Worst VaR Scenarios

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Abstract

The worst-possible Value-at-Risk for a non-decreasing function ψ of n dependent risks is known when $n = 2$ or the copula of the portfolio is bounded from below. In this paper we analyze the properties of the dependence structures leading to this solution, in particular their form and the implied functional dependence between the marginals. Furthermore we criticise the assumption of the worst-possible scenario for VaR-based risk management and we provide an alternative approach supporting comonotonicity.

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1 Introduction

Consider an insurer holding a portfolio consisting of n policies with individual risks $X_1, \dots, X_n$ over a fixed time period. Given some measurable function $\psi : \mathbb{R}^n \to \mathbb{R}$, a relevant task in insurance mathematics is the investigation of the risk position associated with $\psi(X_1, \dots, X_n)$, when the marginal distributions of the single risks are known. Actuarial examples of the function ψ include $\sum_{i=1}^n x_i$, simply characterizing the aggregate claim amount deriving from the policies or $\sum_{i=1}^n h_i(x_i)$ and $h(\sum_{i=1}^n x_i)$, providing the risk positions for a reinsurance treaty with retention functions $h_i, i = 1, \dots, n$ and a global reinsurance treaty with global retention function h, respectively.

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The problem of finding the best-possible lower bound on the distribution function (df) of $\psi(X_1, \dots, X_n)$ has received a considerable interest in insurance mathematics. From a financial risk management point of view, the problem is equivalent to finding the worst-possible Value-at-Risk (VaR) for the corresponding aggregate position. Here we refer to the Introduction in Embrechts and Puccetti (2004) for details.

Modelling the interdependence arising in a random portfolio calls for the use of copulas. If a lower bound on the copula of the vector $(X_1, \dots, X_n)$ is given, the above problem is fully solved and the bounds provided in Embrechts et al. (2003) are sharp. In the no-information case the latter do hold only if $n = 2$. Rather than treating the technical proof of such results, for which we refer to the above cited references, in this paper we analyse in more details the properties of their solutions. We concentrate mainly on the no-information case, when a lower bound on the copula of the portfolio is not available and a solution is known only for two-dimensional portfolios. Without loss of generality, we study the optimizing copula for the sum of two dependent risks, which is well-known to differ from comonotonicity. In particular we discuss its shape, its implications in terms of dependence and we criticise it as not being a rational scenario for an insurance company. Finally, we provide an alternative optimization approach leading to a suitable measure of risk, which supports the assumption of comonotonicity for a prudent evaluation of the VaR for the aggregate position.

2 Preliminaries and fundamental results

In this section we present some well-known concepts about copulas and briefly recall the fundamental results about the problem of bounding the VaR for functions of dependent risks. For more details about copulas, we refer to Nelsen (1999).

2.1 Value-at-Risk and dependence structures

On some probability space $(\Omega, \mathfrak{A}, \mathbb{P})$, let the random vector $\underline{X} := (X_1, \dots, X_n)$ represent a portfolio of risks. Given a measurable function $\psi : \mathbb{R}^n \to \mathbb{R}$ we face the problem of finding the supremum of the VaR for the aggregate position $\psi(\underline{X})$ over the class of possible dfs for $\underline{X}$ having fixed marginals $F_1, \dots, F_n$.

Definition 1 *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function. Its* generalized left continuous inverse *is the function $\varphi^{-1} : \mathbb{R} \to \overline{\mathbb{R}}$ defined by $\varphi^{-1}(y) := \inf\{x \in \mathbb{R} \,|\, \varphi(x) \geq y\}$. For $0 \leq \alpha \leq 1$ the* Value-at-Risk at probability level α *for a random variable Y with distribution function G is its α-quantile, i.e.*

$$\mathrm{VaR}_\alpha(Y) := G^{-1}(\alpha).$$

Of course, quantiles of the df of $\psi(\underline{X})$ can be computed if the joint distribution function $F(x_1, \ldots, x_n) = \mathbb{P}[X_1 \leq x_1, \ldots, X_n \leq x_n]$ is known. At this point, the notion of copula becomes useful.

Definition 2 *An* n -dimensional copula *is an* n*-dimensional distribution function restricted to* $[0,1]^n$ *having standard uniform marginals. We denote with* $\mathfrak{C}^n$ *the family of* n*-dimensional copulas.*

Given a copula $C \in \mathfrak{C}^n$ and a set of univariate marginals $F_1, \ldots, F_n$, we can always define a df F on $\mathbb{R}^n$ having these marginals by

$$F(x_1, \ldots, x_n) := C(F_1(x_1), \ldots, F_n(x_n)). \tag{1}$$

Hence, given n dfs $F_1, \ldots, F_n$, we let $\underline{X}^C = (X_1, \ldots, X_n)$ be the random vector on $\mathbb{R}^n$ having a copula C satisfying (1). Observe that this copula is unique for continuous marginal dfs. Conversely, Sklar's Theorem (Sklar (1973), Theorem 1) states that there always exists $C \in \mathfrak{C}^n$ coupling the marginals of a fixed df F trough (1).

Comparing copulas pointwise and defining the riskiness of a dependence structure through this comparison, we recall that any copula C lies between the *lower* and *upper Fréchet bounds* $W(u_1, \ldots, u_n) := (\sum_{i=1}^n u_i - n + 1)^+$ and $M(u_1, \ldots, u_n) := \min_{1 \leq i \leq n} u_i$, namely

$$W \leq C \leq M. \tag{2}$$

Observe that, contrary to M, the lower Fréchet bound W is not a distribution function for $n > 2$. Random variables coupled through $C = M$ ($C = W$, respectively) are called *comonotonic* (*countermonotonic*). The independence copula is denoted by $\Pi(u_1, \ldots, u_n) := \prod_{i=1}^n u_i$.

Remark 3 *Comonotonicity characterizes the risks of the portfolio as being increasing functions of a common random variable. It is therefore a strong dependence and measure of dependence such as Kendall's* τ *or Spearman's* ρ *will describe* M *as a perfect structure, i.e.* $\tau(M) = \rho(M) = 1$. *It is precisely this representation which motivates the use of the concept of comonotonicity in financial applications. Moreover, assuming comonotonicity leads to almost all the computational benefits of independence, yielding, in addition, a prudent scenario in many contexts as we will emphasize in Section 4. For an in depth discussion of comonotonicity, see Dhaene et al. (2001).*

2.2 Bounds on value at risk for functions of dependent risks

We now recall the two fundamental results being the object of our analysis. For a proof of both theorems and further discussions, we refer to Embrechts and Puccetti

(2004) and the references therein. For a copula C and marginals $F_1, \ldots, F_n$, define

$$\sigma_{C,\psi}(F_1, \ldots, F_n)(s) := \int_{\{\psi<s\}} dC(F_1(x_1), \ldots, F_n(x_n)),$$
$$\tau_{C,\psi}(F_1, \ldots, F_n)(s) := \sup_{x_1,\ldots,x_{n-1}\in\mathbb{R}} C(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), F_n^-(\psi^{\wedge}_{x_{-n}}(s))),$$

where $\psi^{\wedge}_{x_{-n}}(s) := \sup\{x_n \in \mathbb{R} \,|\, \psi(x_{-n}, x_n) < s\}$ for $x_{-n} := (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. In the following, we refer to non-decreasing functions $\psi : \mathbb{R}^n \to \mathbb{R}$ as being non-decreasing in each component.

Remark 4 *Observe that $\sigma_{C,\psi}(F_1, \ldots, F_n)(s) = \mathbb{P}[\psi(\underline{X}^C) < s]$ for $\underline{X}^C$ having marginals $F_1, \ldots, F_n$. In the Appendix, Proposition 23, we show that the operator $\tau_{C,\psi}$ in* (3) *is actually the left-continuous version of a df, i.e. there exists a random variable K with $\mathbb{P}[K < s] = \tau_{C_L,\psi}(F_1, \ldots, F_n)(s)$. This result extends a claim of Denuit et al. (1999), p. 37. As first noted in Schweizer and Sklar (1974) for the sum of two risks, if $C_L \neq M$ there does not exist a measurable real function g such that $K = g(\underline{X})$, with $\underline{X}$ having marginals $F_1, \ldots, F_n$.*

Theorem 5 *Let $\underline{X}^C = (X_1, \ldots, X_n)$ be a random vector on $\mathbb{R}^n$ $(n > 1)$ having marginal distribution functions $F_1, \ldots, F_n$. Assume that there exists a copula C_L such that $C \geq C_L$. If $\psi : \mathbb{R}^n \to \mathbb{R}$ is non-decreasing, then for every real s we have*

$$\sigma_{C,\psi}(F_1, \ldots, F_n)(s) \geq \tau_{C_L,\psi}(F_1, \ldots, F_n)(s). \tag{3}$$

Translated in the language of VaR, the above statement becomes

$$\mathrm{VaR}_\alpha(\psi(X_1, \ldots, X_n)) \leq \tau_{C_L,\psi}(F_1, \ldots, F_n)^{-1}(\alpha)$$

for every α in the unit interval.

The bounds stated in Theorem 5 are pointwise best-possible and cannot be tightened if $n = 2$ or a lower bound $C_L > W$ on the copula of the portfolio $\underline{X}^C$ is assumed.

