Modelling Dependence with Copulas
and Applications to Risk Management

Paul Embrechts, Filip Lindskog* and Alexander McNeil**

Department of Mathematics
ETHZ
CH-8092 Zürich
Switzerland
www.math.ethz.ch/finance

September 10, 2001

*Research of the second author was supported by Credit Suisse Group, Swiss Re and UBS AG through RiskLab, Switzerland. **The third author acknowledges financial support from Swiss Re.

Contents

1 Introduction

Integrated Risk Management (IRM) is concerned with the quantitative description of risks to a financial business. Whereas the qualitative aspects of IRM are extremely important, in the present contribution we only concentrate on the quantitative ones. Since the emergence of Value-at-Risk (VaR) in the early nineties and its various generalisations and refinements more recently, regulators and banking and insurance professionals have build up a huge system aimed at making the global financial system safer. Whereas the steps taken no doubt have been very important towards increasing the overall risk awareness, continuously questions have been asked concerning the quality of the safeguards as constructed.

All quantitative models are based on assumptions vis-a-vis the markets on which they are to be applied. Standard hedging techniques require a high level of liquidity of the underlying instruments, prices quoted for many financial products are often based on "normal" conditions. The latter may be interpreted in a more economic sense, or more specifically referring to the distributional (i.e. normal, Gaussian) behaviour of some underlying data. Especially for IRM, deviations from the "normal" would constitute a prime source of investigation. Hence the classical literature is full of deviations from the so-called random walk (Brownian motion) model and heavy tails appear prominently. The latter has for instance resulted in the firm establishment of Extreme Value Theory (EVT) as a standard tool within IRM. Within market risk management, the so-called stylised facts of econometrics summarise this situation: market data returns tend to be uncorrelated, but dependent, they are heavy tailed, extremes appear in clusters and volatility is random.

Our contribution aims at providing tools for going one step further: what would be the stylised facts of dependence in financial data? Is there a way of understanding so-called normal (i.e. Gaussian) dependence and how can we construct models which allow to go beyond normal dependence? Other problems we would like to understand better are spillover, the behaviour of correlations under extreme market movements, the pros and contras of linear correlation as a measure of dependence, the construction of risk measures for functions of dependent risks. One example concerning the latter is the following: suppose we have two VaR numbers corresponding to two different lines of business. In order to cover the joint position, can we just add the VaR? Under which conditions is this always the upper bound? What can go wrong if these conditions are not fulfilled? A further type of risk where dependence play a crucial role is credit risk: how to define, stress test and model default correlation. The present paper is not solving the above problem, it presents however tools which are crucial towards the construction of solutions.

The notion we concentrate on is that of copula, well known for some time within the statistics literature. The word copula first appeared in the statistics literature 1959, Sklar (1959), although similar ideas and results can be traced back to Hoeffding (1940). Copulas allow us to construct models which go beyond the standard ones at the level of dependence. They yield an ideal tool to stress test a wide variety of portfolios and products in insurance and finance for extreme moves in correlation and more general measures of dependence. As such, they gradually are becoming an extra, but crucial, element of best practice IRM. After Section 2 in which we define the concept of copula in full generality, we turn in Section 3 to an overview of the most important notions of dependence used in IRM. Section 4, 5 and 6 introduces the most important families of copulas, their properties both methodological as well as with respect to simulation. Throughout these sections, we stress the importance of the techniques introduced within an IRM framework. Finally in Section 7 we discuss some specific examples.

We would like to stress that the present paper only gives a first introduction aimed at bringing together from the extensive copula world those results which are immediately usable in IRM. Topics not included are statistical estimation of copulas and the modelling of dependence, through copulas, in a dynamic environment. As such, the topics listed correspond to a one-period point of view. Various extensions are possible; the interested reader is referred to the bibliography for further reading.

2 Copulas

The standard "operational" definition of a copula is a multivariate distribution function defined on the unit cube $[0,1]^n$, with uniformly distributed marginals. This definition is very natural if one considers how a copula is derived from a continuous multivariate distribution function; indeed in this case the copula is simply the original multivariate distribution function with transformed univariate margins. This definition however masks some of the problems one faces when constructing copulas using other techniques, i.e. it does not say what is meant by a multivariate distribution function. For that reason, we start with a slightly more abstract definition, returning to the "operational" one later. Below, we follow Nelsen (1999) in concentrating on general multivariate distributions at first and then studying the special properties of the copula subset.

Throughout this paper, for a function H, we denote by $\operatorname{Dom} H$ and $\operatorname{Ran} H$ the domain and range respectively of H. Furthermore, a function f will be called increasing whenever $x \le y$ implies that $f(x) \le f(y)$. We may also refer to this as f is nondecreasing. A statement about points of a set $S \subset \mathbb{R}^n$, where S is typically the real line or the unit cube $[0,1]^n$, is said to hold almost everywhere if the set of points of S where the statement fails to hold has Lebesgue measure zero.

2.1 Mathematical Introduction

Definition 2.1. Let $S_1, \ldots, S_n$ be nonempty subsets of $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}}$ denotes the extended real line $[-\infty, \infty]$. Let H be a real function of n variables such that $\operatorname{Dom} H = S_1 \times \cdots \times S_n$ and for $\mathbf{a} \le \mathbf{b}$ ($a_k \le b_k$ for all k) let $B = [\mathbf{a}, \mathbf{b}]$ $(= [a_1, b_1] \times \cdots \times [a_n, b_n])$ be an n-box whose vertices are in $\operatorname{Dom} H$. Then the H-volume of B is given by

$$V_H(B) = \sum \operatorname{sgn}(\mathbf{c}) H(\mathbf{c}),$$

where the sum is taken over all vertices $\mathbf{c}$ of B, and $\operatorname{sgn}(\mathbf{c})$ is given by

$$\operatorname{sgn}(\mathbf{c}) = \begin{cases} 1, & \text{if } c_k = a_k \text{ for an even number of } k\text{'s}, \\ -1, & \text{if } c_k = a_k \text{ for an odd number of } k\text{'s}. \end{cases}$$

$\square$

Equivalently, the H-volume of an n-box $B = [\mathbf{a}, \mathbf{b}]$ is the nth order difference of H on B

$$V_H(B) = \triangle_{\mathbf{a}}^{\mathbf{b}} H(\mathbf{t}) = \triangle_{a_n}^{b_n} \ldots \triangle_{a_1}^{b_1} H(\mathbf{t}),$$

where the n first order differences are defined as

$$\triangle_{a_k}^{b_k} H(\mathbf{t}) = H(t_1, \ldots, t_{k-1}, b_k, t_{k+1}, \ldots, t_n) - H(t_1, \ldots, t_{k-1}, a_k, t_{k+1}, \ldots, t_n).$$

Definition 2.2. A real function H of n variables is n-increasing if $V_H(B) \geq 0$ for all n-boxes B whose vertices lie in $\operatorname{Dom} H$. □

Suppose that the domain of a real function H of n variables is given by $\operatorname{Dom} H = S_1 \times \cdots \times S_n$ where each S_k has a smallest element a_k. We say that H is grounded if $H(\mathbf{t}) = 0$ for all $\mathbf{t}$ in $\operatorname{Dom} H$ such that $t_k = a_k$ for at least one k. If each S_k is nonempty and has a greatest element b_k, then H has margins, and the one-dimensional margins of H are the functions H_k with $\operatorname{Dom} H_k = S_k$ and with $H_k(x) = H(b_1, \dots, b_{k-1}, x, b_{k+1}, \dots, b_n)$ for all x in S_k. Higher-dimensional margins are defined in an obvious way. One-dimensional margins are just called margins.

Lemma 2.1. *Let $S_1, \dots, S_n$ be nonempty subsets of $\overline{\mathbb{R}}$, and let H be a grounded n-increasing function with domain $S_1 \times \cdots \times S_n$. Then H is increasing in each argument, i.e., if $(t_1, \dots, t_{k-1}, x, t_{k+1}, \dots, t_n)$ and $(t_1, \dots, t_{k-1}, y, t_{k+1}, \dots, t_n)$ are in $\operatorname{Dom} H$ and $x \leq y$, then $H(t_1, \dots, t_{k-1}, x, t_{k+1}, \dots, t_n) \leq H(t_1, \dots, t_{k-1}, y, t_{k+1}, \dots, t_n)$.*

Lemma 2.2. *Let $S_1, \dots, S_n$ be nonempty subsets of $\overline{\mathbb{R}}$, and let H be a grounded n-increasing function with margins and domain $S_1 \times \cdots \times S_n$. Then, if $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ are any points in $S_1 \times \cdots \times S_n$,*

$$\big|H(\mathbf{x}) - H(\mathbf{y})\big| \leq \sum_{k=1}^{n} \big|H_k(x_k) - H_k(y_k)\big|.$$

For the proof, see Schweizer and Sklar (1983).

Definition 2.3. An n-dimensional distribution function is a function H with domain $\overline{\mathbb{R}}^n$ such that H is grounded, n-increasing and $H(\infty, \dots, \infty) = 1$. □

It follows from Lemma 2.1 that the margins of an n-dimensional distribution function are distribution functions, which we denote $F_1, \dots, F_n$.

Definition 2.4. An n-dimensional copula is a function C with domain $[0,1]^n$ such that

1. C is grounded and n-increasing.
2. C has margins C_k, $k = 1, 2, \dots, n$, which satisfy $C_k(u) = u$ for all u in $[0,1]$.

□

Note that for any n-copula C, $n \geq 3$, each k-dimensional margin of C is a k-copula. Equivalently, an n-copula is a function C from $[0,1]^n$ to $[0,1]$ with the following properties:

1. For every $\mathbf{u}$ in $[0,1]^n$, $C(\mathbf{u}) = 0$ if at least one coordinate of $\mathbf{u}$ is 0, and $C(\mathbf{u}) = u_k$ if all coordinates of $\mathbf{u}$ equal 1 except u_k.
2. For every $\mathbf{a}$ and $\mathbf{b}$ in $[0,1]^n$ such that $a_i \leq b_i$ for all i, $V_C([\mathbf{a}, \mathbf{b}]) \geq 0$.

Since copulas are joint distribution functions (on $[0,1]^n$), a copula C induces a probability measure on $[0,1]^n$ via

$$V_C([0, u_1] \times \cdots \times [0, u_n]) = C(u_1, \dots, u_n)$$

and a standard extension to arbitrary (not necessarily n-boxes) Borel subsets of $[0,1]^n$. A standard result from measure theory says that there is a unique probability measure on the Borel subsets of $[0,1]^n$ which coincides with V_C on the set of n-boxes of $[0,1]^n$. This probability measure will also be denoted V_C.

From Definition 2.4 it follows that a copula C is a distribution function on $[0,1]^n$ with uniformly distributed (on $[0,1]$) margins. The following theorem follows directly from Lemma 2.2.

Theorem 2.1. *Let C be an n-copula. Then for every $\mathbf{u}$ and $\mathbf{v}$ in $[0,1]^n$,*

$$\big|C(\mathbf{v}) - C(\mathbf{u})\big| \leq \sum_{k=1}^{n} \big|v_k - u_k\big|.$$

Hence C is uniformly continuous on $[0,1]^n$.

2.2 Sklar's Theorem

The following theorem is known as Sklar's Theorem. It is perhaps the most important result regarding copulas, and is used in essentially all applications of copulas.

Theorem 2.2. *Let H be an n-dimensional distribution function with margins $F_1, \dots, F_n$. Then there exists an n-copula C such that for all $\mathbf{x}$ in $\overline{\mathbb{R}}^n$,*

$$H(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)). \tag{2.1}$$

If $F_1, \dots, F_n$ are all continuous, then C is unique; otherwise C is uniquely determined on $\operatorname{Ran} F_1 \times \dots \times \operatorname{Ran} F_n$. Conversely, if C is an n-copula and $F_1, \dots, F_n$ are distribution functions, then the function H defined above is an n-dimensional distribution function with margins $F_1, \dots, F_n$.

For the proof, see Sklar (1996).

From Sklar's Theorem we see that for continuous multivariate distribution functions, the univariate margins and the multivariate dependence structure can be separated, and the dependence structure can be represented by a copula.

Let F be a univariate distribution function. We define the generalized inverse of F as $F^{-1}(t) = \inf\{x \in \mathbb{R} \mid F(x) \geq t\}$ for all t in $[0,1]$, using the convention $\inf \emptyset = -\infty$.

Corollary 2.1. *Let H be an n-dimensional distribution function with continuous margins $F_1, \dots, F_n$ and copula C (where C satisfies (2.1)). Then for any $\mathbf{u}$ in $[0,1]^n$,*

$$C(u_1, \dots, u_n) = H(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)).$$

Without the continuity assumption, care has to be taken; see Nelsen (1999) or Marshall (1996).

Example 2.1. Let Φ denote the standard univariate normal distribution function and let Φ_R^n denote the standard multivariate normal distribution function with linear correlation matrix R. Then

$$C(u_1, \dots, u_n) = \Phi_R^n(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n))$$

is the Gaussian or normal n-copula. □

2.3 The Fréchet–Hoeffding Bounds for Joint Distribution Functions

Consider the functions M^n, Π^n and W^n defined on $[0,1]^n$ as follows:

$$\begin{aligned} M^n(\mathbf{u}) &= \min(u_1, \dots, u_n), \\ \Pi^n(\mathbf{u}) &= u_1 \dots u_n, \\ W^n(\mathbf{u}) &= \max(u_1 + \dots + u_n - n + 1, 0). \end{aligned}$$

The functions M^n and Π^n are n-copulas for all $n \geq 2$ whereas the function W^n is not a copula for any $n \geq 3$ as shown in the following example.

Example 2.2. Consider the n-cube $[1/2,1]^n \subset [0,1]^n$.

$$\begin{aligned}
V_{W^n}([1/2,1]^n) &= \max(1+\cdots+1-n+1,0) \\
&\quad -n\max(1/2+1+\cdots+1-n+1,0) \\
&\quad +\binom{n}{2}\max(1/2+1/2+1+\cdots+1-n+1,0) \\
&\quad \cdots \\
&\quad +\max(1/2+\cdots+1/2-n+1,0) \\
&= 1-n/2+0+\cdots+0.
\end{aligned}$$

Hence W^n is not a copula for $n \geq 3$. □

The following theorem is called the Fréchet–Hoeffding bounds inequality (Fréchet (1957)).

Theorem 2.3. *If C is any n-copula, then for every $\mathbf{u}$ in $[0,1]^n$,*

$$W^n(\mathbf{u}) \leq C(\mathbf{u}) \leq M^n(\mathbf{u}).$$

For more details, including geometrical interpretations, see Mikusinski, Sherwood, and Taylor (1992). Although the Fréchet–Hoeffding lower bound W^n is never a copula for $n \geq 3$, it is the best possible lower bound in the following sense.

Theorem 2.4. *For any $n \geq 3$ and any $\mathbf{u}$ in $[0,1]^n$, there is an n-copula C (which depends on $\mathbf{u}$) such that*

$$C(\mathbf{u}) = W^n(\mathbf{u}).$$

For the proof, see Nelsen (1999) p. 42.

We denote by $\overline{C}$ the joint survival function for n random variables with joint distribution function C, i.e., if $(U_1,\ldots,U_n)^T$ has distribution function C, then $\overline{C}(u_1,\ldots,u_n) = \mathbb{P}\{U_1 > u_1,\ldots,U_n > u_n\}$.

Definition 2.5. If C_1 and C_2 are copulas, C_1 is smaller than C_2 (written $C_1 \prec C_2$) if

$$C_1(\mathbf{u}) \leq C_2(\mathbf{u}) \quad \text{and} \quad \overline{C}_1(\mathbf{u}) \leq \overline{C}_2(\mathbf{u}),$$

for all $\mathbf{u}$ in $[0,1]^n$. □

Note that in the bivariate case,

$$\begin{aligned}
\overline{C}_1(u_1,u_2) \leq \overline{C}_2(u_1,u_2) &\Leftrightarrow 1-u_1-u_2+C_1(u_1,u_2) \leq 1-u_1-u_2+C_2(u_1,u_2) \\
&\Leftrightarrow C_1(u_1,u_2) \leq C_2(u_1,u_2).
\end{aligned}$$

The Fréchet–Hoeffding lower bound W^2 is smaller than every 2-copula, and every n-copula is smaller than the Fréchet–Hoeffding upper bound M^n. This partial ordering of the set of copulas is called a concordance ordering. It is a partial ordering since not every pair of copulas is comparable in this order. However many important parametric families of copulas are totally ordered. We call a one-parameter family $\{C_\theta\}$ positively ordered if $C_{\theta_1} \prec C_{\theta_2}$ whenever $\theta_1 \leq \theta_2$. Examples of such one-parameter families will be given later.

2.4 Copulas and Random Variables

Let $X_1, \ldots, X_n$ be random variables with continuous distribution functions $F_1, \ldots, F_n$, respectively, and joint distribution function H. Then $(X_1, \ldots, X_n)^T$ has a unique copula C, where C is given by (2.1). The standard copula representation of the distribution of the random vector $(X_1, \ldots, X_n)^T$ then becomes:

$$H(x_1, \ldots, x_n) = \mathbb{P}\{X_1 \leq x_1, \ldots, X_n \leq x_n\} = C(F_1(x_1), \ldots, F_n(x_n)).$$

The transformations $X_i \mapsto F_i(X_i)$ used in the above representation are usually referred to as the probability-integral transformations (to uniformity) and form a standard tool in simulation methodology.

Since $X_1, \ldots, X_n$ are independent if and only if $H(x_1, \ldots, x_n) = F_1(x_1) \ldots F_n(x_n)$ for all $x_1, \ldots, x_n$ in $\overline{\mathbb{R}}$, the following result follows from Theorem 2.2.

Theorem 2.5. *Let $(X_1, \ldots, X_n)^T$ be a vector of continuous random variables with copula C, then $X_1, \ldots, X_n$ are independent if and only if $C = \Pi^n$.*

One nice property of copulas is that for strictly monotone transformations of the random variables, copulas are either invariant, or change in certain simple ways. Note that if the distribution function of a random variable X is continuous, and if α is a strictly monotone function whose domain contains $\operatorname{Ran} X$, then the distribution function of the random variable $\alpha(X)$ is also continuous.

Theorem 2.6. *Let $(X_1, \ldots, X_n)^T$ be a vector of continuous random variables with copula C. If $\alpha_1, \ldots, \alpha_n$ are strictly increasing on $\operatorname{Ran} X_1, \ldots, \operatorname{Ran} X_n$, respectively, then also $(\alpha_1(X_1), \ldots, \alpha_n(X_n))^T$ has copula C.*

Proof. Let $F_1, \ldots, F_n$ denote the distribution functions of $X_1, \ldots, X_n$ and let $G_1, \ldots, G_n$ denote the distribution functions of $\alpha_1(X_1), \ldots, \alpha_n(X_n)$, respectively. Let $(X_1, \ldots, X_n)^T$ have copula C, and let $(\alpha_1(X_1), \ldots, \alpha_n(X_n))^T$ have copula C_α. Since α_k is strictly increasing for each k, $G_k(x) = \mathbb{P}\{\alpha_k(X_k) \leq x\} = \mathbb{P}\{X_k \leq \alpha_k^{-1}(x)\} = F_k(\alpha_k^{-1}(x))$ for any x in $\overline{\mathbb{R}}$, hence

$$\begin{aligned}
C_\alpha(G_1(x_1), \ldots, G_n(x_n)) &= \mathbb{P}\{\alpha_1(X_1) \leq x_1, \ldots, \alpha_n(X_n) \leq x_n\} \\
&= \mathbb{P}\{X_1 \leq \alpha_1^{-1}(x_1), \ldots, X_n \leq \alpha_n^{-1}(x_n)\} \\
&= C(F_1(\alpha_1^{-1}(x_1)), \ldots, F_n(\alpha_n^{-1}(x_n))) \\
&= C(G_1(x_1), \ldots, G_n(x_n)).
\end{aligned}$$

Since $X_1, \ldots, X_n$ are continuous, $\operatorname{Ran} G_1 = \cdots = \operatorname{Ran} G_n = [0, 1]$. Hence it follows that $C_\alpha = C$ on $[0, 1]^n$. □

From Theorem 2.2 we know that the copula function C "separates" an n-dimensional distribution function from its univariate margins. The next theorem will show that there is also a function, $\widehat{C}$, that separates an n-dimensional survival function from its univariate survival margins. Furthermore this function can be shown to be a copula, and this survival copula can rather easily be expressed in terms of C and its k-dimensional margins.

