Stochastic Processes in Insurance and Finance

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1 Introduction

1.1 The basic building blocks

Coincidence or not? Though the theory of stochastic processes is very much a theory of the 20th century, its first appearance through applications in insurance and finance shows some remarkable similarities. In 1900, Bachelier (1900) wrote in his famous thesis: "Si, à l'égard de plusieurs questions traitées dans cette étude, j'ai comparé les résultats de l'observation à ceux de la théorie, ce n'était pas pour vérifier des formules établies pour les méthodes mathématiques, mais pour montrer seulement que le marché, à son issu, obéit à une loi qui le domine: la loi de la probabilité." The title of Bachelier's thesis is "Théorie de la Spéculation", the theory of speculation. The main point in the above extract is that Bachelier has shown ("montrer") that financial markets are dominated by the laws of probability. More precisely, the erratic behaviour of stockmarket data were so much akin to the motion of small particles suspended in a fluid that a link to a process studied later among others by Einstein and Schmolukowski was obvious. The link between Brownian motion and finance was born. It would take economists another 50 years to realize the importance of this link; today however, nobody doubts the fundamental nature of this observation.

To fix ideas we choose some basic probability space $(\Omega, \mathcal{F}, P)$ on which all stochastic processes we introduce in this paper are defined. The first such process is defined as follows.

Definition 1.1 *Standard Brownian motion $W = (W_t)_{t\geq 0}$ is a real–valued stochastic process which satisfies the following conditions*:

(*a*) *W starts at zero: $W_0 = 0$, a.s.,*

(*b*) *W has independent increments: for any partition $0 \leq t_0 < t_1 < \cdots < t_n < \infty$ and any k, the random variables (rvs) $W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1} \ldots, W_{t_k} - W_{t_{k-1}}$ are independent,*

(*c*) *W has Gaussian increments: for any $t > 0$, W_t is normally distributed with mean 0 and variance t, i.e. $W_t \sim \mathcal{N}(0, t)$, and*

(*d*) *W has a.s. continuous sample paths.*

The conditions (b) and (c) are referred to as: W has stationary and independent increments, moreover, the increments are normally distributed. Processes satisfying the stationary and independent increment property, together with a mild sample path regularity condition, are also referred to as Lévy processes, see Bertoin (1996). As we will see later, they play a crucial role in insurance and finance models. The construction of a process satisfying (a)–(d), i.e. proving existence of Brownian motion, is not trivial. A first systematic treatment actually constructing W was given by Norbert Wiener. For a discussion of this, together with a detailed analysis of further properties of W, see for instance Karatzas and Shreve (1988). Though (d) above states that the sample paths of W are (a.s.) continuous, they show a most erratic behaviour, as shown in the next result.

Proposition 1.2 *Suppose W defined as above, then P–a.s., the sample paths of W are nowhere differentiable.* $\square$

A rather unpleasant consequence of this result is that W has unbounded variation on each interval I, say, i.e.

$$\sup_{\Delta} \sum_{i=1}^{n} |W_{t_i}(\omega) - W_{t_{i-1}}(\omega)| = \infty$$

for P–almost all $\omega \in \Omega$, Δ being a possible partition $\Delta = \{t_0, \ldots, t_n\}$ of I and the sup taken over all such partitions. Consequently, standard integration theory for functions with bounded variation does not work, i.e. the symbol

$$\int_0^t Y_s(\omega)\, dW_s(\omega)\,, \tag{1}$$

for some stochastic process (Y_t) has no immediate meaning. Though at first, the news on W is bad, there is some hope. Indeed the following result due to Paul Lévy will be the clue to "defining" (1) in terms of Itô calculus.

Theorem 1.3 (W has finite quadratic variation)
Suppose W as above, but change in (c) the normality assumption to:
(c') for all t, $W_t \sim \mathcal{N}(0, \sigma^2 t)$, $\sigma > 0$. Then for $n \to \infty$,

$$\sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}|^2 \xrightarrow{L^2} \sigma^2 t\ ,$$

where $\{t_0, \ldots, t_n\}$ is an arbitrary partition of $[0, t]$ such that $\sup_i |t_i - t_{i-1}| \to 0$ and where "$\xrightarrow{L^2}$" denotes convergence in $L^2(\Omega, \mathcal{F}, P)$. Under a slight extra condition on the partitions used, L^2–convergence can be replaced by almost sure convergence. □

Whereas Proposition 1.2 was the "bad news", Theorem 1.3 contains the "good news". It turns out to be the key to a new integration theory with respect to W, giving (1) a meaning at least for so-called predictable integrands.

