VaR-based Risk Management: Sense and (Non-)Sensibility

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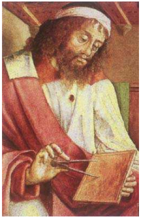
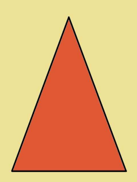
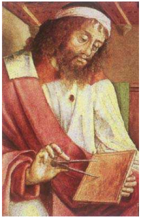




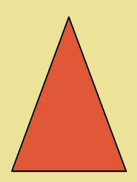
VaR

Jean Dieudonné



Euclid

Freddy Delbaen



Triangle

Jane Austen

Value-at-Risk

Linking the pictures

- Jean Dieudonné

  "À bas le triangle. À bas Euclide!"

- Freddy Delbaen (et al.)

  "*À bas* VaR"

- Jane Austen

  "Sense and Sensibility"

The real title of my talk:

VaR-based Risk Management:
Sense and Nonsensicality

But why *Jane Austen* (besides the play of words)?

Reference: D.D. Skwire (1997). Actuarial issues in the novels of Jane Austen. *North American Actuarial J.* 1(1), 74-83.

"Sense and Sensibility was the first novel Jane Austen published, in 1811, and it contains the most compelling actuarial subject matter."

"Jane Austen was no mathematician, and certainly no actuary, her novels address a wealth of actuarial issues that maintain their relevance for actuaries in the modern world."

A brief history of VaR

- Amendment to Basel I, 1994(+): internal models for Market Risk in the 4^{15} Weatherstone Report
- VaR-based RM for Credit- and Operational Risk, Basel II, 2000(+)
- Idem for Solvency II, not for SST
- Important statement from practice

  "Easy to communicate/understand"

Some recent issues

- In a speech in Dublin, Charlie McCreevy, the European Market Commissioner, denounced

  "the irresponsible lending, blind investing, bad liquidity management, excessive stretching of rating agency brands and defective value-at-risk modelling that prompted the turmoil of recent months" (red. subprime credit crisis).

  (Financial Times, Friday, October 26, 2007)

- Marking to Market ... Marking to Model ... Marking to Myth

- From risk-free return to return-free risk

VaR: the definition

For some P&L distribution function

$$F_{X_T}(x) = \mathrm{P}(X_T \le x),$$

with X_T defined at a future time point (period) T and a confidence level $\alpha \in (0,1)$,

$$\mathrm{VaR}_{\alpha,T}(F_{X_T}) = \mathrm{VaR}_{\alpha,T}(X_T) = F^{\leftarrow}_{X_T}(\alpha)$$

I.e. (we drop T in our notation, hence $X = X_T$, $F_{X_T} = F_X$)

$$\mathrm{P}\big(X > \mathrm{VaR}_{\alpha,T}(X)\big) = 1 - \alpha$$

Typically $\alpha \in \{95\%, 99\%, 99.9\%, 99.97\%\}$

Hence, from a mathematical point of view,

VaR is just a quantile of a df

Some VaR-sense:

- "Finding" F_L is the key work!
- VaR-thinking focuses on the important technical, quantitative side of RM, hopefully without losing sight of the qualitative aspects
- VaR is easy to communicate (sic)
- A key concept within principle-based regulation
- Widely applicable, however . . .

Some VaR-warnings:

- VaR is a frequency measure, not severity
- At high quantiles, difficult to statistically estimate
- Typically very wide confidence intervals
- Scaling from α_1 to α_2
- Scaling from T_1 to T_2
  e.g., how to scale a 10 day 99% VaR to 1 year 99.9% ?

Some comments on these issues

How to calculate

- $X \sim N(\mu, \sigma^2) \implies \mathrm{VaR}_\alpha = \mu + \sigma\Phi^{-1}(\alpha)$
- $\overline{F}_X(x) = 1 - F_X(x) = x^{-1/\xi}L(x)$, L slowly varying, i.e. $\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1$, and $\xi > 0$ (power law)
  $\implies$ (EVT) $\mathrm{VaR}_\alpha = u + \frac{\beta}{\xi}\left(\left(\frac{1-\alpha}{\overline{F}_X(u)}\right)^{-\xi} - 1\right)$
  (here u, β are further model parameters)
- historical simulation
- full Monte-Carlo