Theorem 2.7. *Let $(X_1, \ldots, X_n)^T$ be a vector of continuous random variables with copula $C_{X_1, \ldots, X_n}$. Let $\alpha_1, \ldots, \alpha_n$ be strictly monotone on $\operatorname{Ran} X_1, \ldots, \operatorname{Ran} X_n$, respectively, and* *let $(\alpha_1(X_1), \dots, \alpha_n(X_n))^T$ have copula $C_{\alpha_1(X_1),\dots,\alpha_n(X_n)}$. Furthermore let α_k be strictly decreasing for some k. Without loss of generality let $k = 1$. Then*

$$\begin{aligned} C_{\alpha_1(X_1),\dots,\alpha_n(X_n)}(u_1, u_2, \dots, u_n) &= C_{\alpha_2(X_2),\dots,\alpha_n(X_n)}(u_2, \dots, u_n) \\ &\quad - C_{X_1,\alpha_2(X_2),\dots,\alpha_n(X_n)}(1 - u_1, u_2, \dots, u_n). \end{aligned}$$

Proof. Let $X_1, \dots, X_n$ have distribution functions $F_1, \dots, F_n$ and let $\alpha_1(X_1), \dots, \alpha_n(X_n)$ have distribution functions $G_1, \dots, G_n$. Then

$$\begin{aligned} &C_{\alpha_1(X_1),\alpha_2(X_2),\dots,\alpha_n(X_n)}(G_1(x_1), \dots, G_n(x_n)) \\ &\quad = \mathbb{P}\{\alpha_1(X_1) \le x_1, \dots, \alpha_n(X_n) \le x_n\} \\ &\quad = \mathbb{P}\{X_1 > \alpha_1^{-1}(x_1), \alpha_2(X_2) \le x_2, \dots, \alpha_n(X_n) \le x_n\} \\ &\quad = \mathbb{P}\{\alpha_2(X_2) \le x_2, \dots, \alpha_n(X_n) \le x_n\} \\ &\qquad - \mathbb{P}\{X_1 \le \alpha_1^{-1}(x_1), \alpha_2(X_2) \le x_2, \dots, \alpha_n(X_n) \le x_n\} \\ &\quad = C_{\alpha_2(X_2),\dots,\alpha_n(X_n)}(G_2(x_2), \dots, G_n(x_n)) \\ &\qquad - C_{X_1,\alpha_2(X_2),\dots,\alpha_n(X_n)}(F_1(\alpha_1^{-1}(x_1)), G_2(x_2), \dots, G_n(x_n)) \\ &\quad = C_{\alpha_2(X_2),\dots,\alpha_n(X_n)}(G_2(x_2), \dots, G_n(x_n)) \\ &\qquad - C_{X_1,\alpha_2(X_2),\dots,\alpha_n(X_n)}(1 - G_1(x_1), G_2(x_2), \dots, G_n(x_n)), \end{aligned}$$

from which the conclusion follows directly. □

By using the two theorems above recursively it is clear that the copula $C_{\alpha_1(X_1),\dots,\alpha_n(X_n)}$ can be expressed in terms of the copula $C_{X_1,\dots,X_n}$ and its lower-dimensional margins. This is exemplified below.

Example 2.3. Consider the bivariate case.
Let α_1 be strictly decreasing and let α_2 be strictly increasing. Then

$$\begin{aligned} C_{\alpha_1(X_1),\alpha_2(X_2)}(u_1, u_2) &= u_2 - C_{X_1,\alpha_2(X_2)}(1 - u_1, u_2) \\ &= u_2 - C_{X_1,X_2}(1 - u_1, u_2). \end{aligned}$$

Let α_1 and α_2 be strictly decreasing. Then

$$\begin{aligned} C_{\alpha_1(X_1),\alpha_2(X_2)}(u_1, u_2) &= u_2 - C_{X_1,\alpha_2(X_2)}(1 - u_1, u_2) \\ &= u_2 - \big(1 - u_1 - C_{X_1,X_2}(1 - u_1, 1 - u_2)\big) \\ &= u_1 + u_2 - 1 + C_{X_1,X_2}(1 - u_1, 1 - u_2). \end{aligned}$$

Here $C_{\alpha_1(X_1),\alpha_2(X_2)}$ is the survival copula, $\widehat{C}$, of $(X_1, X_2)^T$, i.e.,

$$\overline{H}(x_1, x_2) = \mathbb{P}\{X_1 > x_1, X_2 > x_2\} = \widehat{C}(\overline{F}_1(x_1), \overline{F}_2(x_2)).$$

□

Note also that the joint survival function of n $U(0, 1)$ random variables whose joint distribution function is the copula C is $\overline{C}(u_1, \dots, u_n) = \widehat{C}(1 - u_1, \dots, 1 - u_n)$.

The mixed kth order partial derivatives of a copula C, $\partial^k C(\mathbf{u})/\partial u_1 \dots \partial u_k$, exist for almost all $\mathbf{u}$ in $[0, 1]^n$. For such $\mathbf{u}$, $0 \le \partial^k C(\mathbf{u})/\partial u_1 \dots \partial u_k \le 1$. For details, see Nelsen (1999) p. 11. With this in mind, let

$$C(u_1, \dots, u_n) = A_C(u_1, \dots, u_n) + S_C(u_1, \dots, u_n),$$

where

$$\begin{aligned} A_C(u_1,\dots,u_n) &= \int_0^{u_1}\dots\int_0^{u_n} \frac{\partial^n}{\partial s_1\dots\partial s_n} C(s_1,\dots,s_n)\,\mathrm{d}s_1\dots\,\mathrm{d}s_n, \\ S_C(u_1,\dots,u_n) &= C(u_1,\dots,u_n) - A_C(u_1,\dots,u_n). \end{aligned}$$

Unlike multivariate distributions in general, the margins of a copula are continuous, hence a copula has no individual points $\mathbf{u}$ in $[0,1]^n$ for which $V_C(\mathbf{u}) > 0$. If $C = A_C$ on $[0,1]^n$, then C is said to be absolutely continuous. In this case C has density $\frac{\partial^n}{\partial u_1\dots\partial u_n}C(u_1,\dots,u_n)$. If $C = S_C$ on $[0,1]^n$, then C is said to be singular, and $\frac{\partial^n}{\partial u_1\dots\partial u_n}C(u_1,\dots,u_n) = 0$ almost everywhere in $[0,1]^n$. The support of a copula is the complement of the union of all open subsets A of $[0,1]^n$ with $V_C(A) = 0$. When C is singular its support has Lebesgue measure zero and conversely. However a copula can have full support without being absolutely continuous. Examples of such copulas are so-called Marshall-Olkin copulas which are presented later.

Example 2.4. Consider the bivariate Fréchet–Hoeffding upper bound M given by $M(u,v) = \min(u,v)$ on $[0,1]^2$. It follows that $\frac{\partial^2}{\partial u\partial v}M(u,v) = 0$ everywhere on $[0,1]^2$ except on the main diagonal (which has Lebesgue measure zero), and $V_M(B) = 0$ for every rectangle B in $[0,1]^2$ entirely above or below the main diagonal. Hence M is singular. $\square$

One of the main aims of this paper is to present effective algorithms for random variate generation from the various copula families studied. The properties of the specific copula family is often essential for the efficiency of the corresponding algorithm. We now present a general algorithm for random variate generation from copulas. Note however that in most cases it is not an efficient one to use.

Consider the general situation of random variate generation from the n-copula C. Let

$$C_k(u_1,\dots,u_k) = C(u_1,\dots,u_k,1,\dots,1), \quad k = 2,\dots,n-1,$$

denote k-dimensional margins of C, with $C_1(u_1) = u_1$ and $C_n(u_1,\dots,u_n) = C(u_1,\dots,u_n)$. Let $U_1,\dots,U_n$ have joint distribution function C. Then the conditional distribution of U_k given the values of $U_1,\dots,U_{k-1}$, is given by

$$\begin{aligned} C_k(u_k \mid u_1,\dots,u_{k-1}) &= \mathbb{P}\{U_k \le u_k \mid U_1 = u_1,\dots,U_{k-1} = u_{k-1}\} \\ &= \frac{\partial^{k-1}C_k(u_1,\dots,u_k)}{\partial u_1\dots\partial u_{k-1}} \Big/ \frac{\partial^{k-1}C_{k-1}(u_1,\dots,u_{k-1})}{\partial u_1\dots\partial u_{k-1}}, \end{aligned}$$

given that the numerator and denominator exist and that the denominator is not zero. The following algorithm generates a random variate $(u_1,\dots,u_n)^T$ from C. As usual, let $U(0,1)$ denote the uniform distribution on $[0,1]$.

Algorithm 2.1.

- Simulate a random variate u_1 from $U(0,1)$.
- Simulate a random variate u_2 from $C_2(\cdot \mid u_1)$.

$\vdots$

- Simulate a random variate u_n from $C_n(\cdot \mid u_1,\dots,u_{n-1})$.

$\square$

This algorithm is in fact a particular case of what is called "the standard construction". The correctness of the algorithm can be seen from the fact that for independent $U(0,1)$ random variables $Q_1, \dots, Q_n$,

$$(Q_1, C_2^{-1}(Q_2 \mid Q_1), \dots, C_n^{-1}(Q_n \mid Q_1, C_2^{-1}(Q_2 \mid Q_1), \dots))^T$$

has distribution function C. To simulate a value u_k from $C_k(\cdot \mid u_1, \dots, u_{k-1})$ in general means simulating q from $U(0,1)$ from which $u_k = C_k^{-1}(q \mid u_1, \dots, u_{k-1})$ can be obtained through the equation $q = C_k(u_k \mid u_1, \dots, u_{k-1})$ by numerical rootfinding. When $C_k^{-1}(q \mid u_1, \dots, u_{k-1})$ has a closed form (and hence there is no need for numerical rootfinding) this algorithm can be recommended.

Example 2.5. Let the copula C be given by $C(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$, for $\theta > 0$. Then

$$\begin{aligned} C_{2|1}(v \mid u) &= \frac{\partial C}{\partial u}(u,v) = -\frac{1}{\theta}(u^{-\theta} + v^{-\theta} - 1)^{-1/\theta-1}(-\theta u^{-\theta-1}) \\ &= (u^{\theta})^{\frac{-1-\theta}{\theta}}(u^{-\theta} + v^{-\theta} - 1)^{-1/\theta-1} = (1 + u^{\theta}(v^{-\theta} - 1))^{\frac{-1-\theta}{\theta}}. \end{aligned}$$

Solving the equation $q = C_{2|1}(v \mid u)$ for v yields

$$C_{2|1}^{-1}(q \mid u) = v = \left((q^{\frac{-\theta}{1+\theta}} - 1)u^{-\theta} + 1\right)^{-1/\theta}.$$

The following algorithm generates a random variate $(u,v)^T$ from the above copula C.

- Simulate two independent random variates u and q from $U(0,1)$.
- Set $v = ((q^{\frac{-\theta}{1+\theta}} - 1)u^{-\theta} + 1)^{-1/\theta}$.

□

3 Dependence Concepts

Copulas provide a natural way to study and measure dependence between random variables. As a direct consequence of Theorem 2.6, copula properties are invariant under strictly increasing transformations of the underlying random variables. Linear correlation (or Pearson's correlation) is most frequently used in practice as a measure of dependence. However, since linear correlation is not a copula-based measure of dependence, it can often be quite misleading and should not be taken as the canonical dependence measure. Below we recall the basic properties of linear correlation, and then continue with some copula based measures of dependence.

3.1 Linear Correlation

Definition 3.1. Let $(X,Y)^T$ be a vector of random variables with nonzero finite variances. The linear correlation coefficient for $(X,Y)^T$ is

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}, \tag{3.1}$$

where $\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$ is the covariance of $(X,Y)^T$, and $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$ are the variances of X and Y. □

Linear correlation is a measure of linear dependence. In the case of perfect linear dependence, i.e., $Y = aX + b$ almost surely for $a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}$, we have $|\rho(X,Y)| = 1$. More important is that the converse also holds. Otherwise, $-1 < \rho(X,Y) < 1$. Furthermore linear correlation has the property that

$$\rho(\alpha X + \beta, \gamma Y + \delta) = \operatorname{sign}(\alpha\gamma)\rho(X,Y),$$

for $\alpha, \gamma \in \mathbb{R} \setminus \{0\}, \beta, \delta \in \mathbb{R}$. Hence linear correlation is invariant under strictly increasing *linear* transformations. Linear correlation is easily manipulated under linear operations. Let A, B be $m \times n$ matrices; $a, b \in \mathbb{R}^m$ and let $\mathbf{X}, \mathbf{Y}$ be random n-vectors. Then

$$\operatorname{Cov}(A\mathbf{X} + a, B\mathbf{Y} + b) = A\operatorname{Cov}(\mathbf{X}, \mathbf{Y})B^T.$$

From this it follows that for $\alpha \in \mathbb{R}^n$,

$$\operatorname{Var}(\alpha^T \mathbf{X}) = \alpha^T \operatorname{Cov}(\mathbf{X})\alpha,$$

where $\operatorname{Cov}(\mathbf{X}) := \operatorname{Cov}(\mathbf{X}, \mathbf{X})$. Hence the variance of a linear combination is fully determined by pairwise covariances between the components, a property which is crucial in portfolio theory.

Linear correlation is a popular but also often misunderstood measure of dependence. The popularity of linear correlation stems from the ease with which it can be calculated and it is a natural scalar measure of dependence in elliptical distributions (with well known members such as the multivariate normal and the multivariate t-distribution). However most random variables are not jointly elliptically distributed, and using linear correlation as a measure of dependence in such situations might prove very misleading. Even for jointly elliptically distributed random variables there are situations where using linear correlation, as defined by (3.1), does not make sense. We might choose to model some scenario using heavy-tailed distributions such as t_2-distributions. In such cases the linear correlation coefficient is not even defined because of infinite second moments.

3.2 Perfect Dependence

For every n-copula C we know from the Fréchet–Hoeffding inequality (Theorem 2.3) that

$$W^n(u_1, \ldots, u_n) \leq C(u_1, \ldots, u_n) \leq M^n(u_1, \ldots, u_n).$$

Furthermore, for $n = 2$ the upper and lower bounds are themselves copulas and we have seen that W and M are the bivariate distributions functions of the random vectors $(U, 1-U)^T$ and $(U, U)^T$, respectively, where $U \sim U(0,1)$ (i.e. U is uniformly distributed on $[0,1]$). In this case we say that W describes perfect negative dependence and M describes perfect positive dependence.

Theorem 3.1. *Let $(X,Y)^T$ have one of the copulas W or M. Then there exist two monotone functions $\alpha, \beta : \mathbb{R} \longrightarrow \mathbb{R}$ and a random variable Z so that*

$$(X,Y) =_d (\alpha(Z), \beta(Z)),$$

with α increasing and β decreasing in the former case (W) and both α and β increasing in the latter case (M). The converse of this result is also true.

For a proof, see Embrechts, McNeil, and Straumann (1999). In a different form this result was already in Fréchet (1951).

Definition 3.2. If $(X, Y)^T$ has the copula M then X and Y are said to be comonotonic; if it has the copula W they are said to be countermonotonic. $\square$

Note that if any of F and G (the distribution functions of X and Y, respectively) have discontinuities, so that the copula is not unique, then W and M are possible copulas. In the case of F and G being continuous, a stronger version of the result can be stated:

$$\begin{aligned} C = W &\quad \Leftrightarrow \quad Y = T(X) \text{ a.s., } T = G^{-1} \circ (1 - F) \text{ decreasing,} \\ C = M &\quad \Leftrightarrow \quad Y = T(X) \text{ a.s., } T = G^{-1} \circ F \text{ increasing.} \end{aligned}$$

Other characterizations of comonotonicity can be found in Denneberg (1994).

3.3 Concordance

Let $(x, y)^T$ and $(\tilde{x}, \tilde{y})^T$ be two observations from a vector $(X, Y)^T$ of continuous random variables. Then $(x, y)^T$ and $(\tilde{x}, \tilde{y})^T$ are said to be concordant if $(x - \tilde{x})(y - \tilde{y}) > 0$, and discordant if $(x - \tilde{x})(y - \tilde{y}) < 0$.

The following theorem can be found in Nelsen (1999) p. 127. Many of the results in this section are direct consequences of this theorem.

Theorem 3.2. *Let $(X, Y)^T$ and $(\tilde{X}, \tilde{Y})^T$ be independent vectors of continuous random variables with joint distribution functions H and $\tilde{H}$, respectively, with common margins F (of X and $\tilde{X}$) and G (of Y and $\tilde{Y}$). Let C and $\tilde{C}$ denote the copulas of $(X, Y)^T$ and $(\tilde{X}, \tilde{Y})^T$, respectively, so that $H(x, y) = C(F(x), G(y))$ and $\tilde{H}(x, y) = \tilde{C}(F(x), G(y))$. Let Q denote the difference between the probability of concordance and discordance of $(X, Y)^T$ and $(\tilde{X}, \tilde{Y})^T$, i.e. let*

$$Q = \mathbb{P}\{(X - \tilde{X})(Y - \tilde{Y}) > 0\} - \mathbb{P}\{(X - \tilde{X})(Y - \tilde{Y}) < 0\}.$$

Then

$$Q = Q(C, \tilde{C}) = 4 \iint_{[0,1]^2} \tilde{C}(u, v) \, \mathrm{d}C(u, v) - 1.$$

Proof. Since the random variables are all continuous, $\mathbb{P}\{(X - \tilde{X})(Y - \tilde{Y}) < 0\} = 1 - \mathbb{P}\{(X - \tilde{X})(Y - \tilde{Y}) > 0\}$ and hence $Q = 2\mathbb{P}\{(X - \tilde{X})(Y - \tilde{Y}) > 0\} - 1$. But $\mathbb{P}\{(X - \tilde{X})(Y - \tilde{Y}) > 0\} = \mathbb{P}\{X > \tilde{X}, Y > \tilde{Y}\} + \mathbb{P}\{X < \tilde{X}, Y < \tilde{Y}\}$, and these probabilities can be evaluated by integrating over the distribution of one of the vectors $(X, Y)^T$ or $(\tilde{X}, \tilde{Y})^T$. Hence

$$\begin{aligned} \mathbb{P}\{X > \tilde{X}, Y > \tilde{Y}\} &= \mathbb{P}\{\tilde{X} < X, \tilde{Y} < Y\} \\ &= \iint_{\mathbb{R}^2} \mathbb{P}\{\tilde{X} < x, \tilde{Y} < y\} \, \mathrm{d}C(F(x), G(y)) \\ &= \iint_{\mathbb{R}^2} \tilde{C}(F(x), G(y)) \, \mathrm{d}C(F(x), G(y)), \end{aligned}$$

Employing the probability-integral transforms $u = F(x)$ and $v = G(y)$ then yields

$$\mathbb{P}\{X > \tilde{X}, Y > \tilde{Y}\} = \iint_{[0,1]^2} \tilde{C}(u, v) \, \mathrm{d}C(u, v).$$

Similarly,

$$\begin{aligned}
\mathbb{P}\{X < \tilde{X}, Y < \tilde{Y}\} &= \iint_{\mathbb{R}^2} \mathbb{P}\{\tilde{X} > x, \tilde{Y} > y\} \, \mathrm{d}C(F(x), G(y)) \\
&= \iint_{\mathbb{R}^2} \{1 - F(x) - G(y) + \tilde{C}(F(x), G(y))\} \, \mathrm{d}C(F(x), G(y)) \\
&= \iint_{[0,1]^2} \{1 - u - v + \tilde{C}(u, v)\} \, \mathrm{d}C(u, v).
\end{aligned}$$

But since C is the joint distribution function of a vector $(U, V)^T$ of $U(0, 1)$ random variables, $\mathbb{E}(U) = \mathbb{E}(V) = 1/2$, and hence

$$\mathbb{P}\{X < \tilde{X}, Y < \tilde{Y}\} = 1 - \frac{1}{2} - \frac{1}{2} + \iint_{[0,1]^2} \tilde{C}(u, v) \, \mathrm{d}C(u, v) = \iint_{[0,1]^2} \tilde{C}(u, v) \, \mathrm{d}C(u, v).$$

Thus

$$\mathbb{P}\{(X - \tilde{X})(Y - \tilde{Y}) > 0\} = 2 \iint_{[0,1]^2} \tilde{C}(u, v) \, \mathrm{d}C(u, v),$$

and the conclusion follows. $\square$

Corollary 3.1. *Let C, $\tilde{C}$, and Q be as given in Theorem 3.2. Then*

1. *Q is symmetric in its arguments: $Q(C, \tilde{C}) = Q(\tilde{C}, C)$.*
2. *Q is nondecreasing in each argument: if $C \prec C'$, then $Q(C, \tilde{C}) \leq Q(C', \tilde{C})$.*
3. *Copulas can be replaced by survival copulas in Q, i.e. $Q(C, \tilde{C}) = Q(\widehat{C}, \widehat{\tilde{C}})$.*

The following definition can be found in Scarsini (1984).