Theorem 6 *Further to the hypotheses of Theorem 5, we assume that ψ is also right-continuous in its last argument. Define the copula $C_\alpha : [0,1]^n \to [0,1]$ as*

$$C_\alpha(u) := \begin{cases} \max\{\alpha, C_L(u)\}, & \textit{if } u = (u_1, \ldots, u_n) \in [\alpha, 1]^n, \\ \min\{u_1, \ldots, u_n\}, & \textit{otherwise}, \end{cases}$$

where $\alpha = \tau_{C_L,\psi}(F_1, \ldots, F_n)(s)$. Then this copula attains bound (3)*, i.e.*

$$\sigma_{C_\alpha,\psi}(F_1, \ldots, F_n)(s) = \alpha. \tag{4}$$

The latter theorem motivates the investigation of the dependence structures leading to the worst-case VaR scenario when $n = 2$ or $C_L > W$. For $C_L = W$ and $n > 2$ the bound stated in (3) is still valid but no more sharp.

3 Analysis of the worst-case portfolios

The aim of the present paper is to give more insight into the shape of the copula yielding the worst-possible VaR for $\psi(\underline{X}^C) = \psi(X_1, \dots, X_n)$ and to understand the implied dependence between the marginals. Under all possible dependence structures, the worst-case scenario for the VaR at level α is given by the copula minimizing $\mathbb{P}[\psi(\underline{X}^C) < s]$ over s-regions depending on α. Indeed, according to Definition 1 with

$$m_\psi(s) := \inf_{C \in \mathfrak{C}^n} \{\mathbb{P}[\psi(\underline{X}^C) < s]\}, \quad s \in \mathbb{R}, \tag{5}$$

we have that $\mathrm{VaR}_\alpha(\psi(\underline{X}^C)) \leq m_\psi^{-1}(\alpha), \alpha \in [0,1]$. The problem at hand becomes also the characterization of the copula minimizing m_ψ, or equivalently maximizing

$$\overline{m}_\psi(s) = 1 - m_\psi(s) = \sup_{C \in \mathfrak{C}^n} \{\mathbb{P}[\psi(\underline{X}^C) \geq s]\}, \quad s \in \mathbb{R}. \tag{6}$$

Such a copula will be referred to as a worst-case *scenario* for the aggregate position $\psi(\underline{X}^C)$. We use the term scenario to indicate a (possibly degenerate) set of probability measures in line with Artzner et al. (1999). Analogous to the above definitions, in the presence of partial information, we write $m_{C_L,\psi}$ (respectively $\overline{m}_{C_L,\psi}$) and the infimum (supremum) is taken over all $C \in \mathfrak{C}^n$ satisfying the boundary condition $C \geq C_L$.

In the next subsections, we concentrate on the sum of risks (generalizations to non-decreasing continuous functions ψ being straightforward) and we choose $C_L = W$. See however Section 5 for some comments on the latter choice of "no dependence information".

3.1 Two-dimensional portfolios

If we take two risks, the bound given in Theorem 5 cannot be tightened and there always exists a two-dimensional copula meeting that bound at a given point s. We restate Theorem 6 in this particular case.

Theorem 7 *Let $\underline{X}^C = (X_1, X_2)$ be a random vector on $\mathbb{R}^2$ having marginal distribution functions F_1, F_2. Define the copula $C_\alpha : [0,1]^2 \to [0,1]$,*

$$C_\alpha(u) := \begin{cases} \max\{\alpha, W(u)\}, & \text{if } u = (u_1, u_2) \in [\alpha, 1]^2, \\ \min\{u_1, u_2\}, & \text{otherwise}, \end{cases}$$

where $\alpha = \tau_{W,+}(F_1, F_2)(s)$. Then this copula attains bound (3)*, i.e.*

$$\sigma_{C_\alpha,+}(F_1, F_2)(s) = \alpha. \tag{7}$$

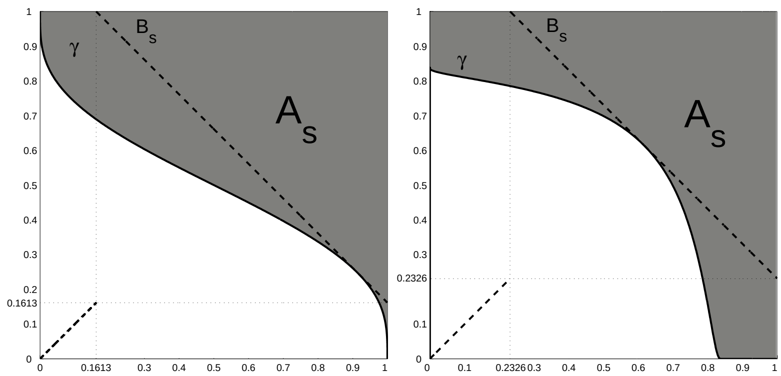

Fig. 1. Support of the copula C_α, sets A_s, B_s and curve γ for: $N(0,1)$-$N(1,2)$-normal marginals and $s = 1$ (which gives $\alpha = 0.1613$) (left); $LN(0.4,1)$-$LN(0.4,1)$-marginals and $s = 4$ ($\alpha = 0.2306$) (right).

Proofs of Theorem 7 can be found in Frank et al. (1987) and Rüschendorf (1982). Our aim here is to restate the problem of maximizing (6) from a geometric point of view and illustrate the properties of the optimizing copulas leading to the worst-case scenario for VaR. Without loss of generality, in what follows, we take continuous, increasing marginals. Let moreover

$$G_s := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 \geq s\}$$

and

$$h : \mathbb{R}^2 \to [0,1]^2, \quad h(x_1, x_2) := (F_1(x_1), F_2(x_2)).$$

The basic idea is to use the function h to transport the optimization problem on the unit square $[0,1]^2$. In fact, $\underline{U}^C := h(\underline{X}^C)$ is a random vector, with distribution function C on $[0,1]^2$. The function h is invertible, hence we have that

$$\mathbb{P}[\underline{X}^C \in G_s] = \mathbb{P}[h(\underline{X}^C) \in h(G_s)] = \mu_C(A_s),$$

where μ_C is the measure corresponding to C on $[0,1]^2$ and

$$A_s := h(G_s) = \{(u_1, u_2) \in [0,1]^2 \mid F_1^{-1}(u_1) + F_2^{-1}(u_2) \geq s\}.$$

The maximization function (6) can now be rewritten as

$$\overline{m}_+(s) = \sup_{C \in \mathfrak{C}^2} \{\mu_C(A_s)\}. \tag{8}$$

For $\alpha = 1$, (3) leads to $\sigma_{C,+}(F_1, F_2)(s) = 1$ for every copula C, hence take $\alpha \in [0,1)$. The boundary of A_s is the image of the curve

$$\gamma : \mathbb{R} \to [0,1]^2, \quad \gamma(t) := (F_1(t), F_2(s-t)).$$

In Figure 1 the curve γ delimiting the set A_s is drawn, with the support of the copula C_α, in case of normal (N) and log-normal (LN) marginals.

The copula C_α is uniformly distributed on its support, hence, defining

$$B_s := \{(u_1, u_2) \in [0,1]^2 \,|\, u_1 + u_2 = 1 + \alpha\}$$

we have $\mu_{C_\alpha}(B_s) = 1 - \alpha$. As noted in Nelsen (1999), p. 187, this is the crucial property leading to the statement of Theorem 7. In fact, when $0 < \alpha < 1$, the continuity of the F_i's implies that

$$\alpha = \tau_{W,+}(F_1, F_2)(s) = F_1(x_1') + F_2(s - x_1') - 1 \tag{9}$$

for some x_1'. Hence the curve γ meets the segment B_s at least in one point. The technical (and for general n and $C_L > W$ rather laborious) part of the proof consists in showing that γ always lies below the segment B_s, hence $A_s \supset B_s$ and

$$\mu_{C_\alpha}(A_s) \geq \mu_{C_\alpha}(B_s) = 1 - \alpha.$$

Noting that $\mu_{C_\alpha}(A_s) \leq 1 - \alpha$, from Theorem 5 we obtain (7). For $\alpha = 0$, instead, the existence of a tangent point between γ and B_s is not necessary, since the copula W yields the theorem. Analogous geometric considerations can be given for the case $C_L > W$ and for non-decreasing continuous ψ.

Remark 8 *Observe that the geometric properties of the support of C_α, illustrated in Figure 1, can be extended to a whole family of copulas, which implies that the dependence structure leading to the worst-case VaR is not unique.*

Let $\widehat{\mathfrak{C}}^2_\alpha$ and $\mathfrak{C}^2_\alpha$ denote the family of copulas leading to the worst possible VaR and the family of copulas sharing their support on $[\alpha, 1]^2$ with C_α, respectively. Formally:

$$\begin{aligned} \widehat{\mathfrak{C}}^2_\alpha &:= \{C \in \mathfrak{C}^2 \,|\, \sigma_{C,+}(F_1, F_2)(s) = \alpha\}, \\ \mathfrak{C}^2_\alpha &:= \{C \in \mathfrak{C}^2 \,|\, C(u_1, u_2) = C_\alpha(u_1, u_2) \text{ for } \alpha \leq u_1, u_2 \leq 1\}. \end{aligned}$$

Observe that we can write $\widehat{\mathfrak{C}}^2_\alpha = \{C \in \mathfrak{C}^2 \,|\, \mu_C(A_s) = \mu_{C_\alpha}(A_s)\}$. In particular, it trivially follows that, every copula in $\mathfrak{C}^2_\alpha$ attains bound (3), since $\mathfrak{C}^2_\alpha \subset \widehat{\mathfrak{C}}^2_\alpha$.

