

Aggregation-Robustness and Model Uncertainty of Regulatory Risk Measures

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Abstract

Research related to aggregation, robustness, and model uncertainty of regulatory risk measures, for instance, Value-at-Risk (VaR) and Expected Shortfall (ES), is of fundamental importance within quantitative risk management. In risk aggregation, marginal risks and their dependence structure are often modeled separately, leading to uncertainty arising at the level of a joint model. In this paper, we introduce a notion of qualitative robustness for risk measures, concerning the sensitivity of a risk measure to the uncertainty of dependence in risk aggregation. It turns out that coherent risk measures, such as ES, are more robust than VaR according to the new notion of robustness. We also give approximations and inequalities for aggregation and diversification of VaR under dependence uncertainty, and derive an asymptotic equivalence for worst-case VaR and ES under general conditions. We obtain that for a portfolio of a large number of risks VaR generally has a larger uncertainty spread compared to ES. The results warn that unjustified diversification arguments for VaR used in risk management need to be taken with much care, and potentially support the use of ES in risk aggregation. This in particular reflects on the discussions in the recent consultative documents by the Basel Committee on Banking Supervision.

Key-words: Value-at-Risk; Expected Shortfall; dependence uncertainty; risk aggregation; robustness; inhomogeneous portfolio; Basel III.

1 Introduction

Risk measurement, with its crucial importance for financial institutions such as banks, insurance companies and investment funds, has drawn a lot of attention in both academia and

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industry over the past several decades. Although a financial risk, often modeled by a probability distribution, cannot be characterized by a single number, sometimes one needs. The determination of regulatory capital is one such example, the ranking of risks another. For such purposes, quantitative tools that map risks to numbers were introduced, and they are called *risk measures*.

Over the past three decades, Value-at-Risk (VaR) became the *benchmark* (Jorion (2006)). Expected Shortfall (ES), an alternative to VaR which is *coherent*, (Artzner et al. (1999)), is arguably the second most popular risk measure in use. In two recent consultative documents BCBS (2012, 2013), the Basel Committee on Banking Supervision proposed to take a move from VaR to ES for risk measurement in banking. Under Solvency 2 and the Swiss Solvency Test, the same discussion takes place within insurance regulation; see for instance Sandström (2010) and SCOR (2008). As a consequence, there have been extensive debates on issues related to diversification, aggregation, economical interpretation, extreme behavior, robustness, and backtesting of VaR and ES. We omit a detailed analysis here and refer to Embrechts et al. (2014), Emmer et al. (2014) and the reference therein.

Here are some of the issues raised: VaR is not coherent, but it is *elicitable* (Gneiting (2011)), easy to backtest and more robust with respect to statistical uncertainty, as argued in Gneiting (2011) and Cont et al. (2010); ES is coherent, but not elicitable, difficult to backtest and less robust. There have been extensive discussions on the problematic diversification and aggregation issues of VaR due to its lack of subadditivity; see for example Embrechts et al. (2013). Danielsson et al. (2005) argue that the violation of subadditivity for VaR is rare in practice. VaR, being a quantile, does not address the crucial “what if” question. Whereas this was clear since its introduction within the financial industry in 1994, it took some serious financial crises to bring this issue fully onto the regulatory agenda.

The importance of robustness properties of risk measures has only fairly recently become a focal point of regulatory attention. By now, numerous academic as well as applied papers address the topic. Conflicting views typically result from different notions of robustness; Embrechts et al. (2014) contains a brief discussion and some references. In this paper, issues related to both aggregation and robustness for VaR and ES will be discussed. We will show that ES indeed enjoys a new notion of robustness which VaR generally does not.

The mathematical property of (non-)subadditivity of a risk measure becomes relevant upon analyzing the aggregate position of a portfolio. As often is the case in practice, the dependence structure among individual risks in a portfolio is difficult to obtain from a statistical point of view, while the marginal distributions of the individual risks (assets) may typically be easier to model; see for instance Embrechts et al. (2013) and Bernard et al. (2014). Modeling a high-dimensional dependence structure is well-known to be data-costly, and dimension reduction

techniques such as vine copulas, hierarchical structures, and very specific parametric models often have to be implemented. Whereas such simplifying techniques in general create computational and modeling ease, they typically involve considerable model uncertainty. This leads to a notion of *dependence uncertainty* (DU) in risk aggregation, a concept of main interest for this paper.

From a mathematical or statistical point of view it is clearly better to look at the robustness properties of a model at the level of the joint distribution of the risk factors. The main reason for separating the two (marginals, dependence) is because of processes in practice, where indeed the two are often modeled separately. This is particularly true in a stress testing environment.

Hence for this paper, we introduce the notion of *aggregation-robustness* to study the robustness properties of risk measures for aggregation in the presence of dependence uncertainty. The new notion of robustness is consistent with the classic notion of robustness for statistical functionals in e.g. [Huber and Ronchetti \(2009\)](#), and is weaker than the robustness for risk measures for instance defined in [Cont et al. \(2010\)](#). However, as opposed to the conclusions in the latter paper, we show that when the model uncertainty lies in the dependence structure, coherent distortion risk measures (such as ES) are robust, whereas VaR in general is not. This result supports the use of ES for risk aggregation, especially when statistical information on marginal distributions is reliable.

Under DU, the values of VaR and ES lie in an interval. This interval can be seen as a measurement of model uncertainty for a particular risk measure. When a risk measure is applied to an aggregate position of a portfolio, the ratio between the risk measure of the aggregate risk and the summation of the risk measures of the marginal risks, is called a *diversification ratio*. The diversification ratio measures how good the risks in a portfolio hedge (compensate for) each other. With only models for marginal distributions available, the diversification ratio also lies in a DU-interval.

To study the DU-interval of VaR and ES, and their diversification ratios, one needs to calculate the worst-case and best-case values of VaR and ES under dependence uncertainty. Due to the subadditivity of ES, the worst-case value of ES is the summation of the ES of the marginal risks. However, the other three quantities (best- and worst-case VaR, best-case ES) are, in general, unknown. Partial results do exist. The worst-case values of VaR for $n = 2$ were given in [Makarov \(1981\)](#) based on early results in multivariate probability theory. [Embrechts and Puccetti \(2006\)](#) gave a dual bound for the worst-case VaR for $n \geq 3$ in the homogeneous model, i.e. all marginal risks have the same distribution. Partial solutions for the worst-case and best-case values of VaR are to be found in [Wang et al. \(2013\)](#), [Puccetti and Rüschendorf \(2013\)](#) and [Bernard et al. \(2014\)](#), based on the notion of complete mixability introduced in [Wang and Wang \(2011\)](#). A fast algorithm to numerically calculate the worst-case and best-case values of

VaR under general conditions was introduced in [Embrechts et al. \(2013\)](#). For the best-case ES, some partial analytical results can be found in [Bernard et al. \(2014\)](#) and [Cheung and Lo \(2013\)](#), and a numerical procedure was proposed by [Puccetti \(2013\)](#).

In most of the existing analytical results, it is assumed that the marginal distributions have to be identical (homogeneous case), with some extra conditions on the shape of the underlying risk factor densities (assumed to exist). In this paper, we relax the assumptions on the marginal distributions. Instead of explicit values for the worst-case and best-case VaR, we obtain approximations. The new results obtained can be used within a discussion on capital requirement; they moreover yield a DU-interval for VaR and its diversification ratio.

Further understanding of the worst-case VaR can be obtained through the asymptotic behavior as the number of risks in the portfolio grows to infinity, i.e. a large portfolio regime. In the homogeneous case, [Puccetti and Rüschendorf \(2014\)](#) obtained an asymptotic equivalence between the worst-case VaR and the worst-case ES under dependence uncertainty, and this under a strong condition of complete mixability. The condition on the identical marginal distributions was later weakened by [Puccetti et al. \(2013\)](#) (based on further results on complete mixability) and [Wang \(2014\)](#) (based on a duality theory in [Rüschendorf \(1982\)](#)). It was finally removed by [Wang and Wang \(2014\)](#) (based on the new notion of extremely negative dependence). When the marginal distributions are not identical, [Puccetti et al. \(2013\)](#) also obtained the asymptotic equivalence under the assumption that only finitely many different choices of the marginal distributions can appear; this mathematically allows a reduction to the case of identical marginal distributions.

In this paper, we give a unifying result on this asymptotic equivalence, by allowing the marginal distributions to be arbitrary. Only weak uniformity conditions on the moments of the marginal distributions are required for our result to hold. These conditions are easily justified in practice, and are necessary for the most general equivalence to hold. The new results lead to the asymptotic DU-spread of VaR and ES, and show that VaR in general yields a larger DU-spread compared to ES.

We hope that the results in this paper would successfully deliver the following message:

With respect to dependence uncertainty in aggregation, VaR is less robust compared to ES.

The rest of the paper is organized as follows. In [Section 2](#) we introduce the notion of aggregation-robustness and show that ES is aggregation-robust but VaR is not. In [Section 3](#) we give new bounds on the diversification ratios under dependence uncertainty, and establish an asymptotic equivalence between VaR and ES under a worst-case scenario. The dependence uncertainty spread of VaR and that of ES are derived and compared in [Section 4](#). In [Section](#)

5, numerical examples are presented to illustrate our results. Section 6 draws some conclusions. All proofs are put in the Appendix.

