Dynamic copula models for multivariate high-frequency data in finance

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Abstract

The stylized facts of univariate high-frequency data in finance are well known; see Dacorogna et al. (2001). In Breymann et al. (2003) we analyzed bivariate high frequency forex data as a function of the sampling frequency, however treating the data as iid. In the present paper, using the data from Breymann et al. (2003), we model the dynamics as GARCH type processes and investigate the stylized facts of the bivariate residuals. As a function of the sampling frequency, we test for tail-dependence and ellipticity. We also investigate clustering of extremes and change-points.

Keywords: change-point; copula; dynamic copula; high-frequency foreign exchange data; tail-dependence; GARCH.

1 Introduction

In Breymann et al. (2003) we investigated the stylized facts of the dependence structure in a set of high-frequency data, namely tick-by-tick observations of foreign exchange (FX) spot rates for USD Dollar quoted against German Mark (USD/DEM) and quoted against Japanese Yen (USD/JPY). After an initial data deseasonalization, bivariate log-return time series for

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six different time horizons were considered, from one hour up to one day. At each frequency we evaluated the dependence structure fitting copula-based models by the pseudo log-likelihood method introduced by Genest et al. (1995), see also Chen and Fan (2002). We further tested for ellipticity across frequencies and modelled tail-dependence for the hourly returns. It is important to stress that the above analysis assumed the vectors to be independent identically distributed (iid). We know however that this assumption is violated in practice due for instance to volatility effects; papers like Fortin and Kuzmics (2002), Patton (2002), Rockinger and Jondeau (2001) discuss this issue. In the present paper we therefore start from the deseasonalized FX data and investigate dependence between the (residual) vector components after some dynamic model has been fitted. In Section 2 we first filter the data through univariate GARCH models and analyze the copula function of the residuals. Ellipticity is tested, spectral densities are estimated and the so-called extreme tail-dependence copula is modelled. Based on the models from Section 2, Section 3 is devoted to test the existence of change points in the dependence parameters, estimating the size and time of the changes. Whereas the above sections use time invariant copulae, models allowing for a dynamic time-varying copula are used in Section 4. Finally, Section 5 concludes.

2 The copula of USD/DEM and USD/JPY spot rate returns across time frequencies and after volatility filtering

Recall from Breymann et al. (2003) that the data considered are the bivariate log-returns of FX spot rates USD/DEM and USD/JPY after being deseasonalized. The observations cover the period from 27 April 1986 until 25 October 1998. The six different time horizons considered are one, two, four, eight, twelve hour and one day periods.

2.1 Time dependence filtering

Among the empirical stylized facts for univariate financial returns are weak linear dependence in time, heteroscedasticity and non-normal innovations. The FX observations of USD/DEM and USD/JPY are no exception. We ran the Jarque-Bera test and normality is rejected at the usual probability levels for all considered time series. Moreover, the test for the absence of ARCH effects (see Engle (1982) and Zivot and Wang (2003) for

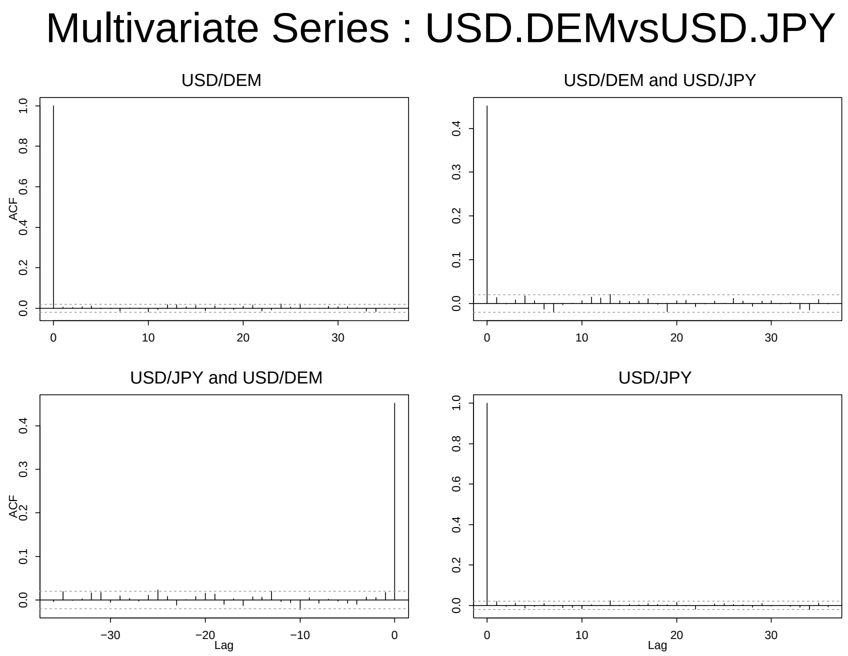

Figure 1: Sample autocorrelograms for the absolute values of the eight hour USD/DEM and USD/JPY residuals, respectively top left and bottom right, and cross-correlograms for USD/DEM on past USD/JPY (top right) and USD/JPY on past USD/DEM (bottom left).

S+FinMetrics implementation) is also rejected. In our discrete-time setting, we model stochastic volatility effects by GARCH type models; see Shephard (1996) for an overview on volatility models. In particular, we fit univariate ARMA-GARCH models to each of the marginal series with innovations assuming come from a t distribution. We used the S+FinMetrics software in order to fit the models, perform the tests and obtain the standardized residuals. Though there exist several multivariate GARCH models in the literature, like CCC-GARCH, DVEC, matrix-diagonal GARCH, BEKK, principal components GARCH, in our first analysis we did not want to bias our investigation of the dependence structure by imposing a specific analytic model on it. In Section 4, we will reanalyze the data using a matrix-diagonal model.

In Figure 1 we plot the sample autocorrelograms and cross-correlograms for the absolute values of the bivariate filtered eight hour FX returns; namely absolute values of the eight hour USD/DEM and USD/JPY residual vectors

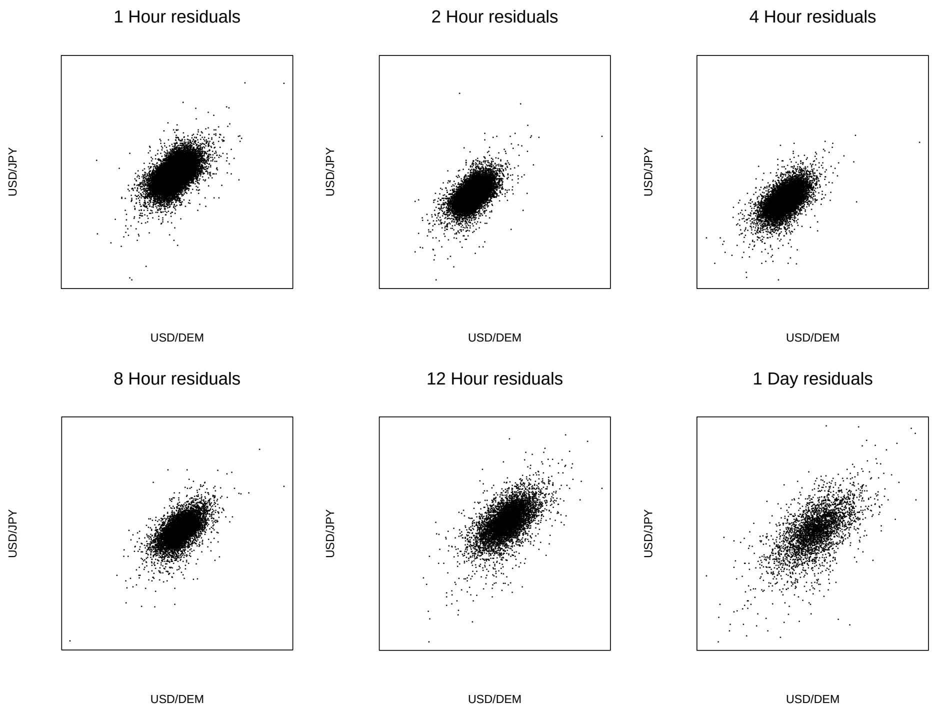

Figure 2: FX spot rates for USD/DEM and USD/JPY. The figure displays the scatter-plots of the filtered returns for the several time frequencies.

resulting from the above marginal ARMA-GARCH fitting. In these plots there seems to be no evidence against serial independence of the absolute residual values. This is in contrast to the static analysis in Breymann et al. (2003), Figure 5. Only the contemporaneous cross-dependence remains (see lag zero in the cross-correlograms of Figure 1) and that is exactly where our interest lies.

2.2 Copulae for USD/DEM and USD/JPY residuals

Figure 2 shows the scatter-plots of the USD/DEM and USD/JPY residuals obtained through the fitting of Section 2.1. Suppose that the USD/DEM residuals are represented by the random variable X_1 and the USD/JPY by the random variable X_2. Assume that (X_1, X_2) has multivariate distribution function F and continuous univariate marginal distribution functions F_1 and F_2. In order to investigate the residual dependence, we fit copula-based models of the type

$$F(x_1, x_2; \boldsymbol{\theta}) = C(F_1(x_1), F_2(x_2); \boldsymbol{\theta}), \tag{1}$$

where C is a copula function, which we know to exist uniquely by Sklar's Theorem (Sklar (1959)), parameterized by the vector $\boldsymbol{\theta} \in \mathbb{R}^q$ with $q \in \mathbb{N}$. Denote by $\mathbf{X} = \{(X_{1i}, X_{2i}) : i = 1, 2, \ldots, n\}$ a random sample of n bivariate observations. The dependence parameter $\boldsymbol{\theta}$ of C is estimated by the pseudo log-likelihood estimator introduced by Genest et al. (1995) where the marginal distribution functions F_i, $i = 1, 2$, are estimated by the rescaled empirical distribution functions

$$F_{in}(x) = \frac{1}{n+1} \sum_{j=1}^{n} \mathbb{I}_{\{y \in \mathbb{R} \,:\, y \leq x\}}(X_{ij}).$$

As usual $\mathbb{I}_A$ denotes the indicator function of the set A. After the marginal transformations to so-called pseudo observations

$$(F_{1n}(X_{1i}), F_{2n}(X_{2i}))$$

for $i = 1, 2, \ldots, n$, the copula family C is fitted. Suppose that its density exists, we then maximize the pseudo log-likelihood function

$$L(\boldsymbol{\theta}; \mathbf{X}) = \sum_{i=1}^{n} \log c(F_{1n}(X_{1i}), F_{2n}(X_{2i}); \boldsymbol{\theta}) \qquad (2)$$

where c is the copula density of model (1) and is given by

$$c(u_1, u_2; \boldsymbol{\theta}) = \frac{\partial^2 C(u_1, u_2; \boldsymbol{\theta})}{\partial u_1 \, \partial u_2}, \quad (u_1, u_2) \in [0, 1]^2.$$

The pseudo log-likelihood estimator $\hat{\boldsymbol{\theta}}$ that maximizes (2) is known to be consistent and asymptotically normally distributed; see Genest et al. (1995).

The copula families fitted to the USD/DEM and USD/JPY spot rate residuals are: t, Frank, Plackett, Gaussian, Gumbel, Clayton and the mixtures Gumbel with survival Gumbel, Clayton with survival Clayton, Gumbel with Clayton and survival Gumbel with survival Clayton; for details on these classes see Embrechts et al. (2002), Joe (1997) and Nelsen (1999). Denoting the copula family A with parameter $\boldsymbol{\theta}$ by $C^A(\cdot, \cdot; \boldsymbol{\theta})$, the fitted mixtures have distribution functions of the form

$$C(u_1, u_2; \boldsymbol{\theta}) = \theta_3 \, C^A(u_1, u_2; \theta_1) + (1 - \theta_3) \, C^B(u_1, u_2; \theta_2).$$

The above choice of time invariant copula models is partly based on previous analysis but also on tractability and flexibility for investigating tail-dependence. The Gaussian copula is included mainly for comparison. The t copula has been proven to be useful in finance. Reasons for specific choices of copulae will became apparent throughout the text.

We fitted all the listed copula models to the USD/DEM and USD/JPY residuals to obtain the dependence parameter estimates $\hat{\boldsymbol{\theta}}$ for the several frequencies. The choice of a best fit is based on two procedures. First, the models are ranked by their Akaike information value. From the maximized log-likelihood we compute for each family the Akaike information criterion:

$$\text{AIC} = -2L(\hat{\boldsymbol{\theta}}; \mathbf{x}) + 2q$$

where q is the number of parameters of the family fitted. The smaller the Akaike information value the better the model fits to the data. Secondly, a goodness of fit test is performed to the best ranked models for each time frequency. Parameter estimates and the approximated standard errors (s.e.) for all fitted models are listed in Tables 1 and 2. For the t copula the parameters θ_1 and θ_2 in Table 2 represent respectively the degrees of freedom and the correlation.

From the models fitted to the residuals, the one which has the best AIC is the t copula for almost all the frequencies. The exception are the daily observations where the mixture of roughly 0.5 of Gumbel with 0.5 survival Gumbel performs slightly better than the t model. To enable an easy comparison among the AIC values obtained for each frequency we plotted in Figure 3 the relative differences between the AIC for the t model and the AIC for all the other models. The results for the Clayton and survival Clayton are not included in the plot because they are significantly worse for all time frequencies.