Some comments on these issues

How to scale

Answer: both in α and T model dependent

Practice: $\sqrt{T}$ - scaling

Severity risk measure: Expected Shortfall

$$\begin{aligned}\mathrm{ES}_\alpha &= \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_q(F_X)\,dq \\ &= \mathbb{E}(X \,|\, X > \mathrm{VaR}_\alpha) \qquad (F_X \text{ continuous})\end{aligned}$$

$$X \sim N(\mu, \sigma^2) \implies \lim_{\alpha \uparrow 1} \frac{\mathrm{ES}_\alpha}{\mathrm{VaR}_\alpha} = 1 \qquad \text{(OK)}$$

$$X \sim t_\nu \implies \lim_{\alpha \uparrow 1} \frac{\mathrm{ES}_\alpha}{\mathrm{VaR}_\alpha} = \frac{\nu}{\nu - 1} = \frac{1}{1-\xi} > 1 \qquad \text{(!)}$$

Some VaR-nonsense

The non-coherence of VaR is only academically important and has no practical consequences

In most relevant cases in practice, VaR is sub-additive

$$\mathrm{VaR}_\alpha(X_1 + X_2) \leq \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2) \tag{1}$$

Diversification effect!

(1) holds true for

- multivariate normal portfolios (why?)
- more generally, elliptical portfolios

but typically fails for

- very heavy-tailed data
- very skew data
- any type of marginals (F_{X_1}, F_{X_2}) but with special dependence

- Heavy-tailed data:
  X_1, X_2 independent, $P(X_i > x) = x^{-1/2}$, $x \geq 1$
  (infinite mean case!)

- Skew data:
  X_i, $i = 1, \ldots, 100$ i.i.d. loans
  2% yearly coupons
  1% yearly default rate, zero recoverable
  100 nominal value
  Then $\mathrm{VaR}_{95\%}\left(\sum_{i=1}^{100} X_i\right) > \mathrm{VaR}_{95\%}(100X_1)$

- Special dependence:
  $X_1, X_2 \sim N(0,1)$ but with special copula for the joint distribution of (X_1, X_2)

Is this relevant for practice?

Operational Risk

Basel II Definition

The risk of loss resulting from inadequate or failed internal processes, people and systems or from external events. This definition includes legal risk, but excludes strategic and reputational risk.

For the LDA within the AMA framework:

- The internal data

$$\mathfrak{X} = \{X_k^{T-i,b,\ell} : i = 1, \ldots, n;\ b = 1, \ldots, 8;\ \ell = 1, \ldots, 7;\ k = 1, \ldots, N^{T-i,b,\ell}\}$$

$$L^{T-i,b} = \sum_{\ell=1}^{7} \sum_{k=1}^{N^{T-i,b,\ell}} X_k^{T-i,b,\ell} \qquad L^{t-i} = \sum_{b=1}^{8} L^{T-i,b}$$

- Moreover: external data, expert opinion

- The loss random variable:

$$L^T = \sum_{b=1}^{8} \sum_{\ell=1}^{7} \sum_{k=1}^{N^{T,b,\ell}} X_k^{T,b,\ell}$$

- The risk measure:

$$\mathrm{VaR}_{\alpha,T}, \ \alpha = 99.9\%, \ T = 1\,\text{year},$$

i.e.

a 1 in 1000 year event

- Numerous models are being proposed, including EVT

Summary for OR

- Data are very heavy-tailed, even infinite mean in some cases, consequences: . . .
- Data are skew
- Little is known on the interdependence of operational risk losses

Conclusion:

Value-at-Risk for $\alpha = 99.9\%$, T=1 year for operational risk is very difficult (if at all possible) to estimate.

Further complication: risk aggregation

$$\mathrm{VaR}^{MR}_{\alpha_1,T_1} + \mathrm{VaR}^{CR}_{\alpha_2,T_2} + \mathrm{VaR}^{OR}_{\alpha_3,T_3} \qquad (+\,?)$$

From VaR to stress testing

- A. Greenspan, 1985:

  "Improving the characterization of the distribution of extreme values is of paramount concern."

- J. Meriwether, 2000:

  "With globalization increasing, you'll see more crises. Our whole focus is on the extremes now – what's the worst that can happen to you in any situation."