Throughout the paper, we let $(\Omega, \mathcal{A}, \mathbb{P})$ be a standard atomless probability space and $L^0 := L^0(\Omega, \mathcal{A}, \mathbb{P})$ be the set of all real-valued random variables (rvs) on that probability space. Elements of L^0 , rvs, will often be referred to as risks. Their distribution functions we simply refer to as distributions. We write $X \sim F$ to denote $F(x) = \mathbb{P}(X \leq x)$, $x \in \mathbb{R}$. We also denote the generalized inverse function of F by $F^{-1}(p)$, that is $F^{-1}(p) = \inf\{t \in \mathbb{R} : F(t) \geq p\}$ for $p \in (0, 1]$, and $F^{-1}(0) = \inf\{t \in \mathbb{R} : F(t) > 0\}$.

2 Robustness of VaR and ES for risk aggregation

2.1 Robustness of risk measures

The robustness of a statistical functional or an estimation procedure describes the sensitivity to underlying model deviations and/or data changes. Different definitions and interpretations of robustness exist in the literature; see for example [Huber and Ronchetti \(2009\)](#) from a pure statistical perspective, [Hansen and Sargent \(2007\)](#) in the context of economic decision making, and [Ben-Tal et al. \(2009\)](#) on robust optimization. In statistics, robustness mainly concerns the so-called *distributional* (or *Hampel-Huber*) *robustness*: the statistical consequences when the shape of the actual underlying distribution deviates slightly from the assumed model.

A risk measure ρ is a function which maps a risk in L^0 to a number, $\rho : L^0 \rightarrow \mathbb{R} \cup \{+\infty\}$. A risk measure is law-invariant if it only depends on the distribution of the risk. We omit the general introduction of risk measures, and refer the interested reader to [Föllmer and Schied \(2011\)](#). Since law-invariant risk measures are a specific type of statistical functionals, their robustness properties are already extensively studied in the statistical literature; see e.g. [Huber and Ronchetti \(2009\)](#).

In this paper, we focus on the two most popular risk measures: Value-at-Risk (VaR) at confidence level p , defined as

$$\text{VaR}_p(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq p\}, \quad p \in (0, 1), \quad X \in L^0, \quad (2.1)$$

and the Expected Shortfall (ES) at confidence level p , defined as

$$\text{ES}_p(X) = \frac{1}{1-p} \int_p^1 \text{VaR}_q(X) dq, \quad p \in (0, 1), \quad X \in L^0. \quad (2.2)$$

It is obvious that $\text{VaR}_p(X) = F^{-1}(p)$ for $p \in (0, 1)$ where $X \sim F$. Though typically in (2.2) it is assumed that $\mathbb{E}[|X|] < \infty$, we may occasionally allow that $\text{ES}_p(X) = \infty$ for some X . On the other hand, $\text{VaR}_p(X)$ is always a finite number for $X \in L^0$.

It is often argued in the literature that quantile-based risk measures, such as VaR, are more robust as compared to mean-based risk measures, such as ES; the notion of robustness used most often is Hampel's (Hampel et al. (1986)). ES is only robust with respect to stronger metrics (e.g. the Wasserstein distance, Dobrushin (1970)); arguments of this type can be found in, for instance, Cont et al. (2010), Kou and Peng (2014) and Emmer et al. (2014). It is well-known that the qualitative robustness of a statistical estimator, as in Hampel et al. (1986), is equivalent to the continuity of the corresponding risk measure at the true distribution. Thus, to analyze statistical robustness, one typically studies the continuity at distributions of a risk measure. Based on such consideration, we say that a law-invariant risk measure is robust at a distribution F if it is continuous at F in some metric. To be precise, ρ is robust if for all $\varepsilon > 0$, there exists $\delta > 0$, such that $|\rho(X) - \rho(Y)| < \varepsilon$ whenever $d(F, G) < \delta$, where d is some distance between the distributions F and G of X and Y . For example, the Lévy distance in Huber and Ronchetti (2009) is used in Cont et al. (2010) to measure the difference between any two univariate distributions F and G :

$$d(F, G) := \inf\{\varepsilon > 0 : F(x - \varepsilon) - \varepsilon < G(x) < F(x + \varepsilon) + \varepsilon, \quad \forall x \in \mathbb{R}\}. \quad (2.3)$$

For rvs S and T , we also denote $d(S, T) := d(F, G)$ where F and G are the distributions of S and T , respectively. Note that the Lévy distance metrizes the weak topology on the set of distributions. Other metrics can also be used for the analysis of robustness; see Emmer et al. (2014). It was shown in Cont et al. (2010) that VaR_p is continuous at F if F^{-1} is continuous at p , while ES_p is not continuous at any distribution F . This result is often used to argue that VaR has better robustness properties than ES. Note that if F^{-1} is not continuous at p , then VaR_p is not continuous at F in the Lévy metric. In Krätschmer et al. (2014) it is argued that Hampel's notion of (statistical) robustness is less relevant for risk management. Using a different definition, they introduce a continuous scale of robustness.

In the following we will introduce a new, in our opinion practically relevant notion of robustness for risk aggregation, which favors ES over VaR.

2.2 Aggregation-robustness

In this section, we show that if model uncertainty lies at the level of the dependence structure but not at single risks, then VaR is not robust but ES is. For single risks X_i , $i = 1, \dots, n$, the aggregate risk S is simply defined as $S = X_1 + \dots + X_n$. Often in practice, a joint model of X_1, \dots, X_n is modeled in two stages: n marginal distributions F_1, \dots, F_n and a dependence structure (often through a copula C). As the modeling for marginal distributions is relatively easy, the dependence structure can be really difficult to model, statistically estimate and test.

Considerable model uncertainty, which is often different in nature from the model uncertainty of marginal distributions, arises from modeling the dependence structure. In the following, we study robustness with respect to uncertainty in the dependence structure; for the purpose of this paper we assume the marginal distributions F_1, \dots, F_n are given.

When the dependence structure between the risks is unknown, the possible distributions of S form a class of distributions. We denote the *admissible class* as

$$\mathfrak{S}_n(F_1, \dots, F_n) = \{X_1 + \dots + X_n : X_i \sim F_i, i = 1, \dots, n\},$$

which for simplicity we further denote as $\mathfrak{S}_n = \mathfrak{S}_n(F_1, \dots, F_n)$ if (F_1, \dots, F_n) is clear from the context. \mathfrak{S}_n is the set of all possible aggregate risks. Note that for notational convenience, we left out portfolio weight factors; these can easily be reintroduced when necessary. Risk aggregation with dependence uncertainty concerns the probabilistic and statistical behavior of $S \in \mathfrak{S}_n$; see [Bernard et al. \(2014\)](#). We say that an admissible class is *compatible with* a risk measure ρ if $\rho(X_i) < \infty$ and $\mathbb{E}[X_i \mathbb{I}_{\{X_i < 0\}}] < \infty$ for $X_i \sim F_i, i = 1, \dots, n$ (the profit of each risk does not have an infinite mean; note that here we allow $\mathbb{E}[X_i \mathbb{I}_{\{X_i \geq 0\}}] = \infty$, i.e. extremely heavy-tailed risks).

Definition 2.1 (Aggregation-robustness). A law-invariant risk measure $\rho : L^0 \rightarrow \mathbb{R} \cup \{+\infty\}$ is *aggregation-robust*, if for all $n \in \mathbb{N}$ and admissible classes \mathfrak{S}_n compatible with ρ , and all $\varepsilon > 0$, there exists $\delta > 0$, such that $|\rho(S) - \rho(T)| < \varepsilon$, for all $S, T \in \mathfrak{S}_n$ such that $d(S, T) < \delta$.

The robustness character of Definition 2.1 is intuitively clear. If the joint distributions of (X_1, \dots, X_n) and (Y_1, \dots, Y_n) are close according to some metric, say, then the distributions of $X_1 + \dots + X_n$ and $Y_1 + \dots + Y_n$ are also close according to some metric. As a consequence, ρ is insensitive to small perturbations of the joint distribution of the underlying risk factors, keeping the marginal distributions of the individual risks fixed. It is clear that Hampel's robustness, as discussed in [Cont et al. \(2010\)](#), without the restriction of S, T being in a common admissible class, implies aggregation-robustness. When the dependence structure is modeled by copulas, our definition of robustness implies that a risk measure is insensitive to the copula of the individual risks when the marginal distributions are assumed to be known. The fact that in Definition 2.1 we look at $S, T \in \mathfrak{S}_n$ reflects our interest in aggregation and diversification. One could of course look at other functional-robustness definitions beyond aggregation (summation).

Example 2.1 (VaR is not aggregation-robust). For $t \in [0, 1]$, let X_t and Y_t have joint distribution C_t ,

$$C_t(x, y) = txy + (1-t)(\max\{\min\{x, 1/2\} + \min\{y, 1/2\} - 1/2, 0\} + \max\{x+y-3/2, 0\}), \quad x, y \in [0, 1].$$

It is easy to see that X_t and Y_t are both $U[0, 1]$ distributed, hence C_t is a copula, $t \in [0, 1]$. Note that C_t , $t \in (0, 1)$ is a mixture of the independent copula C_1 and another copula

$$C_0 : [0, 1]^2 \rightarrow [0, 1], (x, y) \mapsto \max\{\min\{x, 1/2\} + \min\{y, 1/2\} - 1/2, 0\} + \max\{x + y - 3/2, 0\}.$$

C_0 is the *ordinal sum* of two Fréchet lower copulas; see [Neslen \(2006, Section 3.2.2\)](#).

