Paul Embrechts

Sharp bounds on the VaR for sums of dependent risks

joint work with
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and Ludger Rüschendorf (university of Freiburg, Germany)

Mathematical Problem

Assumptions:

$L_1, \ldots, L_d$ one period risks with statistically estimated marginals.

$L_1 + \cdots + L_d$ total loss exposure.

$\mathrm{VaR}_\alpha(L_1 + \cdots + L_d)$ amount of capital to be reserved.

(if $\mathrm{VaR}_\alpha(L_1 + \cdots + L_d) = s$, then $P(L_1 + \cdots + L_d \geq s) \leq 1 - \alpha$)

Task: for a fixed (high) level of probability α, calculate:

$$\overline{\mathrm{VaR}}_\alpha = \sup\left\{\mathrm{VaR}_\alpha(L_1 + \cdots + L_d) : L_j \sim F_j, 1 \leq j \leq d\right\}$$

$$\underline{\mathrm{VaR}}_\alpha = \inf\left\{\mathrm{VaR}_\alpha(L_1 + \cdots + L_d) : L_j \sim F_j, 1 \leq j \leq d\right\}$$

Motivation (QRM)

$$\boxed{\begin{array}{c} L_1 \sim F_1, \quad L_2 \sim F_2, \quad \ldots, \quad L_d \sim F_d \\ \text{marginal distributions} \end{array}} \quad d \approx 600$$

+

$$\boxed{\text{dependence model}}$$

=

$$\mathrm{VaR}_\alpha(L_1 + \cdots + L_d)$$

Motivation (QRM)

$$L_1 \sim F_1, \quad L_2 \sim F_2, \quad \dots, \quad L_d \sim F_d$$

marginal distributions

$d \approx 600$

+

dependence model

=

$$\mathrm{VaR}_\alpha(L_1 + \cdots + L_d)$$

$\underline{\mathrm{VaR}}_\alpha$ $\quad$ $\mathrm{VaR}_\alpha^+ = \sum_{j=1}^d \mathrm{VaR}_\alpha(L_j)$ $\quad$ $\overline{\mathrm{VaR}}_\alpha$

Known results

- In the homogeneous case $F_j = F,\ 1 \leq j \leq d$, the bound $\overline{\mathrm{VaR}}_\alpha$ has been recently given for $d > 2$ in [PR11] and [WW11] under different assumptions.

- In the homogeneous case, $\overline{\mathrm{VaR}}_\alpha$ is very easy to calculate in arbitrary dimensions.

- In the *inhomogeneous* case, the computation of $\overline{\mathrm{VaR}}_\alpha$ poses serious problems. And the computation of $\underline{\mathrm{VaR}}_\alpha$ is not possible.

Homogeneous marginals, $d \geq 3$

In the case that $L_1, \ldots, L_d$ are ***identically distributed***, we have

$$M(s) = \sup\left\{P(L_1 + \cdots + L_d \geq s); L_j \sim F, 1 \leq j \leq d\right\}$$

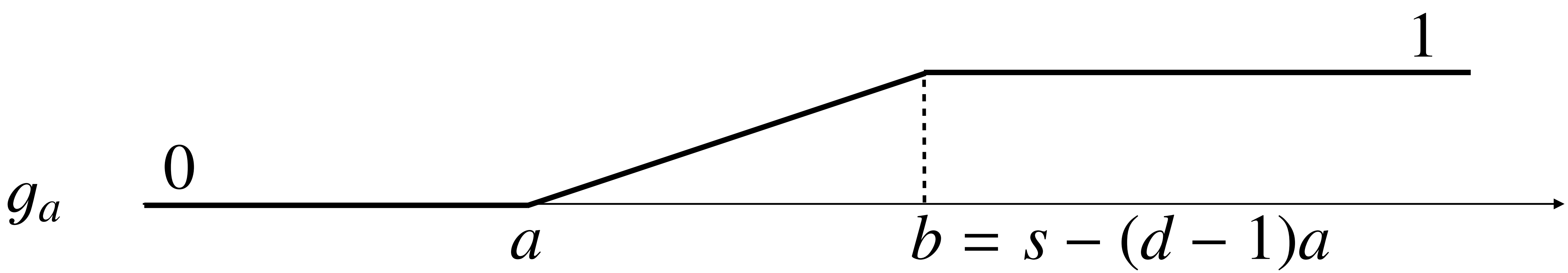
Duality theorem (reduced)

$$M(s) = \inf\left\{d \int g \, dF; g \in \mathcal{A}(s)\right\}$$, where

$$\mathcal{A}(s) = \{g : \mathbb{R} \to [0,1] \text{ such that}$$
$$g(x_1) + \cdots + g(x_d) \geq \mathbf{1}\{x_1 + \cdots + x_d \geq s\}\}$$

Dual bounds

Embrechts and Puccetti (2006) introduce the following class of piecewise-linear functions for $a < s/d$



$$M(s) \le D(s) = \inf_{a<s/d} \left(d \int g_a \, dF \right) = d \inf_{a<s/d} \frac{\int_a^b \overline{F}(x)\, dx}{s - da}.$$

The dual bound $D(s)$ is better than the standard bound produced by choosing piecewise-constant dual functions.

Referee report on Embrechts and Puccetti (2006)

It is interesting to see that such a trivial two-dimensional problem (Makarov (1981)) had sparked so much attention. A.N.Kolmogorov was just curious to see that simple mass-transportation result without knowing that much of the theory was already done by Sudakov, Kantorovich and others. (He gave this problem to Makarov in one of his walks with students listen to theirs topics of research.) The extension to the multivariate case (n>2) was non-trivial as it the upper Fréchet is not attainable by a distribution function, but that is to be expected.

Small remark: re: page 2, l.6-7 "A full solution of the general problem seems still out of reach." . My advice - Forget it! It is terribly hard (if at all possible) and it is not worth.

