

# On EVT, aggregation and diversification in finance

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Consider two heavy-tailed risks  $X_1, X_2$  with arbitrary dependence structure.

### Questions:

- When does **subadditivity** of VaR hold:

$$\text{VaR}_\alpha(X_1 + X_2) \leq \text{VaR}_\alpha(X_1) + \text{VaR}_\alpha(X_2) ?$$

- Or equivalently, when is the **measure of concentration**  $\leq 1$ :

$$C_\alpha(X_1, X_2) := \frac{\text{VaR}_\alpha(X_1 + X_2)}{\text{VaR}_\alpha(X_1) + \text{VaR}_\alpha(X_2)} \leq 1 ?$$

- Or, at least, when does **asymptotic subadditivity** hold:

$$\lim_{\alpha \nearrow 1} C_\alpha(X_1, X_2) \leq 1 ?$$

## MRV = Multivariate Regular Variation

**Property (Daniélsson et al., 2005):**  
**(Under regular variation: see later)**

Finite mean  $\Rightarrow$  asymptotic **subadditivity**

**Fallacy:**

Infinite mean  $\Rightarrow$  asymptotic **superadditivity**

# Example

$(X_1, X_2)$  **meta- $t$**  distributed with

- $t_{0.9}$  margins (**infinite “mean”!**),
- $t_\nu$  copula.

## Model 1:

$t_{0.9}$  copula  $\Rightarrow (X_1, X_2)$  **elliptically** distributed  
 $\Rightarrow$  **subadditivity** (well-known fact)

## Model 2:

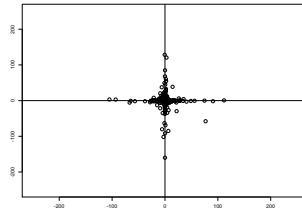
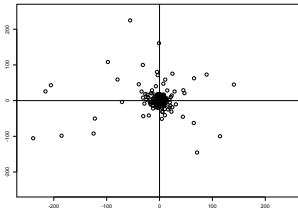
$t_\infty$  copula = Gauss copula  $\Rightarrow (X_1, X_2)$  asymptotically **independent**  
 $\Rightarrow$  **superadditivity** (well-known fact)

$t_{0.9}$  margins with:

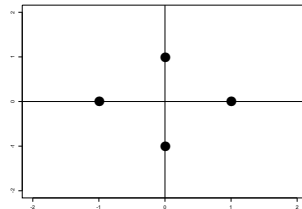
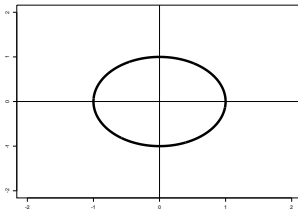
$t_{0.9}$  copula

$t_{\infty}$ /Gauss copula

simulated  
sample  
cloud



spectral  
measure



elliptical = subadditive

independent = superadditive

# Intuition

Asymptotic subadditivity for infinite mean models

if and only if

(very) high values of one risk are **sufficiently compensated**  
by (very) low values of the other risk.

**Remark:** This intuition is indeed correct and can be made mathematically precise.

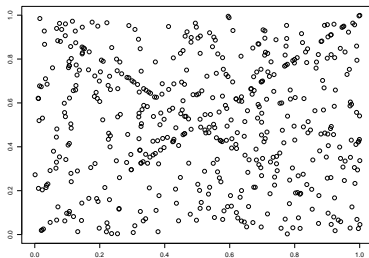
⇒ **Look at upper left and lower right corner!**

# Tail dependence and subadditivity

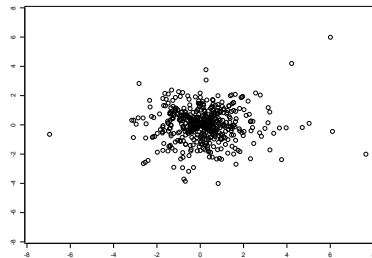
## Fréchet family

$$C_{p_1, p_2}(u, v) = p_1(u \wedge v) + p_2(u + v - 1)^+ + (1 - p_1 - p_2)uv.$$

Copula:



$t_6$ -marginals:



## Precise formulation (for two-sided risk, e.g., market risk)

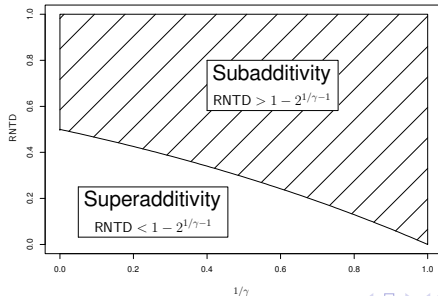
$$\lambda^+ = P(\text{risk 2 high} | \text{risk 1 high})$$

$$\lambda^- = P(\text{risk 2 low} | \text{risk 1 high})$$

$\text{RNTD} = \lambda^- / (1 - \lambda^+)$ : relative negative tail dependence (RNTD)

$\xi$ : heavy-tailedness (i.e. the bigger  $\xi$ , the heavier the tails)

**Theorem (ELW, 2009)** Under certain conditions (Fréchet family) the following holds asymptotically:





# Elliptical vs Archimedean copulas

## Negative tail dependence

- Elliptical distributions (under MRV) **always** have negative tail dependence.
- (Strict) Archimedean copulas **never** have negative tail dependence.

## Consequences

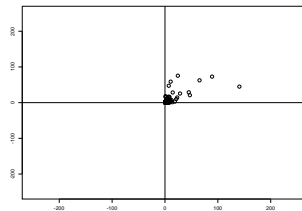
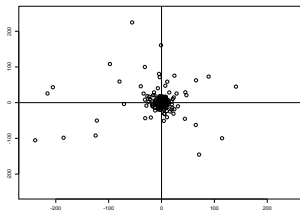
- for spectral measure
- for VaR properties

$t_{0.9}$  margins with:

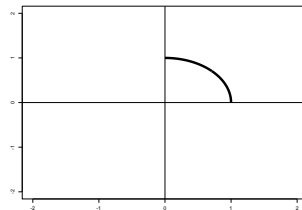
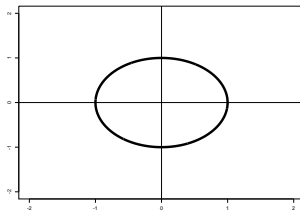
$t_{0.9}$  copula

only positive values

simulated  
sample  
cloud



spectral  
measure



elliptical = subadditive

NOT subadditive!

## Proposition (ELW, 2009)

Let  $\mathbf{X} = (X_1, X_2)$  be elliptically distributed. Then (under MRV):

- a) VaR asymptotically subadditive for  $\mathbf{X}$  in **finite and infinite** mean models.
- b) VaR asymptotically subadditive for  $\mathbf{X} | \mathbf{X} > \mathbf{0}$  **only** in **finite** mean models.

## Multivariate regular variation

**Definition** A random vector  $\mathbf{X} = (X_1, \dots, X_n)$  is **multivariate regularly varying** with index  $-1/\xi < 0$ , if there exists a probability measure  $\mu$ , a measurable function  $b : (0, \infty) \rightarrow (0, \infty)$  with  $\lim_{t \rightarrow \infty} b(t) = \infty$  and a scalar  $q = q(b) > 0$  such that for all  $r > 0$ ,

$$\lim_{t \rightarrow \infty} t P \left( \|\mathbf{X}\| > rb(t), \frac{\mathbf{X}}{\|\mathbf{X}\|} \in G \right) = qr^{-1/\xi} \mu(G), \quad (1)$$

for any Borel set  $G \subset \mathbb{N}_{\|\cdot\|}^{n-1} = \{\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n \mid \|\mathbf{x}\| = 1\}$ . We write  $\mathbf{X} \in \text{MRV}_n(-1/\xi)$ .

## First observations

- For  $\mathbf{X} \in \text{MRV}_n(-1/\xi)$  with identically distributed marginals and  $\Psi$  measurable

$$q(\xi, \Psi) := \lim_{x \rightarrow \infty} \frac{P(\Psi(\mathbf{X}) > x)}{P(X_1 > x)} = \lim_{\alpha \nearrow 1} \left( \frac{\text{VaR}_\alpha(\Psi(\mathbf{X}))}{\text{VaR}_\alpha(X_1)} \right)^{1/\xi}.$$

- If  $\Psi(\mathbf{X}) = \|\mathbf{X}\|$ , then  $q(\xi, \Psi)$  corresponds to the  $q$  in (1).
- (One) relevant case:  $\Psi(\mathbf{X}) = \sum_{i=1}^n X_i$ .
- Problem:  $\sum_{i=1}^n X_i$  is only a norm if  $\mathbf{X} \geq \mathbf{0}$ .