It is immediate that the distribution of $X_t + Y_t$ for $t \in (0, 1]$ is symmetric, centered at 1, with positive density on the interval $(1/2, 3/2)$. Thus, $\text{VaR}_{1/2}(X_t + Y_t) = 1$. It is also straightforward that $X_0 + Y_0$ is a degenerate rv on $\{1/2, 3/2\}$ with $\text{VaR}_{1/2}(X_0 + Y_0) = 1/2$. As a consequence, $\text{VaR}_{1/2}(X_0 + Y_0) \neq \lim_{t \rightarrow 0} \text{VaR}_{1/2}(X_t + Y_t)$. Based on the simple fact that $d(X_t + Y_t, X_0 + Y_0) \rightarrow 0$ as t goes to zero, we conclude that $\text{VaR}_{1/2}$ is not aggregation-robust. A similar example can be constructed for VaR_p , $p \in (0, 1)$. \square

Note that in the above example, the joint distribution C_t with a small $t > 0$ can be seen as the joint distribution C_0 influenced by a small perturbation. It is moreover worth noting that in [Example 2.1](#), the marginal distributions are continuous with positive densities. Hence, even if the true marginal distributions are known to have positive densities, VaR can still be discontinuous in aggregation. This observation weakens the robustness of VaR in aggregation. On the other hand, we will see that ES is aggregation-robust, although it is well-known to be non-robust in Hampel's sense since it is discontinuous at any distribution with respect to the weak topology.

For generality, we study the aggregation-robustness of *distortion risk measures*, defined as

$$\rho(X) = \int_0^1 \text{VaR}_t(X) dh(t), \quad X \in L^0 \text{ such that the integral is well-defined,}$$

where h is a probability measure on $(0, 1]$; h is called the *distortion function* of ρ . See [Wang et al. \(1997\)](#) for distortion risk measures in the context of insurance premium calculations, [Kusuoka \(2001\)](#) for their connection with coherent risk measures, and [Cont et al. \(2010\)](#) for their robustness properties. A distortion risk measure ρ is coherent if and only if h is convex, in which case ρ is called a *spectral risk measure*; see [Acerbi \(2002\)](#). Distortion risk measures are also closely related to L -statistics; see [Huber and Ronchetti \(2009, Section 3.3\)](#). For $p \in (0, 1)$, VaR_p and ES_p are special cases of distortion risk measures, with distortion functions $h(t) = \mathbb{I}_{\{t \geq p\}}$, $t \in [0, 1]$ and $h(t) = \mathbb{I}_{\{t \geq p\}}(t - p)/(1 - p)$, $t \in [0, 1]$, respectively. Our main result on robustness now becomes:

Theorem 2.1. *All coherent distortion risk measures are aggregation-robust.*

The main contribution of [Theorem 2.1](#) is to justify that, when the model uncertainty lies at the level of dependence but not at the level of the marginal distributions, coherent risk measures, such as ES, are more robust than VaR.

Remark 2.1. Cont et al. (2010) also introduced the notion of \mathcal{C} -robustness, where \mathcal{C} is a set of distributions. A risk measure ρ is \mathcal{C} -robust if ρ is continuous in \mathcal{C} with respect to the Lévy distance; see Cont et al. (2010, Proposition 2). Using this notion, VaR_p is \mathcal{C}_p -robust, where \mathcal{C}_p is the set of distributions F for which F^{-1} is continuous at p . If we denote by $\mathfrak{D}(\mathfrak{S}_n)$ the set of all possible distributions of an admissible class \mathfrak{S}_n , then ρ is aggregation-robust if and only if ρ is $\mathfrak{D}(\mathfrak{S}_n)$ -robust for all possible choices of $n \in \mathbb{N}$ and $\mathfrak{D}(\mathfrak{S}_n)$, in which \mathfrak{S}_n is compatible with ρ .

Our result can be interpreted using weak convergence in the admissible class \mathfrak{S}_n . For $S, S_1, S_2, \dots \in \mathfrak{S}_n$ and $S_k \rightarrow S$ weakly as $k \rightarrow \infty$, we have that $\text{ES}_p(S_k) \rightarrow \text{ES}_p(S)$ as $k \rightarrow \infty$ by Theorem 2.1. As illustrated by Example 2.1, the convergence $\text{VaR}_p(S_k) \rightarrow \text{VaR}_p(S)$ as $k \rightarrow \infty$ may fail to hold.

Finally, we remark that it would be of much interest to characterize the class of aggregation-robust statistical functionals (risk measures). Such a characterization is beyond the scope of this paper and we leave it for future work.

3 Bounds on VaR aggregation

Given the considerable statistical difficulty in modeling inter-dependency of risks, one may worry about, in addition to the sensitivity of risk measures to a small change in the dependence structure as discussed in Section 2, the uncertainty of a misspecified dependence structure which is relatively far away from that of the true underlying portfolio. In this section, we study the worst-case values of VaR when the dependence structure in a portfolio with given marginal distributions is unknown. These values can be used to analyze extreme scenarios of risk aggregation and determine conservative capital requirement under model uncertainty.

3.1 Aggregation and diversification under dependence uncertainty

We start with the motivating notion of *diversification ratio*, which is closely related to the aggregation of VaR. Given a portfolio consisting of individual risks X_1, \dots, X_n , the diversification ratio of VaR at confidence level $p \in (0, 1)$ is defined as

$$\Delta_n^p = \frac{\text{VaR}_p(X_1 + \dots + X_n)}{\sum_{i=1}^n \text{VaR}_p(X_i)}.$$

The diversification ratio measures a kind of *diversification benefit*, and is for instance widely used in operational risk (see examples in Embrechts et al. (2013)). In the latter context, X_i corresponds to next year's operational risk loss in business line i , $i = 1, \dots, n$; often explicit models for the loss-dependence among business lines are not available. For capital charge purposes, one estimates the total capital requirement for the superposition of the risks in each business line.

One then typically adds up the risk measures across all business lines, and multiplies by a factor which is an estimate of Δ_n^p . For this purpose, one needs a joint model of the risks X_1, \dots, X_n .

With a known joint distribution of (X_1, \dots, X_n) , Δ_n^p may be calculated theoretically. If $\Delta_n^p \leq 1$, we say there is a *diversification benefit* in the portfolio; if $\Delta_n^p \geq 1$, we say there is a *diversification penalty* in the portfolio. When F_1, \dots, F_n are known and the joint model of (X_1, \dots, X_n) is unspecified, the *worst diversification ratio* is defined as

$$\overline{\Delta}_n^p = \frac{\sup\{\text{VaR}_p(X_1 + \dots + X_n) : X_i \sim F_i, i = 1, \dots, n\}}{\sum_{i=1}^n \text{VaR}_p(X_i)} = \frac{\sup\{\text{VaR}_p(S) : S \in \mathfrak{S}_n\}}{\sum_{i=1}^n \text{VaR}_p(X_i)}.$$

By definition $\overline{\Delta}_n^p \geq 1$ if $\sum_{i=1}^n \text{VaR}_p(X_i) > 0$. In the following we denote the comonotonic VaR by $\text{VaR}_p^+(S_n)$, i.e.

$$\text{VaR}_p^+(S_n) = \sum_{i=1}^n \text{VaR}_p(X_i).$$

Note here that S_n is symbolic and does not represent a particular rv. The calculation of $\overline{\Delta}_n^p$, as a measure of the worst-case diversification effect of VaR, serves two purposes:

- Conservative capital requirement. $\overline{\Delta}_n^p \text{VaR}_p^+(S_n)$ can be used as the most conservative capital requirement in the case of given (or estimated) marginal distributions F_1, \dots, F_n of the individual risks.
- Measurement of model uncertainty. If $\overline{\Delta}_n^p$ is small, then the model uncertainty is small, and the risk measure VaR is considered as less problematic in risk aggregation; capital requirement principles based on VaR_p^+ become more plausible. If $\overline{\Delta}_n^p$ is large, then the model uncertainty is severe, and arguments of diversification benefit need to be taken with care.

The *best diversification ratio*, replacing the sup by an inf, can be studied similarly. Since we are more interested in the worst-case (corresponding to a conservative capital requirement), we omit a discussion of the best diversification ratio.

In the recent literature, it was shown that the value of $\overline{\Delta}_n^p$ is closely related to the risk measure ES. Denote the worst-case ES by $\overline{\text{ES}}_p(S_n) = \sup\{\text{ES}_p(S) : S \in \mathfrak{S}_n\}$; since ES is subadditive and comonotonic additive, we have that

$$\overline{\text{ES}}_p(S_n) = \sum_{i=1}^n \text{ES}_p(X_i) = \text{ES}_p^+(S_n),$$

where the latter +-notation is in line with the notation used for the comonotonic VaR case. Since VaR is bounded by ES, the worst-case VaR is bounded by the worst-case ES. If $\text{VaR}_p^+(S_n) > 0$, we have the following direct upper bound for $\overline{\Delta}_n^p$:

$$1 \leq \overline{\Delta}_n^p \leq \frac{\text{ES}_p^+(S_n)}{\text{VaR}_p^+(S_n)} = \frac{\overline{\text{ES}}_p(S_n)}{\text{VaR}_p^+(S_n)}. \quad (3.1)$$

See also [Embrechts et al. \(2014\)](#) for a discussion on this upper bound. Later in this section we will show that the second inequality in [\(3.1\)](#) is asymptotically sharp as $n \rightarrow \infty$.