According to the AIC criterion, the mixture models and the t model perform better than the one-parameter models. Both copulae with very asymmetric tails give poor fits. We should remark that the Plackett copula is the best of the one parameter models except for daily residuals where the Gaussian is better. It is not surprising that the Gaussian and the t copulae have very different AIC values because the t having between 4.7 and 6 estimated degrees of freedom is still far from its Gaussian limit. Nevertheless, the Gaussian copula approaches the t for a decreasing time frequency (central limit effect).

From this analysis we can conclude that the filtered residuals on USD/DEM and USD/JPY spot rates can be modelled well by the t model or by a mixture between the Gumbel and survival Gumbel copulae. These are always the two best models. Combining the ARMA-GARCH marginal fits with the copula modelled residuals we end up with a dynamic, bivariate model for which functionals of interest can be estimated/simulated.

Comparing these results with those obtained previously using an unconditional, static iid model in Breymann et al. (2003) the conclusions are similar. There the t model was always the best for the several frequencies of returns; the Gumbel mixture model was however not included.

Frequency	Copula model	$\hat{\theta}$ (s.e.)	AIC
1 hour	Clayton	0.859 (0.006)	-23401.101
	Frank	3.979 (0.024)	-27032.306
	Gaussian	0.550 (0.002)	-28267.108
	Gumbel	1.562 (0.004)	-28146.727
	Plackett	6.503 (0.061)	-29324.002
2 hour	Clayton	0.913 (0.009)	-12730.906
	Frank	4.200 (0.035)	-14806.051
	Gaussian	0.571 (0.002)	-15483.028
	Gumbel	1.605 (0.006)	-15506.480
	Plackett	7.038 (0.093)	-16020.855
4 hour	Clayton	0.944 (0.013)	-6652.147
	Frank	4.341 (0.050)	-7821.259
	Gaussian	0.584 (0.004)	-8176.122
	Gumbel	1.634 (0.009)	-8251.189
	Plackett	7.361 (0.137)	-8446.380
8 hour	Clayton	0.984 (0.019)	-3536.107
	Frank	4.563 (0.072)	-4260.632
	Gaussian	0.603 (0.005)	-4413.271
	Gumbel	1.669 (0.013)	-4412.292
	Plackett	7.752 (0.201)	-4533.552
12 hour	Clayton	1.025 (0.024)	-2487.922
	Frank	4.659 (0.088)	-2941.280
	Gaussian	0.615 (0.006)	-3092.874
	Gumbel	1.681 (0.016)	-3007.219
	Plackett	7.949 (0.252)	-3113.152
1 day	Clayton	1.034 (0.035)	-1252.289
	Frank	4.599 (0.124)	-1446.464
	Gaussian	0.617 (0.009)	-1552.695
	Gumbel	1.679 (0.023)	-1500.065
	Plackett	7.772 (0.350)	-1526.993

Table 1: Residuals on USD/DEM and USD/JPY log-returns. Estimates and standard errors of dependence parameters in Clayton, Frank, Gaussian, Gumbel and Plackett models. For each model fitted we provide the AIC value. The reading of this table must be complemented with Table 2.

Freq.	Copula model	$\hat{\theta}_1$ (s.e.)	$\hat{\theta}_2$ (s.e.)	$\hat{\theta}_3$ (s.e.)	AIC
	Cl & s. Cl	1.125 (0.025)	1.171 (0.027)	0.516 (0.007)	-29924.80
	Cl & Gumbel	1.568 (0.014)	1.363 (0.066)	0.659 (0.009)	-30642.91
1 hour	s.Cl & s.Gum	1.552 (0.010)	1.510 (0.065)	0.701 (0.008)	-30665.31
	Gum & s.Gum	2.038 (0.030)	1.405 (0.009)	0.421 (0.010)	-31061.35
	t	4.935 (0.108)	0.558 (0.002)	–	-31517.70
	Cl & s. Cl	1.164 (0.033)	1.316 (0.041)	0.517 (0.010)	-16430.36
	Cl & Gumbel	1.674 (0.034)	1.233 (0.114)	0.650 (0.014)	-16803.66
2 hour	s.Cl & s.Gum	1.576 (0.013)	1.723 (0.086)	0.695 (0.011)	-16801.39
	Gum & s.Gum	2.109 (0.039)	1.420 (0.013)	0.441 (0.013)	-17015.86
	t	4.822 (0.147)	0.580 (0.003)	–	-17192.73
	Cl & s. Cl	1.238 (0.048)	1.325 (0.051)	0.499 (0.014)	-8653.704
	Cl & Gumbel	1.682 (0.032)	1.359 (0.128)	0.674 (0.017)	-8863.199
4 hour	s.Cl & s.Gum	1.640 (0.024)	1.535 (0.115)	0.669 (0.016)	-8847.032
	Gum & s.Gum	1.501 (0.028)	1.991 (0.064)	0.545 (0.020)	-8932.445
	t	4.748 (0.201)	0.593 (0.004)	–	-9088.884
	Cl & s. Cl	1.265 (0.060)	1.472 (0.071)	0.502 (0.018)	-4607.028
	Cl & Gumbel	1.771 (0.053)	1.265 (0.170)	0.667 (0.024)	-4722.141
8 hour	s.Cl & s.Gum	1.663 (0.028)	1.710 (0.140)	0.668 (0.021)	-4713.239
	Gum & s.Gum	1.991 (0.072)	1.534 (0.040)	0.496 (0.027)	-4764.398
	t	5.323 (0.343)	0.612 (0.006)	–	-4818.328
	Cl & s. Cl	1.492 (0.095)	1.286 (0.084)	0.503 (0.024)	-3157.357
	Cl & Gumbel	1.653 (0.031)	1.893 (0.184)	0.679 (0.025)	-3242.862
12 hour	s.Cl & s.Gum	1.787 (0.060)	1.307 (0.206)	0.673 (0.030)	-3248.558
	Gum & s.Gum	1.556 (0.046)	2.018 (0.088)	0.511 (0.033)	-3281.252
	t	5.837 (0.505)	0.621 (0.007)	–	-3304.250
	Cl & s. Cl	1.548 (0.120)	1.280 (0.099)	0.494 (0.032)	-1599.798
	Cl & Gumbel	1.665 (0.045)	1.844 (0.249)	0.671 (0.037)	-1629.394
1 day	s.Cl & s.Gum	1.816 (0.071)	1.234 (0.195)	0.656 (0.039)	-1632.435
	Gum & s.Gum	1.588 (0.072)	1.952 (0.117)	0.501 (0.048)	-1642.460
	t	6.012 (0.786)	0.620 (0.010)	–	-1640.061

Table 2: Residuals on USD/DEM and USD/JPY log-returns. Estimates and standard errors of parameters for the t model and for the four mixture models considered. In case of the mixture models, θ_1 and θ_2 are the dependence parameters respectively for the first and second terms of the mixture. θ_3 is the mixture parameter which gives the proportion of the first term. For the t model, θ_1 are the degrees of freedom and θ_2 is the correlation. For each model fitted we provide the AIC. The reading of this table must be complemented with Table 1.

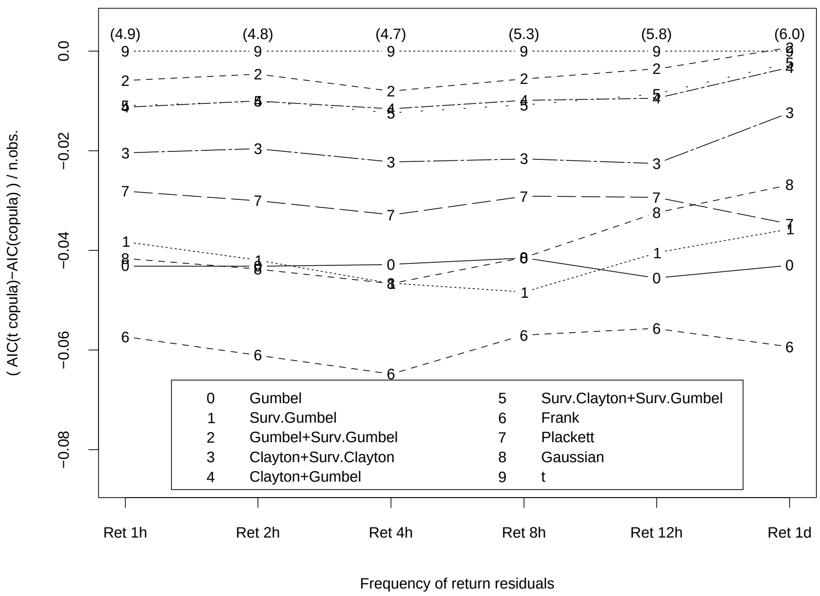

Figure 3: Plot of the AIC values relative to the t copula and to the sample size for each model and time frequency.

Finally for this section we give in Table 3 the p-values for the two best copula-based models for each time frequency. The reported p-values are computed using the probability integral test as discussed in Breymann et al. (2003). The low p-values for one hour up to four hour periods for both models are believed to be more due to the sample size than to a poor fitting power. Therefore, care has to be taken in comparing p-values across frequencies.

Frequency	sample size	t model	Gumbel mixture
1 hour	78,239	0	0
2 hour	39,119	0	0
4 hour	19,559	0.0348	0.0006
8 hour	9,779	0.3808	0.1079
12 hour	6,519	0.2471	0.1949
1 day	3,259	0.7211	0.6775

Table 3: P-values for a goodness-of-fit test of the fitted t and Gumbel mixture models to the residual returns on USD/DEM and USD/JPY spot rates.

2.3 Tail-dependence coefficient

The fitted models can now be used to estimate the tail-dependence coefficient, λ. Recall its definition from Embrechts et al. (2002):

Definition 1 *Let X_1 and X_2 be random variables with distribution functions F_1 and F_2 respectively, such that*

$$\lim_{u \to 0^+} P(X_2 \leq F_2^{-1}(u) | X_1 \leq F_1^{-1}(u)) = \lambda_L$$

exists. If $\lambda_L \in (0, 1]$ then (X_1, X_2) has lower tail-dependence coefficient λ_L and (X_1, X_2) has no lower tail-dependence if $\lambda_L = 0$. Similarly, if

$$\lim_{u \to 1^-} P(X_2 > F_2^{-1}(u) | X_1 > F_1^{-1}(u)) = \lambda_U$$

exists, (X_1, X_2) has upper tail-dependence coefficient λ_U if $\lambda_U \in (0, 1]$ and has no upper tail-dependence if $\lambda_U = 0$.

One can show that these coefficients only depend on the copula C of (X_1, X_2); see Joe (1997). For instance, in the important special case of the t copula,

$$\lambda_L = \lambda_U = 2\bar{t}_{\nu+1} \left(\sqrt{(\nu + 1)(1 - \rho)/(1 + \rho)} \right), \tag{3}$$

where $\bar{t}_\nu$ denotes the tail of a standard univariate t distribution with ν degrees of freedom and ρ is the correlation parameter of the t copula. This result can be found in Embrechts et al. (2002).

The Gumbel mixture model has copula function of the form

$$C(u_1, u_2; \boldsymbol{\theta}) = \theta_3 C^{Gu}(u_1, u_2; \theta_1) + (1 - \theta_3)(u_1 + u_2 - 1 + C^{Gu}(1 - u_1, 1 - u_2; \theta_2))$$

where C^{Gu} is a Gumbel copula and $(u_1, u_2) \in [0, 1]^2$. Using the definition of a Gumbel copula with parameter greater than one, we obtain that

$$C(u_1, u_1; \boldsymbol{\theta}) = \theta_3 u_1^{2^{1/\theta_1}} + (1 - \theta_3) \left(2u_1 - 1 + (1 - u_1)^{2^{1/\theta_2}} \right). \tag{4}$$

Straightforward calculations yield

$$\lambda_L = (1 - \theta_3) \left(2 - 2^{1/\theta_2} \right) \quad \text{and} \quad \lambda_U = \theta_3 \left(2 - 2^{1/\theta_1} \right).$$

Table 4 has the estimated values for the tail-dependence coefficients. Here we could use jackknife or bootstrap with Monte Carlo simulation to estimate the variance of the λ estimators.

The results in Table 4 are good in the sense that both models give very similar tail-dependence coefficient estimates. From the obtained values itself we can say that tail-dependence is still present in the residuals. Nevertheless, we should notice that we are estimating an asymptotic tail feature using a model fitted with the entire data set. Meaning that most of the influence in the fitting process comes from observations in the center of the distribution. Approaches focused only on the tail data will be discussed in Sections 2.5 and 2.6.

	t copula	Gumbel mixture	
Frequency	λ	λ_L	λ_U
1 hour	0.242	0.209	0.250
2 hour	0.261	0.207	0.269
4 hour	0.273	0.265	0.225
8 hour	0.261	0.216	0.289
12 hour	0.247	0.288	0.224
1 day	0.240	0.286	0.226

Table 4: Lower and upper tail-dependence coefficients for the residual returns on USD/DEM and USD/JPY spot rates given by the fitted t and Gumbel mixture models.

2.4 Testing for the ellipticity

Whereas in Breymann et al. (2003) we tested on the ellipticity of the log-return data itself, now we will perform such a test on the residuals. We use the test discussed in Manzotti et al. (2002). First we test ellipticity for the original residuals and secondly for the residuals where we transform the margins into standard univariate t distributed observations using the degrees of freedom estimated for the corresponding t copula.