We now focus on the dependence implied by the copulas in $\mathfrak{C}^2_\alpha$. The support

$$R_\alpha := \{(u_1, u_2) \in [0, \alpha)^2 \,|\, u_1 = u_2\} \cup \{(u_1, u_2) \in [\alpha, 1]^2 \,|\, u_1 + u_2 = 1 + \alpha\}$$

of the copula C_α implicitly defines the dependence of the coupled random variables by the substitution $u_i = F_i(x_i), i = 1, 2$. In fact, if the copula C_α couples X_1 and X_2 into the random vector $\underline{X}^{C_\alpha}$ and if we assume F_1, F_2 to be increasing on their domain, then we have $X_2 = g(X_1)$, where the function $g : \mathbb{R} \to \mathbb{R}$ is defined as

$$g(x) := \begin{cases} F_2^{-1}(F_1(x)), & \text{if } x < F_1^{-1}(\alpha), \\ F_2^{-1}(1 + \alpha - F_1(x)), & \text{otherwise.} \end{cases} \tag{10}$$

Analogously, every other copula in $\widehat{\mathfrak{C}}^2_\alpha$ defines a functional dependence identical to that of g for $x \geq F_1^{-1}(\alpha)$. For example, the copula C^1_α given by

$$C^1_\alpha(u_1,u_2) := \begin{cases} \max\{C_L(u_1,u_2),\alpha\}, & \text{when } (u_1,u_2) \in [\alpha,1]^2, \\ \frac{u_1 u_2}{\alpha}, & \text{otherwise}, \end{cases}$$

couples two marginals, which are independent if the first lies below the threshold $F_1^{-1}(\alpha)$ and behaves like C_α otherwise. Figure 2 compares R_α with the support of C^1_α.

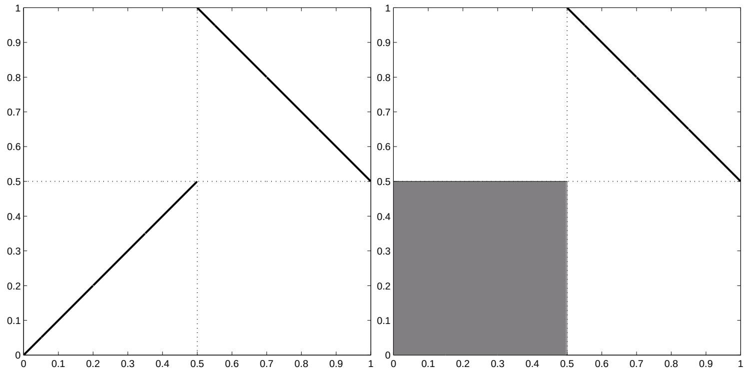

Fig. 2. Supports of the copulas $C_{0.5}$(left) and $C^1_{0.5}$(right).

Merging the two marginals by C_α is therefore equivalent to letting $X_2 = g(X_1)$. Hence, the two risks are *mutually completely dependent*. Moreover, the copula C_α is a so-called *shuffle-of-M* and hence implies that X_1 and X_2 are *strongly piecewise strictly monotone* functions of each other, in the sense defined in Mikusiński et al. (1991). Nevertheless, measures of dependence such as Kendall's τ or Spearman's ρ describe C_α as a non-perfect structure when $0 \leq \alpha < 1$, i.e. $\tau(C_\alpha), \rho(C_\alpha) < 1$. This is due to the fact that this copula only represents piecewise comonotonicity.

Mathematically, the dependence structure induced by C_α is, however, as strong ad the one induced by M, since the two variables coupled by C_α are in a one-to-one correspondence. Finally, note that every df on $\mathbb{R}^2$ defined by applying a $\widehat{\mathfrak{C}}^2_\alpha$-copula to the given set of marginals has a *singular component*, i.e. is mixed with a continuous distribution having zero derivative except for a set of Lebesgue measure zero. For instance, C_α is singular on its whole domain, whereas C^1_α only on $[\alpha, 1]^2$. For more details about singular distribution functions see Billingsley (1995), Section 31 and Nelsen (1999), p. 23.

At this point, it is relevant to note that, in general, $M \notin \widehat{\mathfrak{C}}^2_\alpha$ when $0 \leq \alpha < 1$, the case $\alpha = 1$ being the trivial one in which $\widehat{\mathfrak{C}}^2_\alpha = \mathfrak{C}^2$. This provides a further geometric proof that comonotonicity does not lead to the worst possible scenario for VaR and emphasizes the non-coherence of VaR as stated in Artzner et al. (1999). Suppose that X_1 and X_2 are identically distributed with unbounded, absolutely continuous df having positive density f. If f is eventually decreasing, it is easy to show that for s large enough we have that $\alpha = 2F(s/2) - 1$, while

$$\sigma_{M,+}(F,F) = F(s/2) > \alpha. \tag{11}$$

A necessary condition for M to be in $\widehat{\mathfrak{C}}^2_\alpha$ is that the point (α, α) lies in $[0,1]^2 \setminus A_s$. Equation (11) implies that this condition is not satisfied for s large enough. Finally, $M \in \widehat{\mathfrak{C}}^2_0$ if and only if $A_s = [0,1]^2$, i.e. the sum $X_1 + X_2$ is $\mathbb{P}$-a.s. bounded from below by the threshold s. In this case the problem of bounding the VaR for the sum does not arise. We conclude that, apart from pathological cases of no actuarial importance, we have that

$$\sigma_{M,+}(F_1, F_2)(s) > m_+(s)$$

when $0 \leq \alpha < 1$. This equation shows that the assumption of comonotonicity among the risks of the portfolio may lead to a dangerous under-valuation of the VaR for the aggregate position. At first, the worst dependence scenario could seem to be the one implied by M, since under comonotonicity it is indeed known that every random variable is a non-decreasing function of the other, so that high values for the first imply high values for the second. Theorem 6 provides a deeper view on this issue, stating, instead, that for every threshold s such that $\alpha < 1$, there exists a copula C_α yielding a value for the VaR which is higher than that of comonotonicity. The following example further stresses the fact that M does not belong, in general, to $\widehat{\mathfrak{C}}^2_\alpha$.

Example 9 *Let X_1 be normally distributed $N(0,1)$ with df Φ and put $X_2 = -X_1$ to obtain $\mathbb{P}[X_1 + X_2 = 0] = 1$. The copula describing this dependence is the countermonotonic copula W, under which X_2 is a non-increasing function of X_1. According to Theorem 6, $m_+(0) = 0$. In this set-up, the maximizing solution of* (8) *is then the structure of dependence which is opposite to comonotonicity (note that happens whenever $\alpha = 0$), for which we have instead: $\sigma_{M,+}(\Phi, \Phi)(0) = 1/2$. Figure 3 (left) illustrates how, in the case of standard normal marginals, for every positive $s \in \mathbb{R}$, there exists a copula $C \in \widehat{\mathfrak{C}}^2_\alpha$ such that $\sigma_{C,+}(\Phi, \Phi)(s) < \sigma_{M,+}(\Phi, \Phi)(s)$. In the same figure (right) we also provide the shape of the bivariate distribution obtained by applying C_α to standard normal marginals for $s = 4.898$ ($\alpha = 0.9857$). The reader should compare this figure to Figure 2 (right).*

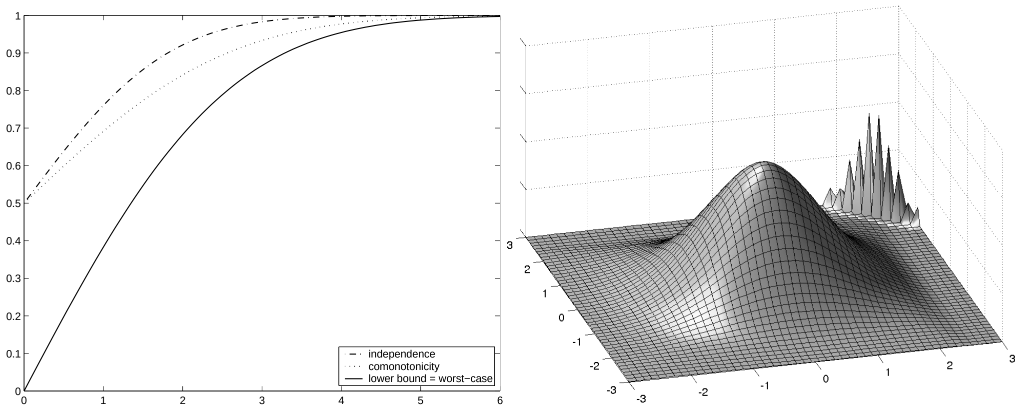

Fig. 3. Range for $\mathbb{P}[X_1 + X_2 < s]$ for a $N(0,1)$-$N(0,1)$-portfolio. Together with the independence and comonotonic value we plot the lower bound $m_+(s)$ (left); the density of the distribution of (X_1, X_2) obtained by applying the copula $C_{0.9857}$ to a $N(0,1)$-$N(0,1)$-portfolio is given in the figure on the right.

3.2 *Two-dimensional uniform portfolios*

We now state some simple results for uniform marginals that will turn out to be useful in understanding the n-dimensional case.

Proposition 10 *Let the hypotheses of Theorem 7 be satisfied with F_1, F_2 uniformly distributed on the unit interval. Then*

$$\widehat{\mathfrak{C}}^2_\alpha = \mathfrak{C}^2_\alpha. \tag{12}$$

Proof If $\alpha = 1$, $\widehat{\mathfrak{C}}^2_1 = \mathfrak{C}^2_1 = \mathfrak{C}^2$. Let $0 \leq \alpha = s - 1 < 1$ and $C \in \widehat{\mathfrak{C}}^2_\alpha$.