By definition, calculation of the worst diversification ratio is equivalent to the calculation of the worst-case VaR

$$\overline{\text{VaR}}_p(S_n) := \sup\{\text{VaR}_p(S) : S \in \mathfrak{S}_n\}. \quad (3.2)$$

For the history and a general discussion on problems related to [\(3.2\)](#) from the perspective of quantitative risk management, we refer to [Embrechts et al. \(2014\)](#). When $F_1 = F_2 = \dots = F_n =: F$, i.e. the homogeneous case, [Wang et al. \(2013\)](#) obtained $\overline{\text{VaR}}_p(S_n)$ for F with a tail-decreasing density. If F_1, \dots, F_n are not identical, explicit calculations of $\overline{\text{VaR}}_p(S_n)$ and $\overline{\Delta}_n^p$ are not available in general. [Embrechts et al. \(2013\)](#) introduced the Rearrangement Algorithm to numerically calculate $\overline{\text{VaR}}_p(S_n)$ based on a discretized approximation.

Regarding the asymptotic behavior of $\overline{\text{VaR}}_p(S_n)$ and $\overline{\Delta}_n^p$, [Puccetti and Rüschendorf \(2014\)](#) obtained that, as $n \rightarrow \infty$,

$$\frac{\overline{\text{VaR}}_p(S_n)}{\overline{\text{ES}}_p(S_n)} \rightarrow 1, \quad (3.3)$$

in the homogeneous case under a condition of complete mixability for the marginal distributions. See also [Wang \(2014\)](#) and [Wang and Wang \(2014\)](#) for weaker conditions so that [\(3.3\)](#) holds. [Puccetti et al. \(2013\)](#) considered the case when there are finitely many different marginal distributions in the sequence F_1, F_2, \dots and obtained the same equivalence [\(3.3\)](#). A consequence of [\(3.3\)](#) is that

$$\lim_{n \rightarrow \infty} \overline{\Delta}_n^p = \lim_{n \rightarrow \infty} \frac{\overline{\text{ES}}_p(S_n)}{\overline{\text{VaR}}_p^+(S_n)}, \quad (3.4)$$

given that the right-hand limit exists. That is, the second inequality in [\(3.1\)](#) is asymptotically sharp. However, as mentioned above, the existing results only deal with the (almost) homogeneous case, and some specific assumptions on the marginal distributions need to be imposed. Later in this section, we will provide analytical approximations for $\overline{\text{VaR}}_p(S_n)$ and $\overline{\Delta}_n^p$. Based on these results, we will give a proof of [\(3.3\)](#) and [\(3.4\)](#) under very general conditions and, moreover, obtain a rate of convergence.

3.2 Bounds on VaR aggregation for a finite number of risks

In this section, we will give inequalities for the worst-case and best-case VaR and its diversification ratio. For the rest of Sections [3](#) and [4](#), we assume that the distributions F_i , $i \in \mathbb{N}$ are continuous; this constitutes no real restriction, and is only assumed for the ease of notation. For a continuous distribution F_i , define

$$\mu_{p,q}^{(i)} = \frac{1}{q-p} \int_p^q F_i^{-1}(t) dt,$$

for $1 \geq q > p \geq 0$, $i = 1, \dots, n$. Note that $\mu_{0,q}^{(i)}$ and $\mu_{p,1}^{(i)}$ might be infinite. Using the above notation, it is immediate that

$$\overline{\text{ES}}_p(S_n) = \sum_{i=1}^n \text{ES}_p(X_i) = \sum_{i=1}^n \mu_{p,1}^{(i)}.$$

For future discussion, we also denote the best-case VaR by $\underline{\text{VaR}}_p(S_n)$, that is

$$\underline{\text{VaR}}_p(S_n) = \inf_{S \in \mathfrak{S}_n} \text{VaR}_p(S),$$

and the best-case ES by $\underline{\text{ES}}_p(S_n)$, that is

$$\underline{\text{ES}}_p(S_n) = \inf_{S \in \mathfrak{S}_n} \text{ES}_p(S).$$

Analytical formulas for each of $\overline{\text{VaR}}_p(S_n)$, $\underline{\text{VaR}}_p(S_n)$ and $\underline{\text{ES}}_p(S_n)$ are not available under general assumptions on the marginal distributions; see [Bernard et al. \(2014\)](#) and [Embrechts et al. \(2014\)](#) for existing results on $\overline{\text{VaR}}_p(S_n)$, $\underline{\text{VaR}}_p(S_n)$ and $\underline{\text{ES}}_p(S_n)$.

The following theorem contains our main result regarding approximations of $\overline{\text{VaR}}_p(S_n)$ and $\underline{\text{VaR}}_p(S_n)$.

Theorem 3.1. *Suppose F_1, \dots, F_n are continuous distributions. We have for $p \in (0, 1)$,*

$$\sup_{q \in (p, 1]} \left\{ \sum_{i=1}^n \mu_{p,q}^{(i)} - \max_{i=1, \dots, n} (F_i^{-1}(q) - F_i^{-1}(p)) \right\} \leq \overline{\text{VaR}}_p(S_n) \leq \overline{\text{ES}}_p(S_n), \quad (3.5)$$

and

$$\sum_{i=1}^n \mu_{0,p}^{(i)} \leq \underline{\text{VaR}}_p(S_n) \leq \inf_{q \in [0, p)} \left\{ \sum_{i=1}^n \mu_{q,p}^{(i)} + \max_{i=1, \dots, n} (F_i^{-1}(q) - F_i^{-1}(p)) \right\}. \quad (3.6)$$

In particular, if F_1, \dots, F_n are supported on $[a, b]$, $a < b$, $a, b \in \mathbb{R}$, then

$$\overline{\text{ES}}_p(S_n) - (b - a) \leq \overline{\text{VaR}}_p(S_n) \leq \overline{\text{ES}}_p(S_n). \quad (3.7)$$

Note that in the case when all marginal distributions are bounded, $\overline{\text{VaR}}_p(S_n)$ and $\overline{\text{ES}}_p(S_n)$ differ by at most a constant which does not depend on n . Theorem 3.1 can also be formulated for the worst diversification ratio of VaR.

Corollary 3.2. *Suppose F_1, \dots, F_n are continuous distributions, and $\text{VaR}_p^+(S_n) > 0$. We have for $p \in (0, 1)$,*

$$\sup_{q \in (p, 1]} \left\{ \frac{\sum_{i=1}^n \mu_{p,q}^{(i)} - \max_{i=1, \dots, n} (F_i^{-1}(q) - F_i^{-1}(p))}{\text{VaR}_p^+(S_n)} \right\} \leq \overline{\Delta}_n^p \leq \frac{\overline{\text{ES}}_p(S_n)}{\text{VaR}_p^+(S_n)}. \quad (3.8)$$

In particular, if F_1, \dots, F_n are supported in $[a, b]$, $a < b$, $a, b \in \mathbb{R}$, then

$$\frac{\overline{\text{ES}}_p(S_n)}{\text{VaR}_p^+(S_n)} - \frac{b - a}{\text{VaR}_p^+(S_n)} \leq \overline{\Delta}_n^p \leq \frac{\overline{\text{ES}}_p(S_n)}{\text{VaR}_p^+(S_n)}. \quad (3.9)$$

In the homogeneous case, i.e. $F := F_1 = F_2 = \dots$, the left-hand side and right-hand side of (3.9) both converge to $\frac{\text{ES}_p(X)}{\text{VaR}_p(X)}$ as $n \rightarrow \infty$, where $X \sim F$, assuming $\text{VaR}_p(X) \neq 0$. In the following, we will study the limit of the worst- and best-case VaR and its diversification ratio under general marginal assumptions, as n goes to infinity.

3.3 Asymptotic equivalence and limit of the worst diversification ratio

Based on Theorem 3.1, we now derive the asymptotic equivalence between the worst-case VaR and the worst-case ES under very weak general conditions. For an asymptotic analysis, some uniformity conditions on F_i , $i \in \mathbb{N}$ need to be imposed. In what follows, X_i is any rv with distribution F_i , $i \in \mathbb{N}$. Define the following conditions, for some $p \in (0, 1)$ and $k > 1$:

- (a) $\mathbb{E}[|X_i - \mathbb{E}[X_i]|^k] < M$ for some $M > 0$;
- (b) $\liminf_{n \rightarrow \infty} n^{-1/k} \sum_{i=1}^n \text{ES}_p(X_i) = +\infty$, and
- (b*) $C_0 := \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \text{ES}_p(X_i) > 0$.

The above conditions only concern the moments of F_i , $i \in \mathbb{N}$, and hence they are quite weak and commonly satisfied. Condition (a) is a uniform boundedness condition, ensuring that the aggregate portfolio S_n does not contain a single risk with a too heavy tail that dominates the other risks. Condition (b) is assumed to guarantee that the average ES of the sequence of risks does not vanish to zero too fast. Without (a) or (b), the limiting portfolio would exhibit a finite- n behavior. Hence, in view of an asymptotic analysis, both conditions are reasonable. The condition (b*) is a stronger version of (b). In particular, in the homogeneous case when F_i , $i \in \mathbb{N}$ are identical, $\text{ES}_p(X_1) > 0$ implies (b*) and hence it also implies (b). We also remark that condition (a*) below is stronger than condition (a):

- (a*) $\mathbb{E}[|X_i|^k]$ is uniformly bounded.

