# Rating-Based Credit Risk Models

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# One-year rating transition probabilities

Standard and Poor's 1981-1991, 'no rating' eliminated, in percentages

| | AAA | AA | A | BBB | BB | B | CCC | D |
|---|---|---|---|---|---|---|---|---|
| AAA | 89.10 | 9.63 | 0.78 | 0.19 | 0.30 | 0 | 0 | 0 |
| AA | 0.86 | 90.10 | 7.47 | 0.99 | 0.29 | 0.29 | 0 | 0 |
| A | 0.09 | 2.91 | 88.94 | 6.49 | 1.01 | 0.45 | 0 | 0.09 |
| BBB | 0.06 | 0.43 | 6.56 | 84.27 | 6.44 | 1.60 | 0.18 | 0.45 |
| BB | 0.04 | 0.22 | 0.79 | 7.19 | 77.64 | 10.43 | 1.27 | 2.41 |
| B | 0 | 0.19 | 0.31 | 0.66 | 5.17 | 82.46 | 4.35 | 6.85 |
| CCC | 0 | 0 | 1.16 | 1.16 | 2.03 | 7.54 | 64.93 | 23.19 |
| D | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 100 |

# The Transition Probability Matrix

We want to represent the probabilities of moving from one to another rating class in a convenient form.

Assumptions:

1. $K$ different states (rating classes): $1, 2, \ldots K$
2. $R(t)$ rating at time $t$
3. $\{P(t,T)\}_{ij}$ transition probability from state $i$ at time $t$ to state $j$ at time $T$.

$$P(t,T)_{ij} = \mathbf{P}\,[\ R(T) = j \mid R(t) = i\ ]$$

4. (Markov property): $P(t,T)_{ij}$ only depends on $i$, $j$ and $(t,T)$.
   *No ratings momentum!*

5. Obviously $P(t,t) = I$,

$$P(t,T)_{ij} \geq 0, \qquad \sum_{k=1}^{K} P(t,T)_{ik} = 1$$

6. Transition probabilities are independent of calendar time:

$$P(t,T) = P(T-t)$$

If property 4 is satisfied, the rating process is called a *Markov Process*.

If property 6 is satisfied, the rating process is called *time-homogeneous*.

If there is only a discrete state-space (property 1) we have a *Markov Chain.*

# Time-Homogeneous Markov Chains

Probability of transition from $i$ to $j$ in *one* year:

$$P(1)_{ij}$$

Probability of transition from $i$ to $j$ in *two* years:

$$P(2)_{ij} = \sum_{k=1}^{K} P(1)_{ik} P(1)_{kj}$$

i.e. Probability of going from $i$ to state $k$ in year one and then from $k$ to $j$ in year two.
(Two-year Chapman-Kolmogorov equations.)

In Matrix writing

$$P(2) = P(1)P(1) = P(1)^2.$$

In general: Transition matrix for $T-t$ years:

$$P(T-t) = P(1)^{T-t}.$$

# The Chapman-Kolmogorov Equations

For all $s \in [t, T]$

$$P(t,T)_{ij} = \sum_{k=1}^{K} P(t,s)_{ik} P(s,T)_{kj}$$

Probability of going from $i$ in $t$ to $j$ in $T$ equals:

Probability of going from $i$ to intermediate state $k$ at intermediate time $s$ and from state $k$ at time $s$ to state $j$ at time $T$.

Summed up over all possible intermediate states $k$.

In matrix multiplication: For all $s \in [t, T]$

$$P(t,T) = P(t,s)P(s,T).$$

For time-homogeneous:

$$P(s+t) = P(s)P(t)$$

# The Generating Matrix

Know $P(0) = I$.
Approximate transition matrix for $\Delta t$:

$$P(\Delta t) = I + \Delta t A.$$

$A$ is called the *generating matrix*.

- Interpretation: For $i \neq j$ element $A_{ij}$ is the intensity of the Poisson Process that triggers the transition from class $i$ to class $j$.

$$A_{ij} = \frac{1}{\Delta t} P_{ij}(\Delta t) \geq 0$$

- Conservation of probability:

$$\sum_k P_{ik}(\Delta t) = 1 \qquad \Rightarrow \qquad \sum_k A_{ik} = 0$$

- Probability of remaining at $i$ equals one minus probability of moving:

$$1 - \Delta t A_{ii} = 1 - \sum_{k \neq i} \Delta t A_{ik} \quad \Rightarrow \quad A_{ii} = - \sum_{k \neq i} A_{ik}$$

 

# Properties of $A$ and $P$

- $A_{ij}$: intensity of Poisson Process triggering transition from $i$ to $j$
- Holding time of state $i$ exponentially distributed with parameter $A_{ii}$.
- Probability of transition to state $j$ given there is a jump: $-\frac{a_{ij}}{a_{ii}}$.
- $\pi = (\pi_1, \ldots, \pi_K)$ stationary distribution iff

$$\pi = \pi P \qquad \pi A = 0.$$

Note: A generator matrix does not always exist.

# The Generator Matrix:

. . . (approximation) to the previous example

| | AAA | AA | A | BBB | BB | B | CCC | D |
|---|---|---|---|---|---|---|---|---|
| AAA | -11.59 | 10.75 | 0.42 | 0.13 | 0.29 | 0.00 | 0.00 | 0.00 |
| AA | 0.95 | -10.61 | 8.32 | 0.81 | 0.26 | 0.27 | 0.00 | 0.00 |
| A | 0.08 | 3.24 | -12.14 | 7.46 | 0.90 | 0.40 | 0.00 | 0.06 |
| BBB | 0.06 | 0.36 | 7.56 | -17.75 | 7.91 | 1.40 | 0.13 | 0.33 |
| BB | 0.04 | 0.22 | 0.58 | 8.85 | -26.12 | 12.95 | 1.36 | 2.08 |
| B | 0 | 0.21 | 0.27 | 0.47 | 6.40 | -19.98 | 5.90 | 6.73 |
| CCC | 0 | 0.04 | 1.44 | 1.36 | 2.46 | 10.13 | -43.53 | 28.10 |
| D | 0 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |

# Historical Default Probabilities vs. Credit Spreads

- Historical default rates do not justify the discounts observed on traded bond prices.
- Credit spreads are too low. (Particularly for higher quality debt.)
- Reasons: Risk Premia, liquidity premia, institutional effects, tax effects (?)
- Historical default transition matrices are *observed frequencies* and *not* probabilities.
  - a zero entry does not mean this transition is impossible: it was just not observed in the relevant period
  - we can have inconsistencies: downgrade AAA-BB more frequent than AAA-BBB.
  - or upgrades: CCC-A more frequent than BB-A.

- Need to make adjustments to the historical transition matrix in order to use it for pricing:
  - incorporate risk premia
  - remove zero entries
  - adjust to achieve consistency and monotonicity

# References

[1] Robert A. Jarrow and Stuart M. Turnbull. Pricing derivatives on financial securities subject to credit risk. *Journal of Finance*, 50:53–85, 1995.

[2] David Lando. On Cox processes and credit risky bonds. Working paper, Institute of Mathematical Statistics, University of Copenhagen, March 1994. revised December 1994.