Spread-Based Credit Risk Models

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FINANCIAL RISK ANALYSIS

Part II

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In this Unit You Will Learn. . .

- the pricing building blocks
  - ⋆ unconditional payoffs
  - ⋆ payoffs in survival
  - ⋆ payoffs at default
- how to price standard instruments: coupon bonds, CDS
- how to extract from the credit spread curve:
  - ⋆ implied default and survival probabilities
  - ⋆ conditional survival probabilities
- the calibration of a credit spread curve

What are We Going to Do?

We are looking for a probability distribution for the time of default which is consistent with observed market prices.

There are many such possible probability distributions.

Here, the modeller's / user's intuition is needed, to eliminate implausible default probability distributions.

Therefore we need a setup where it is easy to build an intuition. A setup where the model quantities have a direct relationship to real-world quantities that are known and on which we have experience (and therefore also an opinion).

Mathematically, we are choosing a martingale measure in an incomplete market.

Implied Probabilities

We use:

- Default-free zero coupon bond (ZCB) curve
- Defaultable zero coupon bond (ZCB) curve
- risk-neutral probabilities
- independence of:
  - ⋆ [defaults and credit spreads dynamics]
  - ⋆ [interest-rate dynamics]

Default-Free Zero Coupon Bonds:

Payoff: $= 1$ at time T

Value: $=$ discounted expected payoff

$$B(t,T) = \mathbf{E}\,[\ \beta_{t,T} \cdot 1\ ] = \mathbf{E}\,[\ \beta_{t,T}\ ]\,.$$

$\beta_{t,T}$ is the **discount factor** over $[t,T]$.

$$\beta_{t,T} = \exp\{-\int_t^T r(s)ds\}$$

where $r(s)$ is the continuously compounded short rate at time t.

Defaultable Zerobonds:

$$\text{Payoff} = \mathbf{1}_{\{\tau > T\}} = \begin{cases} 1 & \text{if default after } T\text{, i.e. } \tau > T, \\ 0 & \text{if default before } T\text{, i.e. } \tau \leq T. \end{cases}$$

Value = discounted expected payoff

$$\overline{B}(t,T) = \mathbf{E}\left[\ \beta_{t,T}\mathbf{1}_{\{\tau > T\}}\ \right].$$

$\mathbf{1}_{\{A\}}$ is the indicator function of event A:
$\mathbf{1}_{\{A\}} = 1$ if A is true, and $\mathbf{1}_{\{A\}} = 0$ if A is false.

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If there is no correlation:

$$\begin{aligned}\overline{B}(t,T) &= \mathbf{E}\left[\ \beta_{t,T}\mathbf{1}_{\{\tau>T\}}\ \right] \\ &= \mathbf{E}\left[\ \beta_{t,T}\ \right]\mathbf{E}\left[\ \mathbf{1}_{\{\tau>T\}}\ \right] \\ &= B(t,T)\mathbf{E}\left[\ \mathbf{1}_{\{\tau>T\}}\ \right] \\ &= B(t,T)P(t,T)\end{aligned}$$

$P(t,T)$ is the implied probability of survival in $[t,T]$.

($\beta_{t,T}$ is the discount-factor over $[t,T]$.)

The Implied Survival Probability

the implied survival probability is the ratio of the ZCB prices:

$$P(t,T) = \frac{\overline{B}(t,T)}{B(t,T)}$$

- Default Probability = 1- Survival Probabilities
- initially at one

$$P(t,t) = 1$$

- eventually there is a default:

$$P(t,\infty) = 0$$

- $P(t,T)$ is decreasing in T.

Conditional Survival Probabilities

Want to focus on *one time interval in the future.*

Probability of survival in $[T_1, T_2]$ . . .
. . . given that there was no default until T_1.:

$$P(t, T_1, T_2) = \frac{P(t, T_2)}{P(t, T_1)} = \frac{\overline{B}(t, T_2)}{B(t, T_2)} \frac{B(t, T_1)}{\overline{B}(t, T_2)}.$$

- Survival to T = survival to s AND survival from s to T

$$\boxed{P(t, T) = P(t, s) P(t, s, T)}$$

- $P(t, T, T) = 1$, unless default is 'scheduled' at T.