Theorem 3.3. *Suppose that the continuous distributions F_i , $i \in \mathbb{N}$, satisfy (a) and (b) for some $p \in (0, 1)$ and $k > 1$, then*

$$\lim_{n \rightarrow \infty} \frac{\overline{\text{VaR}}_p(S_n)}{\overline{\text{ES}}_p(S_n)} = 1. \quad (3.10)$$

If, in addition, (b) is replaced by (b), then for sufficiently large n ,*

$$1 \geq \frac{\overline{\text{VaR}}_p(S_n)}{\overline{\text{ES}}_p(S_n)} \geq 1 - Cn^{-1+1/k}, \quad (3.11)$$

where

$$C = \left(\frac{1}{1-p} \frac{k}{k-1} + 1 \right) \frac{M^{1/k}}{C_0} > 0,$$

M is given in (a) and C_0 is given in (b*).

Theorem 3.3 establishes the asymptotic equivalence of the worst-case ES and the worst-case VaR for risk aggregation for general, possibly inhomogeneous portfolios. As mentioned in Section 3.1, homogeneous or almost-homogeneous cases for (3.10) to hold were previously obtained in the literature. While existing methods of proof were mainly based on the theory of complete mixability, an extension using the same techniques to arbitrarily many different marginal distributions was not possible.

Similarly to Theorem 3.3, we can obtain the limit of the best-case VaR bounds. In the following we define the left-tail ES (LES) as

$$\text{LES}_p(X) = \frac{1}{p} \int_0^p \text{VaR}_q(X) dq = -\text{ES}_p(-X),$$

and denote its best-case value under dependence uncertainty by

$$\underline{\text{LES}}_p(S_n) := \inf_{S \in \mathfrak{S}_n} \text{LES}_p(S) = \sum_{i=1}^n \text{LES}_p(X_i) = \sum_{i=1}^n \mu_{0,p}^{(i)},$$

where the second equality can be seen from the symmetry between ES and LES. For the best-case VaR bounds, we use a slightly different set of conditions, for some $p \in (0, 1)$ and $k > 1$:

- (c) $\liminf_{n \rightarrow \infty} n^{-1/k} \sum_{i=1}^n \text{LES}_p(X_i) = +\infty$, and
- (c*) $C_0 := \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \text{LES}_p(X_i) > 0$.

The following corollary is obtained from Theorem 3.3 by symmetry.

Corollary 3.4. *Suppose that the continuous distributions F_i , $i \in \mathbb{N}$, satisfy (a) and (c) for some $p \in (0, 1)$ and $k > 1$, then*

$$\lim_{n \rightarrow \infty} \frac{\text{VaR}_p(S_n)}{\underline{\text{LES}}_p(S_n)} = 1. \quad (3.12)$$

If, in addition, (c) is replaced by (c*), then for sufficiently large n ,

$$1 \geq \frac{\text{VaR}_p(S_n)}{\underline{\text{LES}}_p(S_n)} \geq 1 - Cn^{-1+1/k}, \quad (3.13)$$

where

$$C = \left(\frac{1}{1-p} \frac{k}{k-1} + 1 \right) \frac{M^{1/k}}{C_0} > 0,$$

M is given in (a) and C_0 is given in (c*).

Remark 3.1. The conditions (c) and (c*) are slightly stronger than (b) and (b*), respectively, and this asymmetry is due to the fact that we mainly consider the cases when the aggregate risk measures LTE and ES are positive. The asymmetry can be trivially removed by assuming $\liminf_{n \rightarrow \infty} |\frac{1}{n} \sum_{i=1}^n \text{LES}_p(X_i)| > C_0$ instead of (c).

Finally, we remark that the limit of $\overline{\Delta}_n^p$ as $n \rightarrow \infty$ can be obtained directly from Theorem 3.3. Suppose the continuous distributions F_i , $i \in \mathbb{N}$ satisfy (a) and (b) for some $p \in (0, 1)$ and $k > 1$, then, as $n \rightarrow \infty$,

$$\overline{\Delta}_n^p \frac{\text{VaR}_p^+(S_n)}{\overline{\text{ES}}_p(S_n)} \rightarrow 1.$$

If in addition, $R_p := \lim_{n \rightarrow \infty} \frac{\overline{\text{ES}}_p(S_n)}{\text{VaR}_p^+(S_n)}$ exists in $[1, \infty]$, then $\Delta_n^p \rightarrow R_p$ as $n \rightarrow \infty$.

4 Uncertainty spread of VaR and ES

In addition to the distribution-wise continuity as discussed in Section 2, in this section we study the uncertainty spread of VaR and ES when the dependence structure is completely unknown, based on results obtained in Section 3. This quantifies the magnitude of dependence uncertainty in a model for risk aggregation. We show that VaR generally exhibits a larger spread compared to ES. This result suggests that VaR is more sensitive to dependence uncertainty compared to ES and can be seen as a supporting argument for Theorem 2.1. For $p \in (0, 1)$ we define the dependence uncertainty spread (DU-spread) of VaR_p as

$$\overline{\text{VaR}}_p(S_n) - \underline{\text{VaR}}_p(S_n),$$

and of ES_p as

$$\overline{\text{ES}}_p(S_n) - \underline{\text{ES}}_p(S_n).$$

See Embrechts et al. (2014) for a discussion on the DU-spread of VaR and its relevance in risk management.

By definition $\text{ES}_p(X) \geq \text{VaR}_p(X)$ for any risk X and the inequality is strict when X is continuous. Naturally, when switching from VaR to ES for the purpose of capital requirement, one should consider a lower confidence level for ES. In the most recent consultative document BCBS (2013), it was proposed that for internal risk models, $\text{VaR}_{0.99}$ should be replaced by $\text{ES}_{0.975}$ which often yields a similar value to $\text{VaR}_{0.99}$ for light-tailed risks. Under the Swiss Solvency Test (SST), $\text{VaR}_{0.995}$ is used to compare with $\text{ES}_{0.99}$ to calculate the capital requirement for the change in the Risk Bearing Capital (RBC) over a one-year period; see (EIOPA, 2011, p.32). Kou and Peng (2014) also proposed that, in order to compare with ES_p , one could use the corresponding Median Shortfall (MS), which is the median of the conditional tail distribution above VaR_p , and hence satisfies

$$\text{MS}_p(X) = \text{VaR}_{(p+1)/2}(X);$$

thus, it is consistent with the SST regime. Hence, it may be useful to compare the DU-spread of VaR_q and ES_p for $q \geq p$. The following proposition compares the DU-spread of VaR_q and that

of ES_p in the asymptotic sense. In what follows, we denote by μ_n the summation of the means of F_1, \dots, F_n , assumed to exist.

Proposition 4.1. *Suppose $1 > q \geq p > 0$.*

(i) *Suppose that the continuous distributions F_i , $i \in \mathbb{N}$, satisfy (a) and (c), then*

$$\liminf_{n \rightarrow \infty} \frac{\overline{\text{VaR}}_q(S_n) - \underline{\text{VaR}}_q(S_n)}{\overline{\text{ES}}_p(S_n) - \underline{\text{ES}}_p(S_n)} = \liminf_{n \rightarrow \infty} \frac{\overline{\text{ES}}_q(S_n) - \underline{\text{ES}}_q(S_n)}{\overline{\text{ES}}_p(S_n) - \underline{\text{ES}}_p(S_n)} \geq \liminf_{n \rightarrow \infty} \frac{\overline{\text{ES}}_q(S_n) - \mu_n}{\overline{\text{ES}}_p(S_n) - \mu_n} \geq 1. \quad (4.1)$$

(ii) *Suppose that the distributions F_i , $i \in \mathbb{N}$, are identical and equal to F , and $\mathbb{E}[|X|] < \infty$ where $X \sim F$, then*

$$\lim_{n \rightarrow \infty} \frac{\overline{\text{VaR}}_q(S_n) - \underline{\text{VaR}}_q(S_n)}{\overline{\text{ES}}_p(S_n) - \underline{\text{ES}}_p(S_n)} = \frac{\text{ES}_q(X) - \text{LES}_q(X)}{\text{ES}_p(X) - \mathbb{E}[X]} \geq 1. \quad (4.2)$$

Proposition 4.1 suggests that VaR is overall more sensitive to dependence uncertainty for large n , compared to ES. Numerical evidence of the comparison of DU-spread for VaR and ES at the same level can be found in Section 5, even for small numbers of n . Note that, although the DU-spread of ES is smaller than that of VaR asymptotically, both risk measures have a rather large uncertainty spread in general, suggesting that dependence uncertainty in risk aggregation must be taken with care no matter whether ES or VaR is chosen as the underlying risk measure.

Remark 4.1. In the homogeneous case, for any continuous distribution F , the limit of the DU-spread ratio in (4.2) is strictly greater than 1 since $\text{LES}_q(X) < \mathbb{E}[X]$ and $\text{ES}_q(X) > \text{ES}_p(X)$. In the case $q = p$, we note that, for light-tailed risks X , $\text{LES}_p(X)$ is slightly smaller than $\mathbb{E}[X]$; for heavy-tailed risks X , $\text{LES}_p(X)$ can be significantly smaller than $\mathbb{E}[X]$, leading to a much larger DU-spread of VaR. From Proposition 4.1, we can also see that, approximately, the VaR_q interval under DU is $[\sum_{i=1}^n \text{LES}_q(X_i), \sum_{i=1}^n \text{ES}_q(X_i)]$ and the ES_p interval under DU is $[\mu_n, \sum_{i=1}^n \text{ES}_p(X_i)]$. Note that when F_i , $i \in \mathbb{N}$ are the same, no conditions other than a finite first moment of F are needed for the asymptotic equivalence in Theorem 3.3 and Corollary 3.4 to hold. This was also suggested by Corollary 3.7 of Wang and Wang (2014). In this case, the limit of the ratio between the two uncertainty spreads is explicitly given in (4.2).