Timeline to the result

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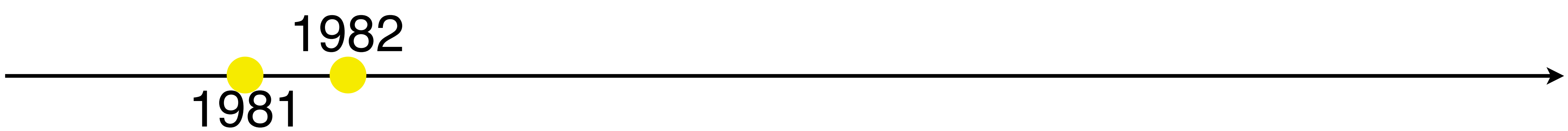
Makarov gives the optimal coupling for the sum of two risks answering a question by Kolmogorov

Timeline to the result

Rüschendorf gives independently the same optimal coupling and the dual solution



Makarov gives the optimal coupling for the sum of two risks answering a question by Kolmogorov

Timeline to the result

1981

Makarov gives the optimal coupling for the sum of two risks answering a question by Kolmogorov

1982

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2006

Embrechts and Puccetti introduce dual bounds in the homogeneous case

Timeline to the result

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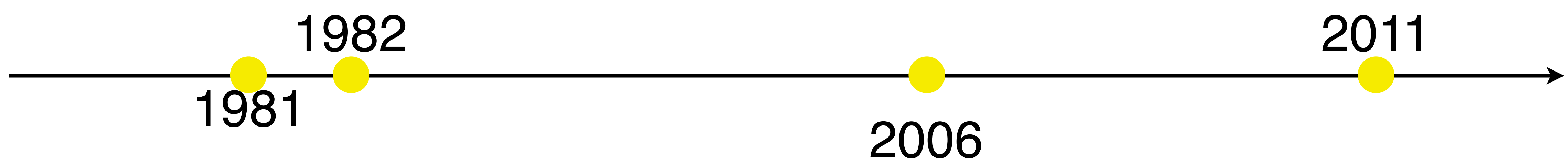
Wang and Wang gives optimal couplings for the sum of arbitrary risks in some specific examples



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2011

Wang and Wang gives optimal couplings for the sum of arbitrary risks in some specific examples

2012

sharpness of dual bounds is stated for a general class of distributions

The Rearrangement Algorithm (RA)

a new numerical approximation procedure

Game

$$X = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 3 \\ 4 & 2 \\ 5 & 5 \end{bmatrix} \quad \begin{matrix} \color{blue}{\mathbf{2}} \\ \mathbf{6} \\ \mathbf{6} \\ \mathbf{6} \\ \color{red}{\mathbf{10}} \end{matrix}$$

Game

$$X = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 3 \\ 4 & 2 \\ 5 & 5 \end{bmatrix} \quad \begin{matrix} \mathbf{2} \\ \mathbf{6} \\ \mathbf{6} \\ \mathbf{6} \\ \mathbf{10} \end{matrix}$$

Rearrange the second column to obtain sum with minimal variance

Game

$$X = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 3 \\ 4 & 2 \\ 5 & 5 \end{bmatrix} \quad \begin{matrix} \mathbf{2} \\ \mathbf{6} \\ \mathbf{6} \\ \mathbf{6} \\ \mathbf{10} \end{matrix} \qquad \begin{bmatrix} 1 & 5 \\ 2 & 4 \\ 3 & 3 \\ 4 & 2 \\ 5 & 1 \end{bmatrix} \quad \begin{matrix} \mathbf{6} \\ \mathbf{6} \\ \mathbf{6} \\ \mathbf{6} \\ \mathbf{6} \end{matrix}$$

Rearrange the second column to obtain sum with minimal variance

Game (more difficult)

$$X = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 1 \\ 3 & 3 & 4 \\ 4 & 2 & 3 \\ 5 & 5 & 5 \end{bmatrix} \quad \begin{matrix} \mathbf{\color{blue}{4}} \\ \mathbf{7} \\ \mathbf{10} \\ \mathbf{9} \\ \mathbf{\color{red}{15}} \end{matrix} \qquad \begin{bmatrix} & 1 & 2 \\ & 4 & 1 \\ & 3 & 4 \\ & 2 & 3 \\ & 5 & 5 \end{bmatrix} \begin{matrix} \mathbf{\color{gray}{3}} \\ \mathbf{\color{gray}{5}} \\ \mathbf{\color{gray}{7}} \\ \mathbf{\color{gray}{5}} \\ \mathbf{\color{gray}{10}} \end{matrix}$$

Rearrange the entries within each column in order to obtain sum with minimal variance

Game (more difficult)

$$X = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 1 \\ 3 & 3 & 4 \\ 4 & 2 & 3 \\ 5 & 5 & 5 \end{bmatrix} \begin{array}{c} \mathbf{4} \\ \mathbf{7} \\ \mathbf{10} \\ \mathbf{9} \\ \mathbf{15} \end{array} \qquad \begin{bmatrix} & 1 & 2 \\ & 4 & 1 \\ & 3 & 4 \\ & 2 & 3 \\ 1 & 5 & 5 \end{bmatrix} \begin{array}{c} \mathbf{3} \\ \mathbf{5} \\ \mathbf{7} \\ \mathbf{5} \\ \mathbf{10} \end{array}$$

Rearrange the entries within each column in order to obtain sum with minimal variance

Game (more difficult)

$$X = \left[\begin{array}{c|c|c} 1 & 1 & 2 \\ 2 & 4 & 1 \\ 3 & 3 & 4 \\ 4 & 2 & 3 \\ 5 & 5 & 5 \end{array}\right] \begin{array}{c} \color{blue}{\mathbf{4}} \\ \mathbf{7} \\ \mathbf{10} \\ \mathbf{9} \\ \color{red}{\mathbf{15}} \end{array} \qquad \left[\begin{array}{c|c|c} & 1 & 2 \\ & 4 & 1 \\ 2 & 3 & 4 \\ & 2 & 3 \\ 1 & 5 & 5 \end{array}\right] \begin{array}{c} \color{gray}{\mathbf{3}} \\ \color{gray}{\mathbf{5}} \\ \color{gray}{\mathbf{7}} \\ \color{gray}{\mathbf{5}} \\ \color{gray}{\mathbf{10}} \end{array}$$

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Game (more difficult)

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Rearrange the entries within each column in order to obtain sum with minimal variance

Game (more difficult)

$$X = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 1 \\ 3 & 3 & 4 \\ 4 & 2 & 3 \\ 5 & 5 & 5 \end{bmatrix} \begin{array}{c} \mathbf{4} \\ \mathbf{7} \\ \mathbf{10} \\ \mathbf{9} \\ \mathbf{15} \end{array} \qquad \begin{bmatrix} 5 & 1 & 2 \\ 3 & 4 & 1 \\ 2 & 3 & 4 \\ 4 & 2 & 3 \\ 1 & 5 & 5 \end{bmatrix} \begin{array}{c} \mathbf{8} \\ \mathbf{8} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{11} \end{array}$$