There are various ways to derive Brownian motion as a key building block of financial time series modeling. First of all, just looking at some pictures of financial data reveals the same erratic behaviour as is observed in simulated data from W, see Figure 1 below.

Moreover, W is only the first building block. Later in Section 3 we shall investigate more carefully how more realistic models in finance come about. Before we move to the next process, this time born out of insurance modeling considerations, we would like to indicate a further reason why Brownian motion is a natural model-candidate for financial (stockmarket) data. As we know, the normal distribution enters as a non–degenerate limit of normalized partial sums of independent, identically distributed (iid) rvs. The latter is often described (in a process context) as: the value W_t is obtained via a large "bombardment" of small, independent shocks. If we interpret these shocks as small price changes (up, down) coming from many individual trades, it should not surprise us that stockmarket and Brownian motion should go hand in hand. A formal microeconomic approach to diffusion models for stock prices which is based on this idea has been proposed by Föllmer and Schweizer (1993) .

Around the same time as Bachelier was working on Brownian motion as a basic limit model for financial data, in Sweden in 1903, Filip Lundberg published a remarkable thesis (Lundberg (1903)) providing a mathematical foundation to non–life insurance. In his model, the key ingredients were the so–called premiums and claims. The latter he proposed to model through a homogeneous Poisson process as defined below.

Definition 1.4 *The stochastic counting process $N = (N(t))_t$ is a homogeneous Poisson process with rate (intensity) $\lambda > 0$ if:*

(a) $N(0) = 0$ *a.s.,*

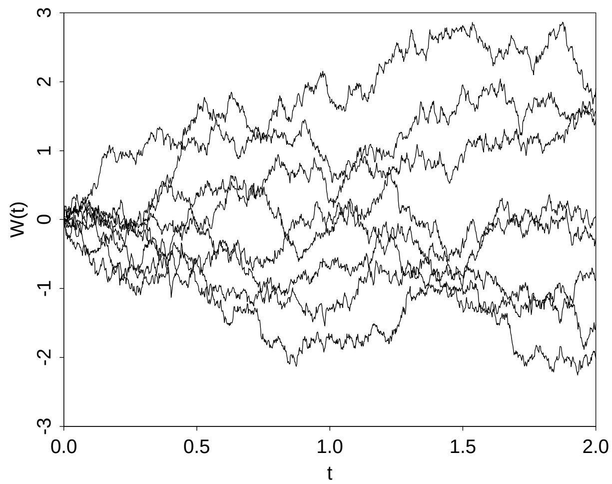

Figure 1: Simulations of standard Brownian motion.

(*b*) *N has stationary, independent increments, and*

(*c*) *for all* $0 \le s < t < \infty : N(t) - N(s) \sim \mathrm{POIS}(\lambda(t-s))$, *i.e.*

$$P(N(t) - N(s) = k) = e^{-\lambda(t-s)} \frac{(\lambda(t-s))^k}{k!} , \quad k \in \mathbb{N} . \tag{2}$$

First of all, the above definition shows a remarkable similarity with Definition 1.1 of Brownian motion. Both processes are Lévy processes. The key difference lies in the sample path behaviour: Brownian motion has continuous sample paths, whereas the Poisson Process is a as a counting process a jump process (for typical realizations, see Figure 2 below).