Table 5 has the p-values obtained for the ellipticity test using the two kinds of margins. The second column has the results of the test on the original residuals plotted in Figure 2 and the third column has the results of testing on the marginal t distributed residuals, displayed in Figure 4.

For the original margins, the ellipticity hypothesis is rejected from one hour up to eight hour periods and not rejected for twelve hour and one day frequencies. After transforming the margins, ellipticity is rejected for one hour and two hour frequencies and not rejected for all the remaining time horizons. Relatively to the results obtained in Breymann et al. (2003) on the returns without filtering we can say that we gained one frequency. Before, the ellipticity was also rejected for the twelve hour period with original margins and for the four hour frequency with transformed margins. This may be due to the gain of information resulting of passing from clearly non-independent samples of returns to much less dependent samples of filtered returns. Also here we stress the fact that our test results are based on varying sample sizes.

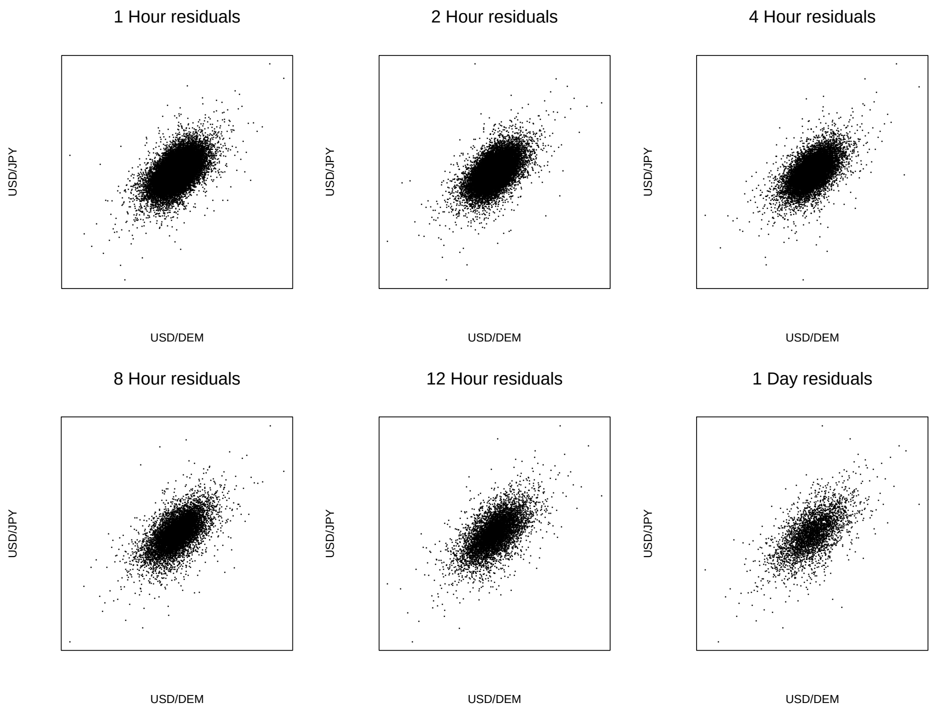

Figure 4: FX spot rates for USD/DEM and USD/JPY. The figure displays the scatter-plots of the filtered returns for the several time frequencies and with margins transformed into standard t distributed observations with the degrees of freedom estimated for the corresponding t copula.

2.5 Spectral density estimates

In this section we study the bivariate tail-dependence of the USD/DEM and USD/JPY residuals through the estimation of the spectral densities for each time frequency considered. In contrast to the method in Section 2.3, here we concentrate the analysis only on relevant tail data. The theoretical background is to be found in Resnick (2002) and Stărică (1999). Below we highlight the main definitions and notation. Let $\| \cdot \|$ denote the usual Euclidean L_2 norm on $\mathbb{R}^d$ and $\mathbb{S}^{d-1}$ be the unit sphere,

$$\mathbb{S}^{d-1} := \{\mathbf{x} \in \mathbb{R}^d : \| \mathbf{x} \| = 1\}.$$

Suppose that the d-dimensional random vector $\mathbf{X}$ has a regularly varying tail distribution. This means that the tail behavior of $\mathbf{X}$ is characterized by a tail index α and the limit

$$\frac{P(\| \mathbf{X} \| > tx, \mathbf{X}/ \| \mathbf{X} \| \in \cdot)}{P(\| \mathbf{X} \| > t)} \quad \xrightarrow{v} \quad x^{-\alpha} P(\Theta \in \cdot), \tag{5}$$

Frequency	Original margins	t margins
1 hour	0	0
2 hour	0	0
4 hour	0	0.348
8 hour	0.001	0.069
12 hour	0.145	0.501
1 day	0.389	0.451

Table 5: P-values for the ellipticity test for the filtered returns on USD/DEM and USD/JPY spot rates with the original and with the t-transformed margins.

where $x > 0$, $t \to \infty$ exists. The convergence is said to be vague and Θ is a random vector on the space $(\mathbb{S}^{d-1}, \mathcal{B}(\mathbb{S}^{d-1}))$. The distribution function of Θ is referred to as the spectral distribution of $\mathbf{X}$. Definition (5) is equivalent to the existence of a measure ν and a positive sequence (a_n), $a_n \to \infty$, such that for $n \to \infty$,

$$nP(a_n^{-1}\mathbf{X} \in \cdot) \quad \xrightarrow{v} \quad \nu(\cdot). \tag{6}$$

For a more precise and detailed treatment on this see for instance Resnick (1987). The measure ν has the following scaling property:

$$\nu(vS) = v^{-\alpha}\nu(S), \tag{7}$$

for any Borel set $S \subset [-\infty, \infty]^d \backslash \{0\}$. This property will be useful in order to find an estimator for the spectral distribution. Intuitively, α indicates the heaviness of the multivariate tails whereas Θ measures in which parts of the space extremes cluster.

Define for $\mathbf{x} \in \mathbb{R}^d$ and $B \in \mathcal{B}(\mathbb{R}^d)$

$$\epsilon_x(B) = \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \in B^c. \end{cases}$$

Then a consistent estimator of $c\nu$, for some $c > 0$, is given by

$$\nu_n := \frac{1}{k_n} \sum_{i=1}^{n} \epsilon_{\mathbf{X}_i / b(n/k_n)},$$

where $b(.)$ is the quantile function $b(t) := F^{\leftarrow}(1 - 1/t)$, for $t > 1$, of the random variable $\parallel \mathbf{X} \parallel$. Here $F^{\leftarrow}$ denotes the (generalized) inverse of F. As usual in extreme value theory, $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$; see

Resnick (2002). If we estimate the quantile function with the corresponding empirical estimator

$$\hat{b}\left(\frac{n}{k_n}\right) = \| \mathbf{X} \|_{k_n,n},$$

where $\| \mathbf{X} \|_{k_n,n}$ is the k-th largest value of the one-dimensional set $\{\| \mathbf{X}_i \|: 1 \le i \le n\}$, we obtain as estimator of the spectral measure

$$\hat{P}(\Theta \in S) = \frac{1}{k_n} \sum_{i=1}^{n} \epsilon_{\mathbf{X}_i/\|\mathbf{X}\|_{k_n,n}}(V(S)), \tag{8}$$

where $V(S) = \{\mathbf{x} \in \mathbb{S}_+^{d-1} : \mathbf{x}/ \| \mathbf{x} \| \in S\}$ and $\mathbb{S}_+^{d-1} := \{\mathbf{x} : \| \mathbf{x} \| > 1\}$. As in the one-dimensional case, the performance of this estimator very much depends on the choice of k_n. Here we use the scaling property (7) and choose k_n such that $\hat{\nu}_n(u\mathbb{S}_+^{d-1})/(u^{-\alpha}\hat{\nu}_n(\mathbb{S}_+^{d-1})) \approx 1$ for values of u in a neighborhood of 1. We plot the set of values

$$\left\{\left(u, \frac{\hat{\nu}_n(u\mathbb{S}_+^{d-1})}{u^{-\hat{\alpha}}\hat{\nu}_n(\mathbb{S}_+^{d-1})}\right) : 0 < u < 2\right\}$$

for several values of k_n and choose the one corresponding to the plot for which these values are closer to 1 around $u = 1$. For more on this procedure see Stărică (1999). We use the Hill estimator to get the tail index estimate $\hat{\alpha}$; see Embrechts et al. (1997). These values are reported in Table 6. Standard errors can be calculated; we do not report them as for the analysis of extremal clustering we only need the point estimates.

Time frequency	Tail index estimate $\hat{\alpha}$
1 hour	4.176
2 hour	3.827
4 hour	4.171
8 hour	4.094
12 hour	3.906
1 day	3.799

Table 6: Tail index estimates for the bivariate tail residual returns on USD/DEM and USD/JPY spot rates using the Hill estimator.

We now estimate the spectral density of the bivariate residuals $\mathbf{X}$ for the USD/DEM and USD/JPY data at a given time frequency using (8). First, choose k_n as described above and consider the points

$$\left\{\theta_i \in [0, 2\pi[: (\cos\theta_i, \sin\theta_i) = \frac{\mathbf{x}_i}{\| \mathbf{x}_i \|}, \ \| \mathbf{x}_i \| > \| \mathbf{x} \|_{k_n,n}, i = 1, \ldots, n\right\}.$$

We then plot a non–parametric density estimate for these angular observations using a smoothed kernel estimator with Gaussian weights and bandwidth 0.2π. The estimated densities are plotted in Figure 5. A more detailed analysis would yield confidence bands. In these plots we can see that USD/DEM and USD/JPY return residuals cluster in the first ($\pi/4$) and in the third ($5\pi/4$) quadrants. These results agree with those obtained in previous sections; see in particular Figure 2 and Table 4. We would like to stress again that a tail-dependence analysis based on the spectral density estimation for regularly varying vectors uses only observations far in the tails and is not "biased" by observations in the center of the data. As such, this approach is akin to a multivariate extreme value analysis as discussed in Resnick (2002).

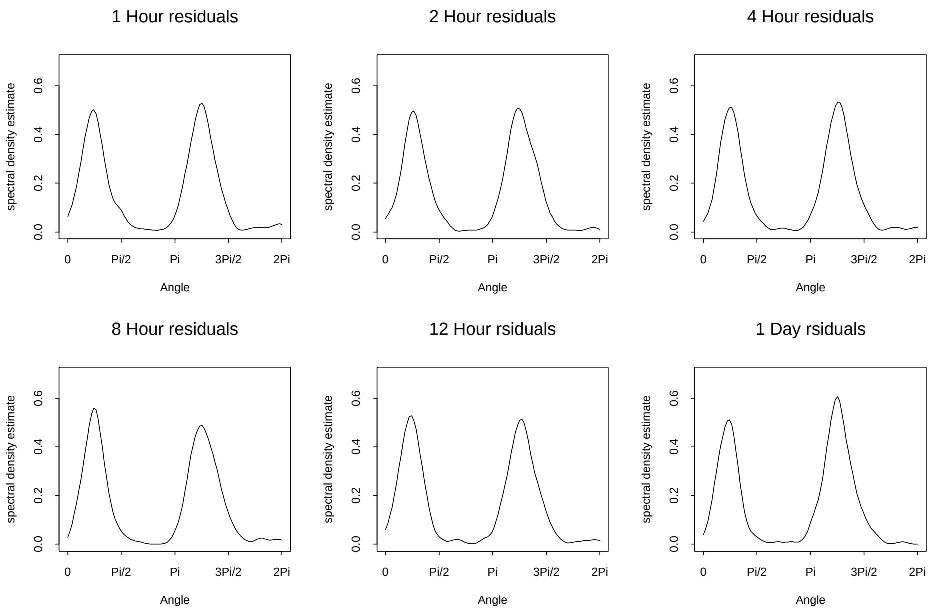

Figure 5: Estimated spectral densities of USD/DEM and USD/JPY return residuals for the six different time horizons.

2.6 Extreme tail-dependence copula

Another point of view can be taken in analyzing multivariate tails. Given a high (low) threshold we can look at the dependence structure of simultaneously large (small) multivariate observations. As in the spectral estimation case above, also here the study is based exclusively on tail observations. No observations from the bulk of the distribution can distort the conclusions. In the univariate case, the Balkema-de Haan-Pickands Theorem gives the generalized Pareto as the limiting distribution for the excesses above a high threshold; see Theorem 3.4.13(b) in Embrechts et al. (1997). In the same spirit, Juri and Wüthrich (2002) showed that for a wide class of so-called Archimedean dependence structures, the extreme tail-dependence copula has the Clayton copula as a limit. This limit procedure has to be interpreted as a bivariate threshold moves to infinity.