- M. Scholes, 2000:

  "Now is the time to encourage the BIS and other regulatory bodies to support studies on stress testing and concentration methodologies. Planning for crises is more important than VaR *analysis."*

A primer on the subprime crisis

- **The facts**: US housing boom → mortgage brokers → subprime loans → mortgage firms → investment banks → packaging into CDOs → rating agencies → (off balance) conduits and SIVs → funding mismatch → client portfolios
- Newspaper clips on mathematics and the crisis:
  - *"UBS - Mathematik hat versagt"*
  - *"Doppelte Niederlage für Wall Street - Mathematik"*
  - *"Doch statt gesunder Menschenverstand regierte nur noch die Finanzmathematik"*
  - *"Statistical assumptions used to value some structured bonds such as CDOs were wrong"*
  - *"Gauss copula"="a produce scale that not only weighs a bag of apples but estimates the chance that they'll all be rotten in a week"*

A primer on the subprime crisis

- *"It is a worry, though, that Merrill can justify a writedown of $4.5bn one week and $7.9bn just three weeks later. The sense that valuation is still a matter of "pick a number and divide by the chief trader's golf handicap", more than anything else, explains why the "super-SIV" proposed by Citigroup and others has failed to reassure the market"*
- *"The (Gauss copula) model is flawed but easy to grasp (somewhat like Black-Scholes) and hence explosive growth of CDO market"*
- *"Ein weiteres Problem war die oft unkritische Übernahme moderner statistischer Wunderwaffen wie Copulas . . . die in falschen Händen extrem viel Schaden anrichten können"*

- The basic formula underlying the Li Gauss copula model ("the broken heart syndrome")

$$F(x_1, \ldots, x_d) = C_\rho^{Ga}(F_1(x_1), \ldots, F_d(x_d))$$

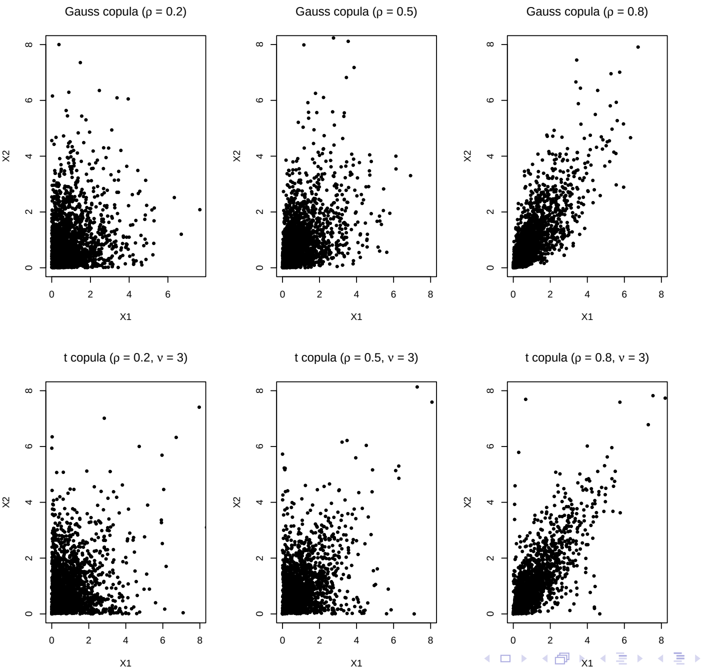
Example: $d = 2$, $F_i = Exp(1)$, $i = 1, 2$ and $\rho \in \{0.2, 0.5, 0.8\}$

Note that in this case:

$$\lim_{\alpha \uparrow 1} \mathrm{P}(X_2 > \mathrm{VaR}_\alpha \,|\, X_1 > \mathrm{VaR}_\alpha) = 0,$$

hence asymptotically no joint extremes.

Replacing C_ρ^{Ga} by a t-copula, $C_\rho^{t,\nu}$ yields extremal dependence . . . , however, dynamic models are needed!

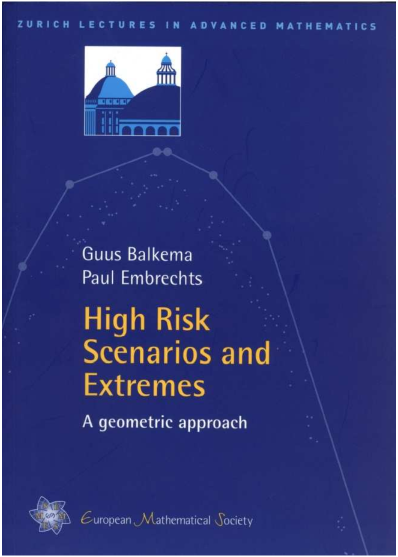

Gauss copula (ρ = 0.2)
Gauss copula (ρ = 0.5)
Gauss copula (ρ = 0.8)
t copula (ρ = 0.2, ν = 3)
t copula (ρ = 0.5, ν = 3)
t copula (ρ = 0.8, ν = 3)
X1
X2

- Some comments from academia
    - *"The Gauss copula is the worst ever invention for credit risk analysis" (LCGR)*
    - *"Everything you can do to deliver the world of finance from copulas, please do" (DD)*
    - *"À bas les copules!" (TM)*
    - Copulas (PE):

      "Pedagogic. Pedagogic. Stress testing."