Definition 3.3. A real valued measure κ of dependence between two continuous random variables X and Y whose copula is C is a measure of concordance if it satisfies the following properties:

1. κ is defined for every pair X, Y of continuous random variables.
2. $-1 \leq \kappa_{X,Y} \leq 1$, $\kappa_{X,X} = 1$ and $\kappa_{X,-X} = -1$.
3. $\kappa_{X,Y} = \kappa_{Y,X}$.
4. If X and Y are independent, then $\kappa_{X,Y} = \kappa_{\Pi} = 0$.
5. $\kappa_{-X,Y} = \kappa_{X,-Y} = -\kappa_{X,Y}$.
6. If C and $\tilde{C}$ are copulas such that $C \prec \tilde{C}$, then $\kappa_C \leq \kappa_{\tilde{C}}$.
7. If $\{(X_n, Y_n)\}$ is a sequence of continuous random variables with copulas C_n, and if $\{C_n\}$ converges pointwise to C, then $\lim_{n\to\infty} \kappa_{C_n} = \kappa_C$.

$\square$

Let κ be a measure of concordance for continuous random variables X and Y. As a consequence of Definition 3.3, if If Y is almost surely an increasing function of X, then $\kappa_{X,Y} = \kappa_M = 1$, and if Y is almost surely a decreasing function of X, then $\kappa_{X,Y} = \kappa_W = -1$. Moreover, if α and β are almost surely strictly increasing functions on $\operatorname{Ran} X$ and $\operatorname{Ran} Y$ respectively, then $\kappa_{\alpha(X),\beta(Y)} = \kappa_{X,Y}$.

3.4 Kendall's tau and Spearman's rho

In this section we discuss two important measures of dependence (concordance) known as Kendall's tau and Spearman's rho. They provide the perhaps best alternatives to the linear correlation coefficient as a measure of dependence for nonelliptical distributions, for which the linear correlation coefficient is inappropriate and often misleading. For more details about Kendall's tau and Spearman's rho and their estimators (sample versions) we refer to Kendall and Stuart (1979), Kruskal (1958), Lehmann (1975) and Capéraà and Genest (1993). For other interesting scalar measures of dependence see Schweizer and Wolff (1981).

Definition 3.4. Kendall's tau for the random vector $(X, Y)^T$ is defined as

$$\tau(X,Y) = \mathbb{P}\{(X - \tilde{X})(Y - \tilde{Y}) > 0\} - \mathbb{P}\{(X - \tilde{X})(Y - \tilde{Y}) < 0\},$$

where $(\tilde{X}, \tilde{Y})^T$ is an independent copy of $(X, Y)^T$. □

Hence Kendall's tau for $(X, Y)^T$ is simply the probability of concordance minus the probability of discordance.

Theorem 3.3. *Let $(X, Y)^T$ be a vector of continuous random variables with copula C. Then Kendall's tau for $(X, Y)^T$ is given by*

$$\tau(X,Y) = Q(C,C) = 4\iint_{[0,1]^2} C(u,v)\,\mathrm{d}C(u,v) - 1.$$

Note that the integral above is the expected value of the random variable $C(U, V)$, where $U, V \sim U(0, 1)$ with joint distribution function C, i.e. $\tau(X, Y) = 4\mathbb{E}(C(U, V)) - 1$.

Definition 3.5. Spearman's rho for the random vector $(X, Y)^T$ is defined as

$$\rho_S(X,Y) = 3(\mathbb{P}\{(X - \tilde{X})(Y - Y') > 0\} - \mathbb{P}\{(X - \tilde{X})(Y - Y') < 0\}),$$

where $(X, Y)^T$, $(\tilde{X}, \tilde{Y})^T$ and $(X', Y')^T$ are independent copies. □

Note that $\tilde{X}$ and Y' are independent. Using Theorem 3.2 and the first part of Corollary 3.1 we obtain the following result.

Theorem 3.4. *Let $(X, Y)^T$ be a vector of continuous random variables with copula C. Then Spearman's rho for $(X, Y)^T$ is given by*

$$\rho_S(X,Y) = 3Q(C,\Pi) = 12\iint_{[0,1]^2} uv\,\mathrm{d}C(u,v) - 3 = 12\iint_{[0,1]^2} C(u,v)\,\mathrm{d}u\,\mathrm{d}v - 3.$$

Hence, if $X \sim F$ and $Y \sim G$, and we let $U = F(X)$ and $V = G(Y)$, then

$$\begin{aligned}
\rho_S(X,Y) &= 12\iint_{[0,1]^2} uv\,\mathrm{d}C(u,v) - 3 = 12\mathbb{E}(UV) - 3 \\
&= \frac{\mathbb{E}(UV) - 1/4}{1/12} = \frac{\mathrm{Cov}(U,V)}{\sqrt{\mathrm{Var}(U)}\sqrt{\mathrm{Var}(V)}} \\
&= \rho(F(X), G(Y)).
\end{aligned}$$

In the next theorem we will see that Kendall's tau and Spearman's rho are concordance measures according to Definition 3.3.

Theorem 3.5. *If X and Y are continuous random variables whose copula is C, then Kendall's tau and Spearman's rho satisfy the properties in Definition 3.3 for a measure of concordance.*

For a proof, see Nelsen (1999) p. 137.

Example 3.1. Kendall's tau and Spearman's rho for the random vector $(X, Y)^T$ are invariant under strictly increasing componentwise transformations. This property does not hold for linear correlation. It is not difficult to construct examples, the following construction is instructive in its own right. Let X and Y be standard exponential random variables with copula C, where C is a member of the Farlie-Gumbel-Morgenstern family, i.e. C is given by

$$C(u, v) = uv + \theta uv(1 - u)(1 - v),$$

for some θ in $[-1, 1]$. The joint distribution function H of X and Y is given by

$$H(x, y) = C(1 - e^{-x}, 1 - e^{-y}).$$

Let ρ denote the linear correlation coefficient. Then

$$\rho(X, Y) = \frac{\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}} = \mathbb{E}(XY) - 1,$$

where

$$\begin{aligned} \mathbb{E}(XY) &= \int_0^\infty \int_0^\infty xy \,\mathrm{d}H(x, y) \\ &= \int_0^\infty \int_0^\infty xy\big((1 + \theta)e^{-x-y} - 2\theta e^{-2x-y} - 2\theta e^{-x-2y} + 4\theta e^{-2x-2y}\big) \,\mathrm{d}x \,\mathrm{d}y \\ &= 1 + \frac{\theta}{4}. \end{aligned}$$

Hence $\rho(X, Y) = \theta/4$. But

$$\begin{aligned} \rho(1 - e^{-X}, 1 - e^{-Y}) &= \rho_S(X, Y) \\ &= 12 \iint_{[0,1]^2} C(u, v) \,\mathrm{d}u \,\mathrm{d}v - 3 \\ &= 12 \iint_{[0,1]^2} (uv + \theta uv(1 - u)(1 - v)) \,\mathrm{d}u \,\mathrm{d}v - 3 \\ &= 12(\frac{1}{4} + \frac{\theta}{36}) - 3 \\ &= \theta/3. \end{aligned}$$

Hence $\rho(X, Y)$ is not invariant under strictly increasing transformations of X and Y and therefore linear correlation is not a measure of concordance. □

Although the properties listed under Definition 3.3 are useful, there are some additional properties that would make a measure of concordance even more useful. Recall that for a random vector $(X, Y)^T$ with copula C,

$$\begin{aligned} C &= M \quad \Longrightarrow \quad \tau_C = \rho_C = 1, \\ C &= W \quad \Longrightarrow \quad \tau_C = \rho_C = -1. \end{aligned}$$

The following theorem states that the converse is also true.

Theorem 3.6. *Let X and Y be continuous random variables with copula C, and let κ denote Kendall's tau or Spearman's rho. Then the following are true:*

1. $\kappa(X, Y) = 1 \iff C = M$.
2. $\kappa(X, Y) = -1 \iff C = W$.

For a proof, see Embrechts, McNeil, and Straumann (1999).

From the definitions of Kendall's tau and Spearman's rho it follows that both are increasing functions of the value of the copula under consideration. Thus they are increasing with respect to the concordance ordering given in Definition 2.5. Moreover, for continuous random variables all values in the interval $[-1, 1]$ can be obtained for Kendall's tau or Spearman's rho by a suitable choice of the underlying copula. This is however not the case with linear correlation as is shown in the following example from Embrechts, McNeil, and Straumann (1999).

Example 3.2. Let $X \sim \mathrm{LN}(0, 1)$ (Lognormal) and $Y \sim \mathrm{LN}(0, \sigma^2), \sigma > 0$. Then $\rho_{\min} = \rho(e^Z, e^{-\sigma Z})$ and $\rho_{\max} = \rho(e^Z, e^{\sigma Z})$, where $Z \sim \mathcal{N}(0, 1)$. $\rho_{\min}$ and $\rho_{\max}$ can be calculated, yielding:

$$\rho_{\min} = \frac{e^{-\sigma} - 1}{\sqrt{e - 1}\sqrt{e^{\sigma^2} - 1}}, \quad \rho_{\max} = \frac{e^{\sigma} - 1}{\sqrt{e - 1}\sqrt{e^{\sigma^2} - 1}},$$

from which follows that $\lim_{\sigma \to \infty} \rho_{\min} = \lim_{\sigma \to \infty} \rho_{\max} = 0$. Hence the linear correlation coefficient can be almost zero, even if X and Y are comonotonic or countermonotonic. □

Kendall's tau and Spearman's rho are measures of dependence between two random variables. However the extension to higher dimensions is obvious, we simply write pairwise correlations in an $n \times n$ matrix in the same way as is done for linear correlation.

3.5 Tail Dependence

The concept of tail dependence relates to the amount of dependence in the upper-right-quadrant tail or lower-left-quadrant tail of a bivariate distribution. It is a concept that is relevant for the study of dependence between extreme values. It turns out that tail dependence between two continuous random variables X and Y is a copula property and hence the amount of tail dependence is invariant under strictly increasing transformations of X and Y.

Definition 3.6. Let $(X, Y)^T$ be a vector of continuous random variables with marginal distribution functions F and G. The coefficient of upper tail dependence of $(X, Y)^T$ is

$$\lim_{u \nearrow 1} \mathbb{P}\{Y > G^{-1}(u) | X > F^{-1}(u)\} = \lambda_U$$

provided that the limit $\lambda_U \in [0, 1]$ exists. If $\lambda_U \in (0, 1]$, X and Y are said to be asymptotically dependent in the upper tail; if $\lambda_U = 0$, X and Y are said to be asymptotically independent in the upper tail. □

Since $\mathbb{P}\{Y > G^{-1}(u) \mid X > F^{-1}(u)\}$ can be written as

$$\frac{1 - \mathbb{P}\{X \leq F^{-1}(u)\} - \mathbb{P}\{Y \leq G^{-1}(u)\} + \mathbb{P}\{X \leq F^{-1}(u), Y \leq G^{-1}(u)\}}{1 - \mathbb{P}\{X \leq F^{-1}(u)\}},$$

an alternative and equivalent definition (for continuous random variables), from which it is seen that the concept of tail dependence is indeed a copula property, is the following which can be found in Joe (1997), p. 33.

Definition 3.7. If a bivariate copula C is such that

$$\lim_{u\nearrow 1}(1-2u+C(u,u))/(1-u)=\lambda_U$$

exists, then C has upper tail dependence if $\lambda_U \in (0,1]$, and upper tail independence if $\lambda_U = 0$. □

Example 3.3. Consider the bivariate Gumbel family of copulas given by

$$C_\theta(u,v)=\exp(-[(-\ln u)^\theta+(-\ln v)^\theta]^{1/\theta}),$$

for $\theta \geq 1$. Then

$$\frac{1-2u+C(u,u)}{1-u}=\frac{1-2u+\exp(2^{1/\theta}\ln u)}{1-u}=\frac{1-2u+u^{2^{1/\theta}}}{1-u},$$

and hence

$$\lim_{u\nearrow 1}(1-2u+C(u,u))/(1-u)=2-\lim_{u\nearrow 1}2^{1/\theta}u^{2^{1/\theta}-1}=2-2^{1/\theta}.$$

Thus for $\theta > 1$, C_θ has upper tail dependence. □

For copulas without a simple closed form an alternative formula for λ_U is more useful. An example is given in the case of the Gaussian copula

$$C_R(u,v)=\int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)}\frac{1}{2\pi\sqrt{1-R_{12}^2}}\exp\{-\frac{s^2-2R_{12}st+t^2}{2(1-R_{12}^2)}\}\,\mathrm{d}s\,\mathrm{d}t,$$

where $-1 < R_{12} < 1$ and Φ is the univariate standard normal distribution function. Consider a pair of $U(0,1)$ random variables (U,V) with copula C. First note that $\mathbb{P}\{V \leq v \mid U=u\} = \partial C(u,v)/\partial u$ and $\mathbb{P}\{V > v \mid U=u\} = 1-\partial C(u,v)/\partial u$, and similarly when conditioning on V. Then

$$\begin{aligned}
\lambda_U &= \lim_{u\nearrow 1}\overline{C}(u,u)/(1-u)\\
&= -\lim_{u\nearrow 1}\frac{\mathrm{d}\overline{C}(u,u)}{\mathrm{d}u}\\
&= -\lim_{u\nearrow 1}(-2+\frac{\partial}{\partial s}C(s,t)\big|_{s=t=u}+\frac{\partial}{\partial t}C(s,t)\big|_{s=t=u})\\
&= \lim_{u\nearrow 1}(\mathbb{P}\{V>u \mid U=u\}+\mathbb{P}\{U>u \mid V=u\}).
\end{aligned}$$

Furthermore, if C is an exchangeable copula, i.e. $C(u,v)=C(v,u)$, then the expression for λ_U simplifies to

$$\lambda_U = 2\lim_{u\nearrow 1}\mathbb{P}\{V>u \mid U=u\}.$$

Example 3.4. Let $(X,Y)^T$ have the bivariate standard normal distribution function with linear correlation coefficient ρ. That is $(X,Y)^T \sim C(\Phi(x),\Phi(y))$, where C is a member of the Gaussian family given above with $R_{12}=\rho$. Since copulas in this family are exchangeable,

$$\lambda_U = 2\lim_{u\nearrow 1}\mathbb{P}\{V>u \mid U=u\},$$

and because Φ is a distribution function with infinite right endpoint,

$$\lim_{u \nearrow 1} \mathbb{P}\{V > u \mid U = u\} = \lim_{x \to \infty} \mathbb{P}\{\Phi^{-1}(V) > x \mid \Phi^{-1}(U) = x\} = \lim_{x \to \infty} \mathbb{P}\{X > x \mid Y = x\}.$$

Using the well known fact that $Y \mid X = x \sim \mathcal{N}(\rho x, 1 - \rho^2)$ we obtain

$$\lambda_U = 2 \lim_{x \to \infty} \overline{\Phi}((x - \rho x)/\sqrt{1 - \rho^2}) = 2 \lim_{x \to \infty} \overline{\Phi}(x\sqrt{1 - \rho}/\sqrt{1 + \rho}),$$

from which it follows that $\lambda_U = 0$ for $R_{12} < 1$. Hence the Gaussian copula C with $\rho < 1$ does not have upper tail dependence. $\square$

The concept of lower tail dependence can be defined in a similar way. If the limit $\lim_{u \searrow 0} C(u, u)/u = \lambda_L$ exists, then C has lower tail dependence if $\lambda_L \in (0, 1]$, and lower tail independence if $\lambda_L = 0$. For copulas without a simple closed form an alternative formula for λ_L is more useful. Consider a random vector $(U, V)^T$ with copula C. Then

$$\begin{aligned}
\lambda_L &= \lim_{u \searrow 0} C(u, u)/u = \lim_{u \searrow 0} \frac{\mathrm{d}C(u, u)}{\mathrm{d}u} = \lim_{u \searrow 0} \Big(\frac{\partial}{\partial s} C(s, t)\big|_{s=t=u} + \frac{\partial}{\partial t} C(s, t)\big|_{s=t=u}\Big) \\
&= \lim_{u \searrow 0} (\mathbb{P}\{V < u \mid U = u\} + \mathbb{P}\{U < u \mid V = u\}).
\end{aligned}$$

Furthermore if C is an exchangeable copula, i.e. $C(u, v) = C(v, u)$, then the expression for λ_L simplifies to

$$\lambda_L = 2 \lim_{u \searrow 0} \mathbb{P}\{V < u \mid U = u\}.$$

Recall that the survival copula of two random variables with copula C is given by

$$\widehat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v),$$

and the joint survival function for two $U(0, 1)$ random variables whose joint distribution function is C is given by

$$\overline{C}(u, v) = 1 - u - v + C(u, v) = \widehat{C}(1 - u, 1 - v).$$

Hence it follows that

$$\lim_{u \nearrow 1} \overline{C}(u, u)/(1 - u) = \lim_{u \nearrow 1} \widehat{C}(1 - u, 1 - u)/(1 - u) = \lim_{u \searrow 0} \widehat{C}(u, u)/u,$$

so the coefficient of upper tail dependence of C is the coefficient of lower tail dependence of $\widehat{C}$. Similarly the coefficient of lower tail dependence of C is the coefficient of upper tail dependence of $\widehat{C}$.

4 Marshall-Olkin Copulas

In this section we discuss a class of copulas called Marshall-Olkin copulas. To be able to derive these copulas and present explicit expressions for rank correlation and tail dependence coefficients without tedious calculations, we begin with bivariate Marshall-Olkin copulas. We then continue with the general n-dimensional case and suggest applications of Marshall-Olkin copulas to the modelling of dependent risks. For further details about Marshall-Olkin distributions we refer to Marshall and Olkin (1967). Similar ideas are contained in Muliere and Scarsini (1987).

4.1 Bivariate Marshall-Olkin Copulas

Consider a two-component system where the components are subject to shocks, which are fatal to one or both components. Let X_1 and X_2 denote the lifetimes of the two components. Furthermore assume that the shocks follow three independent Poisson processes with parameters $\lambda_1, \lambda_2, \lambda_{12} \geq 0$, where the index indicates whether the shocks effect only component 1, only component 2 or both. Then the times Z_1, Z_2 and Z_{12} of occurrence of these shocks are independent exponential random variables with parameters λ_1, λ_2 and λ_{12} respectively. Hence

$$\overline{H}(x_1, x_2) = \mathbb{P}\{X_1 > x_1, X_2 > x_2\} = \mathbb{P}\{Z_1 > x_1\}\mathbb{P}\{Z_2 > x_2\}\mathbb{P}\{Z_{12} > \max(x_1, x_2)\}.$$

The univariate survival functions for X_1 and X_2 are $\overline{F}_1(x_1) = \exp(-(\lambda_1 + \lambda_{12})x_1)$ and $\overline{F}_2(x_2) = \exp(-(\lambda_2 + \lambda_{12})x_2)$. Furthermore, since $\max(x_1, x_2) = x_1 + x_2 - \min(x_1, x_2)$,

$$\begin{aligned} \overline{H}(x_1, x_2) &= \exp(-(\lambda_1 + \lambda_{12})x_1 - (\lambda_2 + \lambda_{12})x_2 + \lambda_{12}\min(x_1, x_2)) \\ &= \overline{F}_1(x_1)\overline{F}_2(x_2)\min(\exp(\lambda_{12}x_1), \exp(\lambda_{12}x_2)). \end{aligned}$$

Let $\alpha_1 = \lambda_{12}/(\lambda_1 + \lambda_{12})$ and $\alpha_2 = \lambda_{12}/(\lambda_2 + \lambda_{12})$. Then $\exp(\lambda_{12}x_1) = \overline{F}_1(x_1)^{-\alpha_1}$ and $\exp(\lambda_{12}x_2) = \overline{F}_2(x_2)^{-\alpha_2}$, and hence the survival copula of $(X_1, X_2)^T$ is given by

$$\widehat{C}(u_1, u_2) = u_1 u_2 \min(u_1^{-\alpha_1}, u_2^{-\alpha_2}) = \min(u_1^{1-\alpha_1}u_2, u_1 u_2^{1-\alpha_2}).$$

This construction leads to a copula family given by

$$C_{\alpha_1,\alpha_2}(u_1, u_2) = \min(u_1^{1-\alpha_1}u_2, u_1 u_2^{1-\alpha_2}) = \begin{cases} u_1^{1-\alpha_1}u_2, & u_1^{\alpha_1} \geq u_2^{\alpha_2}, \\ u_1 u_2^{1-\alpha_2}, & u_1^{\alpha_1} \leq u_2^{\alpha_2}. \end{cases}$$

This family is known as the Marshall-Olkin family. Marshall-Olkin copulas have both an absolutely continuous and a singular component. Since

$$\frac{\partial^2}{\partial u_1 \partial u_2} C_{\alpha_1,\alpha_2}(u_1, u_2) = \begin{cases} u_1^{-\alpha_1}, & u_1^{\alpha_1} > u_2^{\alpha_2}, \\ u_2^{-\alpha_2}, & u_1^{\alpha_1} < u_2^{\alpha_2}, \end{cases}$$

the mass of the singular component is concentrated on the curve $u_1^{\alpha_1} = u_2^{\alpha_2}$ in $[0,1]^2$ as seen in Figure 4.1.