Note: Bayes' Rule:

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}.$$

Here: $A =$ survival until T_2 $\qquad B =$ survival until T_1

$$\begin{aligned} A \cap B &= [\text{survival until } T_2] \text{ AND } [\text{survival until } T_1] \\ &= [\text{survival until } T_2] \end{aligned}$$

Numerical Example

	Yield (%)		ZCB		Probabilities (%)		
Mat	Gov	Issuer	Gov	Issuer	Surv.	CSurv.	CDef/T
0,5	5,75	7	0,9724	0,9667	99,41	99,41	1,17
1	6,10	7,85	0,9425	0,9272	98,38	98,96	2,09
3	6,25	8,25	0,8337	0,7883	94,56	96,12	1,94
5	6,40	8,65	0,7333	0,6605	90,07	95,25	2,38
7	6,78	9,08	0,6318	0,5442	86,14	95,64	2,18
10	6,95	9,70	0,5107	0,3962	77,58	90,06	3,31

Numbers taken from Das (1997)

Forward Rates

define the *default-free forward rate* $F_k(t)$, the *defaultable forward rate* $\overline{F}_k(t)$ and the *discrete forward default intensity* $H_k(t)$ over $[T_k, T_{k+1}]$ as seen from t:

$$F_k(t) = \frac{1}{\delta_k}\left(\frac{B_k(t)}{B_{k+1}(t)} - 1\right).$$

$$\overline{F}_k(t) = \frac{1}{\delta_k}\left(\frac{\overline{B}_k(t)}{\overline{B}_{k+1}(t)} - 1\right).$$

$$H_k(t) = \frac{1}{\delta_k}\left(\frac{P_k(t)}{P_{k+1}(t)} - 1\right).$$

where $\delta_k = T_{k+1} - T_k$ and $P_k(t) = P(t, T_k)$.

Relation to Forward Spreads

the **conditional probability of default** over $[T_k, T_{k+1}]$ is given by:

$$\frac{1}{\delta_k} P^{\mathsf{def}}(t, T_k, T_{k+1}) = \frac{\overline{F}_k(t) - F_k(t)}{1 + \delta_k \overline{F}_k(t)} = \frac{H_k(t)}{1 + \delta_k H_k(t)}$$

[Default Probability]

=[Length of time interval] × [Spread of forward rates]

× [Discounting with defaultable forward rate]

Local Default Probabilities

use **continuously compounded forward rates**:

$$f(t,T) = \lim_{\Delta t \searrow 0} F(t,T,T+\Delta t) = -\frac{\partial}{\partial T} \ln B(t,T)$$

$$\bar{f}(t,T) = \lim_{\Delta t \searrow 0} \overline{F}(t,T,T+\Delta t) = -\frac{\partial}{\partial T} \ln \overline{B}(t,T)$$

The **local probability of default** at time T is:

$$\lim_{\Delta t \searrow 0} \frac{P^{\mathsf{def}}(t,T,T+\Delta t)}{\Delta t} = \bar{f}(t,T) - f(t,T)$$

The probability of default in $[T, T+\Delta t]$ is approximately **proportional** to the length of the interval $[T, T+\Delta t]$
with proportionality factor $(\bar{f}(t,T) - f(t,T))$

Why Poisson Processes?

Ultimate goal:
A mathematical model of defaults that is realistic and tractable and useful for pricing and hedging.

Defaults are

- sudden, usually unexpected
- rare (hopefully :-)
- cause large, *discontinuous* price changes.

Require from the mathematical model the same properties.

Furthermore: Previous section

> The probability of default in a short time interval is approximately **proportional** to the length of the interval.

What is a Poisson Process?

$N(t)$ = value of the process at time t.

- Starts at zero: $N(0) = 0$
- Integer-valued: $N(t) = 0, 1, 2, \ldots$
- Increasing or constant
- Main use: marking points in time
  $T_1, T_2, \ldots$ the jump times of N
- Here **Default:** time of the first jump of N
  $\tau = T_1$
- Jump probability over small intervals proportional to that interval.

- Proportionality factor $= \lambda$

BTW: Except for the last two points, the same notation and properties apply to *Point Processes*, too.