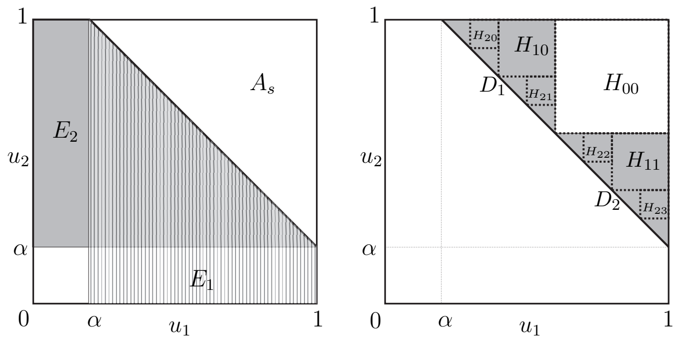

Fig. 4. Sets defined in Proposition 10.

Observe that, for uniform marginals, the boundary of A_s coincides with B_s. For the region underlying such boundary we define

$$E_i := \{(u_1, u_2) \in [0,1]^2 \,|\, u_1 + u_2 < s, u_i \geq \alpha\}, \quad i = 1, 2$$

as illustrated in Figure 4 (left). By the definition of copula and $C \in \widehat{\mathfrak{C}}^2_\alpha$ we have that

$$\mu_C(E_i \cup A_s) = \mu_C(E_i) + \mu_C(A_s) = 1-\alpha, \quad i = 1,2,$$
$$\mu_C(A_s) = 1-\alpha,$$

which implies $\mu_C(E_i) = 0$, $i = 1,2$ and $\mu_C([0,\alpha)^2) = \alpha$. For the upper region we introduce the following partition:

$$H_{ni} := (\alpha + \frac{i(1-\alpha)}{2^n} + \frac{1-\alpha}{2^{n+1}}, \alpha + \frac{(i+1)(1-\alpha)}{2^n}] \times (\alpha + \frac{(2^n-i)(1-\alpha)}{2^n} - \frac{1-\alpha}{2^{n+1}}, \alpha + \frac{(2^n-i)(1-\alpha)}{2^n}]$$

for $n \geq 0$ and $i = 0, \dots, 2^n - 1$. See Figure 4 (right). In particular, consider $H_{00} = (\frac{1+\alpha}{2}, 1]^2$ and let

$$C_i := \{(u_1,u_2) \in [0,1]^2 \mid u_1 + u_2 > 1+\alpha, u_i \leq \frac{1+\alpha}{2}\}, \quad i = 1,2,$$
$$D_1 := \{(u_1,u_2) \in [0,1]^2 \mid u_1 + u_2 = 1+\alpha, \alpha \leq u_1 \leq \frac{1+\alpha}{2}\},$$
$$D_2 := \{(u_1,u_2) \in [0,1]^2 \mid u_1 + u_2 = 1+\alpha, \frac{1+\alpha}{2} < u_1 \leq 1\}.$$

Using the properties of a copula and considering that E_1 and E_2 have zero μ_C-measure, we have that

$$\mu_C(H_{00}) + \mu_C(C_1) + \mu_C(D_1) = 1 - \frac{1+\alpha}{2} = \frac{1-\alpha}{2},$$
$$\mu_C(C_1) + \mu_C(D_1) = \frac{1+\alpha}{2} - \alpha = \frac{1-\alpha}{2}$$

and hence $\mu_C(H_{00}) = 0$. Analogously, applying the same arguments to the upper-right triangles of the squares

$$[\alpha, \frac{1+\alpha}{2}] \times [\frac{1+\alpha}{2}, 1] \quad \text{and} \quad [\frac{1+\alpha}{2}, 1] \times [\alpha, \frac{1+\alpha}{2}],$$

respectively, we obtain that $\mu_C(H_{10}) = \mu_C(H_{11}) = 0$. By iteration we have that $\mu_C(H_{ni}) = 0$ for all $n \geq 0$, $i = 0, \dots, 2^n - 1$ and we trivially obtain

$$\mu_C\left(\bigcup_{n=0}^{\infty} \bigcup_{i=0}^{2^n-1} H_{ni}\right) = 0.$$

Hence the only possibility for C is to assign probability mass $(1-\alpha)$ to the set $D_1 \cup D_2 = B_s$, which implies that $C \in \mathfrak{C}^2_\alpha$. □

Remark 11 *With respect to* (10)*, for* $1 \leq s \leq 2$ *and* $\underline{X}^C = (X_1, X_2)$ *having standard uniform marginals and copula* $C = C_\alpha$*,* $X_2 = g(X_1)$*, where* $g : [0,1] \to$

$[0,1]$ *is the linear function*

$$g(x) = \begin{cases} x, & \textit{if } x < s-1, \\ s-x, & \textit{otherwise.} \end{cases}$$

The above remark, together with Lemma 10, imply that the copula C of a uniform portfolio $\underline{X}^C = (X_1, X_2)$ belongs to $\widehat{\mathfrak{C}}^2_\alpha$ if and only if

$$\mathbb{P}[X_1 + X_2 = s | X_1 + X_2 \geq s] = 1. \tag{13}$$

3.3 Multidimensional portfolios

Though the bound (3) holds in arbitrary dimensions, Theorem 7 fails to be valid if we take $n > 2$. Proposition 12 below shows in a simple way that, if we choose uniformly distributed marginals, it is not always possible to choose a copula C so as to obtain $m_\psi(s) = \tau_{C,+}(F_1, \ldots, F_n)(s) =: \alpha$. Analogously to $\widehat{\mathfrak{C}}^2_\alpha$ in the previous section, we define

$$\widehat{\mathfrak{C}}^n_\alpha = \{C \in \mathfrak{C}^n \,|\, \sigma_{C,+}(F_1, \ldots, F_n)(s) = \alpha\}.$$

Proposition 12 *Let $\underline{X}^C = (X_1, \ldots, X_n)$ be a random vector having marginal dfs uniformly distributed on $[0,1]$. Take $n > 2$ and $n-1 < s < n$. Then $\widehat{\mathfrak{C}}^n_\alpha = \emptyset$.*

Proof Let $S_n := \sum_{i=1}^n X_i$ and note that, for uniform marginals, we have $\alpha = s-n+1$. If there exists $k \in \{1, \ldots, n-2\}$ such that $\mathbb{P}[S_{n-k} < s-k] = 1$ we have $\mathbb{P}[S_n \geq s] = 0$ and the statement trivially holds. Suppose then $\mathbb{P}[S_{n-k} \geq s-k] > 0$ for all $k \in \{1, \ldots, n-2\}$. In this case we have

$$\begin{aligned} \mathbb{P}[S_n \geq s] &= \mathbb{P}[S_n \geq s, S_{n-1} \geq s-1] + \mathbb{P}[S_n \geq s, S_{n-1} < s-1] \\ &= \mathbb{P}[S_n \geq s | S_{n-1} \geq s-1] \cdot \mathbb{P}[S_{n-1} \geq s-1], \end{aligned}$$

since X_n is uniformly distributed on $[0,1]$. Proceeding by iteration we obtain

$$\begin{aligned} \mathbb{P}[S_n \geq s] = \mathbb{P}[S_n \geq s | S_{n-1} \geq s-1] \ldots \mathbb{P}[S_3 \geq s-n+3 | S_2 \geq s-n+2] \\ \cdot \mathbb{P}[S_2 \geq s-n+2 | X_1 \geq s-n+1](n-s). \end{aligned} \tag{14}$$

Assume now that $\widehat{\mathfrak{C}}^n_\alpha \neq \emptyset$, i.e. there exists $\underline{X}^C = (X_1, \ldots, X_n)$ with copula $C \in \widehat{\mathfrak{C}}^n_\alpha$. It immediately follows that $\mathbb{P}[S_n \geq s] = \mathbb{P}[X_1 + \cdots + X_n \geq s] = n-s$ and hence all factors in (14), apart from the last one, must be equal to one. In particular,

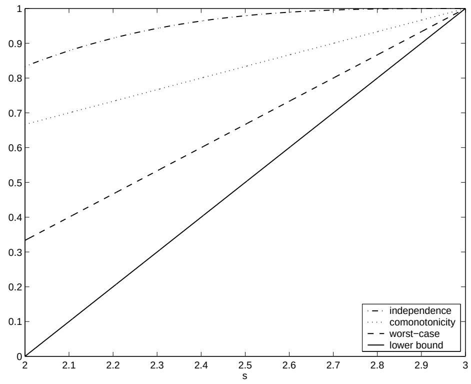

Fig. 5. Range for $\mathbb{P}[X_1 + X_2 + X_3 < s]$ for a standard uniform portfolio. Together with the independence and comonotonic scenario, we plot the worst-case value $m_+(s)$ which differs from the lower bound $\tau_{C_L,+}(s)$ given by (3).

this yields that

$$\mathbb{P}[X_1 + X_2 \geq s - n + 2 | X_1 \geq s - n + 1] = 1, \tag{15}$$
$$\mathbb{P}[S_3 \geq s - n + 3 | S_2 \geq s - n + 2] = 1. \tag{16}$$

According to (13), (15) implies that

$$\mathbb{P}[S_2 = s - n + 2 | S_2 \geq s - n + 2] = 1,$$

which, together with (16), leads to

$$\begin{aligned} 1 = \mathbb{P}[S_3 \geq s - n + 3 | S_2 \geq s - n + 2] &= \mathbb{P}[X_3 \geq 1 | S_2 \geq s - n + 2] \\ &= \frac{\mathbb{P}[X_3 \geq 1, S_2 \geq s - n + 2])}{\mathbb{P}[S_2 \geq s - n + 2]}. \end{aligned}$$

The latter equation is clearly a contradiction to the fact that X_3 is uniformly distributed on $[0, 1]$. □

Remark 13 *The bound given in* (3) *fails to be sharp when $n > 2$ and $C_L = W$. This derives from the fact that W is not a copula for $n > 2$, i.e. for more than two random variables it is impossible for each of them to be almost surely a non-increasing function of each of the remaining ones. In Rüschendorf (1982), the worst-case VaR for uniform and binomial marginals is provided. Till now, this is the only known analytical result. In fact, the optimum dependence for uniform marginals does not solve the general problem, showing that, contrary to the two-dimensional case for $n > 2$, the dependence structure maximizing* (6) *may depend upon the choice of the marginals. In Embrechts and Puccetti (2004), however, an improved bound for the VaR is provided. Figure 5 illustrates the optimum values for uniform portfolios.*

4 Evaluating risk through comonotonicity

In the following, we show that the assumption of comonotonicity among the X_i's may lead to a prudent evaluation of the risk associated with the aggregate position $\psi(\underline{X})$. To this purpose, we first illustrate that such kind of dependence leads to the more dangerous scenario with respect to both stop-loss and supermodular order. Then, changing the optimization approach discussed in the previous sections, we show that comonotonicity also arises as a suitable dependence assumption for our original VaR problem.