Kendall's tau and Spearman's rho are quite easily evaluated for this copula family. For Spearman's rho, applying Theorem 3.4 yields:

$$\begin{aligned} \rho_S(C_{\alpha_1,\alpha_2}) &= 12 \iint_{[0,1]^2} C_{\alpha_1,\alpha_2}(u, v)\,\mathrm{d}u\,\mathrm{d}v - 3 \\ &= 12 \int_0^1 \left(\int_0^{u^{\alpha_1/\alpha_2}} u^{1-\alpha_1} v\,\mathrm{d}v + \int_{u^{\alpha_1/\alpha_2}}^1 u v^{1-\alpha_2}\,\mathrm{d}v \right) \mathrm{d}u - 3 \\ &= \frac{3\alpha_1\alpha_2}{2\alpha_1 + 2\alpha_2 - \alpha_1\alpha_2}. \end{aligned}$$

To evaluate Kendall's tau we use the following theorem, a proof of which is found in Nelsen (1999) p. 131.

Theorem 4.1. *Let C be a copula such that the product $\big(\partial C/\partial u\big)\big(\partial C/\partial v\big)$ is integrable on $[0,1]^2$. Then*

$$\iint_{[0,1]^2} C(u, v)\,\mathrm{d}C(u, v) = \frac{1}{2} - \iint_{[0,1]^2} \frac{\partial}{\partial u} C(u, v) \frac{\partial}{\partial u} C(u, v)\,\mathrm{d}u\,\mathrm{d}v.$$

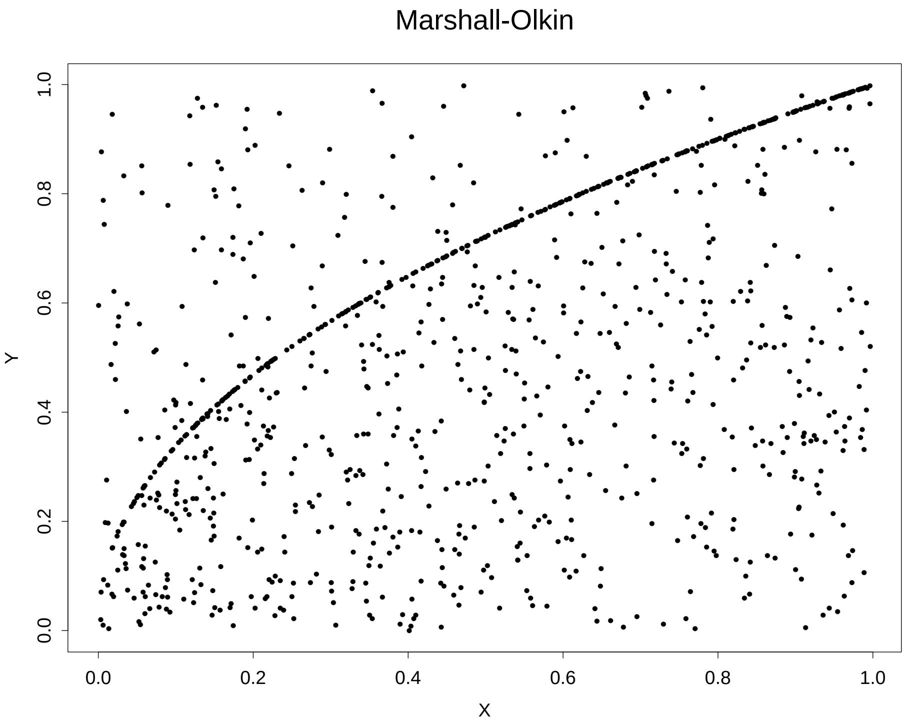

Figure 4.1: A simulation from the Marshall-Olkin copula with $\lambda_1 = 1.1, \lambda_2 = 0.2$ and $\lambda_{12} = 0.6$.

Using Theorems 3.3 and 4.1 we obtain

$$\begin{aligned}
\tau(C_{\alpha_1,\alpha_2}) &= 4\iint_{[0,1]^2} C_{\alpha_1,\alpha_2}(u,v)\,\mathrm{d}C_{\alpha_1,\alpha_2}(u,v) - 1 \\
&= 4\left(\frac{1}{2} - \iint_{[0,1]^2} \frac{\partial}{\partial u}C_{\alpha_1,\alpha_2}(u,v)\frac{\partial}{\partial u}C_{\alpha_1,\alpha_2}(u,v)\,\mathrm{d}u\,\mathrm{d}v\right) - 1 \\
&= \frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1\alpha_2}.
\end{aligned}$$

Thus, all values in the interval $[0,1]$ can be obtained for $\rho_S(C_{\alpha_1,\alpha_2})$ and $\tau(C_{\alpha_1,\alpha_2})$. The Marshall-Olkin copulas have upper tail dependence. Without loss of generality assume that $\alpha_1 > \alpha_2$, then

$$\begin{aligned}
\lim_{u\nearrow 1}\frac{\overline{C}(u,u)}{1-u} &= \lim_{u\nearrow 1}\frac{1-2u+u^2\min(u^{-\alpha_1},u^{-\alpha_2})}{1-u} \\
&= \lim_{u\nearrow 1}\frac{1-2u+u^2u^{-\alpha_2}}{1-u} \\
&= \lim_{u\nearrow 1}(2-2u^{1-\alpha_2}+\alpha_2u^{1-\alpha_2}) = \alpha_2,
\end{aligned}$$

and hence $\lambda_U = \min(\alpha_1,\alpha_2)$ is the coefficient of upper tail dependence.

4.2 A Multivariate Extension

We now present the natural multivariate extension of the bivariate Marshall-Olkin family. Consider an n-component system, where each nonempty subset of components is assigned a shock which is fatal to all components of that subset. Let $\mathcal{S}$ denote the set of nonempty subsets of $\{1, \ldots, n\}$. Let $X_1, \ldots, X_n$ denote the lifetimes of the components, and assume that shocks assigned to different subsets s, $s \in \mathcal{S}$, follow independent Poisson processes with intensities λ_s. Let Z_s, $s \in \mathcal{S}$, denote the time of first occurrence of a shock event for the shock process assigned to subset s. Then the occurrence times Z_s are independent exponential random variables with parameters λ_s, and $X_j = \min_{s:j\in s} Z_s$ for $j = 1, \ldots, n$.

There are in total $2^n - 1$ shock processes, each in one-to-one correspondence with a nonempty subset of $\{1, \ldots, n\}$.

Example 4.1. Let $n = 4$. Then

$$\begin{aligned}
X_1 &= \min(Z_1, Z_{12}, Z_{13}, Z_{14}, Z_{123}, Z_{124}, Z_{134}, Z_{1234}),\\
X_2 &= \min(Z_2, Z_{12}, Z_{23}, Z_{24}, Z_{123}, Z_{124}, Z_{234}, Z_{1234}),\\
X_3 &= \min(Z_3, Z_{13}, Z_{23}, Z_{34}, Z_{123}, Z_{134}, Z_{234}, Z_{1234}),\\
X_4 &= \min(Z_4, Z_{14}, Z_{24}, Z_{34}, Z_{124}, Z_{134}, Z_{234}, Z_{1234}).
\end{aligned}$$

If for example $\lambda_{13} = 0$, then $Z_{13} = \infty$ almost surely. □

We now turn to the question of random variate generation from Marshall-Olkin n-copulas. Order the $l := |\mathcal{S}| = 2^n - 1$ nonempty subsets of $\{1, \ldots, n\}$ in some arbitrary way, $s_1, \ldots, s_l$, and set $\lambda_k := \lambda_{s_k}$ (the parameter of Z_{s_k}) for $k = 1, \ldots, l$. The following algorithm generates random variates from the Marshall-Olkin n-copula.

Algorithm 4.1.

- Simulate l independent random variates $v_1, \ldots, v_l$ from $U(0, 1)$.
- Set $x_i = \min_{1\le k\le l, i\in s_k, \lambda_k\neq 0}(-\ln v_k/\lambda_k), \quad i = 1, \ldots, n.$
- Set $\Lambda_i = \sum_{k=1}^{l} 1\{i \in s_k\}\lambda_k, \quad i = 1, \ldots, n.$
- Set $u_i = \exp(-\Lambda_i x_i), \quad i = 1, \ldots, n.$

Then $(x_1, \ldots, x_n)^T$ is an n-variate from the n-dimensional Marshall-Olkin distribution and $(u_1, \ldots, u_n)^T$ is an n-variate from the corresponding Marshall-Olkin n-copula. Furthermore, Λ_i is the shock intensity "felt" by component i. □

Since the (i, j)-bivariate margin of a Marshall-Olkin n-copula is a Marshall-Olkin copula with parameters

$$\alpha_i = \Big(\sum_{s:i\in s, j\in s} \lambda_s\Big)/\Big(\sum_{s:i\in s} \lambda_s\Big) \quad \text{and} \quad \alpha_j = \Big(\sum_{s:i\in s, j\in s} \lambda_s\Big)/\Big(\sum_{s:j\in s} \lambda_s\Big),$$

the Kendall's tau and Spearman's rho rank correlation matrices are easily evaluated. The (i, j) entries are given by

$$\frac{\alpha_i\alpha_j}{\alpha_i + \alpha_j - \alpha_i\alpha_j} \quad \text{and} \quad \frac{3\alpha_i\alpha_j}{2\alpha_i + 2\alpha_j - \alpha_i\alpha_j},$$

respectively. As seen above, evaluating the rank correlation matrix given the full parameterization of the Marshall-Olkin n-copula is straightforward. However given a (Kandall's tau or Spearman's rho) rank correlation matrix we cannot in general obtain a unique parameterization of the copula. By setting the shock intensities for subgroups with more then two elements to zero, we obtain the perhaps most natural parameterization of the copula in this situation. However this also means that the copula only has bivariate dependence.

4.3 A Useful Modelling Framework

In general the huge number of parameters for high-dimensional Marshall-Olkin copulas make them unattractive for high-dimensional risk modelling. However, we now give an example of how an intuitively appealing and easier parameterized model for modelling dependent loss frequencies can be set up, for which the survival copula of times to first losses is a Marshall-Olkin copula.

Suppose we are interested in insurance losses occurring in several different lines of business or several different countries. In credit-risk modelling we might be interested in losses related to the default of various different counterparties or types of counterparty. A natural approach to modelling this dependence is to assume that all losses can be related to a series of underlying and independent shock processes. In insurance these shocks might be natural catastrophes; in credit-risk modelling they might be a variety of underlying economic events. When a shock occurs this may cause losses of several different types; the common shock causes the numbers of losses of each type to be dependent. It is commonly assumed that the different varieties of shocks arrive as independent Poisson processes, in which case the counting processes of the losses are also Poisson and can be handled easily analytically. In reliability such models are known as *fatal shock models*, when the shock always destroys the component, and *nonfatal shock models*, when components have a chance of surviving the shock. A good basic reference on such models is Barlow and Proschan (1975).

Suppose there are m different types of shocks and for $e = 1, \dots, m$, let $\{N^{(e)}(t), t \geq 0\}$ be a Poisson process with intensity $\lambda^{(e)}$ recording the number of events of type e occurring in $(0, t]$. Assume further that these shock counting processes are independent. Consider losses of n different types and for $j = 1, \dots, n$, let $\{N_j(t), t \geq 0\}$ be a counting process that records the frequency of losses of the jth type occurring in $(0, t]$. At the rth occurrence of an event of type e the Bernoulli variable $I_{j,r}^{(e)}$ indicates whether a loss of type j occurs. The vectors

$$\mathbf{I}_r^{(e)} = (I_{1,r}^{(e)}, \dots, I_{n,r}^{(e)})^T$$

for $r = 1, \dots, N^{(e)}(t)$ are considered to be independent and identically distributed with a multivariate Bernoulli distribution. In other words, each new event represents a new independent opportunity to incur a loss but, for a fixed event, the loss trigger variables for losses of different types may be dependent. The form of the dependence depends on the specification of the multivariate Bernoulli distribution with independence as a special case. We use the following notation for p-dimensional marginal probabilities of this distribution (the subscript r is dropped for simplicity):

$$P(I_{j_1}^{(e)} = i_{j_1}, \dots, I_{j_p}^{(e)} = i_{j_p}) = p_{j_1,\dots,j_p}^{(e)}(i_{j_1}, \dots, i_{j_p}),\ i_{j_1}, \dots, i_{j_p} \in \{0, 1\}.$$

We also write $p_j^{(e)}(1) = p_j^{(e)}$ for one-dimensional marginal probabilities, so that in the special case of conditional independence we have $p_{j_1,\dots,j_p}^{(e)}(1, \dots, 1) = \prod_{k=1}^{p} p_{j_k}^{(e)}$. The counting processes for events and losses are thus linked by

$$N_j(t) = \sum_{e=1}^{m} \sum_{r=1}^{N^{(e)}(t)} I_{j,r}^{(e)}.$$

Under the Poisson assumption for the event processes and the Bernoulli assumption for the loss indicators, the loss processes $\{N_j(t), t \geq 0\}$ are clearly Poisson themselves, since they are obtained by superpositioning m independent (possibly thinned) Poisson processes generated by the m underlying event processes. The random vector $(N_1(t), \dots, N_n(t))^T$ can be thought of as having a *multivariate Poisson* distribution.

The presented nonfatal shock model has an equivalent fatal shock model representation, i.e. of the type presented in Section 4.2. Hence the random vector $(X_1, \dots, X_n)^T$ of times to first losses of different types, where $X_j = \inf\{t \geq 0 \mid N_j(t) > 0\}$, has an n-dimensional Marshall-Olkin distribution whose survival copula is a Marshall-Olkin n-copula. From this it follows that Kendall's tau, Spearman's rho and coefficients of tail dependence for $(X_i, X_j)^T$ can be easily calculated. For more details on this model, see Lindskog and McNeil (2001).

5 Elliptical Copulas

The class of elliptical distributions provides a rich source of multivariate distributions which share many of the tractable properties of the multivariate normal distribution and enables modelling of multivariate extremes and other forms of nonnormal dependences. Elliptical copulas are simply the copulas of elliptical distributions. Simulation from elliptical distributions is easy, and as a consequence of Sklar's Theorem so is simulation from elliptical copulas. Furthermore, we will show that rank correlation and tail dependence coefficients can be easily calculated. For further details on elliptical distributions we refer to Fang, Kotz, and Ng (1987) and Cambanis, Huang, and Simons (1981).

5.1 Elliptical Distributions

Definition 5.1. If $\mathbf{X}$ is a n-dimensional random vector and, for some $\mu \in \mathbb{R}^n$ and some $n \times n$ nonnegative definite, symmetric matrix Σ, the characteristic function $\varphi_{\mathbf{X}-\mu}(\mathbf{t})$ of $\mathbf{X} - \mu$ is a function of the quadratic form $\mathbf{t}^T \Sigma \mathbf{t}$, $\varphi_{\mathbf{X}-\mu}(\mathbf{t}) = \phi(\mathbf{t}^T \Sigma \mathbf{t})$, we say that $\mathbf{X}$ has an elliptical distribution with parameters μ, Σ and ϕ, and we write $\mathbf{X} \sim E_n(\mu, \Sigma, \phi)$. □

When $n = 1$, the class of elliptical distributions coincides with the class of one-dimensional symmetric distributions. A function ϕ as in Definition 5.1 is called a characteristic generator.

Theorem 5.1. $\mathbf{X} \sim E_n(\mu, \Sigma, \phi)$ *with rank*$(\Sigma) = k$ *if and only if there exist a random variable* $R \geq 0$ *independent of* $\mathbf{U}$*, a* k*-dimensional random vector uniformly distributed on the unit hypersphere* $\{\mathbf{z} \in \mathbb{R}^k | \mathbf{z}^T \mathbf{z} = 1\}$*, and an* $n \times k$ *matrix* A *with* $AA^T = \Sigma$*, such that*

$$\mathbf{X} =_d \mu + RA\mathbf{U}.$$

For the proof of Theorem 5.1 and the relation between R and ϕ see Fang, Kotz, and Ng (1987) or Cambanis, Huang, and Simons (1981).

Example 5.1. Let $\mathbf{X} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I}_n)$. Since the components $X_i \sim \mathcal{N}(0,1), i = 1, \dots, n,$ are independent and the characteristic function of X_i is $\exp(-t_i^2/2)$, the characteristic function of $\mathbf{X}$ is

$$\exp\{-\frac{1}{2}(t_1^2 + \cdots + t_n^2)\} = \exp\{-\frac{1}{2}\mathbf{t}^T\mathbf{t}\}.$$

From Theorem 5.1 it then follows that $\mathbf{X} \sim E_n(\mathbf{0}, \mathbf{I}_n, \phi)$, where $\phi(u) = \exp(-u/2)$. □

If $\mathbf{X} \sim E_n(\mu, \Sigma, \phi)$, where Σ is a diagonal matrix, then $\mathbf{X}$ has uncorrelated components (if $0 < \mathrm{Var}(X_i) < \infty$). If $\mathbf{X}$ has independent components, then $\mathbf{X} \sim \mathcal{N}_n(\mu, \Sigma)$. Note that the multivariate normal distribution is the only one among the elliptical distributions where uncorrelated components imply independent components. A random vector $\mathbf{X} \sim E_n(\mu, \Sigma, \phi)$ does not necessarily have a density. If $\mathbf{X}$ has a density it must be of the form $|\Sigma|^{-1/2} g((\mathbf{X} - \mu)^T \Sigma^{-1}(\mathbf{X} - \mu))$ for some nonnegative function g of one scalar variable. Hence the contours of equal density form ellipsoids in $\mathbb{R}^n$. Given the distribution of $\mathbf{X}$, the representation $E_n(\mu, \Sigma, \phi)$ is not unique. It uniquely determines μ but Σ and ϕ are only determined up to a positive constant. More precisely, if $\mathbf{X} \sim E_n(\mu, \Sigma, \phi)$ and $\mathbf{X} \sim E_n(\mu^*, \Sigma^*, \phi^*)$, then

$$\mu^* = \mu, \quad \Sigma^* = c\Sigma, \quad \phi^*(\cdot) = \phi(\cdot/c),$$

for some constant $c > 0$.

In order to find a representation such that $\mathrm{Cov}(\mathbf{X}) = \Sigma$, we use Theorem 5.1 to obtain

$$\mathrm{Cov}(\mathbf{X}) = \mathrm{Cov}(\mu + RA\mathbf{U}) = A\mathbb{E}(R^2)\mathrm{Cov}(\mathbf{U})A^T,$$

provided that $\mathbb{E}(R^2) < \infty$. Let $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I}_n)$. Then $\mathbf{Y} =_d \|\mathbf{Y}\|\mathbf{U}$, where $\|\mathbf{Y}\|$ is independent of $\mathbf{U}$. Furthermore $\|\mathbf{Y}\|^2 \sim \chi_n^2$, so $\mathbb{E}(\|\mathbf{Y}\|^2) = n$. Since $\mathrm{Cov}(\mathbf{Y}) = \mathbf{I}_n$ we see that if $\mathbf{U}$ is uniformly distributed on the unit hypersphere in $\mathbb{R}^n$, then $\mathrm{Cov}(\mathbf{U}) = \mathbf{I}_n/n$. Thus $\mathrm{Cov}(\mathbf{X}) = AA^T\mathbb{E}(R^2)/n$. By choosing the characteristic generator $\phi^*(s) = \phi(s/c)$, where $c = \mathbb{E}(R^2)/n$, we get $\mathrm{Cov}(X) = \Sigma$. Hence an elliptical distribution is fully described by μ, Σ and ϕ, where ϕ can be chosen so that $\mathrm{Cov}(\mathbf{X}) = \Sigma$ (if $\mathrm{Cov}(\mathbf{X})$ is defined). If $\mathrm{Cov}(\mathbf{X})$ is obtained as above, then the distribution of $\mathbf{X}$ is uniquely determined by $\mathbb{E}(\mathbf{X})$, $\mathrm{Cov}(\mathbf{X})$ and the type of its univariate margins, e.g. normal or t_4, say.