In order to check the applicability of this result to our bivariate residual data, a huge number of observations is needed. Because of that we restrict ourselves for this section to hourly observations. We fix a bivariate high (low) threshold u for the USD/DEM and USD/JPY residuals and fit copula-based models to the observations $\{(x_{1i}, x_{2i}) : x_{1i} > u, x_{2i} > u, i = 1, 2, \ldots, n\}$ ($\{(x_{1i}, x_{2i}) : x_{1i} < u, x_{2i} < u, i = 1, 2, \ldots, n\}$) and then proceed with u's more and more in the tails. Hence we model

$$C_{t^-}(u, v) = P(U \leq F_t^{-1}(u), V \leq F_t^{-1}(v) | U \leq t, V \leq t)$$

as well as

$$C_{t^+}(u, v) = P(U \leq F_t^{-1}(u), V \leq F_t^{-1}(v) | U \geq t, V \geq t),$$

using the pseudo log-likelihood method. Tables 7 and 8 have the results of fitting the models Gumbel, survival Gumbel, Clayton, survival Clayton, Gaussian and t which were the more relevant models for this kind of analysis. In Tables 7 and 8 we give the numerical results for thresholds more in the tails, whereas Figure 6 displays the relative AIC values for all the thresholds considered. As the theory predicts, the Clayton copula is the one that best models the bivariate negative tail residual returns. For the other tail, the bivariate positive residuals are best modelled by the survival Clayton. So the conclusion is exactly the same as we obtained for the returns without filtering.

Threshold	Copula model	$\hat{\theta}$ (s.e.)	AIC	p-value
	Gumbel	1.230 (0.034)	-59.181	–
	surv.Gumbel	1.320 (0.036)	-135.818	0.787
0.03^-	Clayton	0.619 (0.062)	-138.912	0.707
(736 obs.)	surv.Clayton	0.294 (0.054)	-33.436	–
	Gaussian	0.351 (0.030)	-92.554	–
	t	5.242 (1.378); 0.340 (0.036)	-109.644	–
	Gumbel	1.205 (0.026)	-81.206	–
	surv.Gumbel	1.289 (0.027)	-194.578	0.992
0.05^-	Clayton	0.568 (0.047)	-200.300	0.992
(1,189 obs.)	surv.Clayton	0.257 (0.042)	-42.751	–
	Gaussian	0.327 (0.024)	-130.255	–
	t	6.077 (1.375); 0.320 (0.028)	-151.591	–
	Gumbel	1.195 (0.019)	-129.482	–
	surv.Gumbel	1.267 (0.020)	-299.681	0.956
0.07^-	Clayton	0.517 (0.035)	-300.765	0.734
(2,044 obs.)	surv.Clayton	0.252 (0.031)	-75.442	–
	Gaussian	0.314 (0.019)	-208.629	–
	t	7.133 (1.440); 0.307 (0.021)	-235.692	–
	Gumbel	1.201 (0.016)	-209.736	–
	surv.Gumbel	1.271 (0.016)	-469.451	0.960
0.1^-	Clayton	0.520 (0.028)	-471.372	0.946
(3,168 obs.)	surv.Clayton	0.265 (0.025)	-129.379	–
	Gaussian	0.324 (0.015)	-347.175	–
	t	9.877 (2.083); 0.321 (0.016)	-372.351	–
	Gumbel	1.194 (0.010)	-485.840	–
	surv.Gumbel	1.268 (0.010)	-1132.761	0.666
0.2^-	Clayton	0.516 (0.017)	-1165.911	0.496
(7,832 obs.)	surv.Clayton	0.250 (0.016)	-284.506	–
	Gaussian	0.320 (0.009)	-845.657	–
	t	12.063 (1.901); 0.320 (0.010)	-890.971	–
	Gumbel	1.235 (0.008)	-1167.899	–
	surv.Gumbel	1.301 (0.008)	-2238.539	0.987
0.3^-	Clayton	0.559 (0.013)	-2235.334	0.608
(13,381 obs.)	surv.Clayton	0.317 (0.012)	-761.293	–
	Gaussian	0.357 (0.007)	-1825.121	–
	t	13.058 (1.652); 0.358 (0.007)	-1898.233	–

Table 7: Bivariate excesses of residuals on USD/DEM and USD/JPY log-returns on the third quadrant of one hour returns for different thresholds. Estimates and standard errors of dependence parameters in Gumbel, survival Gumbel, Clayton, survival Clayton, Gaussian and t models. For each model fitted we provide the AIC value and the p-value for the best two models. For the t copula the first parameter estimate is the correlation and the second is the degrees of freedom and respectively for the s.e.'s. The number of observations associated with each threshold is listed under the threshold value.

Threshold	Copula model	$\hat{\theta}$ (s.e.)	AIC	p-value
	Gumbel	1.334 (0.037)	-139.757	0.715
	surv.Gumbel	1.251 (0.035)	-68.892	–
0.97^+	Clayton	0.339 (0.055)	-43.623	–
(717 obs.)	surv.Clayton	0.636 (0.063)	-140.811	0.654
	Gaussian	0.380 (0.030)	-107.450	–
	t	9.838 (5.232); 0.371 (0.034)	-109.338	–
	Gumbel	1.296 (0.026)	-215.670	0.904
	surv.Gumbel	1.227 (0.025)	-105.082	–
0.95^+	Clayton	0.301 (0.041)	-64.386	–
(1,284 obs.)	surv.Clayton	0.566 (0.045)	-215.589	0.616
	Gaussian	0.348 (0.023)	-161.898	–
	t	8.563 (2.814); 0.340 (0.026)	-170.398	–
	Gumbel	1.295 (0.021)	-325.528	0.760
	surv.Gumbel	1.213 (0.020)	-137.818	–
0.93^+	Clayton	0.276 (0.032)	-82.366	–
(1,941 obs.)	surv.Clayton	0.574 (0.036)	-337.730	0.904
	Gaussian	0.348 (0.018)	-246.270	–
	t	16.115 (7.061); 0.345 (0.020)	-250.269	–
	Gumbel	1.272 (0.016)	-467.118	0.937
	surv.Gumbel	1.188 (0.016)	-175.475	–
0.9^+	Clayton	0.231 (0.025)	-92.082	–
(3,088 obs.)	surv.Clayton	0.540 (0.028)	-495.172	0.962
	Gaussian	0.320 (0.015)	-330.046	–
	t	11.584 (2.835); 0.319 (0.016)	-347.914	–
	Gumbel	1.277 (0.010)	-1168.805	0.965
	surv.Gumbel	1.207 (0.010)	-546.477	–
0.8^+	Clayton	0.268 (0.016)	-316.696	–
(7,685 obs.)	surv.Clayton	0.530 (0.018)	-1195.640	0.891
	Gaussian	0.327 (0.009)	-867.062	–
	t	8.965 (1.073); 0.327 (0.010)	-951.622	–
	Gumbel	1.289 (0.008)	-2135.855	0.906
	surv.Gumbel	1.220 (0.007)	-1049.790	–
0.7^+	Clayton	0.294 (0.012)	-662.247	–
(13,368 obs.)	surv.Clayton	0.544 (0.013)	-2154.319	0.862
	Gaussian	0.343 (0.007)	-1678.013	–
	t	11.517 (1.302); 0.343 (0.007)	-1771.787	–

Table 8: Bivariate excesses of residuals on USD/DEM and USD/JPY log-returns on the first quadrant of one hour returns for different thresholds. Estimates and standard errors of dependence parameters in Gumbel, survival Gumbel, Clayton, survival Clayton, Gaussian and t models. For each model fitted we provide the AIC value and the p-value for the best two models. For the t copula the first parameter estimate is the correlation and the second is the degrees of freedom and respectively for the s.e.'s. The number of observations associated with each threshold is listed under the threshold value.

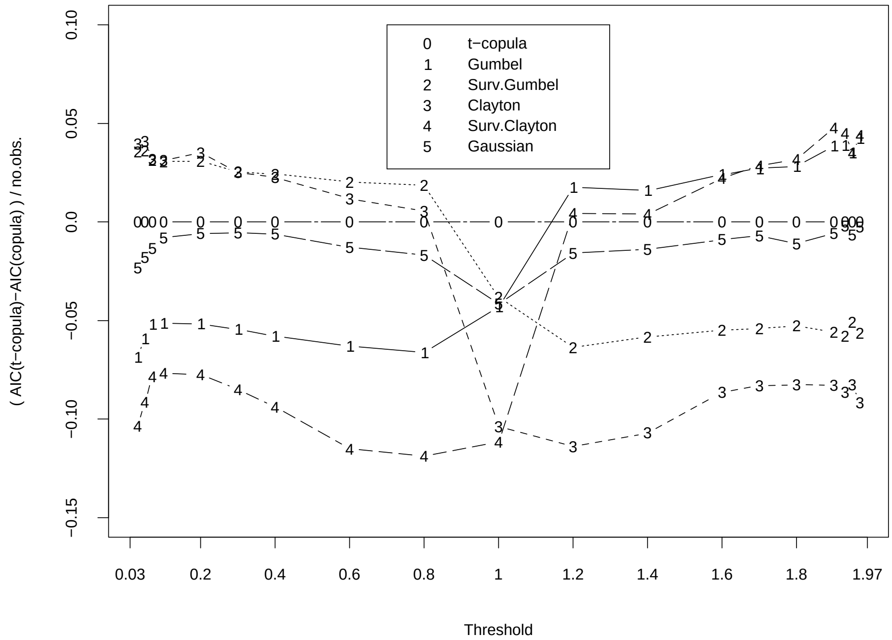

Figure 6: Relative AIC values for the most relevant models in fitting to the positive and negative tail extremes of hourly USD/DEM and USD/JPY return residuals for several thresholds.

2.7 Conclusions on the static copula modelling with volatility models for the margins

Starting from deseasonalized returns on USD/DEM and USD/JPY spot rates we analyzed the dependence structure of the data across different time horizons, namely for one, two, four, eight, twelve hour and one day frequencies. As the returns are time dependent we filtered them using univariate ARMA-GARCH time series models and obtained bivariate residuals of returns showing no evidence against the iid property but which were still contemporaneously dependent. Copula-based models were fitted to the full data sets of residuals at each frequency. The best fit came from the t copula together with a mixture of Gumbel and survival Gumbel, the later only for the daily residuals. The t copula revealed higher degrees of freedom as the time frequency decreases. We investigated the presence of asymptotic tail-dependence by three different approaches. First, the fitted copula-based models gave tail-dependence coefficients significantly different from zero. Allowing for tail index estimation, the data spectral density estimates showed the existence of bivariate extremes. In the case of hourly data, an extreme tail-dependence copula fitting to multivariate excesses above high thresholds revealed a Clayton type dependence structure with parameter estimates between 0.5 and 0.6 once more indicating dependence for the bivariate excesses in the residuals . We also tested for ellipticity and the results yield that lower frequency residuals have a dependence structure suitable to be modelled by elliptical distributions and possibly the higher frequencies too but there the results are less conclusive.

3 Change-point analysis

In Section 2 we modelled the dynamics of each margin with a GARCH type model and the contemporaneous dependence with a time invariant copula model. As the data covers a reasonably long period of time, about twelve years, we may expect that economic factors induced changes if not in the copula family at least in the dependence parameters. This section is devoted to test for the occurrence of such changes in the copula parameters, estimate the size of those changes and the corresponding time of occurrence, namely the change-points. For the theoretical background on change-point analysis we refer to Csörgő and Horváth (1997) and for an application of the methodology to the detection of changes in the copula parameters we refer to Dias and Embrechts (2002).

3.1 The test statistic

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a sequence of independent random vectors in $[0,1]^d$ with univariate uniformly distributed margins and copulae $C(\mathbf{u}; \boldsymbol{\theta}_1, \boldsymbol{\eta}_1)$, $C(\mathbf{u}; \boldsymbol{\theta}_2, \boldsymbol{\eta}_2), \ldots, C(\mathbf{u}; \boldsymbol{\theta}_n, \boldsymbol{\eta}_n)$ respectively, where $\boldsymbol{\theta}_i$ and $\boldsymbol{\eta}_i$ are the copula parameters such that $\boldsymbol{\theta}_i \in \Theta^{(1)} \subseteq \mathbb{R}^p$ and $\boldsymbol{\eta}_i \in \Theta^{(2)} \subseteq \mathbb{R}^q$. We will consider the $\boldsymbol{\eta}_i$ as nuisance parameters and look for one single change-point in $\boldsymbol{\theta}_i$. Formally, we test the null hypothesis

$$H_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_2 = \ldots = \boldsymbol{\theta}_n \qquad \text{and} \qquad \boldsymbol{\eta}_1 = \boldsymbol{\eta}_2 = \ldots = \boldsymbol{\eta}_n$$

versus the alternative

$$H_A : \boldsymbol{\theta}_1 = \ldots = \boldsymbol{\theta}_{k^*} \neq \boldsymbol{\theta}_{k^*+1} = \ldots = \boldsymbol{\theta}_n \qquad \text{and} \qquad \boldsymbol{\eta}_1 = \boldsymbol{\eta}_2 = \ldots = \boldsymbol{\eta}_n.$$

If we reject the null hypothesis, k^* is the time of the change. All the parameters of the model are supposed to be unknown under both hypotheses. If

$k^* = k$ were known, the null hypothesis would be rejected for small values of the likelihood ratio

$$\Lambda_k = \frac{\sup\limits_{(\boldsymbol{\theta},\boldsymbol{\eta})\in\Theta^{(1)}\times\Theta^{(2)}} \prod\limits_{1\leq i\leq n} c(\mathbf{u}_i;\boldsymbol{\theta},\boldsymbol{\eta})}{\sup\limits_{(\boldsymbol{\theta},\boldsymbol{\theta}',\boldsymbol{\eta})\in\Theta^{(1)}\times\Theta^{(1)}\times\Theta^{(2)}} \prod\limits_{1\leq i\leq k} c(\mathbf{u}_i;\boldsymbol{\theta},\boldsymbol{\eta}) \prod\limits_{k<i\leq n} c(\mathbf{u}_i;\boldsymbol{\theta}',\boldsymbol{\eta})}. \tag{9}$$

We assume that C has a density c given by

$$c(\mathbf{u}_i;\boldsymbol{\theta},\boldsymbol{\eta}) = \frac{\partial^d C(u_{1,i}, u_{2,i}\ldots, u_{d,i};\boldsymbol{\theta},\boldsymbol{\eta})}{\partial u_{1,i}\,\partial u_{2,i}\ldots\partial u_{d,i}},$$

with $(u_{1,i}, u_{2,i},\ldots, u_{d,i}) \in [0,1]^d$. The estimation of Λ_k is carried out through maximum likelihood and so all the necessary conditions of regularity and efficiency have to be assumed (see Lehmann and Casella (1998)).