Discrete-time approximation:

- divide $[0, T]$ in n intervals of equal length

$$\Delta t = T/n$$

- Make the jump probability in each interval $[t_i, t_i + \Delta t]$ proportional to Δt:

$$p := \mathbf{P}\,[\; N(t_i + \Delta t) - N(t_i) = 1 \;] = \lambda \Delta t.$$

  (these are independent across intervals.)

- more exact approximation: $p = 1 - e^{-\lambda \Delta t}$
- Let $n \to \infty$ or $\Delta t \to 0$.

Important Properties

Homogeneous Poisson process with intensity λ
Jump Probabilities over interval $[t, T]$:

- No jump:

$$\mathbf{P}\left[\, N(T) = N(t) \,\right] = \exp\{-(T-t)\lambda\}$$

- n jumps:

$$\mathbf{P}\left[\, N(T) = N(t) + n \,\right] = \frac{1}{n!}(T-t)^n \lambda^n e^{-(T-t)\lambda}.$$

- Inter-arrival times $\mathbf{P}\left[\, (T_{n+1} - T_n) \in t\, dt \,\right] = \lambda e^{-\lambda t} dt.$
- Expectation (locally) $\mathbf{E}\left[\, dN \,\right] = \lambda dt.$

Distribution of the Time of the first Jump

T_1 time of first jump.
Distribution: $F(t) := \mathbf{P}[\, T_1 \leq t \,]$

Know probability of no jump until T:

$$\mathbf{P}[\, N(T) = 0 \,] = e^{-\lambda T}$$

$=$ probability of $T_1 > T$. Thus

$$1 - F(t) = e^{-\lambda t}$$
$$F(t) = 1 - e^{-\lambda t}$$
$$F'(t) = f(t) = \lambda e^{-\lambda t}.$$

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- T_1 is exponentially distributed with parameter λ.
- This is also the distribution of the *next* jump, given that there have been k jumps so far.
- Independently of how much time has passed so far:
  It never is 'about time a jump happened'
  or 'nothing has happened, I don't think anything will happen any more. . . '

Inhomogeneous Poisson Process

Imhomogeneous = with **time-dependent** intensity function $\lambda(t)$

Probability of no jumps (survival):

$$\mathbf{P}\,[\, N_T = N_t \,] = \exp\left\{ -\int_t^T \lambda(s)ds \right\}.$$

Probability of n jumps:

$$\mathbf{P}\,[\, N_T - N_t = n \,] = \frac{1}{n!}\left(\int_t^T \lambda(s)ds \right)^n e^{-\int_t^T \lambda(s)ds}.$$

Density of the time of the first jump:

$$\mathbf{P}\,[\,T_1 \in [a,b]\,] = \int_a^b f(t,u)du$$
$$= \int_a^b \lambda(u) e^{-\int_t^u \lambda(s)ds} du.$$

Cox Processes

default rate = Intensity of PP = credit spread.

Credit spreads are stochastic. ☞ Need stochastic intensity

- define a stochastic intensity process λ, e.g.

$$d\lambda = \mu_\lambda dt + \sigma_\lambda dW$$

- $\lambda(t)\Delta t$: default probability over the next time-interval $[t, t + \Delta t]$. (That's all we need to know at t.)

- at $t + \Delta t$: Intensity has changed, $\lambda(t + \Delta t) = \lambda(t) + d\lambda$ is new (local) default probability.

- Conditional on the realisation of the intensity process, the Cox process is an inhomogeneous Poisson process.

Cox Processes: The Conditioning-Trick

The Gods are gambling in a certain sequence:

- First, the full path of the intensity $\lambda(t)$ is drawn from all possible paths for $\lambda(t)$.
- Then they take this $\lambda(t)$ and *use it as intensity for an inhomogeneous Poisson process* N.
  They draw the jumps of $N(t)$ according to this distribution.
- Then the information is revealed to the mortals:
  At time t they may only know $\lambda(s)$ and $N(s)$ for s up to t.

The Conditioning Trick

First, pretend you knew the path of λ, what would the price be? (Depending on λ, of course.)

Then average over the possible paths of λ.