Rearrange the entries within each column in order to obtain sum with minimal variance

$$\begin{bmatrix} 5 & 1 & 2 \\ 3 & 4 & 1 \\ 2 & 3 & 4 \\ 4 & 2 & 3 \\ 1 & 5 & 5 \end{bmatrix} \quad \begin{array}{c} \mathbf{8} \\ \mathbf{8} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{11} \end{array}$$

$$\begin{bmatrix} 5 & 1 & 2 \\ 3 & 4 & 1 \\ 2 & 3 & 4 \\ 4 & 2 & 3 \\ 1 & 5 & 5 \end{bmatrix} \quad \begin{matrix} \mathbf{8} \\ \mathbf{8} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{11} \end{matrix} \qquad\qquad \begin{bmatrix} 5 & 1 & 2 \\ 3 & 5 & 1 \\ 2 & 3 & 4 \\ 4 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \quad \begin{matrix} \mathbf{8} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{10} \end{matrix}$$

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Summary:

For a given matrix, rearrange the entries in the columns until you find an *ordered* matrix, i.e. a matrix in which

each column is oppositely ordered to the sum of the others.

We call a matrix *optimal* if the minimal component of the vector of the sum is maximized and the maximal component of that vector is minimized (min-max problem).

Sudoku

$$X = \begin{array}{cc} \left[\begin{array}{c|c|c} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{array}\right] & \begin{array}{c} \color{blue}{\mathbf{9}} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{9} \\ \color{red}{\mathbf{9}} \end{array} \end{array}$$

Sudoku

$$X = \left[\begin{array}{c|c|c} \bullet & \bullet & 1 \\ \bullet & \bullet & 2 \\ \bullet & \bullet & 3 \\ \bullet & \bullet & 4 \\ \bullet & \bullet & 5 \end{array}\right] \quad \begin{array}{c} \mathbf{9} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{9} \end{array}$$

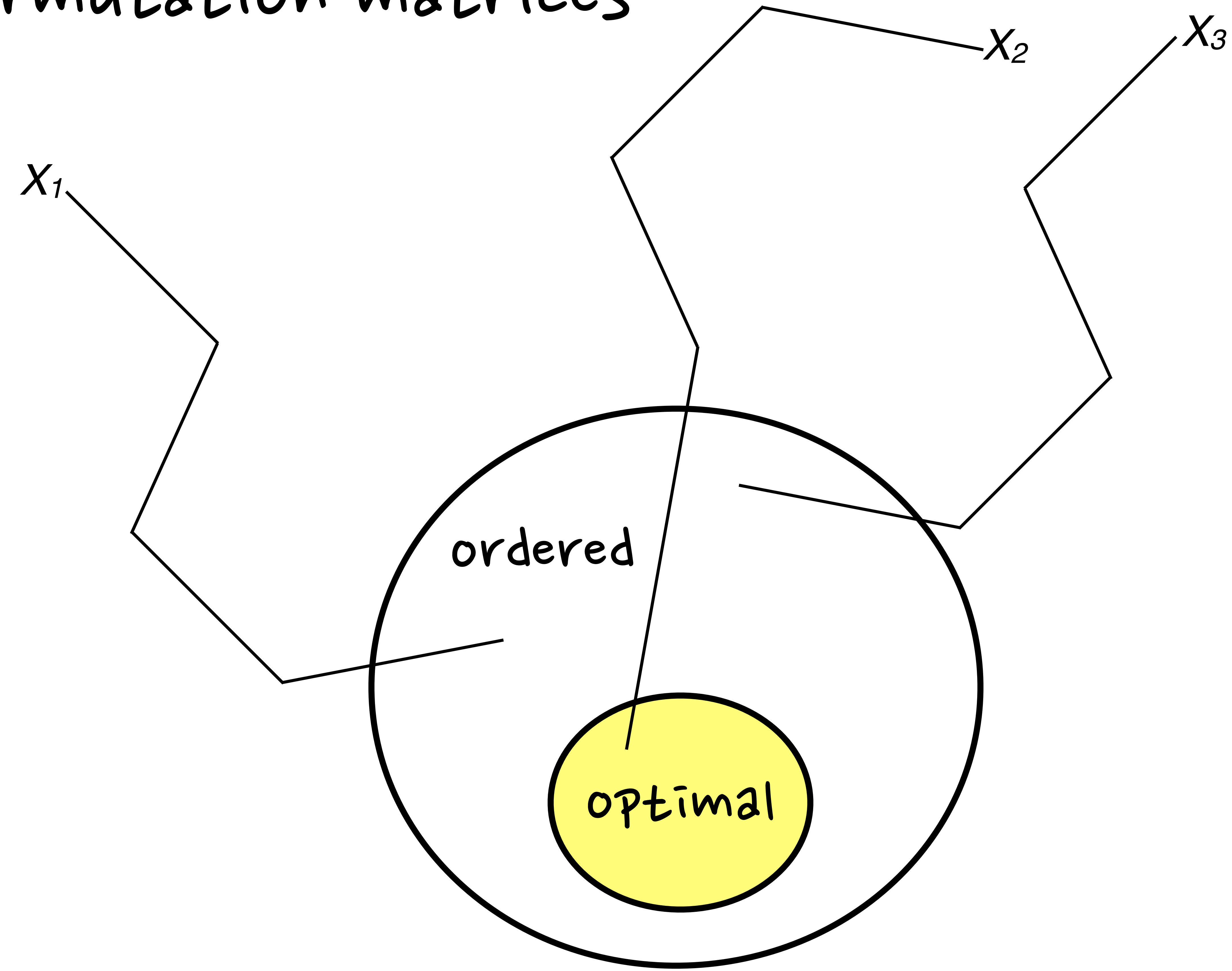
Hint!

Sudoku

$$X = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 4 & 2 \\ 1 & 5 & 3 \\ 4 & 1 & 4 \\ 2 & 2 & 5 \end{bmatrix} \quad \begin{matrix} \mathbf{9} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{9} \\ \mathbf{9} \end{matrix}$$

Solution!

Permutation matrices



BEST/WORST VAR

?