Denote

$$L_k(\boldsymbol{\theta},\boldsymbol{\eta}) = \sum_{1\leq i\leq k} \log c(\mathbf{u}_i;\boldsymbol{\theta},\boldsymbol{\eta})$$

and

$$L_k^*(\boldsymbol{\theta},\boldsymbol{\eta}) = \sum_{k<i\leq n} \log c(\mathbf{u}_i;\boldsymbol{\theta},\boldsymbol{\eta}).$$

Then the likelihood ratio equation can be written as

$$-2\log(\Lambda_k) = 2\left(L_k(\hat{\boldsymbol{\theta}}_k,\hat{\boldsymbol{\eta}}_k) + L_k^*(\boldsymbol{\theta}_k^*,\hat{\boldsymbol{\eta}}_k) - L_n(\hat{\boldsymbol{\theta}}_n,\hat{\boldsymbol{\eta}}_n)\right).$$

As k is unknown, H_0 will be rejected for large values of

$$Z_n = \max_{1\leq k<n}(-2\log(\Lambda_k)). \tag{10}$$

3.1.1 Asymptotic critical values

The asymptotic distribution of $Z_n^{1/2}$ is known but has a very slow rate of convergence; see Csörgő and Horváth (1997), page 22. In the same reference we can also find an approximation for the distribution of $Z_n^{1/2}$ derived to give better small sample rejection regions. Indeed, for $0 < h < l < 1$, the following approximation holds:

$$P\left(Z_n^{1/2} \geq x\right) \approx \frac{x^p \exp(-x^2/2)}{2^{p/2}\Gamma(p/2)} \cdot \left(\log\frac{(1-h)(1-l)}{hl} - \frac{p}{x^2}\log\frac{(1-h)(1-l)}{hl} + \frac{4}{x^2} + O\left(\frac{1}{x^4}\right)\right), \tag{11}$$

as $x \to \infty$ and where h and l can be taken as $h(n) = l(n) = (\log n)^{3/2}/n$. Note that in (11) p is the number of parameters that may change under the alternative. This result turns out to be very accurate as shown in a simulation study in Dias and Embrechts (2002) where it is applied to the Gumbel copula.

3.1.2 The time of the change

If we assume that there is exactly one change–point, then the maximum likelihood estimator for the time of the change is given by

$$\hat{k}_n = \min\{1 \leq k < n : Z_n = -2\log(\Lambda_k)\}. \tag{12}$$

In the case that there is no change, $\hat{k}_n$ will take a value near the limits of the sample. This holds because under the null hypothesis and if all the necessary regularity conditions hold, for $n \to \infty$,

$$\hat{k}_n/n \xrightarrow{d} \xi,$$

where $P(\xi = 0) = P(\xi = 1) = 1/2$; see Csörgő and Horváth (1997), page 51. This behavior was verified in a simulation study for the Gumbel copula under the no-change hypothesis in Dias and Embrechts (2002).

3.1.3 Multiple Changes

The detection of several change–points in multidimensional processes with unknown parameters can be done using the so called binary segmentation procedure. This method was proposed by Vostrikova (1981) and enables to simultaneously detect the number and the location of the change–points. The method consists of first applying the likelihood ratio test for one change. If H_0 is rejected then we have the estimate of the time of the change $\hat{k}_n$. Next, we divide the sample in two subsamples $\{\mathbf{u}_i : 1 \leq i \leq \hat{k}_n\}$ and $\{\mathbf{u}_i : \hat{k}_n < i \leq n\}$ and test H_0 for each one of them. If we find a change in any of the sets we continue this segmentation procedure until we don't reject H_0 in any of the subsamples.

In the next section we use this procedure to estimate change-points in the correlation parameter of a t copula fitted to the residuals for daily USD/DEM and USD/JPY returns. Similar analyses can be performed at other frequencies; these results are not reported here.

3.2 Change-point analysis for the dependence structure parameters

After filtering the univariate returns using GARCH type models as in Section 2.1 the filtered residuals are assumed to be independent in time and we can use (10) and (11) for detecting possible change-points in the parameters of the multivariate contemporaneous distribution and in particular in the copula.

For these residuals, in Section 2 we showed that the t copula yields the best fitting model for the dependence structure between the two series. We use the empirical distribution function to map the residuals into the unit square. Moreover, in a first step, we assume that the degrees of freedom of the copula are constant in time and hence we test for change-points in the correlation parameter. We evaluate Λ_k for $k = 1, 2, \ldots, n$ where $n = 3,259$; see (9). The values obtained are displayed in the top panel of Figure 7. The test statistic (10) takes the value $z^{1/2}_{n\,obs} = 13.26$ and by (11) we have that $P(Z^{1/2}_n > 13.26) \approx 0$. The null hypothesis of no change-point is to be rejected and the estimated time of the change is $\hat{k}_n = 8$ November 1989; corresponding to the fall of the Berlin wall. In the next step, the sample is divided in two sub-samples, one up to 8 November 1989 and another from the estimated time of change onwards. For each sub-sample Λ_k is computed as well as $Z^{1/2}_n$. The middle panel of Figure 7 plots these estimates and Table 9 has the values for $Z^{1/2}_n$ and all the information about the testing procedure. As the obtained p-values are close to zero we reject the null hypothesis of no-change for each sub-sample and estimate two more times of change, 29 December 1986 and 18 June 1997. The later date corresponds to the beginning of the Asia crisis starting with the violent devaluation of the Thai Baht. Each sub-sample is again divided in two and the procedure is repeated yielding the estimates in the bottom panel of Figure 7.

$z^{1/2}_{n\,obs}$	n	$P(Z^{1/2}_n > z^{1/2}_{n\,obs})$	$H_0(0.95)$	Time of change
13.26	3,259	0	reject	8 Nov. 1989
5.96	923	0.0000004	"	29 Dec. 1986
5.31	2,336	0.0000143	"	18 June 1997
2.99	176	0.0689621	not rej.	(23 June 1986)
3.10	747	0.0709747	"	(31 July 1989)
5.86	1,985	0.0000007	reject	23 Oct. 1990
2.36	351	0.3380491	not rej.	(8 Sep. 1998)
2.78	1,736	0.1873493	"	(21 Oct. 1996)
2.86	249	0.1061709	"	(21 Mars 1990)

Table 9: Change-point analysis for USD/DEM and USD/JPY spot rate residuals.

For the results of the analysis showed in the bottom panel of Figure 7 only for the maximum attained at 23 October 1990 the null hypothesis is rejected at a 95% level. So we still have to split this sub-sample further. The first from 8 November 1989 until 23 October 1990 and the second from this date up to 18 June 1997. The $z^{1/2}_{n\,obs}$ obtained in these cases are low, see Table 9, and we do not reject the null hypothesis of no-change in both cases. In Table 9 we give the time where the test statistic is attained for each sub-sample. If the null hypothesis is not rejected the referred date is in parentheses as it is not considered a change-point. In summary we found four change-points: 29 December 1986, 8 November 1989, 23 October 1990 and 18 June 1997. For the five periods between the times of change we estimated for the copula correlation (s.e.): $\hat{\rho}_1 = 0.6513$ (0.0384), $\hat{\rho}_2 = 0.8312$ (0.0113), $\hat{\rho}_3 = 0.3099$ (0.0608), $\hat{\rho}_4 = 0.5752$ (0.0149) and $\hat{\rho}_5 = 0.3505$ (0.0460). To visualize these results we plotted in the bottom panel of Figure 11 the estimated cross-correlation for the five periods between the times of change. No change-points were detected from 23 October 1990 until 18 June 1997 which seems perhaps quite a long period for the correlation to be constant. It may be interesting to note that the former date (23 October 1990) corresponds to the burst in the Japanese asset price bubble. On 18 October 1990, the USD/JPY ended a fall from about 158 to 125.

We continue the dynamic modelling of the dependence structure in the next section refining the modelling from a change-points model, where the parameters are piecewise constant, to models with time-varying parameters, which allow for dependence parameters to vary from observation to observation.

4 Multivariate GARCH models with time-varying dependence parameters

As we saw in the above sections, for the FX spot rates on USD/DEM and USD/JPY the contemporaneous stationary dependence structure between the two residual series was well described by a t copula or by a mixture of a Gumbel with a survival Gumbel copula. The univariate serial dependence was modelled by ARMA-GARCH models with t innovations. These two aspects of dependence were modelled independently in two steps. First the univariate time series models were fitted and then copula-based models, time invariant in Section 2 and piecewise constant in Section 3, were used to model the cross-dependence structure of the residuals resulting from the marginal modelling. In this section, we want to model the dynamics of the time dependence structure as well as the dynamics of the contemporaneous dependence. For that we want to combine two univariate ARMA-GARCH models with a time-varying copula-based model. This is achieved using a copula-based model for the conditional bivariate innovations coupling two ARMA-GARCH processes. With such a procedure we investigate the constancy of the conditional dependence structure on time allowing for time-

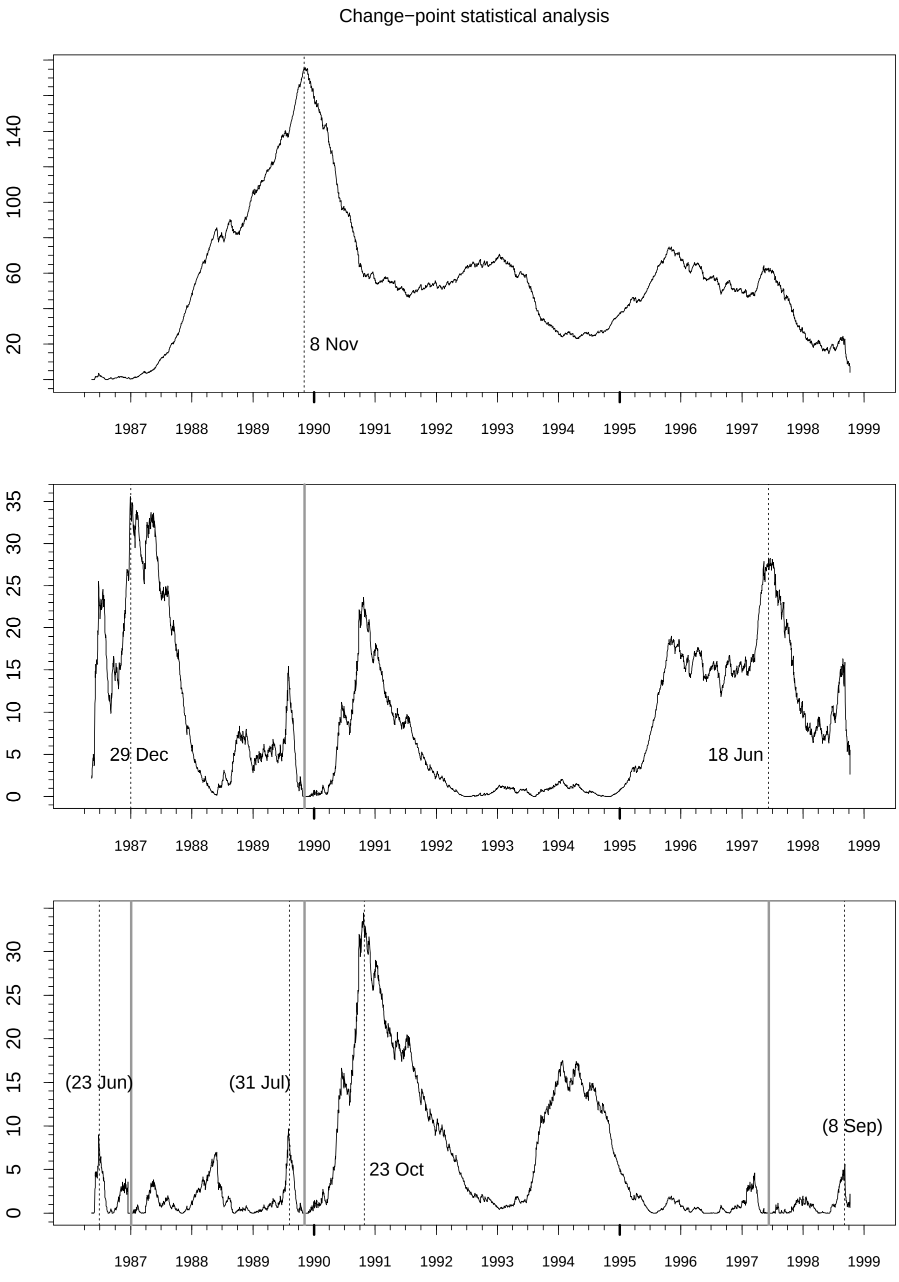

Figure 7: Change-point analysis of daily returns on the FX USD/DEM and USD/JPY spot rates. The three panels display three steps of the change-point analysis. Each panel plots the likelihood ratio values Λ_k for $k = 1, 2, \ldots, n$. In each sub-sample its maximum, the test statistic Z_n, gives the time of the change in case the no-change null hypothesis is rejected. If the null hypothesis is not rejected the moment where Z_n is achieved is put in parentheses.

varying dependence parameters and assuming a fixed copula family.