4.1 Stochastic orders and comonotonicity

In this section we provide some motivation for the assumption of comonotonicity among risks based on stochastic orders. In this framework we illustrate an important application in actuarial mathematics. We first recall some concepts about stochastic orders.

Definition 14 *Let X and Y be two real random variables. We say that X is smaller than Y in* stop-loss *order and we write $X \leq_{sl} Y$ if for all non-decreasing convex functions $g : \mathbb{R} \to \mathbb{R}$ we have*

$$\mathbb{E}[g(X)] \leq \mathbb{E}[g(Y)], \tag{17}$$

provided the expectations $\mathbb{E}[g(X)]$, $\mathbb{E}[g(Y)]$ are finite.

The stop loss-order compares one-dimensional random variables. A multidimensional stochastic order implying stop-loss order is the so-called *supermodular* order, i.e. the order based on a comparison of integrals of supermodular functions $f : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y), \text{ for all } x, y \in \mathbb{R}^n,$$

where $x \vee y$ ($x \wedge y$) is the componentwise maximum (minimum) of x, y.

Definition 15 *Let $\underline{X}$ and $\underline{Y}$ be two n-dimensional random vectors. We say that $\underline{X}$ is smaller than $\underline{Y}$ in* supermodular *order and write $\underline{X} \leq_{sm} \underline{Y}$ if for all supermodular functions $g : \mathbb{R}^n \to \mathbb{R}$,*

$$\mathbb{E}[g(\underline{X})] \leq \mathbb{E}[g(\underline{Y})], \tag{18}$$

provided the expectations $\mathbb{E}[g(\underline{X})]$, $\mathbb{E}[g(\underline{Y})]$ are finite.

The next theorem recalls two important results about supermodular and stop-loss orders. In particular it states that comonotonicity represents the worst possible dependence scenario with respect to both such orders.

Theorem 16 *Let $\underline{X}^C = (X_1, \ldots, X_n)$ be a n-dimensional random vector having marginal distributions $F_1, \ldots, F_n$ and copula C. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a non-decreasing supermodular function. Then*

(a) $\underline{X}^C \leq_{sm} \underline{X}^M$,
(b) $\psi(\underline{X}^C) \leq_{sl} \psi(\underline{X}^M)$.

Proof As noted in Müller (1997), part (a) follows from Theorem 5 in Tchen (1980). Since $\psi(\underline{X}^C) \leq_{sl} \psi(\underline{X}^M)$ holds if and only if (18) holds for all non-decreasing convex functions $g : \mathbb{R} \to \mathbb{R}$ for which expectations exists, to prove part (b) it is sufficient to show that for such a function g the function $g \circ \psi$ is supermodular. This follows from Lemma 2.2(b) in Bäuerle (1997). □

Remark 17 *Note that Theorem 16* (b) *applies to a large class of interesting functionals, including $\psi(x) = \sum_{i=1}^n h_i(x_i)$, where the h_i's are non-decreasing (see also Müller (1997)) and $\psi(x) = h(\sum_{i=1}^n x_i)$ for h non-decreasing and convex. For more examples of supermodular functions or some interesting methods of constructing them, Marshall and Olkin (1979), pp. 150–155 is the standard reference. Here we want to point out that Theorem 16* (b) *does not apply to* (6) *because the indicator function of the set $\{\psi(\underline{X}) \geq s\}$ is not supermodular.*

Consider again a portfolio of risks $\underline{X}^C = (X_1, \ldots, X_n)$. In insurance mathematics if $\psi(\underline{X}^C)$ is to be insured with a retention level d, the net premium $\mathbb{E}[\psi(\underline{X}^C) - d]^+$ is called the *stop-loss* premium. A stop-loss premium is determined once the retention d and the multivariate df of $\underline{X}^C$ are given. Hence we set

$$\pi_{C,\psi}(F_1, \ldots, F_n)(d) := \mathbb{E}[\psi(\underline{X}^C) - d]^+,$$
$$P_\psi(d) := \sup_{C \in \mathfrak{C}^n} \{\pi_{C,\psi}(F_1, \ldots, F_n)(d)\}. \tag{19}$$

Next proposition states that the univariate stop-loss order based on the comparison of integrals of non-decreasing convex functions is equivalent to the stochastic order based on the comparison of stop-loss premiums. It also explains the name stop-loss for the stochastic order $\leq_{sl}$.

Proposition 18 *Let X and Y be two real random variables. Then the following statements are equivalent:*

(a) $X \leq_{sl} Y$,
(b) *For all real d*

$$\mathbb{E}[X - d]^+ \leq \mathbb{E}[Y - d]^+. \tag{20}$$

A simple proof can be found in Müller and Stoyan (2002), Theorem 1.5.7.

According to Proposition 18, part (b) of Theorem 16 is equivalent to

$$P_\psi(d) = \pi_{M,\psi}(F_1, \ldots, F_n)(d) \tag{21}$$

for all non-decreasing supermodular functions ψ, real retention d and arbitrary dimension n. Hence $\pi_{C,\psi}(F_1, \ldots, F_n)(d)$ is maximized over $\mathfrak{C}^n$ when the fixed marginals of the portfolio have a comonotonic joint distribution, provided ψ is a non-decreasing supermodular function. Observe that this is a much stronger result if confronted with Theorem 6. It is remarkable to note that this solution is not unique pointwise. In Figure 6 (right) we plot the density of a df on $\mathbb{R}^2$ which, though differing from comonotonicity (left), maximizes $\pi_{C,+}(\Phi, \Phi)(0)$ over $\mathfrak{C}^n$. However, M is the only dependence structures that attains (19) for all real retentions d.

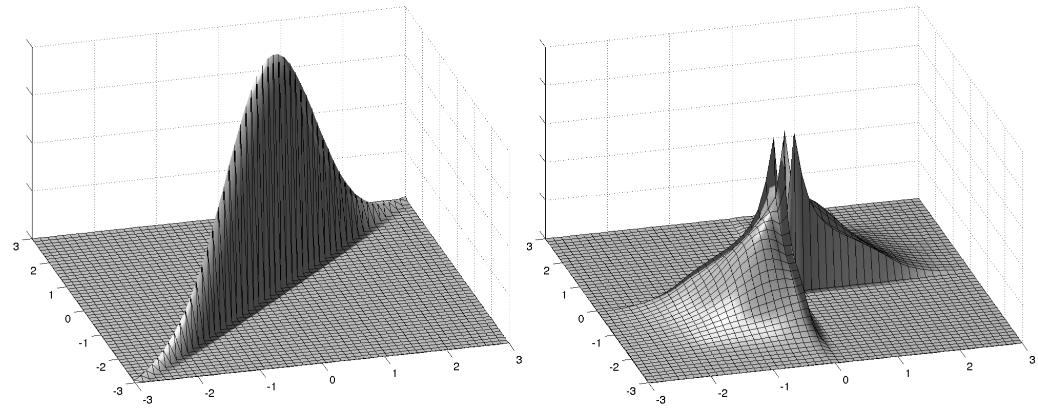

Fig. 6. Densities of two-dimensional distributions obtained by comonotonic dependence (left) and by maximizing $\pi_{C,+}(\Phi, \Phi)(0)$ over $\mathfrak{C}^2$ (right).

4.2 Changing the optimization approach in the VaR problem

Theorem 5 provides a lower bound for the probability $\sigma_{C,\psi}(F_1, \ldots, F_n)(s)$, respectively, an upper bound for the $\mathrm{VaR}_\alpha(\psi(X_1, \ldots, X_n))$. Hence an insurance company holding the risky position $\psi(\underline{X}^C)$ knows that

$$\mathrm{VaR}_\alpha(\psi(\underline{X}^C)) \leq \tau_{W,\psi}(F_1, \ldots, F_n)^{-1}(\alpha), \quad C \in \mathfrak{C}^n.$$

This equation represents the worst-case VaR scenario and can be very expensive for the insurer to handle. Recalling the definition of the family of copulas reaching the above level α, i.e.

$$\widehat{\mathfrak{C}}^n_\alpha = \{C \in \mathfrak{C}^n \mid \sigma_{C,\psi}(F_1, \ldots, F_n)(s) = \alpha\},$$

two quite natural conditions the insurer may ask to be satisfied are

(a) $\widehat{\mathfrak{C}}^n_\alpha \neq \emptyset$,
(b) $\widehat{\mathfrak{C}}^n_\alpha$ *does not depend on* s.

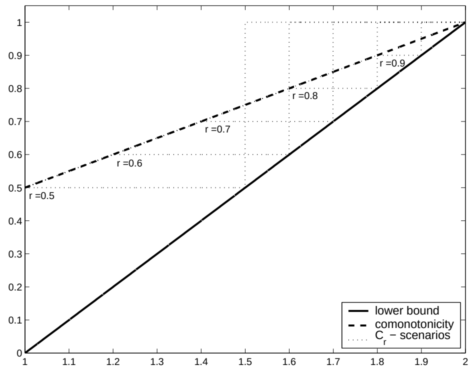

Fig. 7. Range for $\mathbb{P}[X_1 + X_2 < s]$ under different dependence scenarios for a standard uniform portfolio.