BEST/WORST VAR

?

dependence=rearrangement

$\mathrm{VaR}_{0.99}(L_1)$

	1	2	3	Σ
1	9.00000	9.00000	9.00000	27.0000
2	9.17095	9.17095	9.17095	27.5129
3	9.35098	9.35098	9.35098	28.0530
4	9.54093	9.54093	9.54093	28.6228
5	9.74172	9.74172	9.74172	29.2252
6	9.95445	9.95445	9.95445	29.8634
7	10.18034	10.18034	10.18034	30.5410
8	10.42080	10.42080	10.42080	31.2624
9	10.67748	10.67748	10.67748	32.0325
10	10.95229	10.95229	10.95229	32.8569
11	11.24745	11.24745	11.24745	33.7423
12	11.56562	11.56562	11.56562	34.6969
13	11.90994	11.90994	11.90994	35.7298
14	12.28422	12.28422	12.28422	36.8527
15	12.69306	12.69306	12.69306	38.0792
16	13.14214	13.14214	13.14214	39.4264
17	13.63850	13.63850	13.63850	40.9155
18	14.19109	14.19109	14.19109	42.5733
19	14.81139	14.81139	14.81139	44.4342
20	15.51446	15.51446	15.51446	46.5434
21	16.32051	16.32051	16.32051	48.9615
22	17.25742	17.25742	17.25742	51.7723
23	18.36492	18.36492	18.36492	55.0948
24	19.70197	19.70197	19.70197	59.1059
25	21.36068	21.36068	21.36068	64.0820
26	23.49490	23.49490	23.49490	70.4847
27	26.38613	26.38613	26.38613	79.1584
28	30.62278	30.62278	30.62278	91.8683
29	37.72983	37.72983	37.72983	113.1895
30	53.77226	53.77226	53.77226	161.3168
Σ	494.99920	494.99920	494.99920	NA

$\mathrm{VaR}_1(L_1)$

Fix $\alpha \in (0,1)$ and assume that each $F_j^{-1}|[\alpha,1]$ takes only N values all having the same probability $(1-\alpha)/N$.

$$P\left(\sum_{j=1}^{3} L_j \geq \min(\mathrm{rowSums}(X))\right) \geq 1-\alpha$$

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$$\mathrm{VaR}_\alpha(L_1 + \cdots + L_d) \geq \min(\mathrm{rowSums}(X))$$

$$\overline{\mathrm{VaR}}_\alpha = \max_{\tilde{X} \in \mathcal{P}(X)} \min(\mathrm{rowSums}(\tilde{X}))$$

(idea of the proof)

	1	2	3	Σ
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Pareto(2) marginals and $\alpha = 0.99$

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8	10.42080	10.42080	10.42080	31.2624
9	10.67748	10.67748	10.67748	32.0325
10	10.95229	10.95229	10.95229	32.8569
11	11.24745	11.24745	11.24745	33.7423
12	11.56562	11.56562	11.56562	34.6969
13	11.90994	11.90994	11.90994	35.7298
14	12.28422	12.28422	12.28422	36.8527
15	12.69306	12.69306	12.69306	38.0792
16	13.14214	13.14214	13.14214	39.4264
17	13.63850	13.63850	13.63850	40.9155
18	14.19109	14.19109	14.19109	42.5733
19	14.81139	14.81139	14.81139	44.4342
20	15.51446	15.51446	15.51446	46.5434
21	16.32051	16.32051	16.32051	48.9615
22	17.25742	17.25742	17.25742	51.7723
23	18.36492	18.36492	18.36492	55.0948
24	19.70197	19.70197	19.70197	59.1059
25	21.36068	21.36068	21.36068	64.0820
26	23.49490	23.49490	23.49490	70.4847
27	26.38613	26.38613	26.38613	79.1584
28	30.62278	30.62278	30.62278	91.8683
29	37.72983	37.72983	37.72983	113.1895
30	53.77226	53.77226	53.77226	161.3168
Σ	494.99920	494.99920	494.99920	NA

Pareto(2) marginals and $\alpha = 0.99$

	1	2	3	Σ
1	9.00000	9.00000	9.00000	27.0000
2	9.17095	9.17095	9.17095	27.5129
3	9.35098	9.35098	9.35098	28.0530
4	9.54093	9.54093	9.54093	28.6228
5	9.74172	9.74172	9.74172	29.2252
6	9.95445	9.95445	9.95445	29.8634
7	10.18034	10.18034	10.18034	30.5410
8	10.42080	10.42080	10.42080	31.2624
9	10.67748	10.67748	10.67748	32.0325
10	10.95229	10.95229	10.95229	32.8569
11	11.24745	11.24745	11.24745	33.7423
12	11.56562	11.56562	11.56562	34.6969
13	11.90994	11.90994	11.90994	35.7298
14	12.28422	12.28422	12.28422	36.8527
15	12.69306	12.69306	12.69306	38.0792
16	13.14214	13.14214	13.14214	39.4264
17	13.63850	13.63850	13.63850	40.9155
18	14.19109	14.19109	14.19109	42.5733
19	14.81139	14.81139	14.81139	44.4342
20	15.51446	15.51446	15.51446	46.5434
21	16.32051	16.32051	16.32051	48.9615
22	17.25742	17.25742	17.25742	51.7723
23	18.36492	18.36492	18.36492	55.0948
24	19.70197	19.70197	19.70197	59.1059
25	21.36068	21.36068	21.36068	64.0820
26	23.49490	23.49490	23.49490	70.4847
27	26.38613	26.38613	26.38613	79.1584
28	30.62278	30.62278	30.62278	91.8683
29	37.72983	37.72983	37.72983	113.1895
30	53.77226	53.77226	53.77226	161.3168
Σ	494.99920	494.99920	494.99920	NA

	1	2	3	Σ
1	12.28422	21.36068	11.24745	44.8924
2	9.95445	30.62278	9.74172	50.3190
3	21.36068	11.90994	11.56562	44.8362
4	15.51446	14.19109	14.81139	44.5169
5	19.70197	12.28422	13.14214	45.1283
6	17.25742	13.63850	13.63850	44.5344
7	53.77226	9.00000	9.17095	71.9432
8	10.42080	26.38613	10.18034	46.9873
9	13.14214	13.14214	18.36492	44.6492
10	9.17095	53.77226	9.00000	71.9432
11	13.63850	11.24745	19.70197	44.5879
12	18.36492	14.81139	11.90994	45.0862
13	12.69306	19.70197	12.69306	45.0881
14	9.74172	9.95445	30.62278	50.3190
15	10.95229	10.67748	23.49490	45.1247
16	30.62278	9.74172	9.95445	50.3190
17	10.67748	23.49490	10.95229	45.1247
18	14.81139	15.51446	14.19109	44.5169
19	11.56562	17.25742	16.32051	45.1435
20	16.32051	12.69306	15.51446	44.5280
21	37.72983	9.54093	9.35098	56.6217
22	23.49490	10.95229	10.67748	45.1247
23	9.00000	9.17095	53.77226	71.9432
24	10.18034	10.42080	26.38613	46.9873
25	14.19109	18.36492	12.28422	44.8402
26	26.38613	10.18034	10.42080	46.9873
27	9.54093	9.35098	37.72983	56.6217
28	11.24745	16.32051	17.25742	44.8254
29	11.90994	11.56562	21.36068	44.8362
30	9.35098	37.72983	9.54093	56.6217
Σ	494.99920	494.99920	494.99920	NA