In insurance applications, $N(t)$ stands for the number of claims in the time interval $(0, t]$ in a well defined portfolio. If we denote the claim arrival of the nth claim by S_n, then

$$N(t) = \sup\{n \ge 1 : S_n \le t\} , \quad t \ge 0 .$$

The inter–arrival times T_1, $T_k = S_k - S_{k-1}$, $k = 2, 3 \ldots$ are independent, identically exponentially distributed ($\mathrm{EXP}(\lambda)$) with finite mean $EY_1 = 1/\lambda$. The latter property also characterizes the homogeneous Poisson process, see for instance Resnick (1992) . The claim size process $(X_k)_{k\in\mathbb{N}}$ is at first assumed to be iid with distribution function (df) F ($F(0) = 0$) and finite mean $\mu = EX_1$. The rv X_k denotes the claim size occurring at time S_k. In Section 2, various of the above conditions will be relaxed. As a consequence of the above, the total claim amount up to time t is given by $S(t) = \sum_{k=1}^{N(t)} X_k$. The latter rv is referred to as compound Poisson rv. Though its df can be written down easily,

$$G_t(x) = P(S(t) \le x) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} F^{n*}(x) , \quad x, t \ge 0 , \tag{3}$$

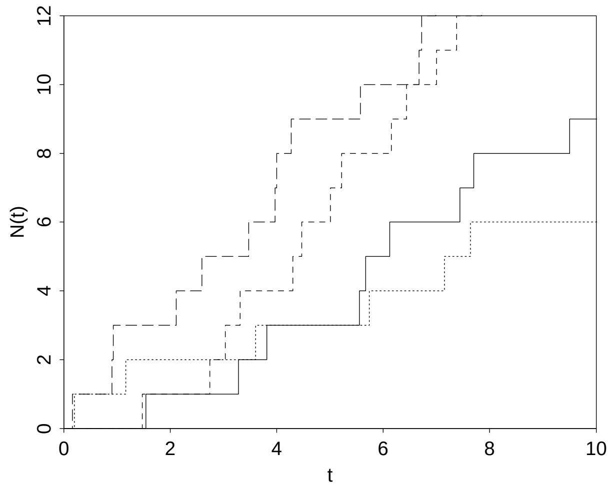

Figure 2: Simulations of homogeneous Poisson processes with intensity $\lambda = 1$.

its precise calculation and indeed statistical estimation in practice form a key area of research in insurance risk theory; see for instance Panjer and Willmot (1992) or the new Klugman, Panjer, and Willmot (1988) and the references therein. The df F^{n*} in (3) denotes the n–fold convolution of F, F^{0*} denotes Dirac measure in 0. Now besides the liability process $(S(t))_{t\geq 0}$, an insurance company cashes premiums in order to compensate the losses. In the above standard (so–called Cramér–Lundberg) model, the premium process $(P(t))_t$ is assumed to be linear (deterministic), i.e. $P(t) = u + ct$, where $u \geq 0$ stands for the initial capital and $c > 0$ is the constant premium rate chosen in such a way that the company (or portfolio) has a fair chance of "survival". The following rv is crucial in this context: denote by τ the *ruin time* of the risk process

$$U(t) = u + ct - S(t)\,, \quad t \geq 0\,. \tag{4}$$

I.e.

$$\tau = \inf\{t \geq 0 : U(t) < 0\} \tag{5}$$

(we always assume that $\inf \emptyset = \infty$). The associated ruin probabilities are defined as

$$\Psi(u, T) = P(\tau \leq T)\,, \quad T \leq \infty\,. \tag{6}$$

For $\Psi(u, \infty)$, the *infinite horizon ruin probability*, we write $\Psi(u)$. It is not difficult to show that, under the so–called *net–profit condition*

$$c - \lambda\mu > 0\,, \tag{7}$$

$\lim_{u\to\infty} \Psi(u) = 0$. Within the Cramér–Lundberg set–up, the condition (7) is always assumed, it says that on the average we obtain a higher premium income than a claim loss. The basic risk process (4) can now be rewritten as

$$U(t) = u + (1 + \vartheta)\lambda\mu t - S(t)\,,$$

where $\lambda\mu t = ES(t)$ and $\vartheta = c/(\lambda\mu) - 1 > 0$ is the so–called safety loading which guarantees "survival". In Figure 3, we have simulated some realizations of (4) for exponentially distributed claims.