Let $X(N)$ be a payoff that we want to value. It depends on the question whether a default occurred ($N = 1$) or not ($N = 0$).

$$\mathbf{E}\,[\; X(N)\;] = \mathbf{E}\,[\;\mathbf{E}\,[\; X(N)\;|\;\lambda(t)\;\;\forall t\,]\;]$$

The inner expectation is easily calculated treating N as an inhomogeneous Poisson Process.

Properties of Cox Processes

Probability of no jumps (survival):

$$\mathbf{P}\left[\, N_T = N_t \,\right] = \mathbf{E}\left[\exp\left\{ -\int_t^T \lambda(s)ds \right\} \right].$$

Probability of n jumps:

$$\mathbf{P}\left[\, N_T - N_t = n \,\right] = \mathbf{E}\left[\frac{1}{n!}\left(\int_t^T \lambda(s)ds \right)^n e^{-\int_t^T \lambda(s)ds} \right].$$

Density of the time of the first jump:

$$\mathbf{P}\left[\, T_1 \in [a,b] \,\right] = \int_a^b f(t,u)du$$

$$= \int_a^b \mathbf{E}\left[\lambda(u) e^{-\int_t^u \lambda(s) ds} \right] du.$$

The expectations always only refer to the realisation of λ.

Models using Poisson-type Default Times

Jarrow / Turnbull (JoF 50(1), p. 53ff.):
default triggered by constant intensity Poisson Process

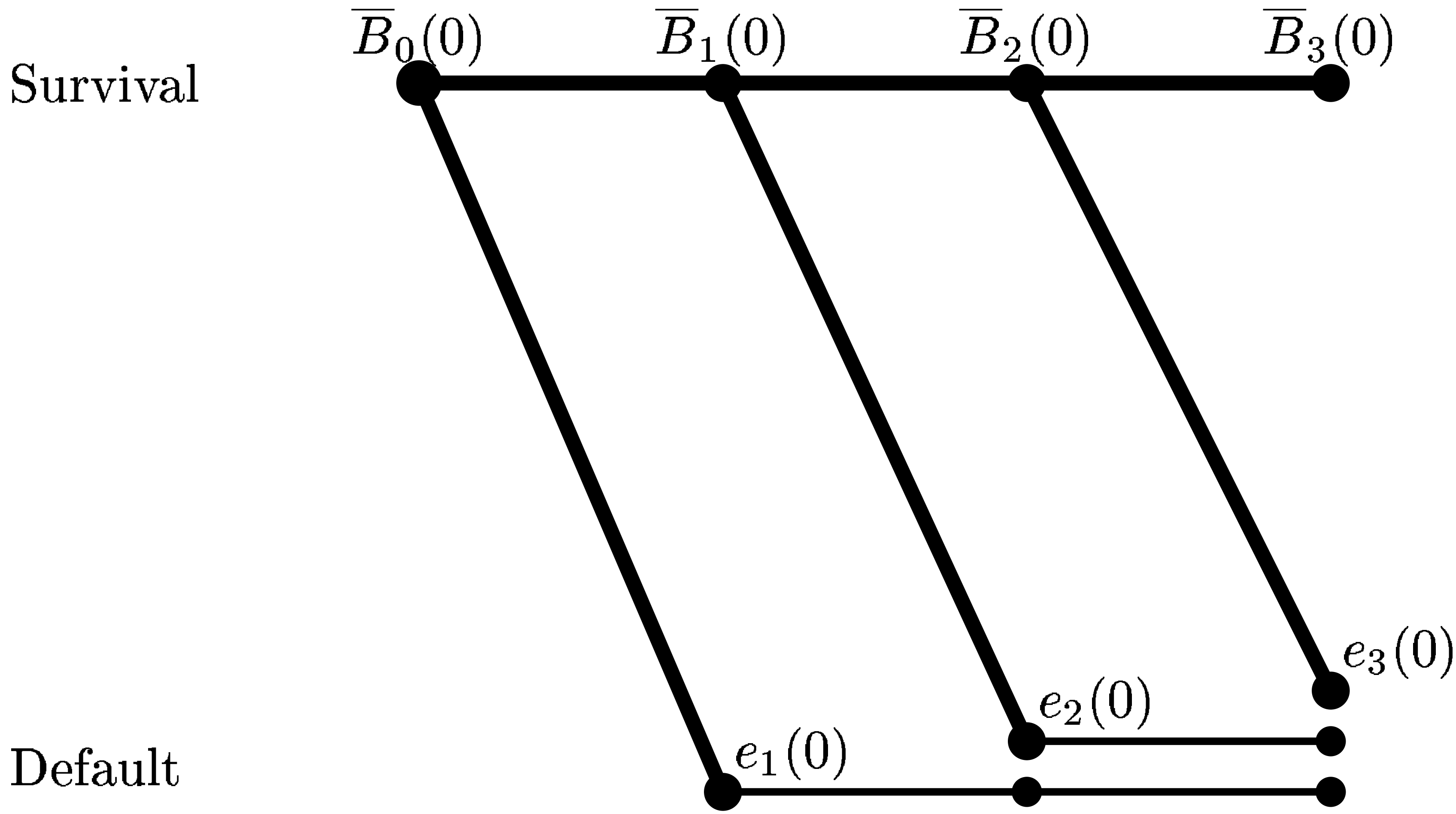
Duffie / Singleton (JoF 52, pp. 1287 ff., 1997):
stochastic-intensity default trigger