Pareto(2) marginals and $\alpha = 0.99$

	1	2	3	Σ
1	9.00000	9.00000	9.00000	27.0000
2	9.17095	9.17095	9.17095	27.5129
3	9.35098	9.35098	9.35098	28.0530
4	9.54093	9.54093	9.54093	28.6228
5	9.74172	9.74172	9.74172	29.2252
6	9.95445	9.95445	9.95445	29.8634
7	10.18034	10.18034	10.18034	30.5410
8	10.42080	10.42080	10.42080	31.2624
9	10.67748	10.67748	10.67748	32.0325
10	10.95229	10.95229	10.95229	32.8569
11	11.24745	11.24745	11.24745	33.7423
12	11.56562	11.56562	11.56562	34.6969
13	11.90994	11.90994	11.90994	35.7298
14	12.28422	12.28422	12.28422	36.8527
15	12.69306	12.69306	12.69306	38.0792
16	13.14214	13.14214	13.14214	39.4264
17	13.63850	13.63850	13.63850	40.9155
18	14.19109	14.19109	14.19109	42.5733
19	14.81139	14.81139	14.81139	44.4342
20	15.51446	15.51446	15.51446	46.5434
21	16.32051	16.32051	16.32051	48.9615
22	17.25742	17.25742	17.25742	51.7723
23	18.36492	18.36492	18.36492	55.0948
24	19.70197	19.70197	19.70197	59.1059
25	21.36068	21.36068	21.36068	64.0820
26	23.49490	23.49490	23.49490	70.4847
27	26.38613	26.38613	26.38613	79.1584
28	30.62278	30.62278	30.62278	91.8683
29	37.72983	37.72983	37.72983	113.1895
30	53.77226	53.77226	53.77226	161.3168
Σ	494.99920	494.99920	494.99920	NA

	1	2	3	Σ
1	12.28422	21.36068	11.24745	44.8924
2	9.95445	30.62278	9.74172	50.3190
3	21.36068	11.90994	11.56562	44.8362
4	15.51446	14.19109	14.81139	44.5169
5	19.70197	12.28422	13.14214	45.1283
6	17.25742	13.63850	13.63850	44.5344
7	53.77226	9.00000	9.17095	71.9432
8	10.42080	26.38613	10.18034	46.9873
9	13.14214	13.14214	18.36492	44.6492
10	9.17095	53.77226	9.00000	71.9432
11	13.63850	11.24745	19.70197	44.5879
12	18.36492	14.81139	11.90994	45.0862
13	12.69306	19.70197	12.69306	45.0881
14	9.74172	9.95445	30.62278	50.3190
15	10.95229	10.67748	23.49490	45.1247
16	30.62278	9.74172	9.95445	50.3190
17	10.67748	23.49490	10.95229	45.1247
18	14.81139	15.51446	14.19109	44.5169
19	11.56562	17.25742	16.32051	45.1435
20	16.32051	12.69306	15.51446	44.5280
21	37.72983	9.54093	9.35098	56.6217
22	23.49490	10.95229	10.67748	45.1247
23	9.00000	9.17095	53.77226	71.9432
24	10.18034	10.42080	26.38613	46.9873
25	14.19109	18.36492	12.28422	44.8402
26	26.38613	10.18034	10.42080	46.9873
27	9.54093	9.35098	37.72983	56.6217
28	11.24745	16.32051	17.25742	44.8254
29	11.90994	11.56562	21.36068	44.8362
30	9.35098	37.72983	9.54093	56.6217
Σ	494.99920	494.99920	494.99920	NA

$$\overline{\mathrm{VaR}}_\alpha = 44.5169 \quad (45.9994)$$

Pareto(2) marginals and $\alpha = 0.99$

	1	2	3	Σ
1	9.00000	9.00000	9.00000	27.0000
2	9.17095	9.17095	9.17095	27.5129
3	9.35098	9.35098	9.35098	28.0530
4	9.54093	9.54093	9.54093	28.6228
5	9.74172	9.74172	9.74172	29.2252
6	9.95445	9.95445	9.95445	29.8634
7	10.18034	10.18034	10.18034	30.5410
8	10.42080	10.42080	10.42080	31.2624
9	10.67748	10.67748	10.67748	32.0325
10	10.95229	10.95229	10.95229	32.8569
11	11.24745	11.24745	11.24745	33.7423
12	11.56562	11.56562	11.56562	34.6969
13	11.90994	11.90994	11.90994	35.7298
14	12.28422	12.28422	12.28422	36.8527
15	12.69306	12.69306	12.69306	38.0792
16	13.14214	13.14214	13.14214	39.4264
17	13.63850	13.63850	13.63850	40.9155
18	14.19109	14.19109	14.19109	42.5733
19	14.81139	14.81139	14.81139	44.4342
20	15.51446	15.51446	15.51446	46.5434
21	16.32051	16.32051	16.32051	48.9615
22	17.25742	17.25742	17.25742	51.7723
23	18.36492	18.36492	18.36492	55.0948
24	19.70197	19.70197	19.70197	59.1059
25	21.36068	21.36068	21.36068	64.0820
26	23.49490	23.49490	23.49490	70.4847
27	26.38613	26.38613	26.38613	79.1584
28	30.62278	30.62278	30.62278	91.8683
29	37.72983	37.72983	37.72983	113.1895
30	53.77226	53.77226	53.77226	161.3168
Σ	494.99920	494.99920	494.99920	NA