As before, we look at several time frequencies for the spot rates considered. The coupling of univariate GARCH models using copulae in the conditional distribution can be found in Rockinger and Jondeau (2001), Patton (2002) and Fortin and Kuzmics (2002).

4.1 Stochastic dependence structure

Stochastic volatility is nowadays accepted as a stylized fact in financial univariate processes. When we want to study several financial time series simultaneously we have to consider the dependence structure among them. If the conditional volatility of a series is not constant there is no reason why we should expect that from the correlation between different time series or more generally from the conditional dependence (copula) structure.

If we want to inspect for time-varying cross-correlation we can construct the so called exponentially weighted covariance estimate; see Foster and Nelson (1996) and Andreou and Ghysels (2002). In the same spirit but more sophisticated are the matrix-diagonal models as proposed by Bollerslev et al. (1994). We choose the later models as a first approach to look at the covariance component of the conditional dependence structure. We also considered the BEKK models from Engle and Kroner (1995) but the increase in the number of parameters did not lead to a fitting improvement.

In a way similar to a one-dimensional GARCH model, the multivariate matrix-diagonal GARCH model for a d-dimensional random vector $\mathbf{X}_t$ is defined as

$$\begin{aligned} \mathbf{X}_t &= \mathbf{c} + \boldsymbol{\epsilon}_t \\ \boldsymbol{\epsilon}_t &= \Sigma_t^{1/2}\mathbf{Z}_t \\ \Sigma_t &= A_0 A_0^t + \sum_{i=1}^{p}(A_i A_i^t) \otimes (\boldsymbol{\epsilon}_{t-i}\boldsymbol{\epsilon}_{t-i}^t) + \sum_{j=1}^{q}(B_j B_j^t) \otimes \Sigma_{t-j} \end{aligned} \tag{13}$$

where A_i for $i = 0, 1, \ldots, p$ and B_j with $j = 1, 2, \ldots, q$ are lower triangular $d \times d$ matrices. Moreover, $\mathbf{c}$ is a vector in $\mathbb{R}^d$ and p and q are like in the univariate model. The d-dimensional vector sequence $\{\mathbf{Z}_t\}_{t\in\mathbb{N}_0}$ is assumed to be iid with zero mean vector and unit variances. The matrix Σ_t stands for the conditional covariance matrix of the vector $\boldsymbol{\epsilon}_t$ and $\Sigma_t^{1/2}$ is obtained through the Cholesky decomposition of Σ_t. In (13), $\otimes$ stands for the Hadamard product, the element by element multiplication. With these models we have the guarantee of obtaining positive semi-definite covariance matrix estimates. To evaluate the standardized residuals we compute $\Sigma_t^{-1/2}\boldsymbol{\epsilon}_t$.

We used S+FinMetrics to fit bivariate AR-GARCH models with matrix-diagonal multivariate specification ((13)) to the six time frequencies considered for the FX spot rate returns. For fitting purposes, the standardized residuals $\mathbf{Z}_t$ are assumed to come from a bivariate t distribution. The estimated degrees of freedom obtained for the t innovations for each time frequency are given in Table 10.

Time frequency	Estimated d.f. $\hat{\nu}(\hat{s.e.})$
1 hour	4.888 (1.318)
2 hour	5.253 (4.358)
4 hour	5.351 (1.450)
8 hour	5.893 (0.930)
12 hour	5.943 (0.429)
1 day	5.998 (0.320)

Table 10: Degrees of freedom of the assumed t innovations in the matrix-diagonal models fitted to the USD/DEM and USD/JPY spot rate returns for the several time horizons.

For each time frequency we plotted in Figures 8 and 9 the estimated conditional cross-correlation. The conditional correlation seems to fluctuate quite a lot, mostly around 0.6 with occasional drops to values that can be negative especially for the higher frequencies. There seems to be evidence for the existence of three regimes, and this consistently at all frequencies. The corresponding periods are first up to the end of 1989, then from the latter date till mid 1993, and finally from this date till the end of 1997. As noted before (Section 3) the shift by the end of 1989 coincides with the start of the German unification while in June of 1997 the Asia crisis began. Assuming that the Berlin wall event (10:30 pm, 9 November 1989) caused the estimated shift from 8 to 9 November 1989 (see Section 3) then this change is visible much more precisely in Figures 8 and 9 at higher frequencies than at the lower frequencies.

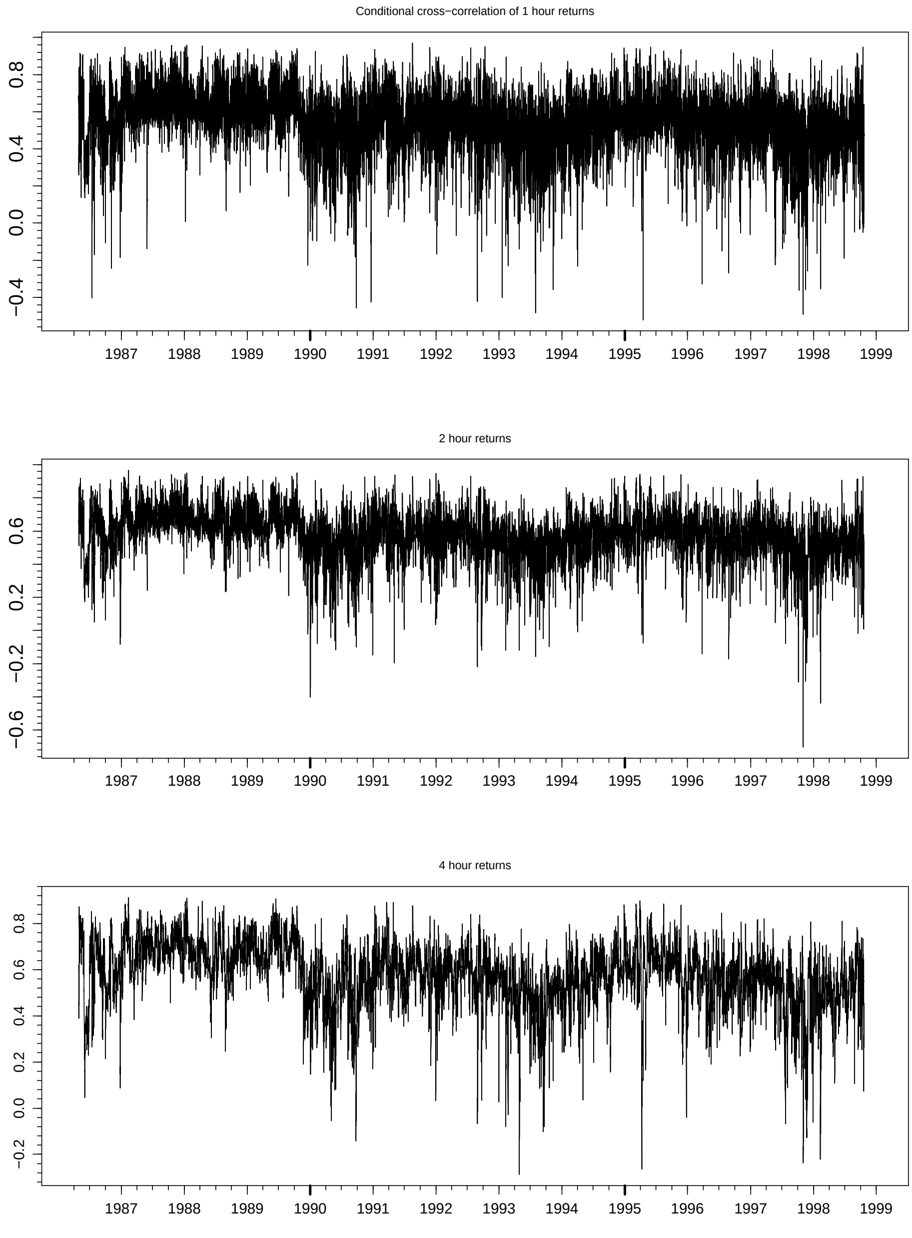

Figure 8: Time-varying cross-correlations estimated by a matrix-diagonal AR-GARCH model for the returns on the FX USD/DEM and USD/JPY spot rates of one, two and four hour frequencies.

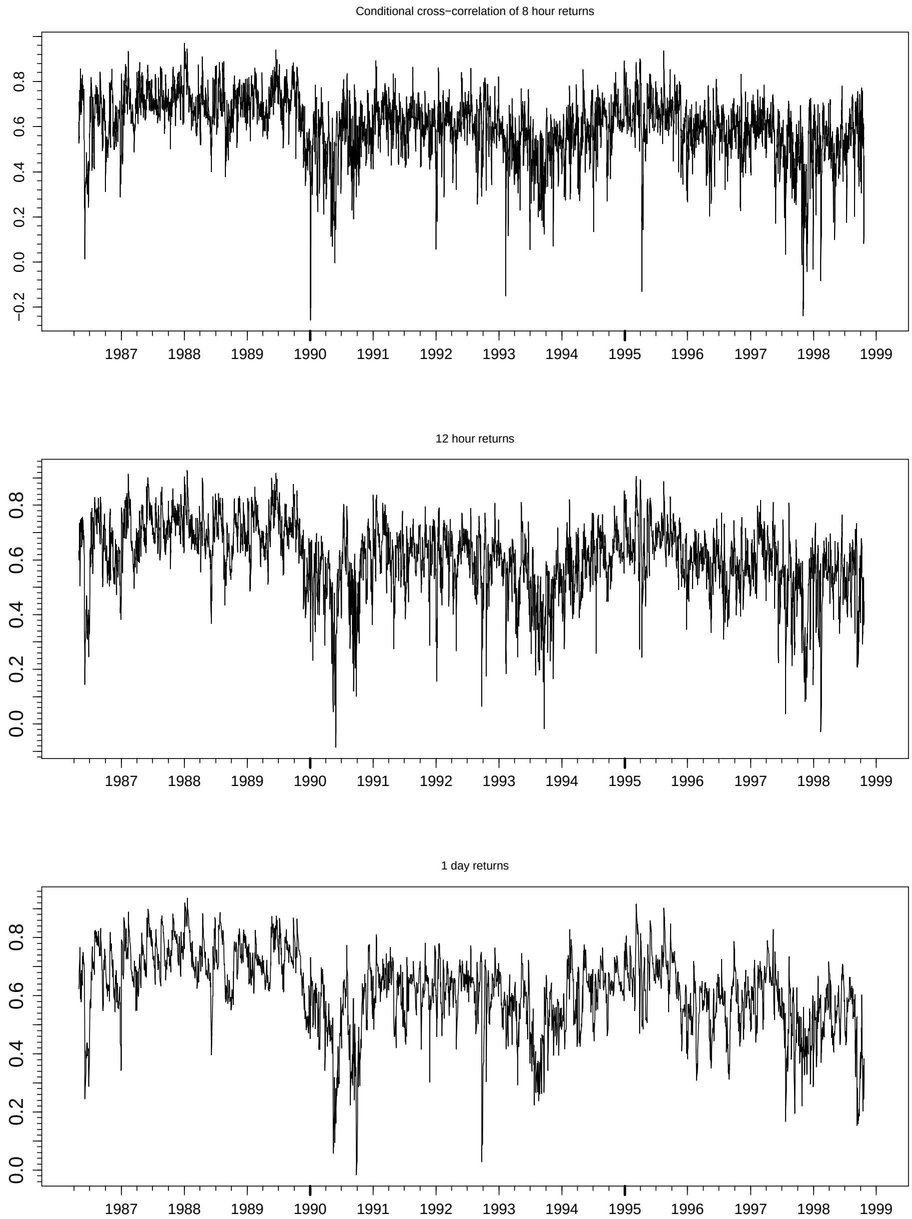

Figure 9: Time-varying cross-correlations estimated by a matrix-diagonal AR-GARCH model for the returns on the FX USD/DEM and USD/JPY spot rates of eight hour, twelve hour and one day frequencies.

4.2 The Multivariate GARCH model with time-varying copula

The estimated covariances in Section 4.1 show that we can not expect a constant type conditional dependence structure between the two return series, USD/DEM and USD/JPY. That means that we should use a model where somehow the conditional dependence variability is incorporated. Given that we want to couple two GARCH type models through a parametric copula, the simplest way is to admit that the conditional copula stays in the same family but the dependence parameters are time-varying.