Madan / Unal: (Review of Derivatives Research 2, pp. 121 ff., 1998)
Default intensity depends on firm's value

Jarrow / Lando / Turnbull:
default intensity is a function of the rating. The rating follows a Markov chain.

Schönbucher (1998, 1999):
stochastic intensity, modelling a full term-structure of future default intensities

A Simple Tree Model



Setup of the Basic Model

discrete points in time

$$0 = T_0, T_1, T_2, \dots, T_N$$

converge to continuous-time when distances $\to 0$

- $B(t, T_k)$ or $B_k(t)$: default-free zerobond price with maturity T_k
- $\overline{B}(t, T_k)$ or $\overline{B}_k(t)$: defaultable zerobond price with maturity T_k
- τ: time of default

The conditional survival probability is the "across" branching probability here.

Price Building Blocks

Payoffs *independent of default* (possibly stochastic)

- 'classical' pricing problem
- building block
  $B_k(t)$ default-free zero coupon bond with maturity T_k

Payoffs *in survival* (possibly stochastic)

- zero recovery (we get the payoff *only* in survival)
- building block
  $\overline{B}_k(t)$ defaultable ZCB, maturity T_k, zero recovery
- $\overline{B}$-prices follow directly from defaultable forward rates $\overline{F}_k$ (defined later)

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Payoffs *at default*

- for example recoveries
- building block:
  $e_k(t)$ price at time t of $\$1$ at default iff default in $[T_{k-1}, T_k]$
- prices $e_k(t)$ are already fully determined by the prices and dynamics of B_k and $\overline{B}_k$