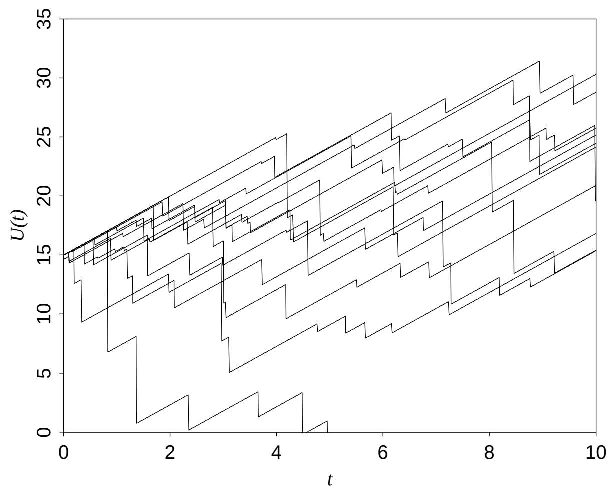

Figure 3: Simulations of a Cramér–Lundberg risk process U with initial capital $u = 15$, premium rate $c = 2.5$, intensity $\lambda = 1$ and exponentially distributed claims with mean $\mu = 2$.

Definition 1.5 *The stochastic process $(U(t))_t$ defined in (4) with the net–profit condition (7) is called the Cramér–Lundberg risk process.*

The following result appears in the applied stochastic process literature under various guises (see for instance Resnick (1992) or Embrechts, Klüppelberg and Mikosch (1997)). We standardly denote $\overline{H} = 1 - H$ for any df H concentrated on $[0, \infty)$.

Theorem 1.6 *Given the Cramér–Lundberg model as above, then*

$$1 - \Psi(u) = (1 - \rho) \sum_{n=0}^{\infty} \rho^n F_I^{n*}(u)\,, \quad u \geq 0\,, \tag{8}$$

where $\rho = \lambda\mu/c < 1$ and the integrated tail df F_I is defined as

$$F_I(x) = \frac{1}{\mu} \int_0^x \overline{F}(y)\, dy\,, \quad x \geq 0\,. \tag{9}$$

□

The fact that the function $\Psi(u)$ in (8) also allows a compound df expression (like in (3)) has important analytic, as well as numerical consequences. The sum in (8) is of compound geometric type. Of course, the compound Poisson process with drift as described in (4) is not comprehensive and does not take into account, for example, the nonlinear premium increase of the capital due to possible investment or inflation and dividend payments to stockholders. However, processes of the form (4) are the basic building blocks of *any* Lévy process Y (without Brownian component) in the sense that Y is the limit (with respect to convergence on compact intervals) of a sequence $Y^{(n)}$ of compound Poisson processes with drift. The conditions underlying the Cramér–Lundberg model are clearly violated in practise. For instance, claims may arrive in clusters. Already early on, actuaries introduced the so–called notion of operational time. The claim–arrival process $(N(t))_t$ is often more realistically modeled as an inhomogeneous Poisson process with intensity measure $\Lambda(t)$, i.e. the process still has independent increments, but for $0 \leq s < t$, $N(t) - N(s) \sim \mathrm{POIS}(\Lambda(t) - \Lambda(s))$. This more realistic situation can be reduced to the homogeneous (standard) case via a time-change $\widetilde{N}(t) = N(\Lambda^{-1}(t))$. In the new, operational time scale, $\widetilde{N}$ is a homogeneous Poisson process. For a discussion of this time transformation, see Bühlmann (1970) and Gerber (1979). More recently, (operational) time considerations are entering stochastic modeling in finance, see e.g. Clark (1973) or Guillaume et al. (1997) and Geman and Ané (1996). The latter papers are mainly based on models coming from the tick–by–tick data world.

In the above discussion, we have seen that the two most important Lévy processes, Brownian motion and the homogeneous Poisson process, appear right at the beginning of stochastic modeling in finance (Bachelier) and insurance (Lundberg). It is remarkable that this development took place well before Kolmogorov created his famous axiomatic theory in the early thirties. Before we discuss in the next sections various generalizations ot the above basic models relevant for insurance and finance, we want to make a little digression into the realm of martingales.

1.2 Some basic martingale theory

Ever since the appearance of Doob (1953), martingales have played a crucial role in probability, so much that in many problems in applied probability the solution could be reduced to "spot the martingale". Below we only give the very basics of martingale theory. All the results, and much more are to be found in the excellent texts Williams (1991), Rogers and Williams (1994, 1987), Karatzas and Shreve (1988), Kopp(1984) and Revuz and Yor (1994). A very readable introduction to the use of martingale methods in insurance is Gerber (1979), for finance Musiela and Rutkowski (1997) is to be recommended. Especially the notion of conditional expectation $E(X|\mathcal{G})$ of a random variable with respect to a σ–algebra $\mathcal{G}$ is crucial in all that follows. Before we can introduce the fundamental notion of a martingale, we need to formalize the concept of information (history).