Let $\{\mathbf{X}_t : t \in \mathbb{N}_0\}$ be a sequence of observable d-dimensional random vectors. Consider the process description given by

$$\begin{aligned} \mathbf{X}_t &= \mathbf{c} + \boldsymbol{\epsilon}_t \\ \boldsymbol{\epsilon}_t &= \boldsymbol{\sigma}_t \mathbf{Z}_t \\ \boldsymbol{\sigma}_t^2 &= A_0 + \sum_{i=1}^{p} A_i \otimes (\boldsymbol{\epsilon}_{t-i}\boldsymbol{\epsilon}_{t-i}^t) + \sum_{j=1}^{q} B_j \otimes \boldsymbol{\sigma}_{t-j}^2 \end{aligned} \tag{14}$$

where A_i for $i = 0, 1, \ldots, p$ and B_j with $j = 1, 2, \ldots, q$ are diagonal $d \times d$ matrices, $\mathbf{c}$ is a vector in $\mathbb{R}^d$ and p and q are positive integers. Moreover, $\{Z_{i,t}\}_{t \in \mathbb{N}_0}$ for $i = 1, 2, \ldots, d$ are assumed to be univariate strict white noise processes with zero mean and unit variance. The set of equations (14) simply defines each marginal process as a univariate GARCH. Now we couple the d processes (14) imposing a copula family to the multivariate distribution of $\mathbf{Z}_t$. Assume that $\mathbf{Z}_t$ has a d-dimensional copula C with time dependent parameter vector $\boldsymbol{\theta}_t = (\theta_{1,t}, \theta_{2,t}, \ldots, \theta_{e,t})$ such that

$$\theta_{m,t} = r_0 + \sum_{i=1}^{r} r_i \prod_{j=1}^{d} Z_{j,t-i} + \sum_{k=1}^{s} s_k \theta_{m,t-k} \tag{15}$$

for $m = 1, 2, \ldots, e$ and where r_i for $i = 0, 1, \ldots, r$ and s_j with $j = 1, 2, \ldots, s$ are scalar model parameters. Equation (15) defines a dynamic structure of GARCH type for the dependence parameters and is motivated by (13). See also Patton (2002) and Rockinger and Jondeau (2001) for similar models. Of course, one can attempt to find more suitable dynamics depending on the interpretation that a specific dependence parameter may have. For instance referring to the matrix-diagonal GARCH, the degrees of freedom for the t innovations were assumed to be constant over time. Asymmetry in the dependence parameters can also be included as for instance for the correlation in the Asymmetric Generalized Dynamic Conditional Correlation GARCH in Cappiello et al. (2003).

4.3 Model estimation

The natural estimation method for (14) and (15) is (conditional) maximum likelihood. Furthermore, the definition of the model suggests a two step estimation procedure. In fact, this is used in similar situations by Patton (2002), Rockinger and Jondeau (2001) and Engle and Sheppard (2001).

Let $(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n)$ be a random sample of d-dimensional vectors. Suppose that $\mathbf{X}_t$, given the past information have continuous marginal distributions F_i parameterized by the vector $\boldsymbol{\alpha}_{i,t}$ for $i = 1, 2, \ldots, d$. Then the multivariate conditional distribution function of $\mathbf{X}_t$ is

$$F(\mathbf{x}; \boldsymbol{\alpha}_{1,t}, \ldots, \boldsymbol{\alpha}_{d,t}, \boldsymbol{\theta}_t) = C(F_1(x_1; \boldsymbol{\alpha}_{1,t}), \ldots, F_d(x_d; \boldsymbol{\alpha}_{d,t}); \boldsymbol{\theta}_t)$$

where C is the copula family of $\mathbf{X}_t$. The conditional density function of $\mathbf{X}_t$ is

$$f(\mathbf{x}; \boldsymbol{\alpha}_{1,t}, \ldots, \boldsymbol{\alpha}_{d,t}, \boldsymbol{\theta}_t) = \\ = c(F_1(x_1; \boldsymbol{\alpha}_{1,t}), \ldots, F_d(x_d; \boldsymbol{\alpha}_{d,t}); \boldsymbol{\theta}_t) \prod_{i=1}^{d} f_i(x_i; \boldsymbol{\alpha}_{i,t}).$$

Here we assume that C has a density c given by

$$c(u_1, u_2, \ldots, u_d; \boldsymbol{\theta}) = \frac{\partial^d C(u_1, u_2 \ldots, u_d; \boldsymbol{\theta})}{\partial u_1 \, \partial u_2 \ldots \partial u_d},$$

with $(u_1, u_2, \ldots, u_d) \in [0,1]^d$ and that F_i has a density f_i for all $i = 1, 2, \ldots, d$. The conditional log-likelihood function of the model then is

$$\sum_{t=m+1}^{n} \left(\log c(F_1(x_{1,t}; \boldsymbol{\alpha}_{1,t}), \ldots, F_d(x_{d,t}; \boldsymbol{\alpha}_{d,t}); \boldsymbol{\theta}_t) + \sum_{i=1}^{d} \log f_i(x_{i,t}; \boldsymbol{\alpha}_{i,t}) \right) \tag{16}$$

where $m = \max(p, r)$. Numerical maximization of (16) gives the maximum likelihood estimates of the model. However, the optimization of the likelihood function with possibly many parameters is numerically difficult and (computer) time consuming. It is more tractable to estimate first the marginal model parameters and then the dependence model parameters using the estimates from the first step. In order to do so, the d marginal likelihood functions

$$\sum_{t=p+1}^{n} \log f_i(x_{i,t}; \boldsymbol{\alpha}_{i,t}) \qquad \text{for } i = 1, 2, \ldots, d, \tag{17}$$

are independently maximized. From here we obtain the estimates $\hat{\boldsymbol{\alpha}}_{1,t}, \ldots, \hat{\boldsymbol{\alpha}}_{d,t}$. These are plugged in (16) where some terms became constant and can be ignored. The final function to maximize becomes

$$\sum_{t=m+1}^{n} \log c(F_1(x_{1,t}; \hat{\boldsymbol{\alpha}}_{1,t}), \ldots, F_d(x_{d,t}; \hat{\boldsymbol{\alpha}}_{d,t}); \boldsymbol{\theta}_t). \tag{18}$$

From this dependence estimates $\hat{\boldsymbol{\theta}}_t$ are obtained and the model is fitted. In Section 4.4 below we highlight the above procedure, however for space considerations restricting attention to the t copula.

4.4 Fitting of the time-varying copula model to the USD/DEM and USD/JPY spot rate returns

For the USD/DEM and USD/JPY spot rate returns we found that the t copula yields a good model for the cross dependence. This was shown first through estimating the stationary bivariate distribution (static models); see Breymann et al. (2003). In Section 2 we arrived at this result though estimating the dependence structure after using time dependent marginal models (fitting copula-models to the filtered returns). In all cases where we fitted univariate GARCH type models, t innovations were used. Now we are going to combine dynamic models for the margins with dynamic copula models. For each marginal time series we assume a GARCH type model. For the dependence structure we use a t copula allowing for time dynamics in the copula parameters which are the degrees of freedom ν and the correlation ρ:

$$\begin{aligned} \nu_t &= \nu \qquad \text{for all } t, \\ \rho_t &= h^{-1}(r_0 + r_1 z_{1,t-1} z_{2,t-1} + s_1 h(\rho_{t-1})), \end{aligned} \tag{19}$$

where $h(\cdot)$ is Fisher's transformation for the correlation

$$h(\rho) = \log\left(\frac{1+\rho}{1-\rho}\right).$$

The choice of model (19) is based on tractability and relevance for practice, at the same time highlighting the general procedure.

4.4.1 Marginal modelling

For every time frequency, each marginal time series is modelled by an ARMA-GARCH model with t innovations.

In each univariate model we included a leverage effect parameter γ in the GARCH dynamics; see for example Bollerslev et al. (1992) and references therein. The introduction of γ attempts to take into account an asymmetric contribution that the innovations seem to have on the volatility in some cases. This improvement is also possible in the model specified in (14); see Zivot and Wang (2003) and Ding et al. (1993) where model (14) is treated as a special case of a power GARCH model. In this case, each line of the last matrix equation in (14) becomes

$$\sigma_{k,t}^2 = \alpha_{k,0} + \sum_{i=1}^{p} \alpha_{k,i}(|\epsilon_{k,t-i}| + \gamma_{k,i}\epsilon_{k,t-i})^2 + \sum_{j=1}^{q} \beta_{k,j}\sigma_{k,t-j}^2$$

where $k = 1, 2, \ldots, d$. In fact, we already used the leverage effect parameter in the marginal modelling in the previous sections. For USD/DEM returns we can not reject the null hypothesis of $\gamma_k = 0$ for the estimated γ parameter for all frequencies. In the case of DEM/JPY the situation is the complete reverse. In this case, we reject the null hypothesis for all frequencies. Rejecting the null hypothesis for the USD/JPY model parameter $\gamma_k = 0$, and using that the estimated values $\hat{\gamma}$ are negative, we have that negative shocks (bad news) have a larger impact on volatility than positive shocks (good news).

Time	$\hat{\nu}$ $(\hat{s.e.})$	
frequency	USD/DEM	USD/JPY
1 hour	3.693 (0.054)	3.654 (0.052)
2 hour	3.708 (0.044)	3.759 (0.077)
4 hour	3.975 (0.105)	3.819 (0.109)
8 hour	4.679 (0.234)	4.357 (0.195)
12 hour	5.385 (0.326)	4.574 (0.251)
1 day	5.797 (0.556)	4.889 (0.412)

Table 11: The degrees of freedom estimated for the marginal conditional distribution t of the innovations and corresponding standard errors.

From the univariate fitting we need the estimated degrees of freedom $\hat{\nu}$ to be used in (18). Table 11 has the estimated marginal degrees of freedom.

4.4.2 Dynamic copula modelling results

Denote by $\{(\hat{z}_{1,t}, \hat{z}_{2,t}) : t = 0, \ldots, n\}$ the standardized residual return series obtained from the univariate filtering performed in Section 4.4.1. The estimated degrees of freedom for the marginal innovation distributions are $\hat{\nu}_1$ and $\hat{\nu}_2$ for USD/DEM and USD/JPY, respectively. Now the standardized residuals are mapped into the unit square by the standard t probability-integral transformation. Considering that t_ν denotes the distribution function of a univariate standard t with ν degrees of freedom, the probability-integral transformation produces the bivariate time series of pseudo observations:

$$\left\{ \left(t_{\hat{\nu}_1} \left(\sqrt{\frac{\hat{\nu}_1}{\hat{\nu}_1 - 2}}\, \hat{z}_{1,t} \right), t_{\hat{\nu}_2} \left(\sqrt{\frac{\hat{\nu}_2}{\hat{\nu}_2 - 2}}\, \hat{z}_{2,t} \right) \right), t = 1, \ldots, n \right\}. \tag{20}$$

This time series, plugged into (18) with C being the t copula function and using (19) for the dynamics of the dependence parameters, gives the maximum likelihood estimates for the copula degrees of freedom and correlation (time-varying) parameters.

The results from fitting the dynamic copula model are in Table 12. We added the t copula parameter estimates obtained with no dynamics in the correlation which corresponds to $r_1 = 0$ and $s_1 = 0$ in (19) or equivalently, to fit a t copula with time invariant parameters to (20).

The AIC of the time-varying copula model is lower than the AIC of the constant copula model. So we have an improvement in the fitting in spite of having two more parameters in the model. The estimate for r_0 can be considered zero for 8 hour, twelve hour and daily returns. But r_1 and s_1 are definitely different from zero for all frequencies. In other words, the estimated (copula) correlation depends on the marginal returns and on the correlation from the previous moment in time. From the estimated parameters for the correlation dynamics we compute, through the second equation of (19), the time-varying estimated correlation which is plotted in Figures 10 and 11. These also show the estimated constant correlation with a 95% confidence interval. Figures 8 and 9 plot the estimated time-varying correlation from a matrix-diagonal GARCH model ((13)) with bivariate t innovations. Comparing the two results the main difference is that the correlation given by the dynamic copula model is much less jagged than the one from the matrix-diagonal GARCH allowing for a more detailed observation of the correlation path.

The number of degrees of freedom estimated is always larger for the time-varying copula model than for the matrix-diagonal model. In both models the copula used is the t and the margins are t distributed. But while in the matrix-diagonal model margins and copula must have the same degrees of freedom in the copula-based model they don't. Actually we can see, for example for the daily returns, from Table 11 that for each margin we have $\hat{\nu}_{USD/DEM} = 5.797$, $\hat{\nu}_{USD/JPY} = 4.889$ and from Table 12 that for the copula $\hat{\nu} = 8.573$. On the other hand, model (13) imposes

$\nu_{USD/DEM} = \nu_{USD/JPY} = \nu$ and gives $\hat{\nu} = 5.998$. Actually the degrees of freedom obtained with the matrix-diagonal model are close to those given by the copula time-invariant model; compare the values listed in Tables 10 and 12. From the different ways the degrees of freedom were estimated there is a common increasing pattern from higher to lower frequencies which may be referred to as a central limit effect.