## Exploiting spectral measures (for positive rvs)

**Theorem (Resnick)**  $\mathbf{X} \in \text{MRV}_n(-1/\xi) \Leftrightarrow$

$\exists$  Radon measure  $\nu_\xi$ :  $\lim_{t \rightarrow \infty} t P(\mathbf{X}/b(t) \in B) = \nu_\xi(B)$ , for all  $B \subset [0, \infty]^n \setminus \{\mathbf{0}\}$  relatively compact with  $\nu_\xi(\partial B) = 0$ .

**Spectral measure:**  $S_{\|\cdot\|} = \nu_1 \{ \mathbf{z} \in [0, \infty]^n \mid \|\mathbf{z}\| > 1, \mathbf{z}/\|\mathbf{z}\| \in \cdot \}$ .

**Theorem (Barbe, Fougères, Genest)**  $\mathbf{X} \in \text{MRV}_n(-1/\xi)$ ,  $\mathbb{R}_+^n$ -valued with identically distributed marginals, then

$$q(\xi, \|\cdot\|) = \lim_{x \rightarrow \infty} \frac{P(\|\mathbf{X}\| > x)}{P(X_1 > x)} = \int_{\mathbb{N}_{+, \|\cdot\|}^{n-1}} \|\mathbf{z}^\xi\|^{1/\xi} S_{\|\cdot\|}(d\mathbf{z}),$$

where  $\mathbb{N}_{+, \|\cdot\|}^{n-1} = \mathbb{N}_{\|\cdot\|}^{n-1} \cap \mathbb{R}_+^n$ .

## Direct consequences

**Corollary** Let  $\Psi = \|\cdot\|_1$  be the  $l_1$ -norm in  $\mathbb{R}_+^n$ . Then:

$$\begin{aligned} n \leq q(\xi, \|\cdot\|_1) &\leq n^{1/\xi} && \text{for } \xi \leq 1, \\ n \geq q(\xi, \|\cdot\|_1) &\geq n^{1/\xi} && \text{for } \xi \geq 1. \end{aligned}$$

**Proof:** (Inverse) Triangular inequality for  $L_{1/\xi}$ -“norm”.

**Risk management interpretation:**  $\mathbf{X} \in \text{MRV}_n(-1/\xi)$ ,  $\mathbb{R}_+^n$ -valued with identically distributed marginals, then  $\text{VaR}_\alpha$  is **asymptotically subadditive** for  $\mathbf{X}$  if and only if  $\xi \leq 1$ .

## Theorem for positive rvs (e.g., Operational Risk) (ELW, 2009)

Under MRV:

a)  $X_1, \dots, X_n \geq 0$  with **finite mean** ( $\xi \leq 1$ ). Then:

$$n^{\xi-1} \leq \lim_{\alpha \nearrow 1} \frac{\text{VaR}_\alpha(\sum_{i=1}^n X_i)}{\sum_{i=1}^n \text{VaR}_\alpha(X_i)} \leq 1.$$

b)  $X_1, \dots, X_n \geq 0$  with **infinite mean** ( $\xi \geq 1$ ). Then:

$$1 \leq \lim_{\alpha \nearrow 1} \frac{\text{VaR}_\alpha(\sum_{i=1}^n X_i)}{\sum_{i=1}^n \text{VaR}_\alpha(X_i)} \leq n^{\xi-1}.$$



## Example:

(Asymptotic) **best/worst case diversification benefit:**

$$1 - \lim_{\alpha \nearrow 1} \frac{\text{VaR}_\alpha(\sum_{i=1}^n X_i)}{\sum_{i=1}^n \text{VaR}_\alpha(X_i)}$$

	$n = 2$	$n = 7$	$n = 8$	$n = 100$
$\xi = 0.1$	46%/0%	83%/0%	85%/0%	98%/0%
$\xi = 0.5$	29%/0%	62%/0%	65%/0%	90%/0%
$\xi = 0.9$	7%/0%	17%/0%	19%/0%	37%/0%
$\xi = 1$	0%/0%	0%/0%	0%/0%	0%/0%
$\xi = 1.1$	0%/ - 7%	0%/ - 21%	0%/ - 23%	0%/ - 58%

## Immediate Corollary (ELW, 2009)

Under MRV:

- a)  $X_1, \dots, X_n \geq 0$  with finite mean  
 $\implies \text{VaR}_\alpha$  is asymptotically subadditive.
  
- b)  $X_1, \dots, X_n \geq 0$  with infinite mean  
 $\implies \text{VaR}_\alpha$  is asymptotically superadditive.