Pricing the Building Blocks $e_k(t)$

Payoff of $e_k = 1$ at T_k if and only if default in $[T_{k-1}, T_k]$

- Probability of default in $[T_{k-1}, T_k]$

$$= P_{k-1}(t) - P_k(t) = \delta_{k-1} H_{k-1}(t) P_k(t)$$

- Discounting until T_k using $B_k(t)$ yields

$$e_k(t) = \delta_{k-1} H_{k-1}(t) \overline{B}_k(t)$$

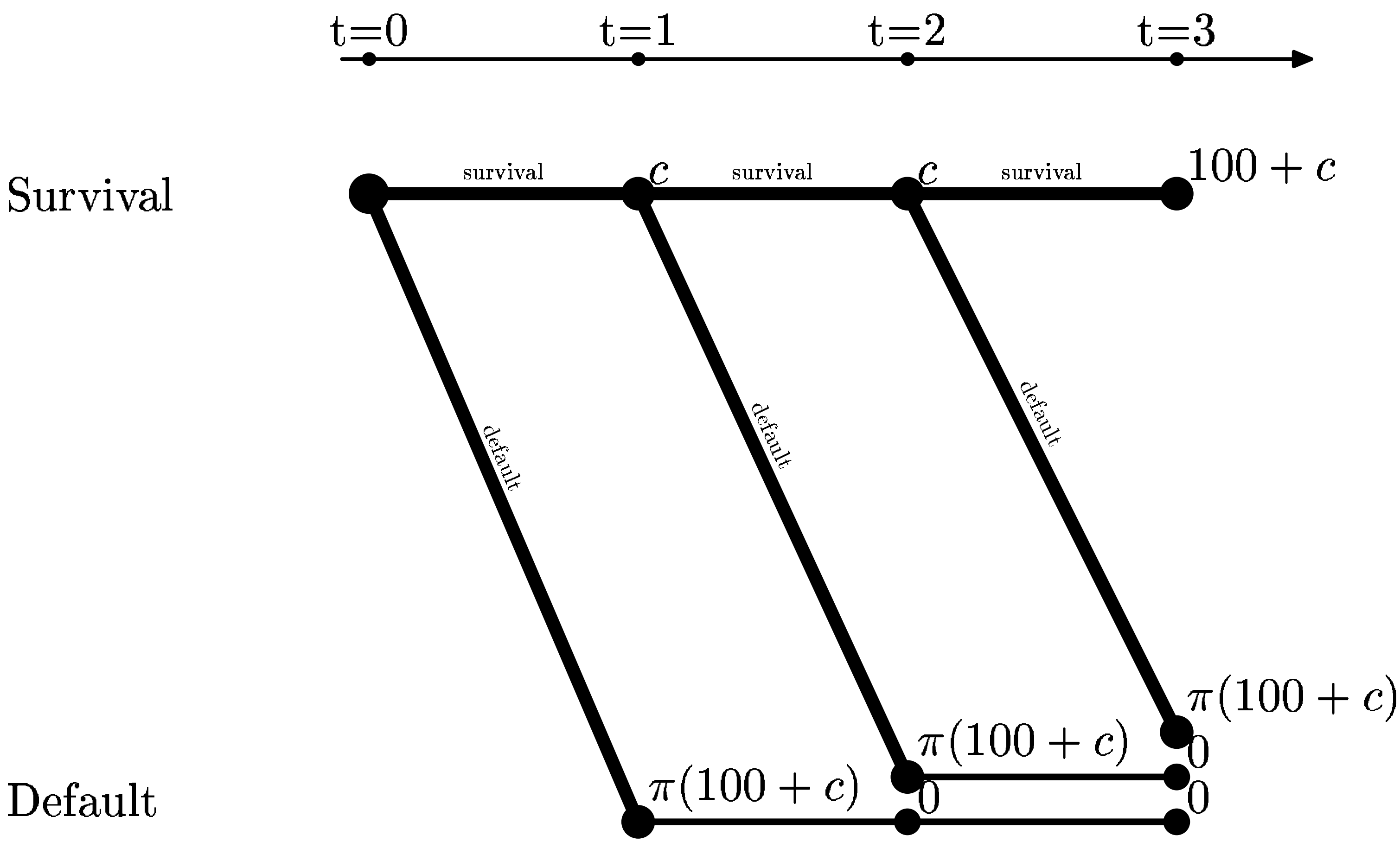
(remember $P_k B_k = \overline{B}_k$)

Here we used independence again. Pricing is more complicated otherwise. (See approximative solutions in paper.)