  Read: "Copulas: A personal view."
  (`www.math.ethz.ch/~embrechts`)

A new book



ZURICH LECTURES IN ADVANCED MATHEMATICS
Guus Balkema
Paul Embrechts
High Risk Scenarios and Extremes
A geometric approach
European Mathematical Society

The mathematical problem

- POT-method within EVT

$$F(x) = \mathrm{P}(X \leq x),$$
$$F_u(x) = \mathrm{P}(X - u \leq x \mid X > u)$$

Equivalent are (for $\xi > 0$):

i) $1 - F(x) = x^{-1/\xi} L(x)$

ii) $\lim_{u \to \infty} |F_u(x) - G_{\xi, \beta(u)}(x)| = 0$

$\Longrightarrow$ high-quantile (VaR) estimation

The mathematical problem

- Balkema and Embrechts (2007) present a multivariate version for a d-dimensional vector $\mathbf{X} = (X_1, \ldots, X_d)'$ of risks and affine transformations β_H, study

  $$\beta_H^{-1}(\mathbf{X}) \xrightarrow[P_H]{d} \mathbf{Z} \text{ for } \mathrm{P}(\mathbf{X} \in H) \to 0,$$

  where H is a general high risk region;

  think of $H_\alpha = \left\{ \sum_{i=1}^{d} w_i X_i > q_\alpha \right\}$, $\alpha \uparrow 1$

Interludium: Some mathematical theorems

1. The Banach-Tarski paradox (1924)

   - The mathematical version: Given any bounded subsets $A, B \subset \mathbb{R}^n$, $n \geq 3$, $int(A) \neq \varnothing$ and $int(B) \neq \varnothing$, then there exist partitions $A = A_1 \cup \ldots \cup A_k$, $B = B_1 \cup \ldots \cup B_k$ such that for all $1 \leq i \leq k$, A_i and B_i are congruent.

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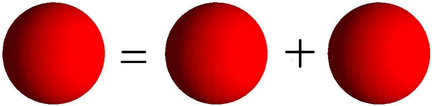
   - The layman's version: given a three-dimensional solid ball, it is possible to cut it into finitely many pieces and reassemble them to form two solid balls, each identical in size to the first.



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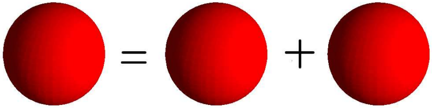

   - The Wall Street version: CDOs, MBSs ...

Interludium: Some mathematical theorems

2. On superadditivity of VaR

- Recall that for (X_1, X_2) such that $\mathrm{P}(X_i > x) = x^{-1/\xi_i} L_i(x)$, $i = 1, 2$ and $\xi_i > 1$, $i = 1, 2$ (infinite mean models), VaR_α is superadditive for α close to 1; i.e.,

$$\mathrm{VaR}_\alpha(X_1 + X_2) > \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2).$$

  Also, for the concentration index:

$$\frac{\mathrm{VaR}_\alpha(X_1 + X_2)}{\mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2)} > 1.$$

- Theorem (Functional Analysis)
  In the spaces L^p, $0 < p < 1$, there exist no convex open sets other than $\varnothing$ and L^p.

- QRM version: There exists no reasonable risk measure on a set of risks with infinite mean. Diversification goes the wrong way.

Final comments

- The quantitative work leading up to VaR calculations is fine
- VaR has been successful for Market Risk but has reached its limits of applicability for Credit Risk and most definitely for Operational Risk
- Hence difficulty in global risk aggregation and economic capital calculation (99.97%)
- Appreciate the endogenous nature of risk and liquidity at the systemic level
- Liquidity and dynamic QRM models
- Industry has to understand more what it means to risk manage in non-normal markets
- Especially as it wants to continue to live on the edge

Any questions?