In the following we give a result for finite n , in the case of bounded risks. A proof can be directly obtained from Theorem 3.1.

Corollary 4.2. *Suppose that $1 > q \geq p > 0$, F_1, \dots, F_n are continuous distributions, supported in $[a, b]$, $a < b$, $a, b \in \mathbb{R}$, and*

$$\sum_{i=1}^n (\text{ES}_q(X_i) + \mathbb{E}[X_i] - \text{ES}_p(X_i) - \text{LES}_q(X_i)) > 2(b - a), \quad (4.3)$$

where $X_i \sim F_i$, $i = 1, \dots, n$, then

$$\frac{\overline{\text{VaR}}_q(S_n) - \underline{\text{VaR}}_q(S_n)}{\overline{\text{ES}}_p(S_n) - \underline{\text{ES}}_p(S_n)} > 1.$$

Note that in Corollary 4.2, since $\text{ES}_q(X_i) \geq \text{ES}_p(X_i)$ and $\mathbb{E}[X_i] \geq \text{LES}_q(X_i)$, the left-hand side of (4.3) is the summation of n non-negative terms while the right-hand side of (4.3) is a constant, hence (4.3) holds for n sufficiently large as long as the summation of the left-hand side of (4.3) diverges as $n \rightarrow \infty$.

We remark that it remains theoretically unclear whether the DU-spread of VaR_q is always larger than (or equal to) ES_p for finite n and $q \geq p$. In all our numerical examples (see Section 5 below), VaR_q always has a larger DU-spread than ES_p .

5 Numerical examples

As suggested by BCBS (2013), the risk measure $\text{ES}_{0.975}$ is a candidate to replace $\text{VaR}_{0.99}$. The SST (see EIOPA, 2011) used $\text{VaR}_{(1+p)/2}$ to compare with ES_p . Based on such considerations, in this section, we provide the worst-case and the best-case values of $\text{VaR}_{0.99}$, $\text{VaR}_{0.9875}$, $\text{VaR}_{0.975}$ and $\text{ES}_{0.975}$ for different portfolios under dependence uncertainty. We compare the dependence uncertainty spread of VaR and ES in each model, and also look at the influence on the number n of risks in the portfolio. The numerical calculation is carried out through the Rearrangement Algorithm (RA) described in Embrechts et al. (2013), with discretization step $\Delta x = 10^{-6}$. The following three models are considered, and the results for $n = 5, 10, 20$ are reported in Tables 1-3.

- (A) (Mixed portfolio) $S_n = X_1 + \dots + X_n$, where $X_i \sim \text{Pareto}(2 + 0.1i)$, $i = 1, \dots, 5$; $X_i \sim \text{Exp}(i - 5)$, $i = 6, \dots, 10$; $X_i \sim \text{Log-Normal}(0, (0.1(i - 10))^2)$, $i = 11, \dots, 20$.
- (B) (Light-tailed portfolio) $S_n = Y_1 + \dots + Y_n$, where $Y_i \sim \text{Exp}(i)$, $i = 1, \dots, 5$; $Y_i \sim \text{Weibull}(i - 5, 1/2)$, $i = 6, \dots, 10$; $Y_i \stackrel{d}{=} Y_{i-10}$, $i = 11, \dots, 20$.
- (C) (Pareto portfolio) $S_n = Z_1 + \dots + Z_n$, where $Z_i \sim \text{Pareto}(1.5)$, $i = 1, \dots, 20$.

From Tables 1-3, we have the following observations.

- (i) The worst-case VaR at level 0.975 and the worst-case ES at level 0.975 are very close, even for small numbers of n , in all models considered (Theorem 3.3, (3.10)).
- (ii) The ratio between the worst-case VaR at level 0.975 and the worst-case ES at level 0.975 goes to 1 as n grows large. In the heavy-tailed model (C), the convergence is relatively slow (Theorem 3.3, (3.11)).

Table 1: Bounds obtained with RA ($\Delta x = 10^{-6}$), Model (A): mixed portfolio.

	$n = 5$			$n = 10$			$n = 20$		
	best	worst	spread	best	worst	spread	best	worst	spread
$\text{ES}_{0.975}(S_n)$	22.48	44.88	22.40	22.52	55.59	33.07	29.15	102.35	73.20
$\text{VaR}_{0.975}(S_n)$	9.79	41.46	31.67	10.04	52.67	42.63	21.44	100.65	79.21
$\text{VaR}_{0.9875}(S_n)$	12.06	56.21	44.16	12.06	69.03	56.98	22.12	126.63	104.51
$\text{VaR}_{0.99}(S_n)$	12.96	62.01	49.05	12.96	75.34	62.38	22.29	136.30	114.01
$\frac{\overline{\text{ES}}_{0.975}(S_n)}{\overline{\text{VaR}}_{0.975}(S_n)}$	1.08			1.06			1.02		

Table 2: Bounds obtained with RA ($\Delta x = 10^{-6}$), Model (B): light-tailed portfolio.

	$n = 5$			$n = 10$			$n = 20$		
	best	worst	spread	best	worst	spread	best	worst	spread
$\text{ES}_{0.975}(S_n)$	4.72	10.71	5.99	24.55	63.19	38.63	31.33	126.38	95.04
$\text{VaR}_{0.975}(S_n)$	3.69	10.57	6.88	13.61	61.41	47.81	13.61	125.73	112.13
$\text{VaR}_{0.9875}(S_n)$	4.38	12.15	7.77	19.20	78.75	59.55	19.20	160.75	141.55
$\text{VaR}_{0.99}(S_n)$	4.61	12.66	8.05	21.21	84.80	63.59	21.21	172.96	151.75
$\frac{\overline{\text{ES}}_{0.975}(S_n)}{\overline{\text{VaR}}_{0.975}(S_n)}$	1.01			1.03			1.01		

Table 3: Bounds obtained with RA ($\Delta x = 10^{-6}$), Model (C): Pareto portfolio.

	$n = 5$			$n = 10$			$n = 20$		
	best	worst	spread	best	worst	spread	best	worst	spread
$\text{ES}_{0.975}(S_n)$	103.8	172.6	68.8	166.2	345.1	178.9	266.2	690.3	424.1
$\text{VaR}_{0.975}(S_n)$	15.7	130.6	114.9	21.8	291.3	269.5	43.5	620.8	577.3
$\text{VaR}_{0.9875}(S_n)$	22.6	207.3	184.7	27.6	462.4	434.8	46.7	985.5	938.8
$\text{VaR}_{0.99}(S_n)$	25.5	240.5	215.0	30.5	536.5	506.0	47.5	1143.6	1096.0
$\frac{\overline{\text{ES}}_{0.975}(S_n)}{\overline{\text{VaR}}_{0.975}(S_n)}$	1.32			1.19			1.11		

- (iii) The DU-spreads of $\text{VaR}_{0.99}$, $\text{VaR}_{0.985}$ and $\text{VaR}_{0.975}$ are larger than those of $\text{ES}_{0.975}$ in all considered models (Proposition 4.1).
- (iv) In the heavy-tailed Model (C), the DU-spreads of VaR are significantly larger than those of ES (Remark 4.1).

6 Conclusion

In this paper, we considered the risk measures VaR and ES under dependence uncertainty. We introduced the notion of aggregation-robustness and showed that all coherent distortion risk measures, including ES, are aggregation-robust while VaR is not. We also derived bounds for the worst-case and best-case VaR in aggregation and its diversification ratio under dependence uncertainty. An asymptotic equivalence between VaR and ES for inhomogeneous portfolios under the weakest known conditions on the marginal distributions was established. It was shown that VaR generally exhibits a larger uncertainty spread compared to ES at the same or a lower confidence level. Numerical examples were provided to support our theoretical results. The main results in this paper suggest that VaR is less robust than ES with respect to dependence uncertainty in aggregation.

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A Proofs

A.1 A useful lemma

Before presenting the main proofs, we first introduce a lemma that is essential to prove the main results in Sections 3 and 4 in this paper. Recall the definitions of the essential supremum and the essential infimum of rvs: for any rv S ,

$$\text{ess-sup}S = \sup\{t : \mathbb{P}(S \leq t) < 1\},$$

and

$$\text{ess-inf}S = \inf\{t : \mathbb{P}(S \leq t) > 0\}.$$

We denote $S_n = X_1 + \dots + X_n$ in the following. We remind the reader that such S_n is different from the symbolic one in the notation of $\overline{\text{VaR}}_p(S_n)$. We hope this will not lead to notational confusion.