	1	2	3	Σ
1	12.28422	21.36068	11.24745	44.8924
2	9.95445	30.62278	9.74172	50.3190
3	21.36068	11.90994	11.56562	44.8362
4	15.51446	14.19109	14.81139	44.5169
5	19.70197	12.28422	13.14214	45.1283
6	17.25742	13.63850	13.63850	44.5344
7	53.77226	9.00000	9.17095	71.9432
8	10.42080	26.38613	10.18034	46.9873
9	13.14214	13.14214	18.36492	44.6492
10	9.17095	53.77226	9.00000	71.9432
11	13.63850	11.24745	19.70197	44.5879
12	18.36492	14.81139	11.90994	45.0862
13	12.69306	19.70197	12.69306	45.0881
14	9.74172	9.95445	30.62278	50.3190
15	10.95229	10.67748	23.49490	45.1247
16	30.62278	9.74172	9.95445	50.3190
17	10.67748	23.49490	10.95229	45.1247
18	14.81139	15.51446	14.19109	44.5169
19	11.56562	17.25742	16.32051	45.1435
20	16.32051	12.69306	15.51446	44.5280
21	37.72983	9.54093	9.35098	56.6217
22	23.49490	10.95229	10.67748	45.1247
23	9.00000	9.17095	53.77226	71.9432
24	10.18034	10.42080	26.38613	46.9873
25	14.19109	18.36492	12.28422	44.8402
26	26.38613	10.18034	10.42080	46.9873
27	9.54093	9.35098	37.72983	56.6217
28	11.24745	16.32051	17.25742	44.8254
29	11.90994	11.56562	21.36068	44.8362
30	9.35098	37.72983	9.54093	56.6217
Σ	494.99920	494.99920	494.99920	NA

$\overline{\mathrm{VaR}}_\alpha = 44.5169 \quad (45.9994)$

$N = 10^5 \Rightarrow \overline{\mathrm{VaR}}_\alpha = 45.99$

Pareto(2) marginals and $\alpha = 0.99$

	1	2	3	Σ
1	9.00000	9.00000	9.00000	27.0000
2	9.17095	9.17095	9.17095	27.5129
3	9.35098	9.35098	9.35098	28.0530
4	9.54093	9.54093	9.54093	28.6228
5	9.74172	9.74172	9.74172	29.2252
6	9.95445	9.95445	9.95445	29.8634
7	10.18034	10.18034	10.18034	30.5410
8	10.42080	10.42080	10.42080	31.2624
9	10.67748	10.67748	10.67748	32.0325
10	10.95229	10.95229	10.95229	32.8569
11	11.24745	11.24745	11.24745	33.7423
12	11.56562	11.56562	11.56562	34.6969
13	11.90994	11.90994	11.90994	35.7298
14	12.28422	12.28422	12.28422	36.8527
15	12.69306	12.69306	12.69306	38.0792
16	13.14214	13.14214	13.14214	39.4264
17	13.63850	13.63850	13.63850	40.9155
18	14.19109	14.19109	14.19109	42.5733
19	14.81139	14.81139	14.81139	44.4342
20	15.51446	15.51446	15.51446	46.5434
21	16.32051	16.32051	16.32051	48.9615
22	17.25742	17.25742	17.25742	51.7723
23	18.36492	18.36492	18.36492	55.0948
24	19.70197	19.70197	19.70197	59.1059
25	21.36068	21.36068	21.36068	64.0820
26	23.49490	23.49490	23.49490	70.4847
27	26.38613	26.38613	26.38613	79.1584
28	30.62278	30.62278	30.62278	91.8683
29	37.72983	37.72983	37.72983	113.1895
30	53.77226	53.77226	53.77226	161.3168
Σ	494.99920	494.99920	494.99920	NA

$$\overline{\mathrm{VaR}}_\alpha = 44.5169 \quad (45.9994)$$

OPTIMAL COUPLING!

	1	2	3	Σ
1	12.28422	21.36068	11.24745	44.8924
2	9.95445	30.62278	9.74172	50.3190
3	21.36068	11.90994	11.56562	44.8362
4	15.51446	14.19109	14.81139	44.5169
5	19.70197	12.28422	13.14214	45.1283
6	17.25742	13.63850	13.63850	44.5344
7	53.77226	9.00000	9.17095	71.9432
8	10.42080	26.38613	10.18034	46.9873
9	13.14214	13.14214	18.36492	44.6492
10	9.17095	53.77226	9.00000	71.9432
11	13.63850	11.24745	19.70197	44.5879
12	18.36492	14.81139	11.90994	45.0862
13	12.69306	19.70197	12.69306	45.0881
14	9.74172	9.95445	30.62278	50.3190
15	10.95229	10.67748	23.49490	45.1247
16	30.62278	9.74172	9.95445	50.3190
17	10.67748	23.49490	10.95229	45.1247
18	14.81139	15.51446	14.19109	44.5169
19	11.56562	17.25742	16.32051	45.1435
20	16.32051	12.69306	15.51446	44.5280
21	37.72983	9.54093	9.35098	56.6217
22	23.49490	10.95229	10.67748	45.1247
23	9.00000	9.17095	53.77226	71.9432
24	10.18034	10.42080	26.38613	46.9873
25	14.19109	18.36492	12.28422	44.8402
26	26.38613	10.18034	10.42080	46.9873
27	9.54093	9.35098	37.72983	56.6217
28	11.24745	16.32051	17.25742	44.8254
29	11.90994	11.56562	21.36068	44.8362
30	9.35098	37.72983	9.54093	56.6217
Σ	494.99920	494.99920	494.99920	NA

$$N = 10^5 \Rightarrow \overline{\mathrm{VaR}}_\alpha = 45.99$$

Optimal coupling yields a dependence in which:

- either the (three) rvs are very close to each other and sum up to something very close to the minimal sum (-> *complete mixability*)

- or one of the components is large and the other (two) are small (-> *mutual exclusivity*)

Complete mixability

Definition

A distribution *F* is called *d*-completely mixable if there exist *d* random variables $X_1, \ldots, X_d$ identically distributed as *F* such that

$$P(X_1 + \cdots + X_d = \text{constant}) = 1$$

Examples

- Gaussian, Cauchy, t

- Uniform

- Binomial(*n,r/s*) is *s*-completely mixable, *n,r,s* integers

- Multivariate extensions can be given [Rüschendorf and Uckelmann (2002)]