Definition 1.7 *A family $\mathbb{F} = (\mathcal{F}_t)_t$ of σ–algebras on $(\Omega, \mathcal{F})$ is called a filtration if $\mathcal{F}_t \subset \mathcal{F}$ for all $t \geq 0$, and for all $s \leq t$, $\mathcal{F}_s \subset \mathcal{F}_t$ (i.e. $\mathbb{F}$ is increasing). A stochastic process $(X_t)_t$ is called $\mathbb{F}$–adapted if X_t is $\mathcal{F}_t$–measurable for all $t \geq 0$. The natural filtration $(\mathcal{F}_t^X)$of a stochastic process $(X_t)_t$ is the smallest filtration such that X is adapted. If a stochastic process X is considered, then if nothing else is mentioned, we use its natural filtration. A filtration is called right–continuous if $\mathcal{F}_{t+} := \cap_{s>t}\mathcal{F}_s = \mathcal{F}_t$ for all $t \geq 0$.*

Definition 1.8 *An $\mathbb{F}$–stopping time T is a random variable with values in $[0, \infty]$ such that* *for all $t \geq 0$, $\{T \leq t\} \in \mathcal{F}_t$. The σ–algebra*

$$\mathcal{F}_T := \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}$$

is called the stopped σ–algebra with respect to T.

The usual interpretation of the natural filtration $\mathbb{F}^X = (\mathcal{F}_t^X)$ is that $\mathcal{F}_t^X$ contains all the information available in the rvs $(X_s)_{s \leq t}$.

Definition 1.9 *A stochastic process $M = (M_t)_t$ on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is an $\mathbb{F}$–martingale (–submartingale, –supermartingale, respectively) if*

(*a*) *M is $\mathbb{F}$–adapted, integrable, and*

(*b*) *for all $0 \leq s \leq t : E(M_t|\mathcal{F}_s) = (\geq, \leq) M_s$, P–a.s.*

We simply say that M is a martingale (submartingale, supermartingale) if it is a martingale (submartingale, supermartingale) with respect to the natural filtration.

The following two results are now key to many applications in insurance and finance.

Theorem 1.10 (Martingale stopping theorem)
Let M be an $\mathbb{F}$–martingale (–submartingale,–supermartingale) and T an $\mathbb{F}$–stopping time. Assume that $\mathbb{F}$ is right continuous. Then also the stopped stochastic process $(M_{T \wedge t} : t \geq 0)$ is an $\mathbb{F}$–martingale (–submartingale, –supermartingale). Moreover, for all $t \geq 0$ $E(M_t|\mathcal{F}_T) = (\geq, \leq)\, M_{T \wedge t}$. □

Theorem 1.10 immediately implies the following important relation:

$$EM_{T \wedge t} = (\geq, \leq)\, EM_0 \,. \tag{10}$$

In various applications, one would like to replace $T \wedge t$ in (10) by T; this result is not true in general, extra (uniform) integrability conditions have to be imposed. The next theorem yields a precise formulation for the often used statement that "all martingales converge".

Theorem 1.11 (Martingale convergence theorem)
Let M be an $\mathbb{F}$–supermartingale such that $\sup_{t \geq 0} EM_t^- < \infty$. If $\mathbb{F}$ is right continuous, then $M_\infty := \lim_{t \to \infty} M_t$ exists P–a.s., moreover $E|M_\infty| < \infty$. □

An immediate consequence of the above is that positive (or indeed negative) martingales converge almost surely. A third important category of results are the so–called martingales inequalities; we refer to the cited literature for examples of the latter.

For our purposes, the following martingales related to Brownian motion and the homogeneous Poisson process are important.

