F^o)$, where F^o has lighter marginals than F:

$$F_j^o(-t) << F_0(-t), \quad 1 - F_j^o(t) << 1 - F_0(t), \quad t \to \infty,\ j = 1, \dots, d,$$

$\Rightarrow$ $\tilde{F}$ and F have the same asymptotics

- dfs $\tilde{G}$ and G may have different asymptotics:
  - $b_n^o \sim b_n$ even though G^o has lighter tails than G
  - suppose sample clouds from G^o converge onto a compact set E^o

then: sample clouds from $\tilde{G}$ converge onto $E \cup E^o$

Mixtures - Implication

Theorem

- Assume the standard set-up
- $A \subset [-1,1]^d$ a star-shaped closed set with continuous boundary and containing the origin as interior point
- Let $E_{00} = \{r\delta \mid 0 \leq r \leq 1,\ \delta \in \{-1,1\}^d\}$ be the diagonal cross

There exists a df $\tilde{F}$ with the same asymptotics as F, but such that the scaled sample clouds from the meta disribution $\tilde{G}$ converge onto the set $E = A \cup E_{00}$

Concluding remarks (1/2)

Question: What does the copula say about the asymptotics?

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Answer 1: Everything, since it determines the df if the marginals are given

Concluding remarks (1/2)

Question: What does the copula say about the asymptotics?

Answer 1: Everything, since it determines the df if the marginals are given

Answer 2: Nothing, since the examples above show that there is no relation between the asymptotics of F^* and the asymptotics of G^* even with the prescribed marginals f_0 and g_0

Concluding remarks (2/2)

- Sensitivity of the limit shape may be radical due to even slight perturbations of the original distribution, perturbations which do not affect the asymptotics of the extremes from the original distribution
- Original and meta distributions have the same copula, yet a relation between the behaviour of their sample clouds is lost in the limit
- The limit shape of sample clouds from light-tailed meta distributions gives a very rough picture

Future work:

- A closer look at the edge of the sample clouds under a more refined scaling
- A full analysis of the asymptotics needs to take into account conditional exceedance distributions and associated excess measures