Does this corollary **fully** solve the subadditivity problem for **positive risks**?

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# NO!

It is **only an asymptotic statement**, i.e., for  $\alpha \rightarrow 1$ .

**And:** asymptotically (for positive risks) only the finite/**infinite mean issue matters** (as seen before).

**But:** **subadditivity** of VaR also typically **fails** (although asymptotic subadditivity may hold!) for

- **very skew** distributions
- very **special dependence** (usually not a problem in practice)

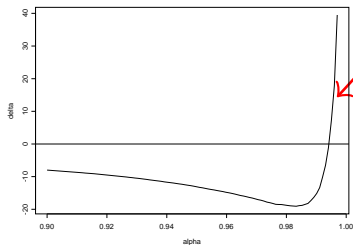
## Counter-Example (Degen, E., L., 2007)

Suppose  $X_1, X_2 \stackrel{\text{iid}}{\sim} g\text{-and-}h$  ( $\stackrel{d}{=} \frac{e^{gZ}-1}{g} e^{hZ^2/2}$ ,  $Z \sim \mathcal{N}(0, 1)$ ), with skewness parameter  $g = 2.4$ , and heavy-tailedness parameter  $h = 0.2$  (typical for Operational Risk).

(Absolute) diversification benefit:

$$\text{delta} = \text{VaR}_\alpha(X_1) + \text{VaR}_\alpha(X_2) - \text{VaR}_\alpha(X_1 + X_2)$$

99.4%

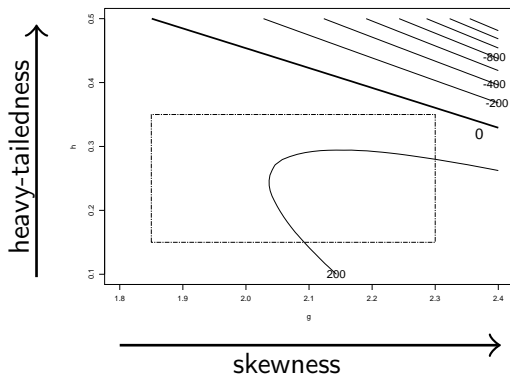


Finite mean:

indeed asymptotically subadditive.

But! **VaR superadditive for  $\alpha < 0.994$ .**

## Diversification benefit for 99.9%-VaR

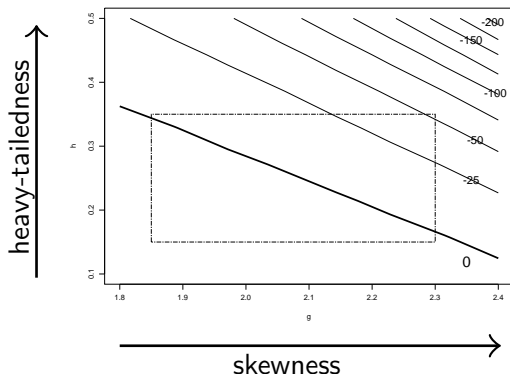


- Typical OpRisk parameters **within subadditivity** range
- Small changes of parameters  $\Rightarrow$  **superadditivity**

What happens when we go **deeper** in the data?

- VaR-estimation at 99.9% and higher: difficult!
- Estimate at lower level (99%, say) and scale.

# Diversification benefit for 99%-VaR







- Substantial fraction of parameter rectangle switched regime
- Far from diversification!



## Main messages to the OpRisk community

- Questions like “*Do we have diversification?*” or “*How big is the diversification benefit?*” is **always** related to the VaR-level  $\alpha$ .
- We may have a high diversification benefit on a 99.9% level and a small diversification benefit on a 99% level.
- Look at upper left and lower right corner!
- **Asymptotically** (for positive risks) only the heavy-tailedness matters (finite/infinite mean).
- In the real (and finite!) world also skewness has to be considered.

## References

-  Embrechts, P., Lambrigger, D. D. and Wüthrich, M. V. (2009) Multivariate extremes and the aggregation of dependent risks: examples and counter-examples. *Extremes*, to appear.
-  Degen, M., Embrechts, P., Lambrigger, D. D. (2007) The quantitative modeling of operational risk: between g-and-h and EVT. *ASTIN Bulletin* **37**(2), 265–291.
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-  Barbe, P., Fougères, A. and Genest, C. (2006) On the tail behavior of sums of dependent risks. *Astin Bulletin* **36**, 361–373.