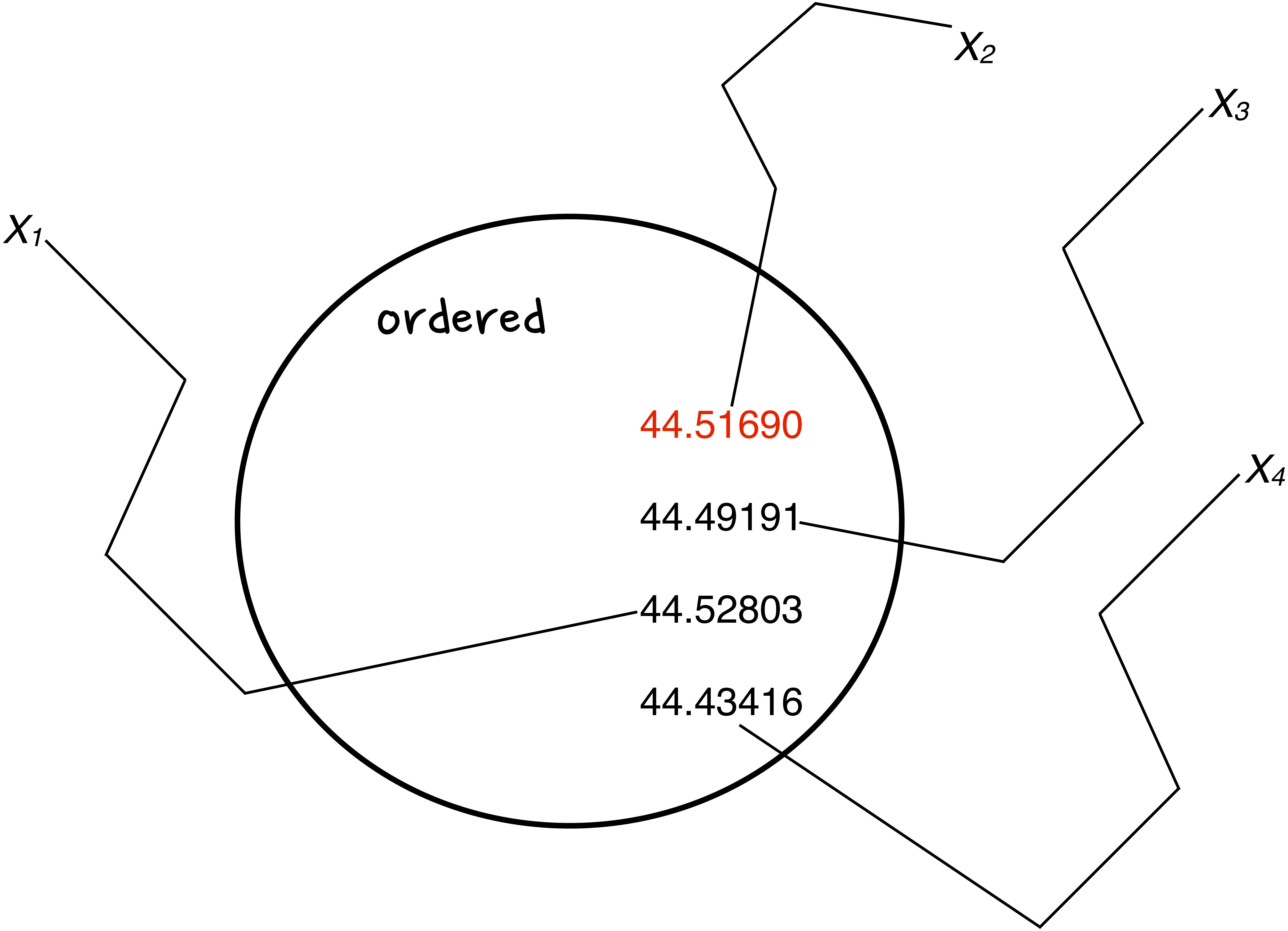
Complete mixability

Sufficient conditions for complete mixability (Wang and Wang (2011), Puccetti, Wang and Wang (2012))

- *F* is continuous with a symmetric and unimodal density. [Rüschendorf and Uckelmann (2002)]

- *F* is continuous with a *monotone* density on a bounded support and satisfies a moderate mean condition. [Wang and Wang (2011)]

- *F* is continuous with a *concave* density on a bounded support. [Puccetti, Wang and Wang (2012)]



With *N=100000*, we obtain the first two decimals of $\overline{\mathrm{VaR}}_\alpha$ in 0.2 sec.

Rearrangement algorithm

	[,1]
[1,]	9.000000
[2,]	9.170953
[3,]	9.350983
[4,]	9.540926
[5,]	9.741723
[6,]	9.954451
[7,]	10.180340
[8,]	10.420805
[9,]	10.677484
[10,]	10.952286
[11,]	11.247449
[12,]	11.565617
[13,]	11.909944
[14,]	12.284223
[15,]	12.693064
[16,]	13.142136
[17,]	13.638501
[18,]	14.191091
[19,]	14.811388
[20,]	15.514456
[21,]	16.320508
[22,]	17.257419
[23,]	18.364917
[24,]	19.701967
[25,]	21.360680
[26,]	23.494897
[27,]	26.386128
[28,]	30.622777
[29,]	37.729833
[30,]	53.772256
[31,]	Inf

1) Approximate the $(1-\alpha)$ upper part of the support of each marginal F_j from above and below:

$$\underline{F}_j \geq F_j \geq \overline{F}_j$$

and create two matrices *X* and *Y* with *N* columns and *d* rows.

2) Iteratively rearrange the column of each matrix until the matrices X^* and Y^* with each column oppositely ordered to the sum of the other columns.

3) $\min(\mathrm{rowSums}(X^*)) \leq \overline{\mathrm{VaR}}_\alpha \leq \min(\mathrm{rowSums}(Y^*))$

4) Run the algorithm with *N* large enough.