Indeed (a) requires the bound to be sharp, i.e. that there exists a structure of dependence which determines the worst-case VaR scenario. Condition (b) requires the worst scenario for the insurance company to be independent of the threshold s or, equivalently, independent of the parameter α that is chosen by the company or by the regulator to evaluate the aggregate risk.

In the previous sections we showed that (a) is violated when $n > 2$ but it holds in the two-dimensional case, while (b) is violated even when $n = 2$. Our aim here is to change the optimization approach so that the solutions satisfy conditions (a) and (b). In order to do this, we define the worst-case VaR scenario over a suitable range for the threshold s, rather than on a single value. Figure 7 explains how that can be done.

In this graph we plot $\sigma_{C_r,+}(F_1, F_2)(s)$ for different values of $r \in (0, 1)$ in case of two uniform marginals together with the best-possible lower bound $m_+(s)$ and the comonotonic curve $\sigma_{M,+}(F_1, F_2)(s)$. As a consequence of Theorem 7, every copula C_r gives a lower bound that meets the curve $m_+(s)$ at the corresponding threshold and then becomes one. The intuition behind this plot is that the comonotonic copula, though never meeting the bound $m_+(s)$, is closer to it than any other copula *on average*.

This idea can now be formalized by introducing a loss function Λ to measure the error committed by evaluating the risky position using a fixed copula $C \in \mathfrak{C}^n$ rather than the appropriate worst-possible structure of dependence. We then integrate the loss function over a suitable set B and we search for the infimum over the class of all n-copulas. Defining

$$r_\psi := \inf_{C \in \mathfrak{C}^n} \left\{ \int_B \Lambda[\sigma_{C,\psi}(F_1, \dots, F_n)(s) - m_\psi(s)] ds \right\} \tag{22}$$

for some non-decreasing functional $\Lambda : [0, 1] \to \mathbb{R}_0^+$, we therefore introduce a measure for the distance between the worst and the comonotonic scenario in Figure 7, where Λ can be viewed as a weighting function. Let $\widehat{\mathfrak{C}^n}(B)$ denote the set of copulas leading to (22). To focus our attention, we choose $B = [d, \infty)$.

Theorem 19 *Let* $\Lambda = Id$ *and* $B = [d, \infty)$*. Then, for every real threshold* d *and non-decreasing supermodular function* ψ *satisfying* $\mathbb{E}[\psi(\underline{X}^M)] < \infty$*, we have that*

$$M \in \widehat{\mathfrak{C}^n}([d, \infty)).$$

Proof Let $\int_d^\infty \overline{m}_\psi(s)ds < \infty$. Note that $\overline{m}_\psi(s)$ depends only on the fixed marginals, so we obtain

$$\begin{aligned} r_\psi &= \inf_{C \in \mathfrak{C}^n} \left\{ \int_d^\infty [\sigma_{C,\psi}(F_1, \dots, F_n)(s) - m_\psi(s)] \, ds \right\} \\ &= - \sup_{C \in \mathfrak{C}^n} \left\{ \int_d^\infty [\mathbb{P}[\psi(\underline{X}^C) \geq s] - \overline{m}_\psi(s)] \, ds \right\} \\ &= \int_d^\infty \overline{m}_\psi(s) \, ds - \sup_{C \in \mathfrak{C}^n} \left\{ \int_d^\infty \mathbb{P}[\psi(\underline{X}^C) \geq s] \, ds \right\} \\ &= \int_d^\infty \overline{m}_\psi(s) \, ds - \sup_{C \in \mathfrak{C}^n} \left\{ \int_d^\infty \mathbb{P}[\psi(\underline{X}^C) > s] \, ds \right\}, \end{aligned}$$

where the last step is obtained since $\mathbb{P}[\psi(\underline{X}^C) = s]$ can be positive at most for countably many values of s, so that the last two integrals contained in the brackets are the same. Finally, recalling that

$$\int_d^\infty \mathbb{P}[\psi(\underline{X}^C) > s] \, ds = \mathbb{E}[\psi(\underline{X}^C) - d]^+,$$

it follows that

$$\begin{aligned} r_\psi &= \int_d^\infty \overline{m}_\psi(s)ds - \sup_{C \in \mathfrak{C}^n} \left\{ \int_d^\infty \mathbb{P}[\psi(\underline{X}^C) > s] \, ds \right\} \\ &= \int_d^\infty \overline{m}_\psi(s)ds - \sup_{C \in \mathfrak{C}^n} \{\mathbb{E}[\psi(\underline{X}^C) - d]^+\} \\ &= \int_d^\infty \overline{m}_\psi(s)ds - P_\psi(d). \end{aligned}$$

Since $\mathbb{E}[\psi(\underline{X}^M)]$ is finite, (21) finally implies that $M \in \widehat{\mathfrak{C}^n}([d, \infty))$. If the integral $\int_d^\infty \overline{m}_\psi(s)ds = \infty$, trivially $\widehat{\mathfrak{C}^n}([d, \infty)) = \mathfrak{C}^n$. □

Remark 20 *Theorem 19 is valid for right-open intervals but it does not hold in full generality. For instance, if we fix a trivial interval consisting of a single point, we go back to the original* VaR *problem. In such case,* M *does not lead to the worst-possible scenario.*

Remark 21 *For the above theorem, the only relevant portfolios* $(X_1, \dots, X_n)$ *are those for which* $\int_d^\infty \overline{m}_\psi(s)ds$ *is finite. This technical condition is satisfied for all marginal distributions of interest. As described in the previous sections and illustrated in Figure 8 for the density of the* $N(0, 1)$*-*$N(0, 1)$*-portfolio of Example 9,* *under the optimizing copula, the mass $\overline{m}_+(s)$ is concentrated on the upper triangle. On the remaining strips, where only one marginal takes high values, the mass is zero. Observe that, by the definition of copula, $\overline{m}_+(s)$ can not exceed the sum of the marginal tails relative to these strips, i.e. $2\overline{\Phi}(s/2)$. In arbitrary dimensions, with the same argument, we have that*

$$\int_d^\infty \overline{m}_\psi(s)\, ds \le \sum_{i=1}^n \int_d^\infty \overline{F}_i(s/n)\, ds < \infty \tag{23}$$

for marginal dfs $F_1, \dots, F_n$ satisfying $\int_{d/n}^\infty \overline{F}_i(s)ds < \infty$. The latter trivially holds if the marginal rvs are integrable. In particular, if $X_1, \dots, X_n$ denote losses and assume only positive values, condition (23) is satisfied provided that their expectations are finite. The finiteness of $\int_d^\infty \overline{m}_\psi(s)ds$ is equivalent to the finiteness of the expectation of the rv K in Proposition 23. We prove this result in full generality for non-decreasing, continuous ψ and increasing marginals in Proposition 24 in the Appendix.

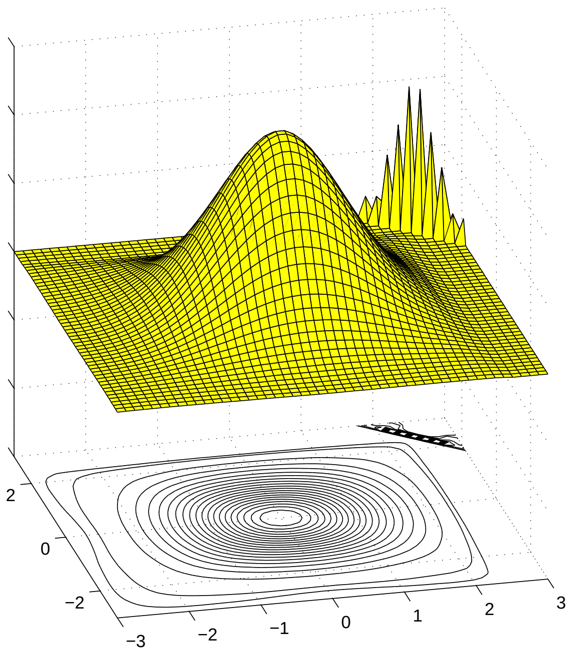

Fig. 8. Density and contour plot of the joint distribution of (X_1, X_2) maximizing $\mathbb{P}[X_1 + X_2 \ge s]$ in Example 9.

The main issue underlying Theorem 19 is that, even if the comonotonic dependence structure does not lead to the worst-case scenario for the original VaR problem, if an insurance company wants to bound well $\sigma_{C,\psi}(F_1, \dots, F_n)(s)$ for all thresholds in $[d, \infty)$ in the sense defined by (22), comonotonicity provides a prudent evaluation for the aggregate risk. The next theorem delivers an insight into an extension of Theorem 19 for general *loss functions* Λ, i.e. every increasing, convex function $\Lambda : [0, 1] \to \mathbb{R}_0^+$ satisfying $\Lambda(0) = 0$. For a copula C, let $e_{C,\psi}(s) := \sigma_{C,\psi}(F_1, \dots, F_n)(s) - m_\psi(s)$ denote the error function in (22).