Theorem 5.2. *Let $\mathbf{X} \sim E_n(\mu, \Sigma, \phi)$, let B be a $q \times n$ matrix and $\mathbf{b} \in \mathbb{R}^q$. Then*

$$\mathbf{b} + B\mathbf{X} \sim E_q(\mathbf{b} + B\mu, B\Sigma B^T, \phi).$$

Proof. By Theorem 5.1, $\mathbf{b} + B\mathbf{X}$ has the stochastic representation

$$\mathbf{b} + B\mathbf{X} =_d \mathbf{b} + B\mu + RBA\mathbf{U}.$$

□

Partition $\mathbf{X}$, μ and Σ into

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

where $\mathbf{X}_1$ and μ_1 are $r \times 1$ vectors and Σ_{11} is a $r \times r$ matrix.

Corollary 5.1. *Let* $\mathbf{X} \sim E_n(\mu, \Sigma, \phi)$. *Then*

$$\mathbf{X}_1 \sim E_r(\mu_1, \Sigma_{11}, \phi), \quad \mathbf{X}_2 \sim E_{n-r}(\mu_2, \Sigma_{22}, \phi).$$

Hence marginal distributions of elliptical distributions are elliptical and of the same type (with the same characteristic generator). The next result states that the conditional distribution of $\mathbf{X}_1$ given the value of $\mathbf{X}_2$ is also elliptical, but in general not of the same type as $\mathbf{X}_1$.

Theorem 5.3. *Let* $\mathbf{X} \sim E_n(\mu, \Sigma, \phi)$ *with* Σ *strictly positive definite. Then*

$$\mathbf{X}_1 \mid \mathbf{X}_2 = \mathbf{x} \sim E_r(\tilde{\mu}, \tilde{\Sigma}, \tilde{\phi}),$$

where $\tilde{\mu} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{x} - \mu_2)$ *and* $\tilde{\Sigma} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. *Moreover,* $\tilde{\phi} = \phi$ *if and only if* $\mathbf{X} \sim \mathcal{N}_n(\mu, \Sigma)$.

For the proof and details about $\tilde{\phi}$, see Fang, Kotz, and Ng (1987). For the extension to the case where $\text{rank}(\Sigma) < n$, see Cambanis, Huang, and Simons (1981).

The following lemma states that linear combinations of independent, elliptically distributed random vectors with the same dispersion matrix Σ (up to a positive constant) remain elliptical.

Lemma 5.1. *Let* $\mathbf{X} \sim E_n(\mu, \Sigma, \phi)$ *and* $\tilde{\mathbf{X}} \sim E_n(\tilde{\mu}, c\Sigma, \tilde{\phi})$ *for* $c > 0$ *be independent. Then for* $a, b \in \mathbb{R}$, $a\mathbf{X} + b\tilde{\mathbf{X}} \sim E_n(a\mu + b\tilde{\mu}, \Sigma, \phi^*)$ *with* $\phi^*(u) = \phi(a^2u)\tilde{\phi}(b^2cu)$.

Proof. By Definition 5.1, it is sufficient to show that for all $\mathbf{t} \in \mathbb{R}^n$

$$\begin{aligned}
\varphi_{a\mathbf{X}+b\tilde{\mathbf{X}}-a\mu-b\tilde{\mu}}(\mathbf{t}) &= \varphi_{a(\mathbf{X}-\mu)}(\mathbf{t})\varphi_{b(\tilde{\mathbf{X}}-\tilde{\mu})}(\mathbf{t}) \\
&= \phi((a\mathbf{t})^T\Sigma(a\mathbf{t}))\tilde{\phi}((b\mathbf{t})^T(c\Sigma)(b\mathbf{t})) \\
&= \phi(a^2\mathbf{t}^T\Sigma\mathbf{t})\tilde{\phi}(b^2c\mathbf{t}^T\Sigma\mathbf{t}).
\end{aligned}$$

□

As usual, let $\mathbf{X} \sim E_n(\mu, \Sigma, \phi)$. Whenever $0 < \text{Var}(X_i), \text{Var}(X_j) < \infty$,

$$\rho(X_i, X_j) := \text{Cov}(X_i, X_j)/\sqrt{\text{Var}(X_i)\text{Var}(X_j)} = \Sigma_{ij}/\sqrt{\Sigma_{ii}\Sigma_{jj}}.$$

This explains why linear correlation is a natural measure of dependence between random variables with a joint nondegenerate ($\Sigma_{ii} > 0$ for all i) elliptical distribution. Throughout this section we call the matrix R, with $R_{ij} = \Sigma_{ij}/\sqrt{\Sigma_{ii}\Sigma_{jj}}$, the linear correlation matrix of $\mathbf{X}$. Note that this definition is more general than the usual one and in this situation (elliptical distributions) makes more sense. Since an elliptical distribution is uniquely determined by μ, Σ and ϕ, the copula of a nondegenerate elliptically distributed random vector is uniquely determined by R and ϕ.

One practical problem with elliptical distributions in multivariate risk modelling is that all margins are of the same type. To construct a realistic multivariate distribution for some given risks, it may be reasonable to choose a copula of an elliptical distribution but different types of margins (not necessarily elliptical). One big drawback with such a model seems to be that the copula parameter R can no longer be estimated directly from data. Recall that for nondegenerate elliptical distributions with finite variances, R is just the usual linear correlation matrix. In such cases, R can be estimated using (robust) linear correlation estimators. One such robust estimator is provided by the next theorem. For nondegenerate nonelliptical distributions with finite variances and elliptical copulas, R does not correspond to the linear correlation matrix. However, since the Kendall's tau rank correlation matrix for a random vector is invariant under strictly increasing transformations of the vector components, and the next theorem provides a relation between the Kendall's tau rank correlation matrix and R for nondegenerate elliptical distributions, R can in fact easily be estimated from data.

Theorem 5.4. *Let $\mathbf{X} \sim E_n(\mu, \Sigma, \phi)$ with $\mathbb{P}\{X_i = \mu_i\} < 1$ and $\mathbb{P}\{X_j = \mu_j\} < 1$. Then*

$$\tau(X_i, X_j) = (1 - \sum_{x \in \mathbb{R}} (\mathbb{P}\{X_i = x\})^2)\, \frac{2}{\pi} \arcsin(R_{ij}), \tag{5.1}$$

where the sum extends over all atoms of the distribution of X_i. If rank$(\Sigma) \geq 2$, then (5.1) *simplifies to*

$$\tau(X_i, X_j) = (1 - (\mathbb{P}\{X_i = \mu_i\})^2)\, \frac{2}{\pi} \arcsin(R_{ij}).$$

For a proof, see Lindskog, McNeil, and Schmock (2001). Note that if $\mathbb{P}\{X_i = \mu_i\} = 0$ for all i, which is true for e.g. multivariate t or normal distributions with strictly positive definite dispersion matrices Σ, then

$$\tau(X_i, X_j) = \frac{2}{\pi} \arcsin(R_{ij})$$

for all i and j.

The nonparametric estimator of R, $\sin(\pi\widehat{\tau}/2)$ (dropping the subscript for simplicity), provided by the above theorem, inherits the robustness properties of the Kendall's tau estimator and is an efficient (low variance) estimator of R for both elliptical distributions and nonelliptical distributions with elliptical copulas.

5.2 Gaussian Copulas

The copula of the n-variate normal distribution with linear correlation matrix R is

$$C_R^{\mathrm{Ga}}(\mathbf{u}) = \Phi_R^n(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)),$$

where Φ_R^n denotes the joint distribution function of the n-variate standard normal distribution function with linear correlation matrix R, and Φ^{-1} denotes the inverse of the distribution function of the univariate standard normal distribution. Copulas of the above form are called Gaussian copulas. In the bivariate case the copula expression can be written as

$$C_R^{\mathrm{Ga}}(u, v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi(1 - R_{12}^2)^{1/2}} \exp\left\{ -\frac{s^2 - 2R_{12}st + t^2}{2(1 - R_{12}^2)} \right\} \mathrm{d}s\, \mathrm{d}t.$$

Note that R_{12} is simply the usual linear correlation coefficient of the corresponding bivariate normal distribution. Example 3.4 shows that Gaussian copulas do not have upper tail dependence. Since elliptical distributions are radially symmetric, the coefficient of upper and lower tail dependence are equal. Hence Gaussian copulas do not have lower tail dependence.

We now address the question of random variate generation from the Gaussian copula C_R^{Ga}. For our purpose, it is sufficient to consider only strictly positive definite matrices R. Write $R = AA^T$ for some $n \times n$ matrix A, and if $Z_1, \dots, Z_n \sim \mathcal{N}(0,1)$ are independent, then

$$\mu + A\mathbf{Z} \sim \mathcal{N}_n(\mu, R).$$

One natural choice of A is the Cholesky decomposition of R. The Cholesky decomposition of R is the unique lower-triangular matrix L with $LL^T = R$. Furthermore Cholesky decomposition routines are implemented in most mathematical software. This provides an easy algorithm for random variate generation from the Gaussian n-copula C_R^{Ga}.

Algorithm 5.1.

- Find the Cholesky decomposition A of R.
- Simulate n independent random variates $z_1, \dots, z_n$ from $\mathcal{N}(0,1)$.
- Set $\mathbf{x} = A\mathbf{z}$.
- Set $u_i = \Phi(x_i), i = 1, \dots, n$.
- $(u_1, \dots, u_n)^T \sim C_R^{\text{Ga}}$.

As usual Φ denotes the univariate standard normal distribution function. □

5.3 t-copulas

If $\mathbf{X}$ has the stochastic representation

$$\mathbf{X} =_d \mu + \frac{\sqrt{\nu}}{\sqrt{S}}\mathbf{Z}, \tag{5.2}$$

where $\mu \in \mathbb{R}^n$, $S \sim \chi^2_\nu$ and $\mathbf{Z} \sim \mathcal{N}_n(\mathbf{0}, \Sigma)$ are independent, then $\mathbf{X}$ has an n-variate t_ν-distribution with mean μ (for $\nu > 1$) and covariance matrix $\frac{\nu}{\nu-2}\Sigma$ (for $\nu > 2$). If $\nu \le 2$ then $\text{Cov}(\mathbf{X})$ is not defined. In this case we just interpret Σ as being the shape parameter of the distribution of $\mathbf{X}$.

The copula of $\mathbf{X}$ given by (5.2) can be written as

$$C^t_{\nu,R}(\mathbf{u}) = t^n_{\nu,R}(t_\nu^{-1}(u_1), \dots, t_\nu^{-1}(u_n)),$$

where $R_{ij} = \Sigma_{ij}/\sqrt{\Sigma_{ii}\Sigma_{jj}}$ for $i,j \in \{1, \dots, n\}$ and where $t^n_{\nu,R}$ denotes the distribution function of $\sqrt{\nu}\mathbf{Y}/\sqrt{S}$, where $S \sim \chi^2_\nu$ and $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{0}, R)$ are independent. Here t_ν denotes the (equal) margins of $t^n_{\nu,R}$, i.e. the distribution function of $\sqrt{\nu}Y_1/\sqrt{S}$. In the bivariate case the copula expression can be written as

$$C^t_{\nu,R}(u,v) = \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \frac{1}{2\pi(1-R_{12}^2)^{1/2}} \left\{1 + \frac{s^2 - 2R_{12}st + t^2}{\nu(1-R_{12}^2)}\right\}^{-(\nu+2)/2} \mathrm{d}s\,\mathrm{d}t.$$

Note that R_{12} is simply the usual linear correlation coefficient of the corresponding bivariate t_ν-distribution if $\nu > 2$.

If $(X_1, X_2)^T$ has a standard bivariate t-distribution with ν degrees of freedom and linear correlation matrix R, then $X_2 \mid X_1 = x$ is t-distributed with $v + 1$ degrees of freedom and

$$\mathbb{E}(X_2 \mid X_1 = x) = R_{12}x, \quad \text{Var}(X_2 \mid X_1 = x) = \left(\frac{\nu + x^2}{\nu + 1}\right)(1 - R_{12}^2).$$

This can be used to show that the t-copula has upper (and because of radial symmetry) equal lower tail dependence:

$$
\begin{aligned}
\lambda_U &= 2 \lim_{x\to\infty} \mathbb{P}(X_2 > x \mid X_1 = x) \\
&= 2 \lim_{x\to\infty} \bar{t}_{\nu+1}\left(\left(\frac{\nu+1}{\nu+x^2}\right)^{1/2} \frac{x - R_{12}x}{\sqrt{1-\rho_l^2}}\right) \\
&= 2 \lim_{x\to\infty} \bar{t}_{\nu+1}\left(\left(\frac{\nu+1}{\nu/x^2+1}\right)^{1/2} \frac{\sqrt{1-R_{12}}}{\sqrt{1+R_{12}}}\right) \\
&= 2\bar{t}_{\nu+1}\left(\sqrt{\nu+1}\sqrt{1-R_{12}}/\sqrt{1+R_{12}}\right).
\end{aligned}
$$

From this it is also seen that the coefficient of upper tail dependence is increasing in R_{12} and decreasing in ν, as one would expect. Furthermore, the coefficient of upper (lower) tail dependence tends to zero as the number of degrees of freedom tends to infinity for $R_{12} < 1$.
Coefficients of upper tail dependence for the bivariate t-copula are:

$\nu \backslash R_{12}$	-0.5	0	0.5	0.9	1
2	0.06	0.18	0.39	0.72	1
4	0.01	0.08	0.25	0.63	1
10	0.00	0.01	0.08	0.46	1
∞	0	0	0	0	1

The last row represents the Gaussian copula, i.e. no tail dependence.

It should be mentioned that the expression given above is just a special case of a general formula for the coefficient(s) of tail dependence for elliptical distributions with tail dependence. It turns out that if $\Sigma_{ii} > 0$ for all i and $-1 < \Sigma_{ij}/\sqrt{\Sigma_{ii}\Sigma_{jj}} < 1$ for all $i \neq j$, then the bivariate marginal distributions of an elliptically distributed random vector $\mathbf{X} =_d \mu + RA\mathbf{U} \sim E_n(\mu, \Sigma, \phi)$ has tail dependence if and only if R is so-called regularly varying (at ∞). For more details, see Hult and Lindskog (2001), and for details about regular variation in general see Resnick (1987) or Embrechts, Mikosch, and Klüppelberg (1997).

Equation (5.2) provides an easy algorithm for random variate generation from the t-copula, $C^t_{\nu,R}$.

Algorithm 5.2.

- Find the Cholesky decomposition A of R.
- Simulate n independent random variates $z_1, \ldots, z_n$ from $\mathcal{N}(0,1)$.
- Simulate a random variate s from χ^2_ν independent of $z_1, \ldots, z_n$.
- Set $\mathbf{y} = A\mathbf{z}$.
- Set $\mathbf{x} = \frac{\sqrt{\nu}}{\sqrt{s}}\mathbf{y}$.
- Set $u_i = t_\nu(x_i), i = 1, \ldots, n$.
- $(u_1, \ldots, u_n)^T \sim C^t_{\nu,R}$.

$\square$

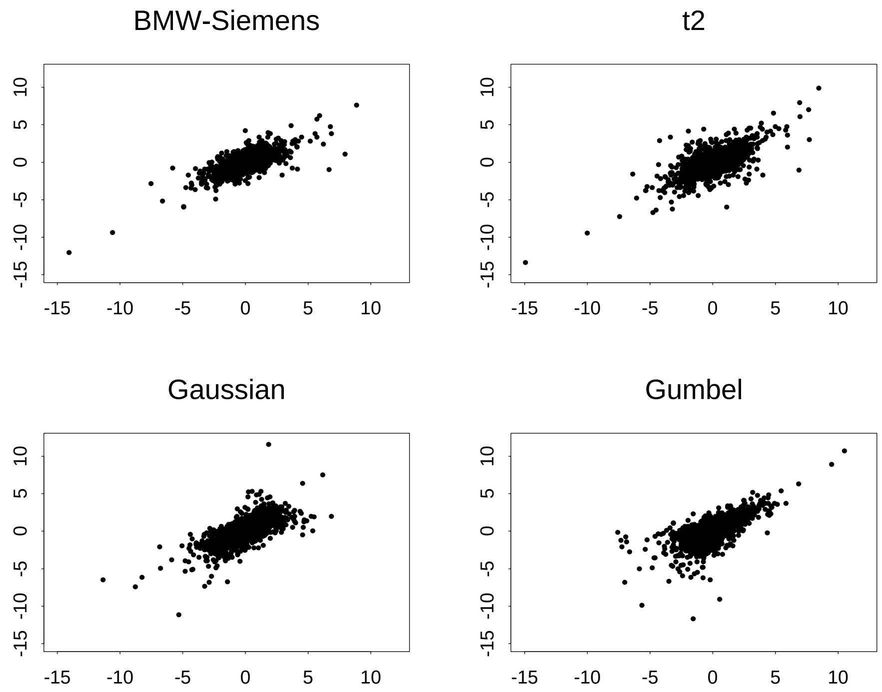

Figure 5.1: The upper left plot shows BMW-Siemens daily log returns from 1989 to 1996. The other plots show samples from bivariate distributions with t_4-margins and Kendall's tau 0.5.

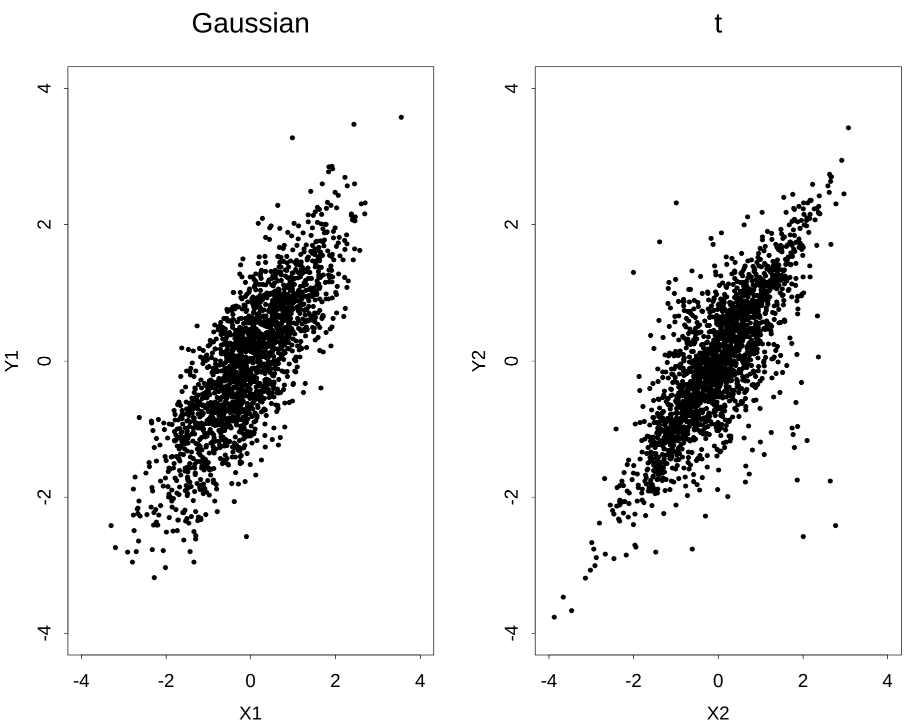

Figure 5.2: Samples from two distributions with standard normal margins, $R_{12} = 0.8$ but different dependence structures. $(X1, Y1)^T$ has a Gaussian copula and $(X2, Y2)^T$ has a t_2-copula.

Figures 5.1 and 5.2 show samples from bivariate distributions with Gaussian and t-copulas. In Figure 5.1, we have contrasted a real example (BMW-Siemens daily return data) with simulated data using marginal t_4 tails, corresponding Kendall's tau (0.5) and varying copulas. Note that the Gaussian copula does not get the extreme joint tail observations clearly present in the real data. The t_2-copula seems to be able to do a much better job in that respect. Indeed the t_2-generated scatter plot shows most of the graphical features in the real data. Note that these examples were only introduced to highlight the simulation procedures and do not constitute a detailed statistical analysis. Figure 5.2 (a simulated example) further highlights the difference between the Gaussian and t-copulas, this time with standard normal margins.