Proposition 1.12 (*a*) *Suppose N is a homogeneous Poisson process with intensity $\lambda > 0$, then $(N(t) - \lambda t)_t$ is a martingale.*

(*b*) *Consider the Cramér–Lundberg model from Definition 1.5. Let*

$$\theta(r) = \lambda(Ee^{rX_1} - 1) - cr \,, \tag{11}$$

for those r–values for which Ee^{rX_1} exists. Then

$$(M_r(t))_t := (\exp\{-rU(t) - \theta(r)t\})_t \tag{12}$$

is a martingale. □

Together with the stopping theorem (Theorem 1.10) this result yields important information on the probabilities of ruin $\Psi(u,T)$, see Section 2. The proof of (12) is fairly easy once we know that $(U(t))_t$ is a (strong) Markov process: For $0 \le s < t$,

$$\begin{aligned}
E(M_r(t)|\mathcal{F}_s) &= E(\exp\{-rU(t)-\theta(r)t\}|\mathcal{F}_s)\\
&= E(\exp\{-r(U(t)-U(s))\}\exp\{-rU(s)\}|\mathcal{F}_s)e^{-\theta(r)t}\\
&= E(\exp\{-r(U(t)-U(s))\}|\mathcal{F}_s)\exp\{-rU(s)-\theta(r)t\}\\
&= E(\exp\{r\textstyle\sum_{i=N(s)+1}^{N(t)} X_i\}|\mathcal{F}_s)\exp\{-rU(s)-\lambda(Ee^{rX_1}-1)t+crs\}\\
&= \exp\{\lambda(Ee^{rX_1}-1)(t-s)\}\exp\{-rU(s)-\lambda(Ee^{rX_1}-1)t+crs\}\\
&= \exp\{-rU(s)-\theta(r)s\}\\
&= M_r(s)\,.
\end{aligned}$$

In the Brownian case, the following results are easily obtained.

Proposition 1.13 *Suppose $W=(W_t)_t$ is standard Brownian motion, then*

(*a*) *W and $(W_t^2-t)_t$ are martingales,*

(*b*) *for any $\mu\in\mathbb{R}$, $\sigma>0$, denote $W_{\mu,\sigma}(t)=\mu t+\sigma W_t$, then $(W_{\mu,\sigma}(t))_t$ is called Brownian motion with drift μ and variance σ^2. For each $\beta\in\mathbb{R}$, the following process*

$$(\exp\{\beta W_{\mu,\sigma}(t)-(\mu\beta+\sigma^2\beta^2/2)t\})_t \tag{13}$$

is a martingale associated to Brownian motion, called the Wald- or exponential martingale.

□

For a nice discussion on how the latter result can be used for deriving properties on models involving Brownian motion see for instance Harrison (1985) .

2 Stochastic processes in insurance

2.1 Some basic results

In Section 1.1 we introduced the Cramér–Lundberg model $U(t)=u+ct-S(t)\,,\quad t\ge 0$; see (4). In Proposition 1.12 we derived a whole family of associated exponential martingales parametrised by those $r\in\mathbb{R}$ for which $m_X(r)=Ee^{rX_1}$ is finite. One easily verifies that the function $\theta(r)$ in (11) is strictly convex, $\theta(0)=0$, $\theta'(0)=\lambda\mu-c<0$ so that the situation depicted in Figure 4 may occur.

This motivates the following

Definition 2.1 (Lundberg coefficient)
Suppose the claim size df F allows for a constant $R>0$ to exist for which $\theta(R)=0$, then R is called the Lundberg– (or adjustment) coefficient of the risk process $(U(t))_t$.

Typical examples where R exists are the exponential and gamma distributions. However, R does *not* exist for Pareto or lognormal distributions.

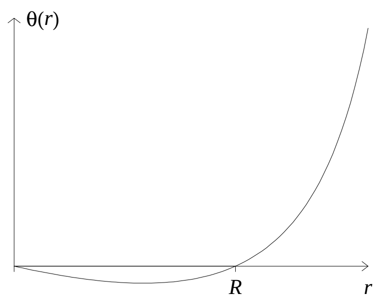

Figure 4: Visualisation of the function $\theta(r)$ in (11).