Theorem 22 *Consider $\underline{X}^C = (X_1, \ldots, X_n)$ with marginal distributions $F_1, \ldots, F_n$ and let ψ be as in Theorem 19 with $\int_0^\infty \overline{m}_\psi(s)ds < \infty$. Then there exists $d_C \in [\underline{d}_C, \overline{d}_C]$ such that for all loss functions Λ:*

$$\int_d^\infty \Lambda[\sigma_{C,\psi}(F_1, \ldots, F_n)(s) - m_\psi(s)]ds \geq \int_d^\infty \Lambda[\sigma_{M,\psi}(F_1, \ldots, F_n)(s) - m_\psi(s)]ds \tag{24}$$

for every $d \geq d_C$, where

$$\begin{aligned} \underline{d}_C &:= \sup\{d \in \mathbb{R} \,|\, e_{C,\psi}(s) \leq e_{M,\psi}(d)\ \forall s \geq d \\ &\qquad\qquad \textit{and } e_{M,\psi}(s) > e_{C,\psi}(s) \textit{ on some interval } (d_C^*, d)\}, \\ \overline{d}_C &:= \inf\{d \in \mathbb{R} \,|\, e_{C,\psi}(s) \geq e_{M,\psi}(s)\ \forall s \geq d\}. \end{aligned}$$

Proof Theorem 19 yields $\int_d^\infty e_{C,\psi}(s)ds \geq \int_d^\infty e_{M,\psi}(s)ds$ for all $d \in \mathbb{R}$. The latter integrals are finite since $\int_0^\infty \overline{m}_\psi(s)ds < \infty$. Denote with $\mathcal{B}(d, \infty)$ and m the Borel sets on (d, ∞) and the Lebesgue measure, respectively. Applying Chong (1974), Theorem 1.6, Theorem 2.1 and Corollary 1.2 to $e_{C,\psi}$ and $e_{M,\psi}$, with $\Phi = \Lambda$ and $(X, \Lambda, \mu) = (X', \Lambda', \mu') = ((d, \infty), \mathcal{B}(d, \infty), m)$, (24) is equivalent to

$$\int_d^\infty [e_{C,\psi}(s) - u]^+ ds \geq \int_d^\infty [e_{M,\psi}(s) - u]^+ ds \tag{25}$$

for all $u \in \mathbb{R}$. By definition, $\underline{d}_C \leq \overline{d}_C$ and $e_{C,\psi} \geq e_{M,\psi}$ on $[\overline{d}_C, \infty)$ implying $d_C \leq \overline{d}_C$. Assume now $-\infty < d_C < \underline{d}_C$ and let $d \in (d_C^*, \underline{d}_C]$. Choosing $u = e_{M,\psi}(\underline{d}_C)$ in (25) we have that

$$\begin{aligned} \int_d^\infty [e_{C,\psi}(s) - u]^+ ds &= \int_d^{\underline{d}_C} [e_{C,\psi}(s) - e_{M,\psi}(\underline{d}_C)]^+ ds + \int_{\underline{d}_C}^\infty [e_{C,\psi}(s) - e_{M,\psi}(\underline{d}_C)]^+ ds \\ &= \int_d^{\underline{d}_C} [e_{C,\psi}(s) - e_{M,\psi}(\underline{d}_C)]^+ ds < \int_d^{\underline{d}_C} [e_{M,\psi}(s) - e_{M,\psi}(\underline{d}_C)]^+ ds \\ &\leq \int_d^\infty [e_{M,\psi}(s) - u]^+ ds, \end{aligned}$$

which concludes the proof. □

With respect to a copula C and any loss function Λ, comonotonicity is hence a suitable dependence scenario on $[d_C, \infty)$ and it provides a prudent VaR on the corresponding α-set. Note that, for a copula C, the set in the definition of $\underline{d}_C$ may be empty and hence $\underline{d}_C$ arbitrarily small. Contrarily, since $\psi(\underline{X}^C) \leq_{sl} \psi(\underline{X}^M)$, we have that $\overline{d}_C < \infty$ if the two dfs cross finitely many times. Unfortunately, in general, $\underline{d}_C$ and $\overline{d}_C$ may become arbitrary large. For instance, if the function ψ is unbounded, Rüschendorf (1981), Theorem 5 yields the existence of a copula $\widehat{C}$ depending on ψ, s and the marginals such that

$$m_\psi(s) = \sigma_{\widehat{C},\psi}(F_1, \ldots, F_n)(s) \leq \sigma_{M,\psi}(F_1, \ldots, F_n)(s),$$

where, in general, $\widehat{C} \neq M$ implying $\sup_{C\in\mathfrak{C}^n} \overline{d}_C = \infty$. Analogously there exist dependence structures leading to $\sup_{C\in\mathfrak{C}^n} \underline{d}_C = \infty$. We therefore conclude that an extension of Theorem 19 to loss functionals can only exist for suitable subclasses of $\mathfrak{C}^n$.

5 The Presence of Information

Throughout this paper we mainly assumed a no-information scenario and put the lower Fréchet bound W as lower bound for the unknown copula C in Theorem 5. From a mathematical point of view this is an unsatisfactory initial assumption, since, for $n > 2$, W is not a copula anymore and the bound provided in Theorem 6 fails to be sharp. However, we want to warn the reader from choosing some a priori assumption such as $C \geq \Pi$ and we emphasize that similar restrictions may lead to a critical undervaluation of the portfolio risk. The assumption $C \geq \Pi$, for instance, is justified by the idea that the worst-VaR scenarios is implied by the so-called *positive lower orthant dependent* risks with $\mathbb{P}[X_1 \leq x_1, \dots, X_n \leq x_n] \geq \mathbb{P}[X_1 \leq x_1] \cdot \ldots \cdot \mathbb{P}[X_n \leq x_n]$. Unfortunately, this is not true and restricting the optimization to the class $\{C \geq \Pi\}$ substantially changes the initial problem, since it does not allow to focus on riskier portfolios, as long as $\overline{m}_+ > \overline{m}_{\Pi,+}$. This is a consequence of the fact that the componentwise ordering in the class $\mathfrak{C}^n$ is not complete and, putting a lower bound on a copula, excludes all copulas not comparable to such bound. To point out this aspect, we observe that countermonotonicity plays a central role in the definition of the family $\widehat{\mathfrak{C}}^n_\alpha$ arising from (6), whereas every copula shuffled with it is not comparable with the independence scenario.

6 Conclusions

In this paper we focus on the copulas leading to the worst-possible VaR for a function of dependent risks and we emphasize that comonotonicity does not lie in this family. Such worst-case scenarios depend upon the level α where the VaR is evaluated and therefore are not reasonable from a practical point of view. Moreover, these solutions are known only for two-dimensional portfolios or in presence of partial information. The investigation of optimal bounds in arbitrary dimensions with no prior information remains open. Therefore, we provide an alternative approach supporting the assumption of comonotonicity in a prudent evaluation of the quantiles of the aggregate position.

Appendix The operator $\tau_{C,\psi}$

In this section we extend a claim of Denuit et al. (1999), p. 37 by showing that the operator

$$\tau_{C,\psi}(F_1,\ldots,F_n)(s) = \sup_{x_1,\ldots,x_{n-1}\in\mathbb{R}} C(F_1(x_1),\ldots,F_{n-1}(x_{n-1}),F_n^-(\psi^{\frown}_{x_{-n}}(s))),$$

with $\psi^{\frown}_{x_{-n}}(s) = \sup\{x_n \in \mathbb{R} \,|\, \psi(x_{-n},x_n) < s\}$, for fixed $x_{-n} \in \mathbb{R}^{n-1}$, is actually the left-continuous version of a df.

Proposition 23 *For $\psi : \mathbb{R}^n \to \mathbb{R}$ non-decreasing, there exists a random variable K such that $\tau_{C,\psi}(F_1,\ldots,F_n)(s) = \mathbb{P}[K < s]$.*

Proof Since ψ is non-decreasing in each component, $\tau_{C,\psi}(s) := \tau_{C,\psi}(F_1,\ldots,F_n)(s)$ is a non-decreasing function. Hence, we have to show that it is also left-continuous and that its right and left limits converge to one and zero, respectively. To prove that $\lim_{s\to\infty}\tau_{C,\psi}(s) = 1$, we fix $\varepsilon > 0$ and define $u^\varepsilon = (u_1^\varepsilon,\ldots,u_n^\varepsilon)$ as a vector satisfying

$$F_i(u_i^\varepsilon) \geq 1 - \frac{\varepsilon}{n}, \quad i = 1,\ldots,n.$$

The existence of such a vector is straightforward, since $F_1,\ldots,F_n$ are non-defective dfs. By definition, the function $\psi^{\frown}_{u^\varepsilon_{-n}}$ is non-decreasing and its right limit is either finite or infinite. Suppose it is finite. For every real s, it follows that

$$\sup\{x_n \in \mathbb{R} \,|\, \psi(u^\varepsilon_{-n},x_n) < s\} \leq \lim_{s\to\infty}\psi^{\frown}_{u^\varepsilon_{-n}}(s) =: R < \infty,$$

which implies

$$\psi(u^\varepsilon_{-n},x_n) \geq s \text{ for all } x_n \geq R.$$

Therefore $\psi(u^\varepsilon_{-n},R) = \infty$, which contradicts ψ having $\mathbb{R}$ as its range. Hence $R = \infty$ and it is always possible to select a real s_ε, depending only on ε, such that $\psi^{\frown}_{u^\varepsilon_{-n}}(s_\varepsilon) > u_n^\varepsilon$ implying

$$F_n(\psi^{\frown}_{u^\varepsilon_{-n}}(s_\varepsilon)) \geq F_n(u_n^\varepsilon) \geq 1 - \frac{\varepsilon}{n}.$$