Lemma A.1. *Suppose that $(F_i, i \in \mathbb{N})$ is a sequence of distributions on $[0, 1]$, then there exist $X_i \sim F_i, i \in \mathbb{N}$, such that for each $n \in \mathbb{N}$,*

$$\text{ess-sup}S_n - \text{ess-inf}S_n \leq 1. \quad (\text{A.1})$$

Proof. We first show that if X and Y are counter-monotonic and both take values in $[0, 1]$, then $\text{ess-sup}(X + Y) - \text{ess-inf}(X + Y) \leq 1$. Since X and Y are counter-monotonic, there exist $U \sim U[0, 1]$ such that $X = F^{-1}(U)$ and $Y = G^{-1}(1 - U)$ where F and G are the distributions of X and Y , respectively. For $u, v \in (0, 1)$, one of $F^{-1}(u) - F^{-1}(v)$ and $G^{-1}(1 - u) - G^{-1}(1 - v)$ is non-positive. Hence,

$$\begin{aligned} & F^{-1}(u) + G^{-1}(1 - u) - (F^{-1}(v) + G^{-1}(1 - v)) \\ &= (F^{-1}(u) - F^{-1}(v)) + (G^{-1}(1 - u) - G^{-1}(1 - v)) \\ &\leq \max\{F^{-1}(u) - F^{-1}(v), G^{-1}(1 - u) - G^{-1}(1 - v)\} \\ &\leq 1. \end{aligned}$$

Thus,

$$\text{ess-sup}(X+Y) - \text{ess-inf}(X+Y) = \sup_{u \in (0,1)} \{F^{-1}(u) + G^{-1}(1-u)\} - \inf_{v \in (0,1)} \{F^{-1}(v) + G^{-1}(1-v)\} \leq 1.$$

Let $X_1 \sim F_1$. For $k \geq 2$, let X_k be counter-monotonic with S_{k-1} . Since $\text{ess-sup}(X_1) - \text{ess-inf}(X_1) \leq 1$, by induction we get that $\text{ess-sup}(S_k) - \text{ess-inf}(S_k) = \text{ess-sup}(S_{k-1} + X_k) - \text{ess-inf}(S_{k-1} + X_k) \leq 1$ for all $k \geq 2$. This completes the proof. \square

Remark A.1. Lemma A.1 is of independent interest in the probability theory of negative dependence. Indeed, it shows that an *extremely negatively dependent* sequence always exists for bounded marginal distributions. The definition of and details on extremely negative dependence can be found in Wang and Wang (2014). In the latter paper, it was shown that an extremely negatively dependent sequence always exists for identical marginal L_1 -distributions. Lemma A.1, as a new contribution, confirms that the same statement holds for inhomogeneous marginal distributions if we assume some uniform boundedness.

The following useful corollary is directly implied by Lemma A.1.

Corollary A.2. *Suppose that $(F_i, i \in \mathbb{N})$ is a sequence of distributions with bounded support, then there exist $X_i \sim F_i, i \in \mathbb{N}$, such that for each $n \in \mathbb{N}$,*

$$|S_n - \mathbb{E}[S_n]| \leq L_n. \quad (\text{A.2})$$

where L_n is the largest length of the support of F_i , $i = 1, \dots, n$, that is,

$$L_n = \max\{\text{ess-sup}X_i - \text{ess-inf}X_i : X_i \sim F_i, i = 1, \dots, n\}.$$

A.2 Proof of Theorem 2.1

Proof. Suppose ρ is a distortion risk measure with distortion function h . Since h is increasing and convex on $(0,1)$, its has left-derivative on $(0, 1)$, denoted as

$$\delta(t) := \lim_{x \rightarrow 0^+} \frac{h(t) - h(t-x)}{x}, \quad t \in (0, 1).$$

Since h may have a jump at 1, we let $\delta(1) = \delta_1(h(1) - h(1-))$, where δ_1 is the Dirac delta function at 1. It follows that $\rho(X) = \int_0^1 \text{VaR}_t(X)\delta(t)dt$. Note that since \mathfrak{S}_n is compatible with a coherent risk measure ρ , we have that $\mathbb{E}[|X_i|] < \infty$, $i = 1, \dots, n$. For $q \in (1/2, 1)$, define

$$\tilde{\rho}_q(X) = \frac{1}{1-h(q)} \int_q^1 \text{VaR}_t(X)\delta(t)dt, \quad X \in L^0.$$

We can easily check that $\tilde{\rho}_q$ is also a coherent distortion risk measure.

For any $S \in \mathfrak{S}_n(F_1, \dots, F_n)$, write $S = X_1 + \dots + X_n$, where $X_i \sim F_i$, $i = 1, \dots, n$. For $q \in (1/2, 1)$, we have that

$$\begin{aligned} \left| \rho(S) - \int_{1-q}^q \text{VaR}_t(S)\delta(t)dt \right| &= \left| \int_0^{1-q} \text{VaR}_t(S)\delta(t)dt + \int_q^1 \text{VaR}_t(S)\delta(t)dt \right| \\ &\leq \left| \int_0^{1-q} \text{VaR}_t(S)\delta(t)dt \right| + |(1-h(q))\tilde{\rho}_q(S)| \\ &\leq \delta(1-q) \int_0^{1-q} |\text{VaR}_t(S)|dt + |(1-h(q))\tilde{\rho}_q(S)|. \end{aligned}$$

Note that

$$|(1-h(q))\tilde{\rho}_q(S)| \leq \left| (1-h(q)) \sum_{i=1}^n \tilde{\rho}_q(X_i) \right| = \left| \sum_{i=1}^n \int_q^1 \text{VaR}_t(X_i)\delta(t)dt \right|.$$

On the other hand, by the comonotonic additivity of VaR_t , $t \in (0, 1)$, we have that

$$\begin{aligned} \int_0^{1-q} |\text{VaR}_t(S)|dt &= \int_0^{1-q} |\text{VaR}_t(SI_{\{S \geq 0\}}) + \text{VaR}_t(SI_{\{S < 0\}})|dt \\ &\leq \int_0^{1-q} \text{VaR}_t(SI_{\{S \geq 0\}})dt + \int_0^{1-q} \text{VaR}_{1-t}(-SI_{\{S < 0\}})dt \\ &\leq \int_0^{1-q} \text{VaR}_t(|S|)dt + \int_0^{1-q} \text{VaR}_{1-t}(|S|)dt \\ &\leq 2(1-q)\text{ES}_q(|S|) \\ &\leq (1-q) \sum_{i=1}^n \text{ES}_q(|X_i|) \\ &= 2 \sum_{i=1}^n \int_q^1 \text{VaR}_t(|X_i|)dt. \end{aligned}$$

Note that for $i = 1, \dots, n$, $\rho(X_i) < \infty$ implies that $\int_q^1 \text{VaR}_t(X_i)\delta(t)dt \rightarrow 0$ as $q \rightarrow 1$, and that $\mathbb{E}[|X_i|] < \infty$ implies that $\int_q^1 \text{VaR}_t(|X_i|)dt \rightarrow 0$ as $q \rightarrow 1$. As a consequence, as $q \rightarrow 1$,

$$\eta(q) := \left| \rho(S) - \int_{1-q}^q \text{VaR}_t(S)\delta(t)dt \right| \rightarrow 0$$

uniformly in $S \in \mathfrak{S}_n$. Therefore, for each $\varepsilon > 0$, there exists $1/2 < q < 1$ such that $\eta(q) < \varepsilon/3$. By Theorem 1 of [Cont et al. \(2010\)](#), the distortion risk measure

$$\hat{\rho}_q(X) := \frac{1}{2q-1} \int_{1-q}^q \text{VaR}_t(X)\delta(t)dt, \quad X \in L^0$$

is continuous at all distributions with respect to the Lévy distance. In summary, for each $\varepsilon > 0$, and $q \in (1/2, 1)$ fixed, there exists $\delta > 0$ such that $|\hat{\rho}_q(S) - \hat{\rho}_q(T)| < \varepsilon/3$ for all S, T such that $d(S, T) < \delta$. Therefore, for all $S, T \in \mathfrak{S}_n$ with $d(S, T) < \delta$, we have that

$$|\rho(S) - \rho(T)| \leq (2q-1)|\hat{\rho}_q(S) - \hat{\rho}_q(T)| + 2\eta(q) < \varepsilon.$$

Thus, ρ is aggregation-robust. □

A.3 Proof of Theorem 3.1

Proof. We will use the following equivalence lemma in [Bernard et al. \(2014\)](#); in the latter paper, an alternative definition of VaR is used:

$$\text{VaR}_p^*(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) > p\}, \quad p \in (0, 1).$$

Lemma A.3 (Lemma 4.3 of [Bernard et al. \(2014\)](#)). *For $p \in (0, 1)$ and a continuous distribution F ,*

$$\sup_{S \in \mathfrak{S}_n} \text{VaR}_p^*(S) = \sup\{\text{ess-inf } S : S \in \mathfrak{S}_n(F_{p,1}, \dots, F_{p,n})\},$$

and

$$\inf_{S \in \mathfrak{S}_n} \text{VaR}_p(S) = \inf\{\text{ess-sup } S : S \in \mathfrak{S}_n(F_1^p, \dots, F_n^p)\},$$

where $F_{p,i}$ is the conditional distribution of $W_i \sim F_i$ on $[F_i^{-1}(p), \infty)$, and F_i^p is the conditional distribution of $W_i \sim F_i$ on $(-\infty, F_i^{-1}(p))$, $i = 1, \dots, n$.