The algorithms presented for the Gaussian and t-copulas are fast and easy to implement. We want to emphasize the potential usefulness of t-copulas as an alternative to Gaussian copulas. Both Gaussian and t-copulas are easily parameterized by the linear correlation matrix, but only t-copulas yield dependence structures with tail dependence.

6 Archimedean Copulas

The copula families we have discussed so far have been derived from certain families of multivariate distribution functions using Sklar's Theorem. We have seen that elliptical copulas are simply the distribution functions of componentwise transformed elliptically distributed random vectors. Since simulation from elliptical distributions is easy, so is simulation from elliptical copulas. There are however drawbacks: elliptical copulas do not have closed form expressions and are restricted to have radial symmetry ($C = \widehat{C}$). In many finance and insurance applications it seems reasonable that there is a stronger dependence between big losses (e.g. a stock market crash) than between big gains. Such asymmetries cannot be modelled with elliptical copulas.

In this section we discuss an important class of copulas called Archimedean copulas. This class of copulas is worth studying for a number of reasons. Many interesting parametric families of copulas are Archimedean and the class of Archimedean copulas allow for a great variety of different dependence structures. Furthermore, in contrast to elliptical copulas, all commonly encountered Archimedean copulas have closed form expressions. Unlike the copulas discussed so far these copulas are not derived from multivariate distribution functions using Sklar's Theorem. A consequence of this is that we need somewhat technical conditions to assert that multivariate extensions of Archimedean 2-copulas are proper n-copulas. A further disadvantage is that multivariate extensions of Archimedean copulas in general suffer from lack of free parameter choice in the sense that some of the entries in the resulting rank correlation matrix are forced to be equal. At the end of this section we present one possible multivariate extension of Archimedean copulas. For other multivariate extensions we refer to Joe (1997).

There is much written about Archimedean copulas. For some background on bivariate Archimedean copulas see Genest and MacKay (1986b). For parameter estimation and a discussion on other statistical questions we refer to Genest and Rivest (1993). Good references on Archimedean copulas in general are Genest and MacKay (1986a), Nelsen (1999) and Joe (1997). See also the webpage http://www.mat.ulaval.ca/pages/genest/ for further related work.

6.1 Definitions

We begin with a general definition of Archimedean copulas, which can be found in Nelsen (1999) p. 90. As our aim is the construction of multivariate extensions of Archimedean 2-copulas, this general definition will later prove to be a bit more general than needed.

Definition 6.1. Let φ be a continuous, strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\varphi(1)=0$. The pseudo-inverse of φ is the function $\varphi^{[-1]}:[0,\infty]\to[0,1]$ given by

$$\varphi^{[-1]}(t)=\begin{cases}\varphi^{-1}(t), & 0\le t\le\varphi(0),\\ 0, & \varphi(0)\le t\le\infty.\end{cases}$$

□

Note that $\varphi^{[-1]}$ is continuous and decreasing on $[0,\infty]$, and strictly decreasing on $[0,\varphi(0)]$. Furthermore, $\varphi^{[-1]}(\varphi(u))=u$ on $[0,1]$, and

$$\varphi(\varphi^{[-1]}(t))=\begin{cases}t, & 0\le t\le\varphi(0),\\ \varphi(0), & \varphi(0)\le t\le\infty.\end{cases}$$

Finally, if $\varphi(0)=\infty$, then $\varphi^{[-1]}=\varphi^{-1}$.

Theorem 6.1. *Let φ be a continuous, strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\varphi(1)=0$, and let $\varphi^{[-1]}$ be the pseudo-inverse of φ. Let C be the function from $[0,1]^2$ to $[0,1]$ given by*

$$C(u,v)=\varphi^{[-1]}(\varphi(u)+\varphi(v)). \tag{6.1}$$

Then C is a copula if and only if φ is convex.

For a proof, see Nelsen (1999) p. 91.

Copulas of the form (6.1) are called Archimedean copulas. The function φ is called a generator of the copula. If $\varphi(0)=\infty$, we say that φ is a strict generator. In this case, $\varphi^{[-1]}=\varphi^{-1}$ and $C(u,v)=\varphi^{-1}(\varphi(u)+\varphi(v))$ is said to be a strict Archimedean copula.

Example 6.1. Let $\varphi(t)=(-\ln t)^\theta$, where $\theta\ge 1$. Clearly $\varphi(t)$ is continuous and $\varphi(1)=0$. $\varphi'(t)=-\theta(-\ln t)^{\theta-1}\frac{1}{t}$, so φ is a strictly decreasing function from $[0,1]$ to $[0,\infty]$. $\varphi''(t)\ge 0$ on $[0,1]$, so φ is convex. Moreover $\varphi(0)=\infty$, so φ is a strict generator. From (6.1) we get

$$C_\theta(u,v)=\varphi^{-1}(\varphi(u)+\varphi(v))=\exp(-[(-\ln u)^\theta+(-\ln v)^\theta]^{1/\theta}).$$

Furthermore $C_1=\Pi$ and $\lim_{\theta\to\infty}C_\theta=M$ (recall that $\Pi(u,v)=uv$ and $M(u,v)=\min(u,v)$). This copula family is called the Gumbel family. As shown in Example 3.3 this copula family has upper tail dependence. □

Example 6.2. Let $\varphi(t)=(t^{-\theta}-1)/\theta$, where $\theta\in[-1,\infty)\setminus\{0\}$. This gives the Clayton family

$$C_\theta(u,v)=\max([u^{-\theta}+v^{-\theta}-1]^{-1/\theta},0).$$

For $\theta>0$ the copulas are strict and the copula expression simplifies to

$$C_\theta(u,v)=(u^{-\theta}+v^{-\theta}-1)^{-1/\theta}. \tag{6.2}$$

The Clayton family has lower tail dependence for $\theta>0$, and $C_{-1}=W$, $\lim_{\theta\to 0}C_\theta=\Pi$ and $\lim_{\theta\to\infty}C_\theta=M$. Since most of the following results are results for strict Archimedean copulas we will refer to (6.2) as the Clayton family. □

Example 6.3. Let $\varphi(t) = -\ln \frac{e^{-\theta t}-1}{e^{-\theta}-1}$, where $\theta \in \mathbb{R} \setminus \{0\}$. This gives the Frank family

$$C_\theta(u,v) = -\frac{1}{\theta} \ln \left(1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right).$$

The Frank copulas are strict Archimedean copulas. Furthermore $\lim_{\theta\to-\infty} C_\theta = W$, $\lim_{\theta\to 0} C_\theta = \Pi$ and $\lim_{\theta\to\infty} C_\theta = M$. Members of the Frank family are the only Archimedean copulas which satisfy the equation $C(u,v) = \widehat{C}(u,v)$ for so-called radial symmetry, see Frank (1979) for details. □

Example 6.4. Let $\varphi(t) = 1 - t$ for t in $[0,1]$. Then $\varphi^{[-1]}(t) = 1 - t$ for t in $[0,1]$, and 0 for $t > 1$; i.e., $\varphi^{[-1]}(t) = \max(1-t, 0)$. Since $C(u,v) = \max(u+v-1, 0) =: W(u,v)$, we see that the bivariate Fréchet–Hoeffding lower bound W is Archimedean. □

6.2 Properties

The results in the following theorem will enable us to formulate multivariate extensions of Archimedean copulas.

Theorem 6.2. *Let C be an Archimedean copula with generator φ. Then*

1. *C is symmetric, i.e. $C(u,v) = C(v,u)$ for all u, v in $[0,1]$.*
2. *C is associative, i.e. $C(C(u,v),w) = C(u, C(v,w))$ for all u, v, w in $[0,1]$.*

Proof. The first part follows directly from (6.1). For 2.,

$$\begin{aligned} C(C(u,v),w) &= \varphi^{[-1]}(\varphi(\varphi^{[-1]}(\varphi(u)+\varphi(v))) + \varphi(w)) \\ &= \varphi^{[-1]}(\varphi(u)+\varphi(v)+\varphi(w)) \\ &= \varphi^{[-1]}(\varphi(u)+\varphi(\varphi^{[-1]}(\varphi(v)+\varphi(w)))) = C(u, C(v,w)). \end{aligned}$$

□

The associativity property of Archimedean copulas is not shared by copulas in general as shown by the following example.

Example 6.5. Let C_θ be a member of the bivariate Farlie-Gumbel-Morgenstern family of copulas, i.e. $C_\theta(u,v) = uv + \theta uv(1-u)(1-v)$, for $\theta \in [-1,1]$. Then

$$C_\theta\left(\frac{1}{4}, C_\theta\left(\frac{1}{2}, \frac{1}{3}\right)\right) \neq C_\theta\left(C_\theta\left(\frac{1}{4}, \frac{1}{2}\right), \frac{1}{3}\right)$$

for all $\theta \in [-1,1]\backslash\{0\}$. Hence the only member of the bivariate Farlie-Gumbel-Morgenstern family of copulas that is Archimedean is Π. □

Theorem 6.3. *Let C be an Archimedean copula generated by φ and let*

$$K_C(t) = V_C(\{(u,v) \in [0,1]^2 \mid C(u,v) \le t\}).$$

Then for any t in $[0,1]$,

$$K_C(t) = t - \frac{\varphi(t)}{\varphi'(t^+)}. \tag{6.3}$$

For a proof, see Nelsen (1999) p. 102.

Corollary 6.1. *If $(U, V)^T$ has distribution function C, where C is an Archimedean copula generated by φ, then the function K_C given by (6.3) is the distribution function of the random variable $C(U, V)$.*

The next theorem will provide the basis for a general algorithm for random variate generation from Archimedean copulas. Before the theorem can be stated we need an expression for the density of an absolutely continuous Archimedean copula. From (6.1) it follows that

$$\begin{aligned}
\varphi'(C(u,v))\frac{\partial}{\partial u}C(u,v) &= \varphi'(u), \\
\varphi'(C(u,v))\frac{\partial}{\partial v}C(u,v) &= \varphi'(v), \\
\varphi''(C(u,v))\frac{\partial}{\partial u}C(u,v)\frac{\partial}{\partial v}C(u,v) + \varphi'(C(u,v))\frac{\partial^2}{\partial u \partial v}C(u,v) &= 0,
\end{aligned}$$

and hence

$$\frac{\partial^2}{\partial u \partial v}C(u,v) = -\frac{\varphi''(C(u,v))\frac{\partial}{\partial u}C(u,v)\frac{\partial}{\partial v}C(u,v)}{\varphi'(C(u,v))} = -\frac{\varphi''(C(u,v))\varphi'(u)\varphi'(v)}{[\varphi'(C(u,v))]^3}.$$

Thus, when C is absolutely continuous, its density is given by

$$\frac{\partial^2}{\partial u \partial v}C(u,v) = -\frac{\varphi''(C(u,v))\varphi'(u)\varphi'(v)}{[\varphi'(C(u,v))]^3}. \tag{6.4}$$

Theorem 6.4. *Under the hypotheses of Corollary 6.1, the joint distribution function $H(s,t)$ of the random variables $S = \varphi(U)/[\varphi(U) + \varphi(V)]$ and $T = C(U, V)$ is given by $H(s,t) = sK_C(t)$ for all (s,t) in $[0,1]^2$. Hence S and T are independent, and S is uniformly distributed on $[0,1]$.*

Proof. (This proof, for the case when C is absolutely continuous, can be found in Nelsen (1999) p. 104. For the general case, see Genest and Rivest (1993).) The joint density $h(s,t)$ of S and T is given by

$$h(s,t) = \frac{\partial^2}{\partial u \partial v}C(u,v)\left|\frac{\partial(u,v)}{\partial(s,t)}\right|,$$

where $\partial^2 C(u,v)/\partial u \partial v$ is given by (6.4) and $\partial(u,v)/\partial(s,t)$ denotes the Jacobian of the transformation $\varphi(u) = s\varphi(t)$, $\varphi(v) = (1-s)\varphi(t)$. But

$$\frac{\partial(u,v)}{\partial(s,t)} = \frac{\varphi(t)\varphi'(t)}{\varphi'(u)\varphi'(v)},$$

and hence

$$h(s,t) = \left(-\frac{\varphi''(t)\varphi'(u)\varphi'(v)}{[\varphi'(t)]^3}\right)\left(-\frac{\varphi(t)\varphi'(t)}{\varphi'(u)\varphi'(v)}\right) = \frac{\varphi''(t)\varphi(t)}{[\varphi'(t)]^2}.$$

Therefore

$$H(s,t) = \int_0^s \int_0^t \frac{\varphi''(y)\varphi(y)}{[\varphi'(y)]^2}dydx = s\left[y - \frac{\varphi(y)}{\varphi'(y)}\right]_0^t = sK_C(t),$$

from which the conclusion follows. $\square$

An application of Theorem 6.4 is the following algorithm for generating random variates $(u, v)^T$ whose joint distribution is an Archimedean copula C with generator φ.

Algorithm 6.1.

- Simulate two independent $U(0,1)$ random variates s and q.
- Set $t = K_C^{-1}(q)$, where K_C is the distribution function of $C(U,V)$.
- Set $u = \varphi^{[-1]}(s\varphi(t))$ and $v = \varphi^{[-1]}((1-s)\varphi(t))$.

Note that the variates s and t correspond to the random variables S and T in Theorem 6.4 and from the proof it follows that this algorithm yields the desired result. □

Example 6.6. Consider the Archimedean copula family given by

$$C_\theta(u,v) = \left(1 + \left[(u^{-1}-1)^\theta + (v^{-1}-1)^\theta\right]^{1/\theta}\right)^{-1}$$

generated by $\varphi_\theta(t) = (t^{-1}-1)^\theta$ for $\theta \geq 1$. To generate a random variate from C we simply apply Algorithm 6.1 with

$$\begin{aligned}
\varphi_\theta(t) &= (t^{-1}-1)^\theta, \\
\varphi_\theta^{-1}(t) &= (t^{1/\theta}+1)^{-1}, \\
K_{C_\theta}^{-1}(t) &= \frac{\theta_i+1}{2} - \sqrt{(\frac{\theta_i+1}{2})^2 - \theta_i s}.
\end{aligned}$$

□

6.3 Kendall's tau Revisited

Recall that Kendall's tau for a copula C can be expressed as a double integral of C. This double integral is in most cases not straightforward to evaluate. However for an Archimedean copula, Kendall's tau can be expressed as an (one-dimensional) integral of the generator and its derivative, as shown in the following theorem from Genest and MacKay (1986a).

Theorem 6.5. *Let X and Y be random variables with an Archimedean copula C generated by φ. Kendall's tau of X and Y is given by*

$$\tau_C = 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}\,\mathrm{d}t. \tag{6.5}$$

Proof. Let U and V be $U(0,1)$ random variables with joint distribution function C, and let K_C denote the distribution function of $C(U,V)$. Then from Theorem 3.3 we have

$$\begin{aligned}
\tau_C &= 4\mathbb{E}(C(U,V)) - 1 = 4\int_0^1 t\,\mathrm{d}K_C(t) - 1 \\
&= 4\left([tK_C(t)]_0^1 - \int_0^1 K_C(t)\,\mathrm{d}t\right) - 1 = 3 - 4\int_0^1 K_C(t)\,\mathrm{d}t.
\end{aligned}$$

From Theorem 6.3 and Corollary 6.1 it follows that

$$K_C(t) = t - \frac{\varphi(t)}{\varphi'(t^+)}.$$

Since φ is convex, $\varphi'(t^+)$ and $\varphi'(t^-)$ exist for all t in $(0,1)$ and the set $\{t \in (0,1) \mid \varphi'(t^+) \neq \varphi'(t^-)\}$ is at most countable (i.e. it has Lebesgue measure zero). Hence

$$\tau_C = 3 - 4\int_0^1 \big(t - \frac{\varphi(t)}{\varphi'(t^+)}\big)\,\mathrm{d}t = 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}\,\mathrm{d}t.$$

□

Example 6.7. Consider the Gumbel family with generator $\varphi(t) = (-\ln t)^\theta$, for $\theta \geq 1$. Then $\varphi(t)/\varphi'(t) = (t\ln t)/\theta$. Using Theorem 6.5 we can calculate Kendall's tau for the Gumbel family.

$$\tau_\theta = 1 + 4\int_0^1 \frac{t\ln t}{\theta}\,\mathrm{d}t = 1 + \frac{4}{\theta}\left(\left[\frac{t^2}{2}\ln t\right]_0^1 - \int_0^1 \frac{t}{2}\,\mathrm{d}t\right) = 1 + \frac{4}{\theta}(0 - 1/4) = 1 - 1/\theta.$$

As a consequence, in order to have Kendall's tau equal to 0.5 in Figure 5.1 (the Gumbel case), we put $\theta = 2$. □

Example 6.8. Consider the Clayton family with generator $\varphi(t) = (t^{-\theta} - 1)/\theta$, for $\theta \in [-1,\infty) \setminus \{0\}$. Then $\varphi(t)/\varphi'(t) = (t^{\theta+1} - t)/\theta$. Using Theorem 6.5 we can calculate Kendall's tau for the Clayton family.

$$\tau_\theta = 1 + 4\int_0^1 \frac{t^{\theta+1} - t}{\theta}\,\mathrm{d}t = 1 + \frac{4}{\theta}\left(\frac{1}{\theta+2} - \frac{1}{2}\right) = \frac{\theta}{\theta+2}.$$

□

Example 6.9. Consider the Frank family presented in Example 6.3. It can be shown that (see e.g. Genest (1987)) Kendall's tau is $\tau_\theta = 1 - 4(1 - D_1(\theta))/\theta$, where $D_k(x)$ is the Debye function, given by

$$D_k(x) = \frac{k}{x^k}\int_0^x \frac{t^k}{e^t - 1}\,\mathrm{d}t$$

for any positive integer k. □

6.4 Tail Dependence Revisited

For Archimedean copulas, tail dependence can be expressed in terms of the generators.

Theorem 6.6. *Let φ be a strict generator such that φ^{-1} belongs to the class of Laplace transforms of strictly positive random variables. If ${\varphi^{-1}}'(0)$ is finite, then*

$$C(u,v) = \varphi^{-1}(\varphi(u) + \varphi(v))$$

does not have upper tail dependence. If C has upper tail dependence, then ${\varphi^{-1}}'(0) = -\infty$ and the coefficient of upper tail dependence is given by

$$\lambda_U = 2 - 2\lim_{s\searrow 0}[{\varphi^{-1}}'(2s)/{\varphi^{-1}}'(s)].$$

Proof. (This proof can be found in Joe (1997), p. 103.) Note that

$$\begin{aligned}\lim_{u\nearrow 1}\overline{C}(u,u)/(1-u) &= \lim_{u\nearrow 1}[1-2u+\varphi^{-1}(2\varphi(u))]/(1-u)\\ &= 2-2\lim_{u\nearrow 1}\varphi^{-1\prime}(2\varphi(u))/\varphi^{-1\prime}(\varphi(u))\\ &= 2-2\lim_{s\searrow 0}[\varphi^{-1\prime}(2s)/\varphi^{-1\prime}(s)].\end{aligned}$$

If $\varphi^{-1\prime}(0)\in(-\infty,0)$, then the limit is zero and C does not have upper tail dependence. Since $\varphi^{-1\prime}(0)$ is the negative of the expectation of a strictly positive random variable, $\varphi^{-1\prime}(0)<0$ from which the conclusion follows. □

The additional condition on the generator φ might seem somewhat strange. It will however prove quite natural when we turn to the construction of multivariate Archimedean copulas. Furthermore, the condition is satisfied by the majority of the commonly encountered Archimedean copulas.

Example 6.10. The Gumbel copulas are strict Archimedean with generator $\varphi(t)=(-\ln t)^{\theta}$. Hence $\varphi^{-1}(s)=\exp(-s^{1/\theta})$ and $\varphi^{-1\prime}(s)=-s^{1/\theta-1}\exp(-s^{1/\theta})/\theta$. Using Theorem 6.6 we get

$$\lambda_U=2-2\lim_{s\searrow 0}[\varphi^{-1\prime}(2s)/\varphi^{-1\prime}(s)]=2-2^{1/\theta}\lim_{s\searrow 0}\left[\frac{\exp(-(2s)^{1/\theta})}{\exp(-s^{1/\theta})}\right]=2-2^{1/\theta},$$

see also Example 3.3. □

Theorem 6.7. *Let φ be as in Theorem 6.6. The coefficient of lower tail dependence for the copula $C(u,v)=\varphi^{-1}(\varphi(u)+\varphi(v))$ is equal to*

$$\lambda_L=2\lim_{s\to\infty}[\varphi^{-1\prime}(2s)/\varphi^{-1\prime}(s)].$$

The proof is similar to that of Theorem 6.6.