$d = 8$	$N = 1.0e05$	*avg time*: 30 secs		
α	$\underline{\mathrm{VaR}}(\alpha)$ (RA range)	$\mathrm{VaR}^+(\alpha)$ (exact)	$\overline{\mathrm{VaR}}(\alpha)$ (exact)	$\overline{\mathrm{VaR}}(\alpha)$ (RA range)
0.99	9.00 – 9.00	72.00	141.67	141.66–141.67
0.995	13.13 – 13.14	105.14	203.66	203.65–203.66
0.999	30.47 – 30.62	244.98	465.29	465.28–465.30
$d = 56$	$N = 1.0e05$	*avg time*: 9 mins		
α	$\underline{\mathrm{VaR}}(\alpha)$ (RA range)	$\mathrm{VaR}^+(\alpha)$ (exact)	$\overline{\mathrm{VaR}}(\alpha)$ (exact)	$\overline{\mathrm{VaR}}(\alpha)$ (RA range)
0.99	45.82 – 45.82	504	1053.96	1053.80–1054.11
0.995	48.60 – 48.61	735.96	1513.71	1513.49–1513.93
0.999	52.56 – 52.58	1714.88	3453.99	3453.49–3454.48
$d = 648$	$N = 5.0e04$	*avg time*: 8 hrs		
α	$\underline{\mathrm{VaR}}(\alpha)$ (RA range)	$\mathrm{VaR}^+(\alpha)$ (exact)	$\overline{\mathrm{VaR}}(\alpha)$ (exact)	$\overline{\mathrm{VaR}}(\alpha)$ (RA range)
0.99	530.12 – 530.24	5832.00	12302.00	12269.74-12354.00
0.995	562.33 – 562.50	8516.10	17666.06	17620.45-17739.60
0.999	608.08 – 608.47	19843.56	40303.48	40201.48-40467.92

TABLE 1. Estimates for $\overline{\mathrm{VaR}}(\alpha)$ and $\underline{\mathrm{VaR}}(\alpha)$ for random vectors of Pareto(2)-distributed risks.

Application: superadditivity ratio

Define the *superadditivity ratio* as:

$$\delta_\alpha(d) = \frac{\overline{\mathrm{VaR}}_\alpha(L_+)}{\mathrm{VaR}_\alpha^+(L_+)}$$

and investigate its properties as a function of the dimension *d*, the level α and the parameters of the underlying model.

Investigate the limit, given it exists,

$$\delta_\alpha = \lim_{d \to +\infty} \delta_\alpha(d)$$

Examples

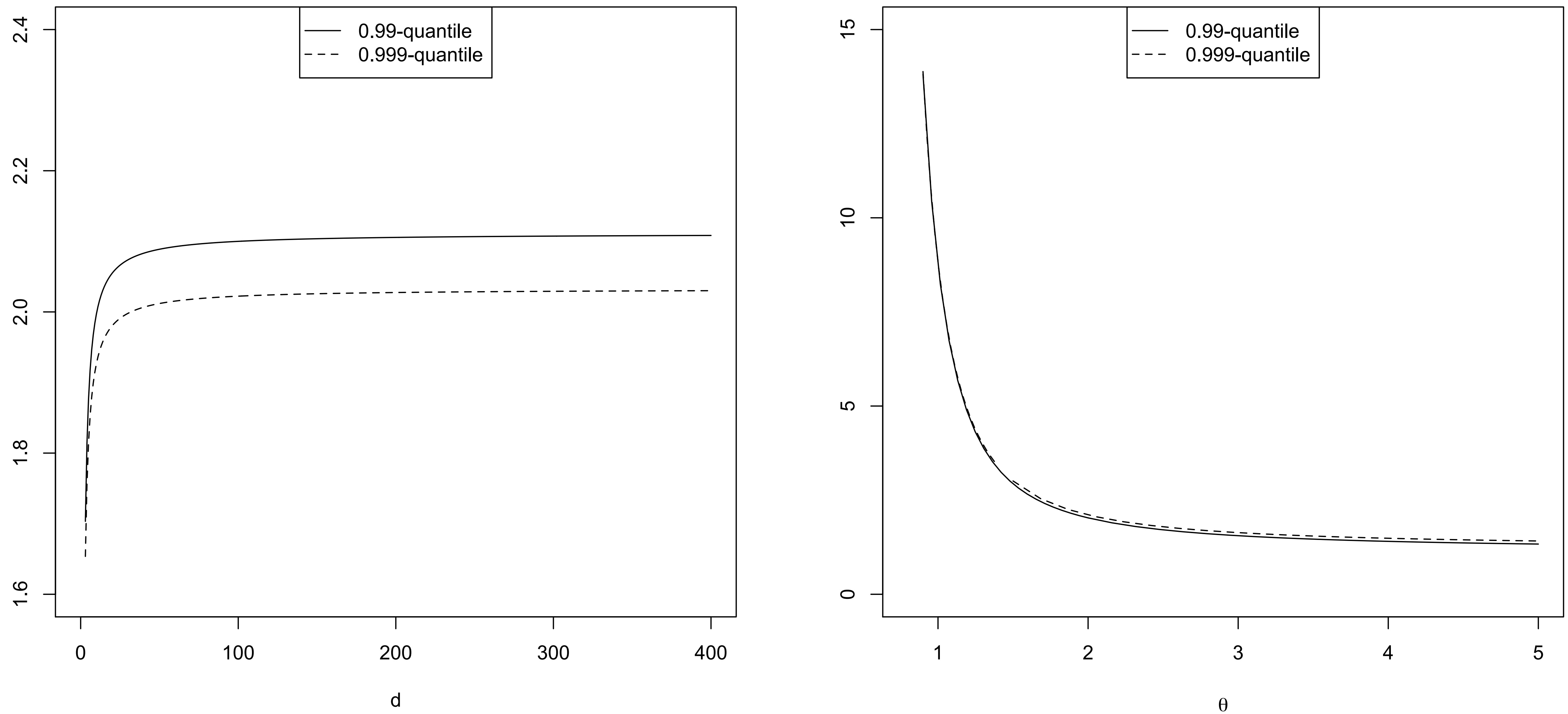


Figure 5: Left: plot of the function $\delta_\alpha(d)$ versus the dimensionality d of the portfolio for a risk vector of Pareto(θ)-distributed risks, for two different quantile levels and $\theta = 2$. Right: Plot of the limit constant δ_α versus the tail parameter θ of the Pareto distribution.

Conclusions

The rearrangement algorithm calculates numerically sharp bounds for the VaR of a sum of dependent random variables.

- it is accurate, fast and computationally less demanding wrt to the methods in the literature.

- can be used with *inhomogeneous* marginals, in high dimensions.

- computes also the *best-possible* Value-at-Risk.

- can be used with *any* marginal df and *any* quantile level.

- can be used also to compute bounds on the distribution function of different operators such as $\times$, $\min$, $\max$.

Further work

- Find optimal couplings for the best VaR
- Interpret these couplings wrt realistic scenarios
- Add statistical uncertainty
- Compute VaR sharp bounds with some additional dependence information
- Compare and contrast with other approaches: Robust Optimization
- ...

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