For $\tau_{C,\psi}(s_\varepsilon)$ we also obtain that

$$\begin{aligned}\tau_{C,\psi}(s_\varepsilon) = \sup_{x_1,\ldots,x_{n-1}\in\mathbb{R}} & C(F_1(x_1),\ldots,F_{n-1}(x_{n-1}),F_n^-(\psi^{\frown}_{x_{-n}}(s_\varepsilon)))\\ &\geq C(F_1(u_1^\varepsilon),\ldots,F_{n-1}(u^\varepsilon_{n-1}),F_n^-(\psi^{\frown}_{u^\varepsilon_{-n}}(s_\varepsilon)))\\ &\geq W(F_1(u_1^\varepsilon),\ldots,F_{n-1}(u^\varepsilon_{n-1}),F_n^-(\psi^{\frown}_{u^\varepsilon_{-n}}(s_\varepsilon)))\\ &\geq (1-\frac{\varepsilon}{n})n - n + 1 = 1 - \varepsilon,\end{aligned}$$

and, since $\tau_{C,\psi}$ is non-decreasing, $\tau_{C,\psi}(s) \geq \tau_{C,\psi}(s_\varepsilon) \geq 1 - \varepsilon$ for every $s \geq s_\varepsilon$. Hence the right limit converges to one. Similarly, for the left limit, we fix $\varepsilon > 0$ and choose u^ε satisfying $F_i(u_i^\varepsilon) < \varepsilon$, $i = 1, \ldots, n$ for which $L := \lim_{s \to -\infty} \psi_{\widehat{u^\varepsilon_{-n}}}(s) = -\infty$. It is always possible to select a real s_ε, depending only on ε, such that $\psi_{\widehat{u^\varepsilon_{-n}}}(s_\varepsilon) < u_n^\varepsilon$ and

$$F_n^-(\psi_{\widehat{u^\varepsilon_{-n}}}(s_\varepsilon)) < F_n(u_n^\varepsilon) < \varepsilon.$$

Let now $A_{u^\varepsilon} := \{x_{-n} \in \mathbb{R}^{n-1} \,|\, x_{i'} \leq u_{i'}^\varepsilon \text{ for some } i' \in \{1, \ldots, n-1\}\}$ and $\overline{A_{u^\varepsilon}} := \{x_{-n} \in \mathbb{R}^{n-1} \,|\, x_i > u_i^\varepsilon \text{ for all } i = 1, \ldots, n-1\}$. Then

$$\begin{aligned} \sup_{x_{-n} \in A_{u^\varepsilon}} & C(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), F_n^-(\psi_{\widehat{x_{-n}}}(s_\varepsilon))) \\ & \leq \sup_{x_{-n} \in A_{u^\varepsilon}} M(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), F_n^-(\psi_{\widehat{x_{-n}}}(s_\varepsilon))) \\ & \leq F_{i'}(u_{i'}^\varepsilon) < \varepsilon. \end{aligned} \tag{26}$$

If $x \in \overline{A_{u^\varepsilon}}$, then $\psi_{\widehat{x_{-n}}}(s_\varepsilon) \leq \psi_{\widehat{u^\varepsilon_{-n}}}(s_\varepsilon)$ and hence

$$\begin{aligned} \sup_{x_{-n} \in \overline{A_{u^\varepsilon}}} & C(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), F_n^-(\psi_{\widehat{x_{-n}}}(s_\varepsilon))) \\ & \leq \sup_{x_{-n} \in \overline{A_{u^\varepsilon}}} C(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), F_n^-(\psi_{\widehat{u^\varepsilon_{-n}}}(s_\varepsilon))) \\ & \leq F_n^-(\psi_{\widehat{u^\varepsilon_{-n}}}(s_\varepsilon)) < \varepsilon. \end{aligned} \tag{27}$$

From (26) and (27) we have that $\tau_{C,\psi}(s_\varepsilon) < \varepsilon$. Since $\tau_{C,\psi}$ is non-decreasing, $\tau_{C,\psi}(s) < \tau_{C,\psi}(s_\varepsilon) < \varepsilon$ for all $s < s_\varepsilon$ and the left limit goes to zero.

It remains to show that $\tau_{C,\psi}$ is left-continuous. For non-decreasing functions $f : \mathbb{R} \to \mathbb{R}$ left-continuity is equivalent to lower-semicontinuity. By Rudin (1974), p. 39, the supremum of any collection of lower-semicontinuous function is lower-semicontinuous. It is sufficient then to show that

$$C(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), F_n^-(\psi_{\widehat{x_{-n}}}(s)))$$

is left-continuous in s for every $x_{-n} \in \mathbb{R}^{n-1}$. By uniform continuity of C, left-continuity and non-decreasingness of F_n^-, and non-decreasingness of $\psi_{\widehat{x_{-n}}}(s)$, the problem is reduced to show that $\psi_{\widehat{x_{-n}}}(s)$ is left-continuous. By definition, it is non-decreasing and hence, for every real s, there exists $l(s) := \lim_{x \to s-} \psi_{\widehat{x_{-n}}}(x)$. Assume now that $\psi_{\widehat{x_{-n}}}(s)$ is not left-continuous. Then there exists s_1 with $l(s_1) < \psi_{\widehat{x_{-n}}}(s_1)$. Let $l(s_1)$ be finite (otherwise there is nothing to prove). Then, for arbitrary positive ε, we have

$$\sup\{x_n \in \mathbb{R} \,|\, \psi(x_{-n}, x_n) < s_1 - \varepsilon\} \leq l(s_1) < \infty$$

and hence, whenever $x_n \geq l(s_1)$, it follows that $\psi(x_{-n}, x_n) \geq s_1 - \varepsilon$. Since ε is arbitrary, it follows that $\psi(x_{-n}, l(s_1)) \geq s_1$, contradicting the fact that

$$\psi(x_{-n}, x_n) < s \text{ for every } x_n < \psi^{\frown}_{x_{-n}}(s_1),$$

which concludes the proof. □

The following proposition states that, under suitable assumptions, the rv K has finite expectation.

Proposition 24 *Let $\underline{X}^C$ have increasing, continuous marginals $F_1, \ldots, F_n$, $\psi : \mathbb{R}^n \to \mathbb{R}$ be non-decreasing continuous and increasing in the last argument. Then, for K as in Proposition 23, $E[K] \leq const + \int_d^\infty \overline{m}_\psi(s)ds < \infty$ for all $d \in \mathbb{R}$ if and only if $\mathbb{E}[\psi(\underline{X}^M)] < \infty$.*

Proof A random variable Y with df F has finite expectation if and only if $\int_{-\infty}^0 F(x)dx < \infty$ and $\int_0^\infty \overline{F}(x)dx < \infty$, which is equivalent to $\mathbb{E}[Y-d]^+ < \infty$ for all $d \in \mathbb{R}$. We therefore have to show that

$$\int_d^\infty \overline{m}_\psi(s)\, ds < \infty \iff \mathbb{E}[\psi(\underline{X}^M) - d]^+ < \infty \tag{28}$$

for all $d \in \mathbb{R}$. By the definition of $\overline{m}_\psi(s)$, we have that for $d \in \mathbb{R}$,

$$\mathbb{E}[\psi(\underline{X}^M) - d]^+ = \int_d^\infty \mathbb{P}[\psi(\underline{X}^M) \geq s]\, ds \leq \int_d^\infty \overline{m}_\psi(s)\, ds$$

and hence ”⇒” immediately follows.

Assume now that the rhs of (28) holds and let U be uniformly distributed on $[0, 1]$. By Dhaene et al. (2001), Theorem 2 we have that

$$\mathbb{E}[\phi(U) - d]^+ = \mathbb{E}[\psi(\underline{X}^M) - d]^+ < \infty,$$

where $\phi : [0, 1] \to \mathbb{R}$, $\phi(u) := \psi(F_1^{-1}(u), \ldots, F_n^{-1}(u))$. Under the assumptions of the theorem, ϕ is continuous and $\phi(\phi^{-1}(s)) = s$. We can write

$$\begin{aligned}
\overline{m}_\psi(s) &\leq 1 - \tau_{W,\psi}(F_1, \ldots, F_n)(s) \\
&= 1 - \sup_{x_{-n} \in \mathbb{R}^{n-1}} \{(F_1(x_1) + \cdots + F_{n-1}(x_{n-1}) + F_n^-(\psi^{\frown}_{x_{-n}}(s)) - n + 1)^+\} \\
&\leq \inf_{x_{-n} \in \mathbb{R}^{n-1}} \{\overline{F}_1(x_1) + \cdots + \overline{F}_{n-1}(x_{n-1}) + \mathbb{P}[X_n \geq \psi^{\frown}_{x_{-n}}(s)]\}.
\end{aligned} \tag{29}$$

Choosing $x_{-n} = (F_1^{-1}(\phi^{-1}(s)), \dots, F_{n-1}^{-1}(\phi^{-1}(s)))$ in (29) and since ψ is increasing in the last argument, $\psi_{\widehat{x_{-n}}}(s) = F_n^{-1}(\phi^{-1}(s))$. Integrating, we finally obtain:

$$\begin{aligned}\int_d^\infty \overline{m}_\psi(s)ds &\le \int_d^\infty \sum_{i=1}^n \mathbb{P}[X_i \ge F_i^{-1}(\phi^{-1}(s))]ds \\ &= \sum_{i=1}^n \int_d^\infty \mathbb{P}[F_i^{-1}(U) \ge F_i^{-1}(\phi^{-1}(s))]ds \\ &= \sum_{i=1}^n \int_d^\infty \mathbb{P}[U \ge \phi^{-1}(s)]ds \le \sum_{i=1}^n \int_d^\infty \mathbb{P}[\phi(U) \ge \phi(\phi^{-1}(s))]ds \\ &= n\int_d^\infty \mathbb{P}[\phi(U) \ge s]ds = n\mathbb{E}[\phi(\underline{X}^M) - d]^+ < \infty.\end{aligned}$$

□

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