Time frequency	Parameter Estimates (s.e.) non-dynamic		dynamic	
1 hour	$\hat{\nu}$	4.935 (0.108)	$\hat{\nu}$	6.330 (0.167)
	$\hat{\rho}$	0.558 (0.002)	$\hat{r}_0$	0.0005 (0.0002)
			$\hat{r}_1$	0.0193 (0.0010)
			$\hat{s}_1$	0.9921 (0.0005)
	AIC	-31517.70	AIC	-34488.72
2 hour	$\hat{\nu}$	4.822 (0.147)	$\hat{\nu}$	6.203 (0.230)
	$\hat{\rho}$	0.580 (0.003)	$\hat{r}_0$	-0.0004 (0.0002)
			$\hat{r}_1$	0.0128 (0.0009)
			$\hat{s}_1$	0.9952 (0.0004)
	AIC	-17192.73	AIC	-19349.29
4 hour	$\hat{\nu}$	4.669 (0.195)	$\hat{\nu}$	6.072 (0.313)
	$\hat{\rho}$	0.592 (0.005)	$\hat{r}_0$	-0.0008 (0.0002)
			$\hat{r}_1$	0.0147 (0.0011)
			$\hat{s}_1$	0.9947 (0.0004)
	AIC	-9085.848	AIC	-10262.23
8 hour	$\hat{\nu}$	5.296 (0.339)	$\hat{\nu}$	7.206 (0.584)
	$\hat{\rho}$	0.612 (0.006)	$\hat{r}_0$	0.0005 (0.0005)
			$\hat{r}_1$	0.0173 (0.0014)
			$\hat{s}_1$	0.9927 (0.0006)
	AIC	-4813.6	AIC	-5456.312
12 hour	$\hat{\nu}$	5.830 (0.499)	$\hat{\nu}$	8.053 (0.884)
	$\hat{\rho}$	0.620 (0.008)	$\hat{r}_0$	0.0002 (0.0008)
			$\hat{r}_1$	-0.0249 (0.0023)
			$\hat{s}_1$	0.9901 (0.0010)
	AIC	-3299.16	AIC	-3744.28
1 day	$\hat{\nu}$	5.945 (0.758)	$\hat{\nu}$	8.573 (1.455)
	$\hat{\rho}$	0.619 (0.011)	$\hat{r}_0$	-0.0023 (0.0017)
			$\hat{r}_1$	-0.0343 (0.0041)
			$\hat{s}_1$	0.9846 (0.0021)
	AIC	-1644.549	AIC	-1881.760

Table 12: Parameter estimates, standard errors and AIC values for the two copula models, without and with dynamics in the correlation, fitted to the hourly up to daily returns on USD/DEM and USD/JPY rates.

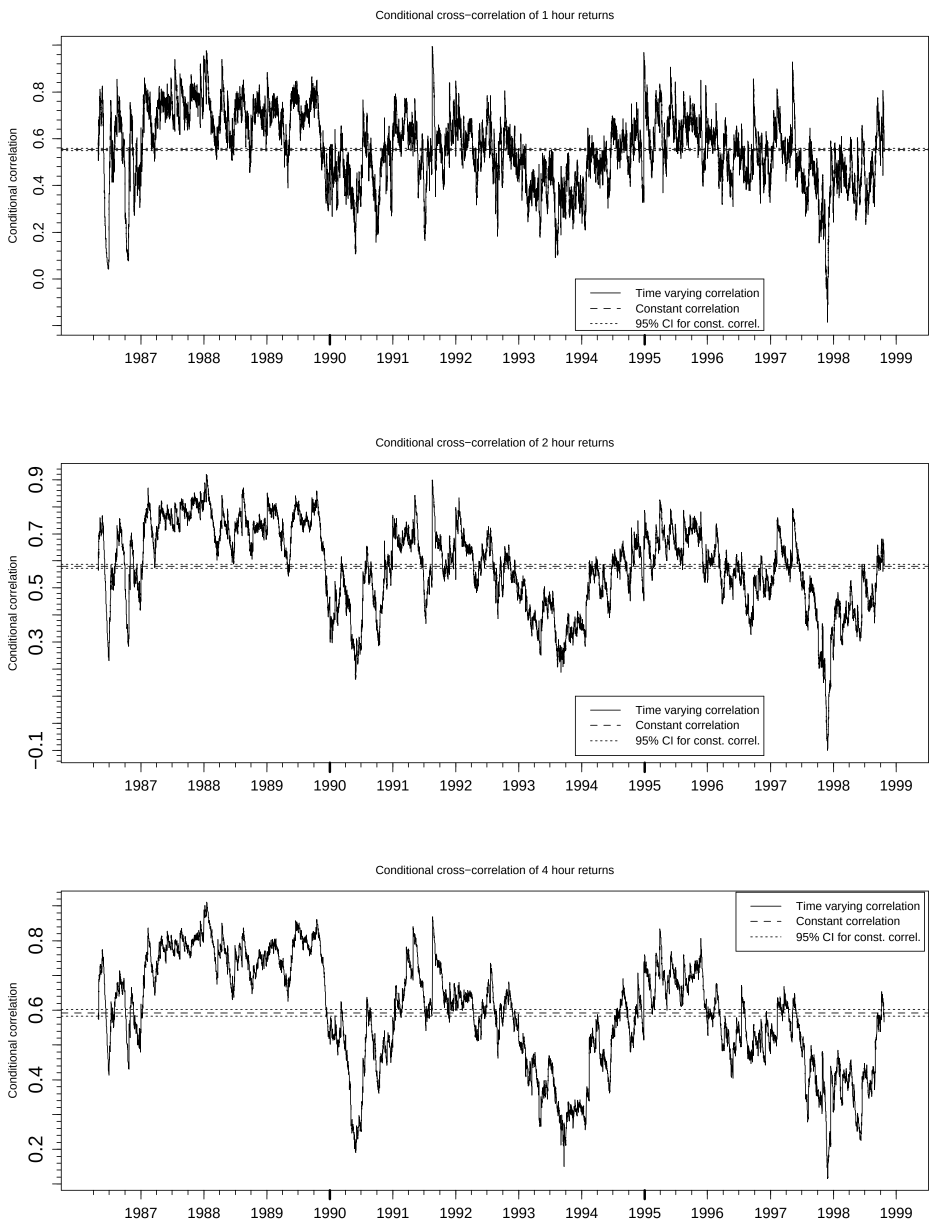

Figure 10: Time-varying cross-correlations estimated by a time-varying copula-based model for the four hour returns on the FX USD/DEM and USD/JPY spot rates.

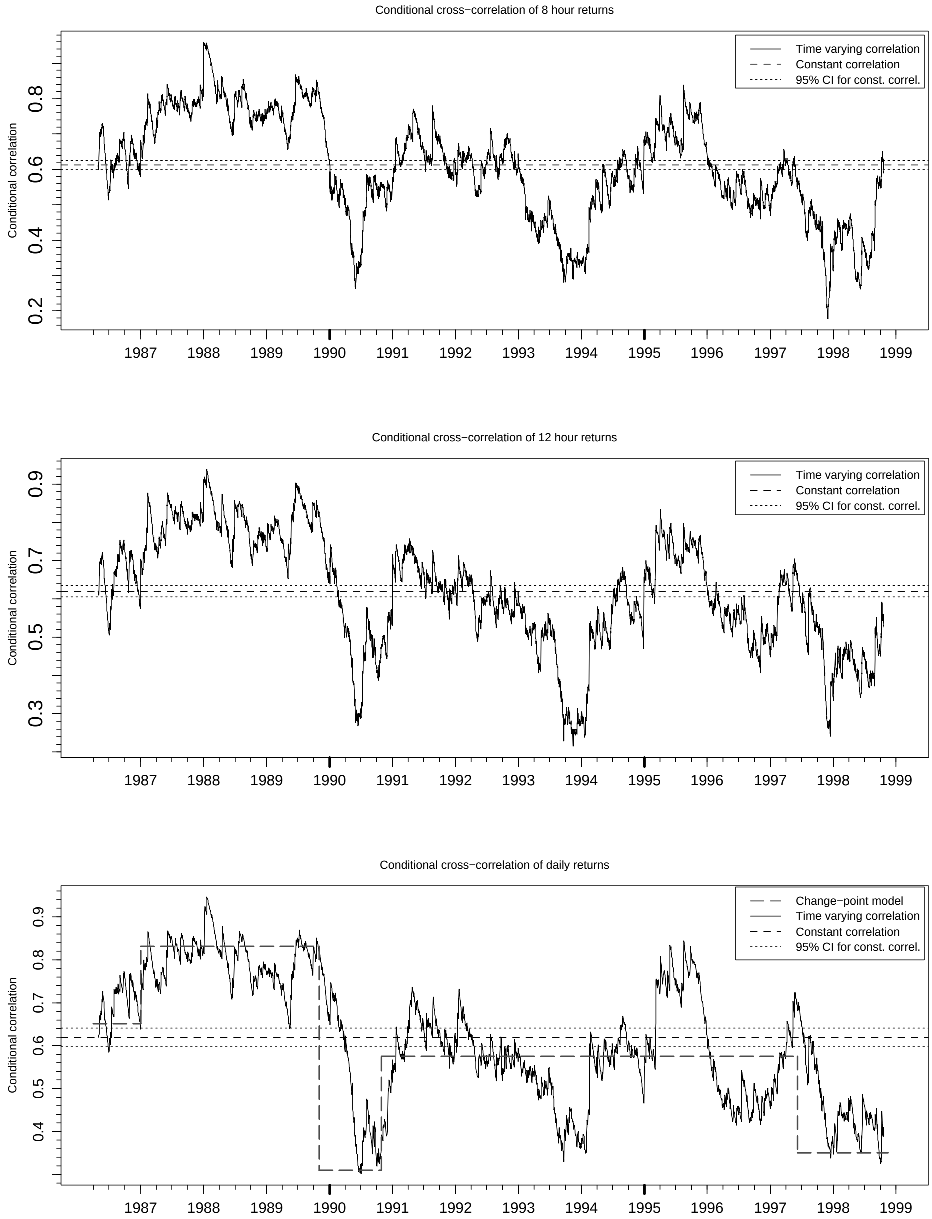

Figure 11: Time-varying cross-correlations estimated by a time-varying copula-based model for the eight hour, twelve hour and daily returns on the FX USD/DEM and USD/JPY spot rates.

5 Conclusion

Our goal in this study is a first statistical analysis of the dependence structure between FX returns on USD/DEM and USD/JPY spot rates across six different frequencies from one hour up to one day. The data used are the deseasonalized returns obtained for the static study performed in Breymann et al. (2003). The analysis is done under three main approaches to the modelling of the (conditional) observations. Firstly, the dependence structure is assumed to be constant in time, secondly, we test for changes in the dependence parameters and finally, the copula family is supposed to be always the same but its parameters are time-varying. In all the approaches each marginal time series is modelled by a GARCH type model producing what is referred to as filtered returns (or residuals).

Assuming a time invariant copula model, in Section 2 a list of copula models are fitted to the filtered returns. The t copula turns out to be a potentially good model followed by a mixture of a Gumbel with a survival Gumbel for all frequencies except for the daily returns where the two models perform almost equally well with slight advantage for the mixture one. The fitted degrees of freedom of the t copula increase as the time frequency decreases just like it happened without marginal GARCH filtering in Breymann et al. (2003). Nevertheless, the degrees of freedom are higher in the time-varying copula model compared with the other models used. A tail-dependence coefficient estimation gives values around 0.25 revealing the presence of asymptotic tail-dependence between the two FX rates even at the residuals level. This fact is confirmed through the estimation of the spectral densities showing clustering of extremes in both tails of equal signs. The extreme tail-dependence copula is well modelled by a Clayton type copula and the parameter estimates imply the existence of dependence in the bivariate residuals over thresholds going further into the first and third quadrant tails. Ellipticity of the data is rejected only for one and two hour frequencies when the margins are t transformed.

As an improvement to the time invariant copula-model we tested for the existence of change-points in the correlation parameter of the t copula for the daily filtered returns. The tests gave four change-points in the considered period which imply considerable jumps in the correlation over time. The largest is estimated at 8 November 1989 with a drop in the correlation from $\hat{\rho} = 0.831$ to $\hat{\rho} = 0.310$. Change-point tests have less power in case of small changes and so we refined the dependence analysis assuming a time-varying behavior for the correlation parameter. In this way matrix-diagonal GARCH models are fitted to all frequencies. From these we can observe a jagged path to the estimated correlations far from being constant or even piecewise constant. Using the t copula model with time varying correlation we get a more flexible distribution for the model innovations which gives estimated correlation processes with much less noise than the matrix-diagonal models. On the other hand, allowing for time-varying dependence parameters, according to the AIC criterion, we obtain a better fitting than with a time invariant dependence structure. Comparing the estimates obtained by the copula-varying model and the other models considered we observe that in the first case the degrees of freedom are considerably higher for all time frequencies. The monotonic behavior for the estimated degrees of freedom is preserved. For the daily returns we can compare the change-point model with the time-varying t copula model. It seems that change-point models only detect bigger changes but apparently sooner than a time-varying parameter model; see the bottom panel in Figure 11.

Clearly more general copula models can be handled similarly. It would in particular be interesting to investigate t copula models with time varying degrees of freedom for the returns on the USD/DEM and USD/JPY spot rates. Moreover, in models with time-varying dependence parameters, the existence of change-points can be tested in the parameters of the equations that define the copula dynamics. We return to some of these questions in further publications.

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