Example 6.11. Consider the Clayton family given by $C_\theta(u,v)=(u^{-\theta}+v^{-\theta}-1)^{-1/\theta}$, for $\theta>0$. This strict copula family has generator $\varphi(t)=(t^{-\theta}-1)/\theta$. It follows that $\varphi^{-1}(s)=(1+\theta s)^{-1/\theta}$. Using Theorem 6.6 and 6.7 shows that $\lambda_U=0$ and that the coefficient of lower tail dependence given by

$$\lambda_L=2\lim_{s\to\infty}[\varphi^{-1\prime}(2s)/\varphi^{-1\prime}(s)]=2\lim_{s\to\infty}\left[\frac{(1+2\theta s)^{-1/\theta-1}}{(1+\theta s)^{-1/\theta-1}}\right]=2\cdot 2^{-1/\theta-1}=2^{-1/\theta}.$$

□

Example 6.12. Consider the Frank family given by

$$C_\theta(u,v)=-\frac{1}{\theta}\ln\left(1+\frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right)$$

for $\theta\in\mathbb{R}\setminus\{0\}$. This strict copula family has generator $\varphi(t)=-\ln\frac{e^{-\theta t}-1}{e^{-\theta}-1}$. It follows that $\varphi^{-1}(s)=-\frac{1}{\theta}\ln(1-(1-e^{-\theta})e^{-s})$ and $\varphi^{-1\prime}(s)=-\frac{1}{\theta}(1-e^{-\theta})e^{-s}/(1-(1-e^{-\theta})e^{-s})$. Since

$$\varphi^{-1\prime}(0)=-\frac{e^{\theta}-1}{\theta}$$

is finite, the Frank family does not have upper tail dependence according to Theorem 6.6. Furthermore, members of the Frank family are radially symmetric, i.e. $C=\widehat{C}$, and hence the Frank family does not have lower tail dependence. □

6.5 Multivariate Archimedean Copulas

In this section we look at the construction of one particular multivariate extension of Archimedean 2-copulas. For other multivariate extensions see Joe (1997). It should be noted that in order to show that other multivariate extensions are proper copulas, we essentially have to go through the same arguments as those given below.

The expression for the n-dimensional product copula Π^n, with $\mathbf{u} = (u_1, \dots, u_n)^T$, can be written as $\Pi^n(\mathbf{u}) = u_1 \dots u_n = \exp(-[(-\ln u_1) + \cdots + (-\ln u_n)])$. This naturally leads to the following generalization of (6.1):

$$C^n(\mathbf{u}) = \varphi^{[-1]}(\varphi(u_1) + \cdots + \varphi(u_n)). \tag{6.6}$$

In the 3-dimensional case,

$$C^3(u_1, u_2, u_3) = \varphi^{[-1]}(\varphi \circ \varphi^{[-1]}(\varphi(u_1) + \varphi(u_2)) + \varphi(u_3)) = C(C(u_1, u_2), u_3),$$

and in the 4-dimensional case,

$$\begin{aligned} C^3(u_1, \dots, u_4) &= \varphi^{[-1]}(\varphi \circ \varphi^{[-1]}(\varphi \circ \varphi^{[-1]}(\varphi(u_1) + \varphi(u_2)) + \varphi(u_3)) + \varphi(u_4)) \\ &= C(C^3(u_1, u_2, u_3), u_4) = C(C(C(u_1, u_2), u_3), u_4). \end{aligned}$$

Whence in general, for $n \geq 3$, $C^n(u_1, \dots, u_n) = C(C^{n-1}(u_1, u_2, \dots, u_{n-1}), u_n)$. This technique of constructing higher-dimensional copulas generally fails. But since Archimedean copulas are symmetric and associative it seems more likely that C^n as defined above, given certain additional properties of φ (and $\varphi^{[-1]}$), is indeed a copula for $n \geq 3$.

Definition 6.2. A function $g(t)$ is completely monotone on the interval I if it has derivatives of all orders which alternate in sign, i.e. if it satisfies

$$(-1)^k \frac{d^k}{dt^k} g(t) \geq 0$$

for all t in the interior of I and $k = 0, 1, 2, \dots$. □

If $g : [0, \infty) \mapsto [0, \infty)$ is completely monotone on $[0, \infty)$ and there is a $t \in [0, \infty)$ such that $g(t) = 0$, then $g(t) = 0$ for all $t \in [0, \infty)$. Hence if the pseudo-inverse $\varphi^{[-1]}$ of an Archimedean generator φ is completely monotone, then $\varphi^{[-1]}(t) > 0$ for all $t \in [0, \infty)$ and hence $\varphi^{[-1]} = \varphi^{-1}$.

The following theorem from Kimberling (1974) gives necessary and sufficient conditions for the function (6.6) to be an n-copula.

Theorem 6.8. *Let φ be a continuous strictly decreasing function from $[0, 1]$ to $[0, \infty]$ such that $\varphi(0) = \infty$ and $\varphi(1) = 0$, and let φ^{-1} denote the inverse of φ. If C^n is the function from $[0, 1]^n$ to $[0, 1]$ given by (6.6), then C^n is an n-copula for all $n \geq 2$ if and only if φ^{-1} is completely monotone on $[0, \infty)$.*

This theorem can be partially extended to the case where φ is nonstrict and $\varphi^{[-1]}$ is m-monotone on $[0, \infty)$ for some $m \geq 2$, that is, the derivatives of $\varphi^{[-1]}$ alter sign up to and including the mth order on $[0, \infty)$. Then the function C^n given by (6.6) is an n-copula for $2 \leq n \leq m$. However, for most practical purposes, the class of strict generators φ such that φ^{-1} is completely monotone is a rich enough class.

The following corollary shows that the generators suitable for extensions to arbitrary dimensions of Archimedean 2-copulas correspond to copulas which can model only positive dependence.

Corollary 6.2. *If the inverse φ^{-1} of a strict generator φ of an Archimedean copula C is completely monotone, then $C \succ \Pi$, i.e. $C(u,v) \geq uv$ for all u, v in $[0,1]$.*

For a proof, see Nelsen (1999) p. 122.

While it is simple to generate n-copulas of the form given by (6.6), they suffer from a very limited dependence structure since all k-margins are identical, they are distribution functions of n exchangeable $U(0,1)$ random variables. One would like to have a multivariate extension of the Archimedean 2-copula given by (6.1) which allows for nonexchangeability. Such multivariate extensions are discussed in Joe (1997). We will now discuss one such extension in detail. Since any multivariate extension should contain (6.6) as a special case, clearly the necessary conditions for (6.6) to be a copula has to be satisfied. In the light of Theorem 6.8, we restrict ourselves to strict generators.

The expression for the general multivariate extension of (6.1) we will now discuss is notationally complex. For that reason we will discuss sufficient conditions for the 3- and 4-dimensional extensions to be proper 3- and 4-copulas respectively. The pattern and conditions indicated generalize in an obvious way to higher dimensions. The 3-dimensional generalization of (6.1) is

$$\varphi_1^{-1}(\varphi_1 \circ \varphi_2^{-1}(\varphi_2(u_1) + \varphi_2(u_2)) + \varphi_1(u_3)), \tag{6.7}$$

where φ_1 and φ_2 are generators of strict Archimedean copulas. The 4-dimensional generalization of (6.1) is

$$\varphi_1^{-1}(\varphi_1 \circ \varphi_2^{-1}(\varphi_2 \circ \varphi_3^{-1}(\varphi_3(u_1) + \varphi_3(u_2)) + \varphi_2(u_3)) + \varphi_1(u_4)), \tag{6.8}$$

where φ_1, φ_2 and φ_3 are generators of strict Archimedean copulas. The expressions (6.7) and (6.8) can be written as $C_1(C_2(u_1,u_2),u_3)$ and $C_1(C_2(C_3(u_1,u_2),u_3),u_4)$, respectively, where C_i denotes an Archimedean copula generated by φ_i.

If generators φ_i are chosen so that certain conditions are satisfied, then multivariate copulas can be obtained such that each bivariate margin has the form (6.1) for some i. However, the number of distinct generators φ_i among the $n(n-1)/2$ bivariate margins is only $n-1$, so that the resulting dependence structure is one of partial exchangeability.

Clearly the generators have to satisfy the necessary conditions for the n-copula given by (6.6) in order to make (6.7) and (6.8) valid copula expressions. What other conditions are needed to make these proper copulas? To answer that question we now introduce function classes $\mathcal{L}_n$ and $\mathcal{L}_n^*$. Let

$$\mathcal{L}_n = \{\phi : [0,\infty) \to [0,1] \mid \phi(0) = 1,\ \phi(\infty) = 0,\ (-1)^j \phi^{(j)} \geq 0,\ j = 1,\dots,n\},$$

$n = 1, 2, \dots, \infty$, with $\mathcal{L}_\infty$ being the class of Laplace transforms of strictly positive random variables.

Also introduce

$$\mathcal{L}_n^* = \{\omega : [0,\infty) \to [0,\infty) \mid \omega(0) = 0,\ \omega(\infty) = \infty,\ (-1)^{j-1} \omega^{(j)} \geq 0,\ j = 1,\dots,n\},$$

$n = 1, 2, \dots, \infty$. Note that $\varphi^{-1} \in \mathcal{L}_1$ if φ is the generator of a strict Archimedean copula. The functions in $\mathcal{L}_n^*$ are usually compositions of the form $\psi^{-1} \circ \phi$ with ψ, $\phi \in \mathcal{L}_1$.
Note also that with this notation, the necessary and sufficient conditions for (6.6) to be a proper copula is that $\varphi^{-1} \in \mathcal{L}_n$ and that, if (6.6) is a copula for all n, then φ^{-1} must be completely monotone and hence be a Laplace transform of a strictly positive random variable.

It turns out that if φ_1^{-1} and φ_2^{-1} are completely monotone (Laplace transforms of strictly positive random variables) and $\varphi_1 \circ \varphi_2^{-1} \in \mathcal{L}_\infty^*$, then (6.7) is a proper copula. Note that (6.7) has $(1,2)$ bivariate margin of the form (6.1) with generator φ_2 and $(1,3)$ and $(2,3)$ bivariate margins of the form (6.1) with generator φ_1. Also (6.6) is the special case of (6.7) with $\varphi_1 = \varphi_2$. The 3-dimensional copula in (6.7) has a $(1,2)$ bivariate margin copula which is larger than the $(1,3)$ and $(2,3)$ bivariate margin copulas (which are identical).

As one would expect, there are similar conditions for the 4-dimensional case. If φ_1^{-1}, φ_2^{-1} and φ_3^{-1} are completely monotone (Laplace transforms of strictly positive random variables) and $\varphi_1 \circ \varphi_2^{-1}$ and $\varphi_2 \circ \varphi_3^{-1}$ are in $\mathcal{L}_\infty^*$, then (6.8) is a proper copula. Note that all 3-dimensional margins of (6.8) have the form (6.7) and all bivariate margins have the form (6.1). Clearly the idea underlying (6.7) and (6.8) generalize to higher dimensions.

Example 6.13. Let $\varphi_i(t) = (-\ln t)^{\theta_i}$ with $\theta_i \geq 1$ for $i = 1, \dots, n$, i.e. the generators of Gumbel copulas. What conditions do we have to impose on $\theta_1, \dots, \theta_n$ in order to obtain an n-dimensional extension of the Gumbel family of the form indicated above (expressions (6.7) and (6.8)). It should first be noted that $\varphi_i^{-1} \in \mathcal{L}_\infty$ for all i, so (6.6) with the above generators gives an n-copula for all $n \geq 2$. Secondly, $\varphi_i \circ \varphi_{i+1}^{-1}(t) = t^{\theta_i/\theta_{i+1}}$. If $\theta_i/\theta_{i+1} \notin \mathbb{N}$, then the nth derivative of $\varphi_i \circ \varphi_{i+1}^{-1}(t)$ is given by

$$\frac{\theta_i}{\theta_{i+1}} \cdots \left(\frac{\theta_i}{\theta_{i+1}} - (n-1) \right) t^{\theta_i/\theta_{i+1} - n}.$$

Hence if $\theta_i/\theta_{i+1} \notin \mathbb{N}$, then $\varphi_i \circ \varphi_{i+1}^{-1} \in \mathcal{L}_\infty^*$ if and only if $\theta_i/\theta_{i+1} < 1$. If $\theta_i/\theta_{i+1} \in \mathbb{N}$, then $\varphi_i \circ \varphi_{i+1}^{-1} \in \mathcal{L}_\infty^*$ if and only if $\theta_i/\theta_{i+1} = 1$. Hence an n-dimensional extension of the Gumbel family of the form indicated above, given by

$$\exp\{-([(-\ln u_1)^{\theta_2} + (-\ln u_2)^{\theta_2}]^{\theta_1/\theta_2} + (-\ln u_3)^{\theta_1})^{1/\theta_1}\}$$

in the 3-dimensional case, is a proper n-copula if $\theta_1 \leq \cdots \leq \theta_n$. □

Example 6.14. Consider the Archimedean copula family given by

$$C_\theta(u,v) = \left(1 + \left[(u^{-1}-1)^\theta + (v^{-1}-1)^\theta\right]^{1/\theta}\right)^{-1}$$

generated by $\varphi_\theta(t) = (t^{-1} - 1)^\theta$ for $\theta \geq 1$. Set $\varphi_i(t) = \varphi_{\theta_i}(t)$ for $i = 1, \dots, n$. Can the above copulas be extended to n-copulas of the form indicated by (6.7) and (6.8), and if so under what conditions on $\theta_1, \dots, \theta_n$? By calculating derivatives of φ_i^{-1} and $\varphi_i \circ \varphi_{i+1}^{-1}$ it follows that $\varphi_i^{-1} \in \mathcal{L}_\infty$ and $\varphi_i \circ \varphi_{i+1}^{-1} \in \mathcal{L}_\infty^*$ if and only if $\theta_i/\theta_{i+1} \leq 1$. Hence the n-dimensional extension of the above copulas are n-copulas if $\theta_1 \leq \cdots \leq \theta_n$.

Copulas of the above form have upper and lower tail dependence, with coefficients of upper and lower tail dependence given by $2 - 2^{1/\theta}$ and $2^{-1/\theta}$ respectively. One limiting factor for the usefulness of this copula family might be that they only allow for a limited range of positive dependence, as seen from the expression for Kendall's tau given by $\tau = 1 - \frac{2}{3\theta}$, for $\theta \geq 1$. □

Note that the results presented in this section hold for strict Archimedean copulas. With some additional constraints most of the results can be generalized to hold also for nonstrict Archimedean copulas. However for practical purposes it is sufficient to only consider strict Archimedean copulas. This basically means (there are exceptions such as the Frank family) that we consider copula families with only positive dependence. Furthermore, risk models are often designed to model positive dependence, since in some sense it is the "dangerous" dependence: assets (or risks) move in the same direction in periods of extreme events.

7 Modelling Extremal Events in Practice

7.1 Insurance Risk

Consider a portfolio consisting of n risks $X_1, \dots, X_n$, representing potential losses in different lines of business for an insurance company. Suppose that the insurance company, in order to reduce the risk in its portfolio, seeks protection against simultaneous big losses in different lines of business. One suitable reinsurance contract might be the one which pays the excess losses $X_i - k_i$ for $i \in B \subseteq \{1, \dots, n\}$ (where B is some prespecified set of business lines), given that $X_i > k_i$ for all $i \in B$. Hence the payout function f is given by

$$f((X_i, k_i); i \in B) = \left(\prod_{i \in B} 1_{\{X_i > k_i\}}\right)\left(\sum_{i \in B}(X_i - k_i)\right). \tag{7.1}$$

In order to price this contract the seller (reinsurer) would typically need to estimate $\mathbb{E}(f((X_i, k_i); i \in B))$. Without loss of generality let $B = \{1, \dots, l\}$ for $l \leq n$. If the joint distribution H of $X_1, \dots, X_l$ could be accurately estimated, calculating the expected value of (7.1) (possibly by using numerical methods) would not be difficult. Unfortunately, accurate estimation of H is seldom possible due to lack of reliable data. It is more realistic, and we will assume this, that the data available allow for estimation of the margins $F_1, \dots, F_n$ of H and pairwise rank correlations. The probability of payout is given by

$$\overline{H}(k_1, \dots, k_l) = \widehat{C}(\overline{F}_1(k_1), \dots, \overline{F}_l(k_l)), \tag{7.2}$$

where $\overline{H}$ and $\widehat{C}$ denotes the joint survival function and survival copula of $X_1, \dots, X_l$. If the thresholds are chosen to be quantiles of the X_is, i.e. if $k_i = F_i^{-1}(\alpha_i)$ for all i, then the right hand side of (7.2) simplifies to $\widehat{C}(1 - \alpha_1, \dots, 1 - \alpha_l)$. In a reinsurance context, these quantile levels are often given as return periods and are known to the underwriter.

For a specific copula family, Kendall's tau estimates can typically be transformed into an estimate of the copula parameters. For Gaussian (elliptical) n-copulas this is due to the relation $R_{ij} = \sin(\pi\tau(X_i, X_j)/2)$, where $R_{ij} = \Sigma_{ij}/\sqrt{\Sigma_{ii}\Sigma_{jj}}$ with Σ being the dispersion matrix of the corresponding normal (elliptical) distribution. For the multivariate extension of the Gumbel family presented in Section 6.5 this is due to the relation $\theta = 1/(1 - \tau(X_i, X_j))$, where θ denotes the copula parameter for the bivariate Gumbel copula of $(X_i, X_j)^T$. Hence, once a copula family is decided upon, calculating the probability of payout or the expected value of the contract is easy. However there is much uncertainty in choosing a suitable copula family representing the dependence between potential losses for the l lines of business. The data may give indications of properties such as tail dependence but it should be combined with careful consideration of the nature of the underlying loss causing mechanisms. To show the relevance of good dependence modelling, we will consider marginal distributions and pairwise rank correlations to be given and compare the effect of the Gaussian and Gumbel copula on the probability of payout and expected value of the contract. To be able to interpret the results more easily, we make some further simplifications: let $X_i \sim F$ for all i, where F is the distribution function of the standard Lognormal distribution $\mathrm{LN}(0, 1)$, let $k_i = k$ for all i and let $\tau(X_i, X_j) = 0.5$ for all $i \neq j$. Then,

$$\overline{H}(k, \dots, k) = 1 + (-1)\binom{l}{1}C_1(F(k)) + \dots + (-1)^l\binom{l}{l}C_l(F(k), \dots, F(k)),$$

where C_m, for $m = 1, \ldots, l-1$, are m-dimensional margins of $C = C_l$ (the copula of $(X_1, \ldots, X_l)$). In the Gaussian case,

$$C_m(F(k), \ldots, F(k)) = \Phi^m_{R_m}(\Phi^{-1}(F(k)), \ldots, \Phi^{-1}(F(k))),$$

where $\Phi^m_{R_m}$ denotes the distribution function of m multivariate normally distributed random variables with linear correlation matrix R_m with off-diagonal entries $\sin(\pi 0.5/2) = 1/\sqrt{2}$. $\Phi^m_{\rho_l}(\Phi^{-1}(F(k)), \ldots, \Phi^{-1}(F(k)))$ can be calculated by numerical integration using the fact that (see Johnson and Kotz (1972) p. 48)

$$\Phi^m_{\rho_l}(a, \ldots, a) = \int_{-\infty}^{\infty} \phi(x) \big[\Phi\left(\frac{a - \sqrt{\rho_l}x}{\sqrt{1-\rho_l}}\right)\big]^m \, \mathrm{d}x,$$

where ϕ denotes the univariate standard normal density function. In the Gumbel case,

$$C_m(F(k), \ldots, F(k)) = \exp\{-[(-\ln F(k))^\theta + \cdots + (-\ln F(k))^\theta]^{1/\theta}\} = F(k)^{m^{1/\theta}},$$

where $\theta = 1/(1-0.5) = 2$.

For illustration, let $l = 5$, i.e. we consider 5 different lines of business. Figure 7.1 shows payout probabilities (probabilities of joint exceedances) for thresholds $k \in [0, 15]$, when the dependence structure among the potential losses are given by a Gaussian copula (lower curve) and a Gumbel copula (upper curve). If we let $k = F^{-1}(0.99) \approx 10.25$, i.e. payout occurs when all 5 losses exceed their respective 99% quantile, then Figure 7.2 shows that if one would choose a Gaussian copula when the true dependence structure between the potential losses $X_1, \ldots, X_5$ is given by a Gumbel copula, the probability of payout is underestimated almost by a factor 8.