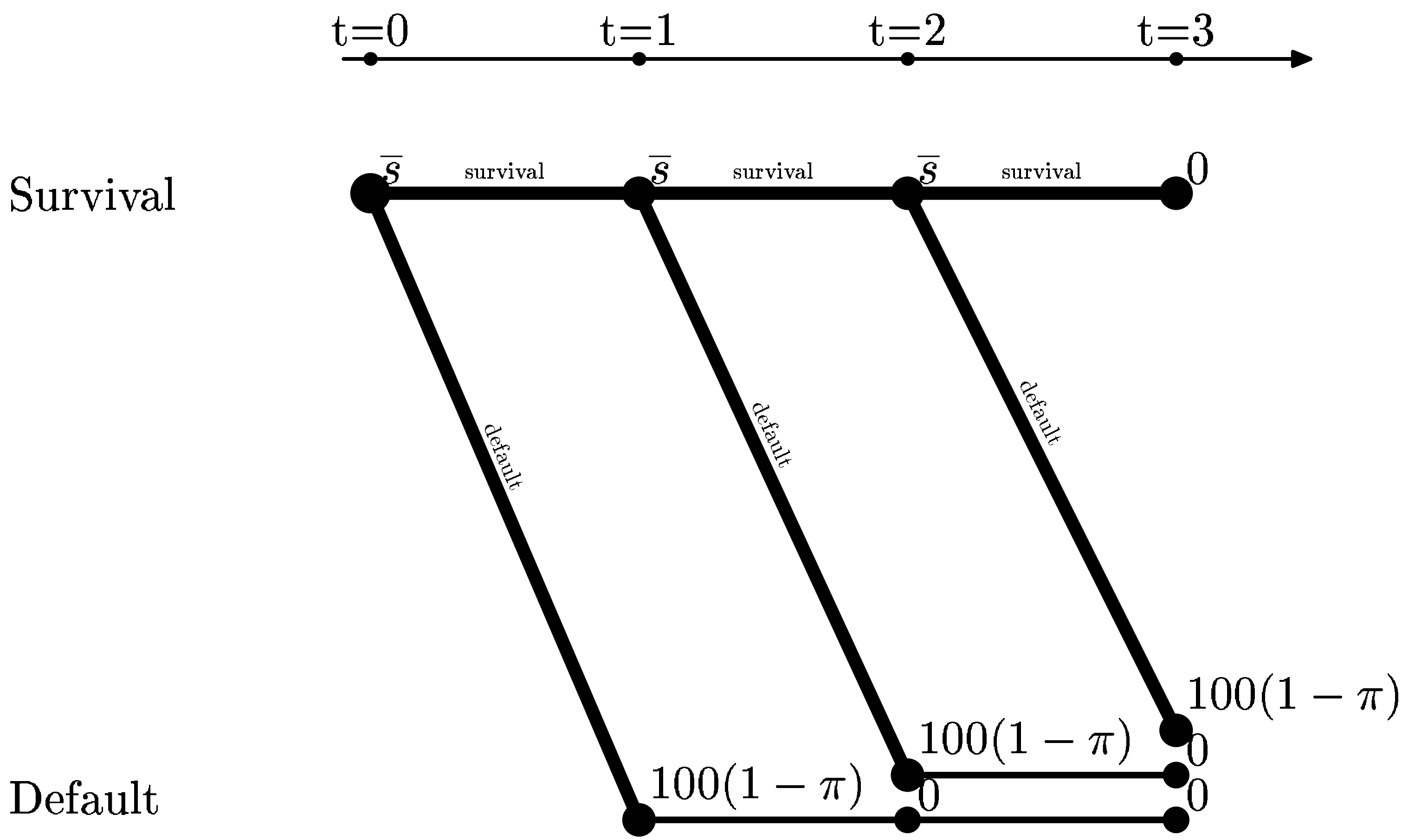
Pricing Defaultable Securities

- Identify the payoffs in events of survival and default
- Payoffs in default:
  - ⋆ recovery rate π
  - ⋆ take care specifying the claim size in default (notional, accrued interest, amortisation)
  - ⋆ observe seniority
- For calibration securities this should be straightforward.
- Model price = weighted sum of building-block prices

Pricing a Defaultable Coupon Bond



Pricing a Credit Default Swap



Pricing with Building Blocks

Fixed Coupon Bonds: coupon c, payable at T_i, recovery π

$$\underbrace{\sum_{i=1}^{N} c\overline{B}_i}_{\text{coupons}} \quad \underbrace{+\overline{B}_N}_{+\text{principal}} \quad \underbrace{+\pi(1+c)\sum_{i=1}^{N} e_i}_{+\text{recovery}}$$

Credit Default Swap: fee s, payable at T_i, payoff at default π

$$\underbrace{-\sum_{i=1}^{N} s\overline{B}_i}_{-\text{fee}} \quad \underbrace{+\pi\sum_{i=1}^{N} e_i}_{+\text{default payoff}}$$

Exercise: Amortisation, Floating-Rate Notes

What have we reached?

- Prices of defaultable securities are given in terms of the building blocks B_k, $\overline{B}_k$ and e_k
- The prices of the building blocks can be represented in terms of "forward-rates´´ F_k and H_k
- Fitting default-free forward rates F_k is standard.
- Can adapt forward rate - fitting methods to fit H_k

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