For the proof of Theorem 3.1, we first show that for $p \in (0, 1)$ and $q \in (p, 1]$,

$$\sup\{\text{ess-inf } S : S \in \mathfrak{S}_n(F_{p,1}, \dots, F_{p,n})\} \geq \sum_{i=1}^n \mu_{p,q}^{(i)} - \max_{i=1, \dots, n} (F_i^{-1}(q) - F_i^{-1}(p)). \quad (\text{A.3})$$

Since the case when $F_i^{-1}(q) = \infty$ for some i is trivial, we suppose that $F_i^{-1}(q) < \infty$ for all $i = 1, \dots, n$.

Let $F_{p,q}^{(i)}$ be the conditional distribution of $W_i \sim F_i$ on $[F_i^{-1}(p), F_i^{-1}(q)]$ for $0 < p < q \leq 1$. By Corollary A.2, there exist rvs $X_i \sim F_{p,q}^{(i)}$, $i = 1, \dots, n$, such that

$$X_1 + \dots + X_n \geq \sum_{i=1}^n \mu_{p,q}^{(i)} - \max_{i=1, \dots, n} (F^{-1}(q) - F^{-1}(p)).$$

Let Z_i , $i = 1, \dots, n$ be any rv with distribution $F_{q,i}$, and let C be a set independent of $X_1, \dots, X_n, Z_1, \dots, Z_n$ and with $\mathbb{P}(C) = (q - p)/(1 - p)$. Define $Y_i = X_i \mathbf{I}_C + Z_i(1 - \mathbf{I}_C)$ for $i = 1, \dots, n$. It is straightforward to check that Y_i has distribution $F_{p,i}$, and

$$Y_1 + \dots + Y_n \geq X_1 + \dots + X_n \geq \sum_{i=1}^n \mu_{p,q}^{(i)} - \max_{i=1, \dots, n} (F_i^{-1}(q) - F_i^{-1}(p)).$$

Thus

$$\text{ess-inf}(Y_1 + \dots + Y_n) \geq \sum_{i=1}^n \mu_{p,q}^{(i)} - \max_{i=1, \dots, n} (F_i^{-1}(q) - F_i^{-1}(p)),$$

and we obtain (A.3). Since $\text{VaR}_p(X) \geq \text{VaR}_r^*(X)$ for any $r < p$ and any rv X , we have that

$$\begin{aligned} \overline{\text{VaR}}_p(S_n) &\geq \lim_{r \rightarrow p^-} \left(\sup_{S \in \mathfrak{S}_n} \text{VaR}_p^*(S) \right) \geq \lim_{r \rightarrow p^-} \left(\sum_{i=1}^n \mu_{r,q}^{(i)} - \max_{i=1, \dots, n} (F_i^{-1}(q) - F_i^{-1}(r)) \right) \\ &= \sum_{i=1}^n \mu_{p,q}^{(i)} - \max_{i=1, \dots, n} (F_i^{-1}(q) - F_i^{-1}(p)). \end{aligned}$$

Note that here we use the fact that F_i^{-1} is left-continuous for each i . On the other hand,

$$\overline{\text{VaR}}_p(S_n) \leq \sup_{S \in \mathfrak{S}_n} \text{VaR}_p^*(S) = \sup\{\text{ess-inf} S : S \in \mathfrak{S}_n(F_p, \dots, F_p)\} \leq \sum_{i=1}^n \mu_{p,1}^{(i)}$$

always holds. Thus we obtain (3.5). We can show (3.6) similarly. \square

A.4 Proof of Theorem 3.3

Proof. First, let us assume that $\mathbb{E}[X_i] = 0$ for all $i \in \mathbb{N}$. Note that $\overline{\text{ES}}_p(S_n) = \sum_{i=1}^n \text{ES}_p(X_i) = \sum_{i=1}^n \mu_{p,1}^{(i)}$ for $X_i \sim F_i$. We use (3.5) and take $q_n = 1 - n^{-1}$ for n large enough such that $q_n > p$. By (b), we have $\sum_{i=1}^n \mu_{p,1}^{(i)} > 0$ for large n .

Note that by (a), $\mathbb{E}[|X_i|^k] \leq M$ uniformly. Therefore, $[F_i^{-1}(t)]^k(1 - t) \leq M$ for $t \in (0, 1)$, and we have

$$F_i^{-1}(t) \leq \left(\frac{M}{1 - t} \right)^{1/k}, \quad t \in (0, 1), \quad i \in \mathbb{N}.$$

Note that for $X_i \sim F_i$,

$$\begin{aligned}
\mu_{p,1}^{(i)} - \mu_{p,q_n}^{(i)} &= \frac{1}{1-p} \mathbb{E}[X_i \mathbf{I}_{\{X_i \geq F_i^{-1}(p)\}}] - \frac{1}{q_n - p} \mathbb{E}[X_i \mathbf{I}_{\{F_i^{-1}(q_n) \geq X_i \geq F_i^{-1}(p)\}}] \\
&\leq \frac{1}{1-p} \mathbb{E}[X_i \mathbf{I}_{\{X_i \geq F_i^{-1}(q_n)\}}] \\
&= \frac{1}{1-p} \int_{q_n}^1 F_i^{-1}(t) dt \\
&\leq \frac{1}{1-p} \int_{q_n}^1 \left(\frac{M}{1-t} \right)^{1/k} dt \\
&= \frac{1}{1-p} \frac{1}{1-1/k} M^{1/k} (1-q_n)^{1-1/k}.
\end{aligned}$$

As a consequence we have

$$\begin{aligned}
\sup_{S \in \mathfrak{S}_n} \text{VaR}_p(S) &\geq \sum_{i=1}^n \mu_{p,q_n}^{(i)} - \max_{i=1,\dots,n} (F_i^{-1}(q_n) - F_i^{-1}(p)) \\
&\geq \sum_{i=1}^n \mu_{p,1}^{(i)} - \sum_{i=1}^n (\mu_{p,1}^{(i)} - \mu_{p,q_n}^{(i)}) - \max_{i=1,\dots,n} F_i^{-1}(q_n) \\
&\geq \sum_{i=1}^n \mu_{p,1}^{(i)} - \sum_{i=1}^n \frac{1}{1-p} \frac{1}{1-1/k} M^{1/k} (1-q_n)^{1-1/k} - \left(\frac{M}{1-q_n} \right)^{1/k} \\
&= \sum_{i=1}^n \mu_{p,1}^{(i)} - \frac{1}{1-p} \frac{1}{1-1/k} M^{1/k} n^{1/k} - M^{1/k} n^{1/k} \\
&= \sum_{i=1}^n \mu_{p,1}^{(i)} - O(n^{1/k}). \tag{A.4}
\end{aligned}$$

By (b), it follows that

$$1 \geq \frac{\overline{\text{VaR}}_p(S_n)}{\sum_{i=1}^n \mu_{p,1}^{(i)}} \geq 1 - \frac{O(n^{1/k})}{\sum_{i=1}^n \mu_{p,1}^{(i)}} \rightarrow 1 \text{ as } n \rightarrow \infty,$$

hence we obtain (3.10).

Now for the case that $\mathbb{E}[X_i] \neq 0$ for some $i \in \mathbb{N}$, we denote by F_i^* the distribution of $X_i - \mathbb{E}[X_i]$, and by

$$\mathfrak{S}_n^* = \{Y_1 + \dots + Y_n : Y_i \sim F_i^*, i = 1, \dots, n\}.$$

Then, by (A.4), with \mathfrak{S}_n replaced by \mathfrak{S}_n^* , we have

$$\begin{aligned}
\sup_{S \in \mathfrak{S}_n} \text{VaR}_p(S) &= \sup_{S \in \mathfrak{S}_n^*} \text{VaR}_p(S) + \sum_{i=1}^n \mathbb{E}[X_i] \\
&= \sum_{i=1}^n (\mu_{p,1}^{(i)} - \mathbb{E}[X_i]) - O(n^{1/k}) + \sum_{i=1}^n \mathbb{E}[X_i] \\
&= \sum_{i=1}^n \mu_{p,1}^{(i)} - O(n^{1/k}).
\end{aligned}$$

Thus, (A.4) still holds for \mathfrak{S}_n in the case $\mathbb{E}[X_i] \neq 0$ for some i .

When (b*) holds, by (A.4), we have that

$$1 \geq \frac{\overline{\text{VaR}}_p(S_n)}{\sum_{i=1}^n \mu_{p,1}^{(i)}} \geq 1 - \frac{\left(\frac{1}{1-p} \frac{k}{k-1} + 1\right) M^{1/k} (n^{1/k})}{\sum_{i=1}^n \mu_{p,1}^{(i)}} \geq 1 - Cn^{-1+1/k},$$

for n sufficiently large. This leads to (3.11) and completes the proof of the theorem. \square

A.5 Proof of Corollary 4.1

Proof. (i) By Theorem 3.3 and Corollary 3.4, we have that

$$\lim_{n \rightarrow \infty} \frac{\overline{\text{VaR}}_p(S_n) - \underline{\text{VaR}}_p(S_n)}{\overline{\text{ES}}_p(S_n) - \underline{\text{ES}}_p(S_n)} = 1,$$

and (4.1) follows directly, noting that $\text{ES}_p(X) \geq \mathbb{E}[X] \geq \text{LES}_p(X)$ for any rv X and any $p \in (0, 1)$.

(ii) Corollary 3.3 of Wang and Wang (2014) implies that

$$\lim_{n \rightarrow \infty} \frac{\overline{\text{ES}}_p(S_n)}{n} = \mathbb{E}[X],$$

and hence we obtain (4.2). \square

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