Figure 7.3 shows estimates of $\mathbb{E}(f(X_1, X_2, k))$ for $k = 1, \ldots, 18$. The lower curve shows estimates for the expectation when $(X_1, X_2)^T$ has a Gaussian copula and the upper curve when $(X_1, X_2)^T$ has a Gumbel copula. The estimates are sample means from samples of size 150 000. Since $F^{-1}(0.99) \approx 10.25$, Figure 7.3 shows that if one would choose a Gaussian copula when the true dependence between the potential losses X_1 and X_2 is given by a Gumbel copula, the expected loss to the reinsurer is underestimated by a factor 2.

7.2 Market Risk

We now consider the problem of measuring the risk of holding an equity portfolio over a short time horizon (one day, say) without the possibility of rebalancing. More precisely, consider a portfolio of n equities with current value given by

$$V_t = \sum_{i=1}^{n} \beta_i S_{i,t},$$

where β_i is the number of units of equity i and $S_{i,t}$ is the current price of equity i. Let $\Delta_{t+1} = -(V_{t+1} - V_t)/V_t$, the (negative) relative loss over time period $(t, t+1]$, be our aggregate risk. Then

$$\Delta_{t+1} = \sum_{i=1}^{n} \gamma_{i,t} \delta_{i,t+1}$$

where $\gamma_{i,t} = \beta_i S_{i,t}/V_t$ is the portion of the current portfolio value allocated to equity i, and $\delta_{i,t+1} = -(S_{i,t+1} - S_{i,t})/S_{i,t}$ is the (negative) relative loss over time period $(t, t+1]$ of equity i.

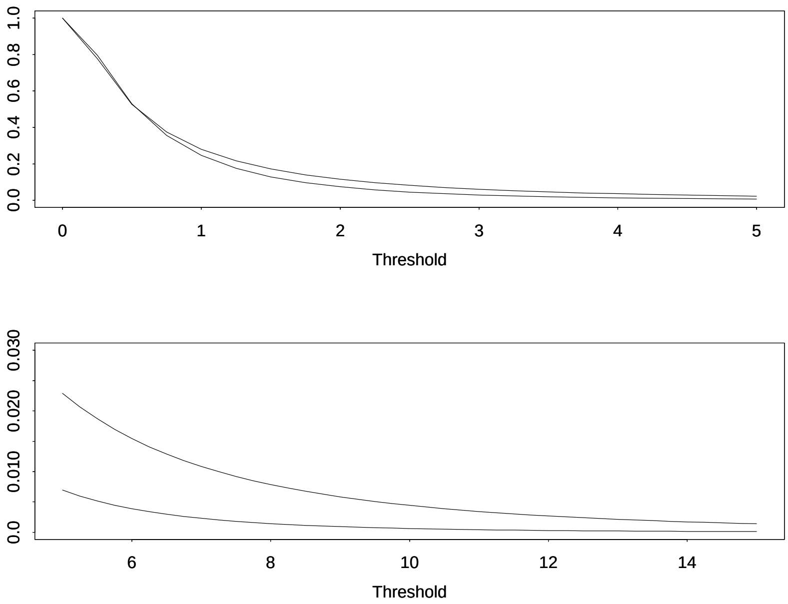

Figure 7.1: Probability of payout for $l = 5$ when the dependence structure is given by a Gaussian copula (lower curve) and Gumbel copula (upper curve).

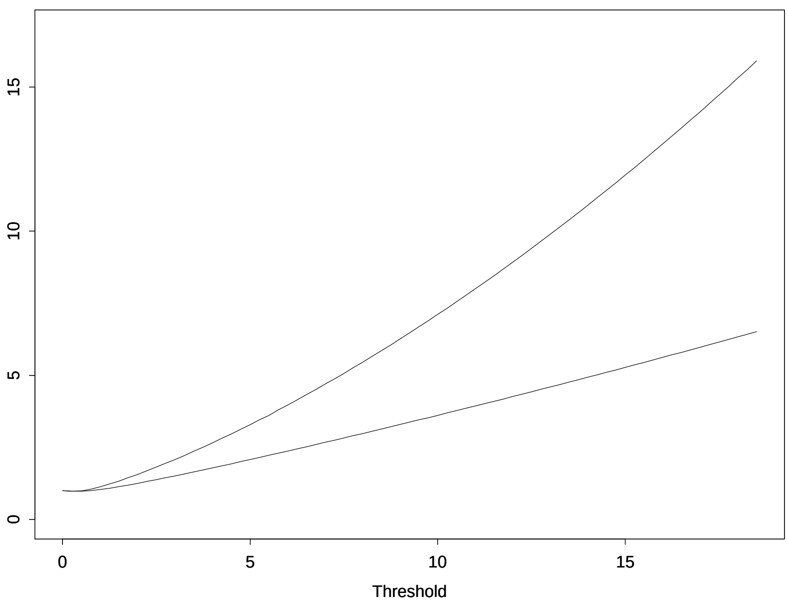

Figure 7.2: Ratios of payout probabilities (Gumbel/Gaussian) for $l = 3$ (lower curve) and $l = 5$ (upper curve).

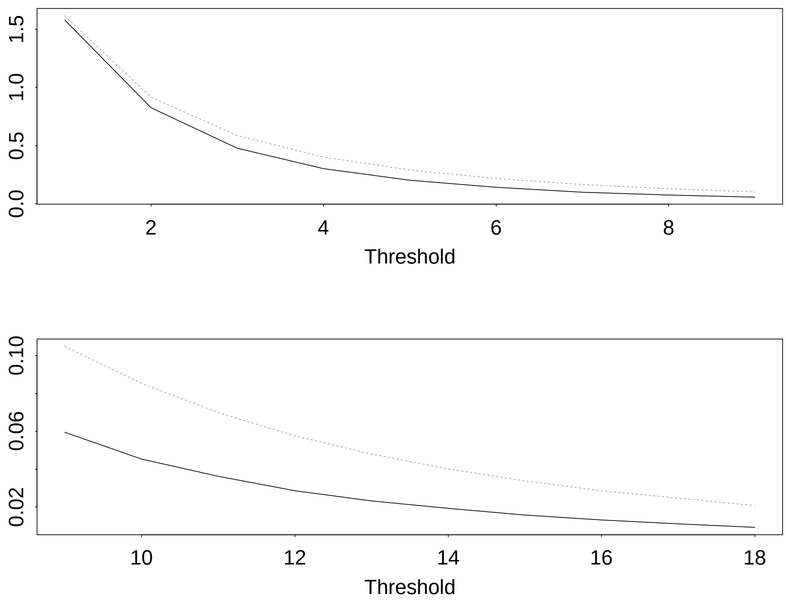

Figure 7.3: Estimates of $\mathbb{E}(f(X_1, X_2, k))$ for Gaussian (lower curve) and Gumbel (upper curve) copulas.

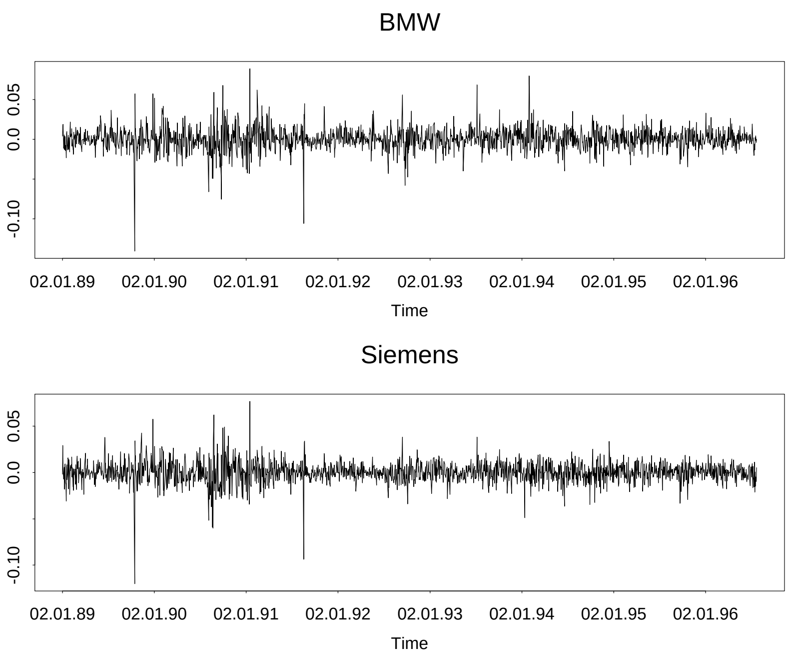

Figure 7.4: Daily log returns from 1989 to 1996.

We will highlight the techniques introduced by studying the effect of different distributional assumptions for $\delta := (\delta_{1,t+1}, \ldots, \delta_{n,t+1})^T$ on the aggregate risk $\Delta := \Delta_{t+1}$. The classical distributional assumption on δ, widely used within market risk management, is that of multivariate normality. However, in general the empirical distribution of δ has (one-dimensional) marginal distributions which are heavier tailed than the normal distribution. Furthermore, there is an even more critical problem with multivariate normal distributions in this context. Extreme falls in equity prices are often joint extremes, in the sense that a big fall in one equity price is accompanied by simultaneous big falls in other equity prices. This is for instance seen in Figure 7.4, an example already encountered in Figure 5.1. Loosely speaking, a problem with the multivariate normal distributions (or models based on them) is that they do not assign a high enough probability of occurrence to the event in which many thing go wrong a the same time - the "perfect storm" scenario. More precisely, daily equity return data often indicate that the underlying dependence structure has the property of tail dependence, a property which we know Gaussian copulas lack.

Suppose δ is modelled by a multivariate normal distribution $\mathcal{N}_n(\mu, \Sigma)$, where μ and Σ are estimated from historical prices of the equities in the portfolio. There seems to be much agreement on the fact that the quantiles of $\Delta = \gamma^T \delta \sim \mathcal{N}(\gamma^T \mu, \gamma^T \Sigma \gamma)$ do not capture the portfolio risk due to extreme market movements; see for instance Embrechts, Mikosch, and Klüppelberg (1997), Embrechts (2000) and the references therein. Therefore, different stress test solutions have been proposed. One such "solution" is to choose μ_s and Σ_s in such a way that $\delta_s \sim \mathcal{N}_n(\mu_s, \Sigma_s)$ represents the distribution of the relative losses of the different equities under more adverse market conditions. The aim is that the quantiles of $\Delta_s = \gamma^T \delta_s \sim \mathcal{N}(\gamma^T \mu_s, \gamma^T \Sigma_s \gamma)$ should be more realistic risk estimates. To judge this approach we note that

$$\frac{F_s^{-1}(\alpha) - \gamma^T \mu_s}{F^{-1}(\alpha) - \gamma^T \mu} = \sqrt{\frac{\gamma^T \Sigma_s \gamma}{\gamma^T \Sigma \gamma}},$$

where F and F_s denotes the distribution functions of Δ and Δ_s respectively. Hence the effect of this is simply a translation and scaling of the quantile curve $F^{-1}(\alpha)$. As a comparison, let δ^* have a t_4-distribution with mean μ and covariance matrix Σ and let Δ^* be the corresponding portfolio return. Furthermore let $n = 10$, $\mu_i = \mu_{s,i} = \mu_i^* = 0$, $\gamma_i = 1/10$ for all i and let $\tau(\delta_i, \delta_j) = \tau(\delta_i^*, \delta_j^*) = 0.4$, $\tau(\delta_{s,i}, \delta_{s,j}) = 0.6$, $\Sigma_{ij} = \sin(\pi \tau(\delta_i, \delta_j)/2)$, $\Sigma_{s,ij} = 1.5 \sin(\pi \tau(\delta_{s,i}, \delta_{s,j})/2)$ for all i, j. Then Figure 7.5 shows from lower to upper the quantile curves of Δ, Δ_s and Δ^* respectively. If Δ^* were the true portfolio return, Figure 7.5 shows that the approach described above would eventually underestimate the quantiles of the portfolio return. It should be noted that this is not mainly due to the heavier tailed t_4-margins. This can be seen in Figure 7.6 which shows quantile curves of Δ^* and $\Delta' = \gamma^T \delta'$, where δ' is a random vector with t_4-margins, a Gaussian copula, $\mathbb{E}(\delta') = \mathbb{E}(\delta)$ and $\mathrm{Cov}(\delta') = \mathrm{Cov}(\delta)$.

There are of course numerous alternative applications of copula techniques to integrated risk management. Besides the references already quoted, also see Embrechts, Hoeing, and Juri (2001) where the calculation of Value-at-Risk bounds for functions of dependent risks is discussed. The latter paper also contains many more relevant references to this important topic.

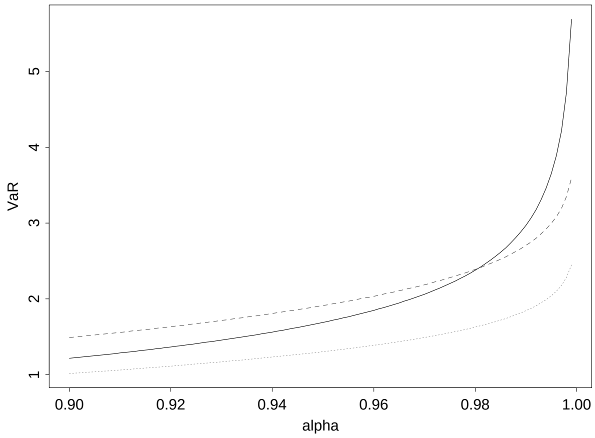

Figure 7.5: Quantile curves: $\mathrm{VaR}_{\Delta}(\alpha)$, $\mathrm{VaR}_{\Delta_s}(\alpha)$ and $\mathrm{VaR}_{\Delta^*}(\alpha)$ from lower to upper.

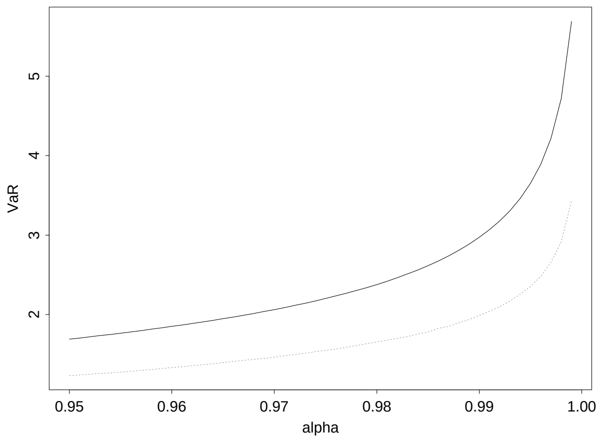

Figure 7.6: Quantile curves: $\mathrm{VaR}_{\Delta'}(\alpha)$ and $\mathrm{VaR}_{\Delta^*}(\alpha)$ from lower to upper.

References

Barlow, R., and F. Proschan (1975): *Statistical Theory of Reliability and Life Testing.* Holt, Rinehart & Winston, New York.

Cambanis, S., S. Huang, and G. Simons (1981): "On the theory of elliptically contoured distributions," *Journal of Multivariate Analysis*, 11, 368–85.

Capéraà, P., and C. Genest (1993): "Spearman's rho is larger than Kendall's tau for positively dependent random variables," *Journal of Nonparametric Statistics*, 2, 183–194.

Denneberg, D. (1994): *Non-Additive Measure And Integral.* Kluwer Academic Publishers, Boston.

Embrechts, P. (2000): "The Bell curve is wrong: so what?," in *Extremes and Integrated Risk Management*, ed. by P. Embrechts, pp. xxv–xxviii. Risk Waters Group.

Embrechts, P., A. Hoeing, and A. Juri (2001): "Using copulae to bound the Value-at-Risk for functions of dependent risks," ETH preprint.

Embrechts, P., A. McNeil, and D. Straumann (1999): "Correlation and Dependence in Risk Management: Properties and Pitfalls," To appear in *Risk Management: Value at Risk and Beyond*, ed. by M. Dempster and H.K. Moffatt, Cambridge University Press (2001).

Embrechts, P., T. Mikosch, and C. Klüppelberg (1997): *Modelling Extremal Events for Insurance and Finance.* Springer, Berlin.

Fang, K.-T., S. Kotz, and K.-W. Ng (1987): *Symmetric Multivariate and Related Distributions.* Chapman & Hall, London.

Frank, M. (1979): "On the simultaneous associativity of $F(x, y)$ and $x + y - F(x, y)$," *Aequationes Mathematicae*, 19, 194–226.

Fréchet, M. (1951): "Sur les tableaux de corrélation dont les marges sont données," *Annales de l'Université de Lyon, Sciences Mathématiques et Astronomie*, 14, 53–77.

Fréchet, M. (1957): "Les tableaux de corrélation dont les marges et des bornes sont données," *Annales de l'Université de Lyon, Sciences Mathématiques et Astronomie*, 20, 13–31.

Genest, C. (1987): "Frank's family of bivariate distributions," *Biometrika*, 74, 549–555.

Genest, C., and J. MacKay (1986a): "The joy of copulas: Bivariate distributions with uniform marginals," *The American Statistician*, 40, 280–283.

Genest, C., and R. MacKay (1986b): "Copules archimédiennes et familles de lois bidimensionell dont les marges sont données," *The Canadian Journal of Statistics*, 14, 145–159.

Genest, C., and L.-P. Rivest (1993): "Statistical inference procedures for bivariate Archimedean copulas," *Journal of the American Statistical Association*, 88, 1034–1043.

Hoeffding, W. (1940): "Massstabinvariante Korrelationstheorie," *Schriften des Mathematischen Seminars und des Instituts für Angewandte Mathematik der Universität Berlin*, 5, 181–233.

Hult, H., and F. Lindskog (2001): "Multivariate extremes, aggregation and dependence in elliptical distributions," ETH preprint.

Joe, H. (1997): *Multivariate Models and Dependence Concepts*. Chapman & Hall, London.

Johnson, N., and S. Kotz (1972): *Distributions in Statistics: Continuous Multivariate Distributions*. Wiley, New York.

Kendall, M., and A. Stuart (1979): *Handbook of Statistics*. Griffin & Company, London.

Kimberling, C. (1974): "A probabilistic interpretation of complete monotonicity," *Aequationes Mathematicae*, 10, 152–164.

Kruskal, W. (1958): "Ordinal measures of association," *Journal of the American Statistical Association*, 53, 814–861.

Lehmann, E. (1975): *Nonparametrics: Statistical Methods Based on Ranks*. Holden-Day, Inc., San Francisco.

Lindskog, F., and A. McNeil (2001): "Common Poisson shock models: applications to insurance and credit risk modelling," ETH preprint.

Lindskog, F., A. McNeil, and U. Schmock (2001): "A note on Kendall's tau for elliptical distributions," ETH preprint.

Marshall, A. (1996): "Copulas, marginals and joint distributions," in *Distributions with Fixed Marginals and Related Topics*, ed. by L. Rüschendorff, B. Schweizer, and M. Taylor, pp. 213–222. Institute of Mathematical Statistics, Hayward, CA.

Marshall, A., and I. Olkin (1967): "A multivariate exponential distribution," *Journal of the American Statistical Association*, 62, 30–44.

Mikusinski, P., H. Sherwood, and M. Taylor (1992): "The Fréchet bounds revisited," *Real Analysis Exchange*, 17, 759–764.

Muliere, P., and M. Scarsini (1987): "Cheracterization of a Marshall-Olkin type class of distributions," *Annals of the Institute of Statistical Mathematics*, 39, 429–441.

Nelsen, R. (1999): *An Introduction to Copulas*. Springer, New York.

Resnick, S. (1987): *Extreme values, Regular variation and Point processes*. Springer, New York.

Scarsini, M. (1984): "On measures of concordance," *Stochastica*, 8, 201–218.

Schweizer, B., and A. Sklar (1983): *Probabilistic Metric Spaces*. North-Holland, New York.

Schweizer, B., and E. Wolff (1981): "On nonparametric measures of dependence for random variables," *Annals of Statistics*, 9, 879–885.

Sklar, A. (1959): "Fonctions de répartition à n dimensions et leurs marges," *Publications de l'Institut de Statistique de l'Université de Paris*, 8, 229–231.

Sklar, A. (1996): "Random variables, distribution functions, and copulas – a personal look backward and forward," in *Distributions with Fixed Marginals and Related Topics*, ed. by L. Rüschendorff, B. Schweizer, and M. Taylor, pp. 1–14. Institute of Mathematical Statistics, Hayward, CA.