Suppose now that the Lundberg coefficient R exists, then by Proposition 1.12, $(M_R(t) = \exp\{-RU(t)\})_t$ is a martingale. Since the ruin time τ is a stopping time for $U(t)$ we can apply Theorem 1.10, hence for $t \geq 0$:

$$E(\exp\{-RU(\tau \wedge t)\}) = E(\exp\{-RU(0)\}) = e^{-Ru} \ .$$

The left hand side can be bounded below by

$$E(\exp\{-RU(\tau \wedge t)\}; \tau \leq t) = E(\exp\{-RU(\tau)\}; \tau \leq t) \,,$$

where, in general, we denote $E(X; A) = \int_A X\, dP$. Using monotone convergence ($t \to \infty$) and $U(\tau) < 0$ we obtain

$$e^{-Ru} \geq E(\exp\{-RU(\tau)\}; \tau < \infty) > P(\tau < \infty) \,.$$

Hence the following, so–called Cramér–Lundberg inequality is obtained for ruin in infinite time:

$$\Psi(u) \leq e^{-Ru} \,. \tag{14}$$

In order to answer the important question on how sharp the estimate in (14) is, one has to resort to more refined arguments. Using renewal theory results like Blackwell's renewal theorem (in the version of Smith's key renewal theorem; see Resnick (1992)) one obtains:

Theorem 2.2 (The Cramér–Lundberg approximation)
Assume that the Lundberg coefficient R in Definition 2.1 exists and that

$$\int_0^\infty x e^{Rx} \overline{F}(x)\, dx < \infty \,, \tag{15}$$

then

$$\lim_{u \to \infty} \Psi(u) e^{Ru} = \frac{c - \lambda\mu}{\lambda m_X'(R) - c} \ . \tag{16}$$

□

The limit result (16) shows that the Lundberg inequality is (asymptotically) sharp. The moment condition (15) is satisfied in all standard examples where R exists. For a discussion on this, see Embrechts (1983). The asymptotic estimate (16) is called the *Cramér–Lundberg approximation.* A key question in risk theory is to what extend the results (14) and (16) carry over to more general risk models. The important assumption in the Cramér–Lundberg approximation is that the exponential moments of the claim size distribution exist for some $r > 0$. This means that the right tail of F decreases at least exponentially fast. However, analysis of insurance and financial data typically indicates the presence of heavy tails; see Embrechts, Klüppelberg, and Mikosch (1997). The main result on asymptotic ruin estimates when the Lundberg coefficient does not exist is based on *subexponentiality* of F_I. Notice that $\varphi(u) := 1 - \Psi(u)$ given in (8) is the df of the random geometric sum $S_N = L_1 + \cdots + L_N$, where (L_i) is a sequence of iid rvs with common df F_I and N is geometrically distributed with parameter $1 - \rho$, independent of (L_i). Now if F_I is "long–tailed", large observations of L_i may occur with high probability and it is not unreasonable to conjecture that the random sum S_N is governed by just one summand. For that reason, it might be possible to relate the tail behaviour of φ to that of F_I. It turns out that the proper class for this purpose is the class $\mathcal{S}$ of subexponential distributions defined below.

Definition 2.3 *A distribution G on $[0, \infty)$ with unbounded support belongs to the class $\mathcal{S}$ of subexponential distributions if*

$$\lim_{x\to\infty} \frac{1 - G^{2*}(x)}{1 - G(x)} = 2 \ . \tag{17}$$

To explain why $\mathcal{S}$ can be used to model large claims we reformulate (17) as follows. If (X_i) are iid rvs with df $G \in \mathcal{S}$, then $P(S_n > x) \sim P(M_n > x)$, $x \to \infty$. Here we mean by $f(x) \sim g(x)$, $x \to \infty$, that $\lim_{x\to\infty} f(x)/g(x) = 1$, and $M_n = \max(X_1, \ldots, X_n)$. The name subexponential stems from the following property: if $G \in \mathcal{S}$, then the right tail of G decreases slower than any exponential, i.e. $\lim_{x\to\infty} e^{\varepsilon x}\, \overline{G}(x) = \infty$, for all $\varepsilon > 0$. A detailed analysis of the class $\mathcal{S}$ and its application to insurance are given in Embrechts, Klüppelberg and Mikosch (1997).

Asymptotic ruin estimates involving the class $\mathcal{S}$ were proposed in Embrechts, Goldie, and Veraverbeke (1979), where it is shown that the distribution of a random geometric sum $L_1 + \cdots + L_N$ belongs to $\mathcal{S}$ if and only if $L_i \in \mathcal{S}$, yielding the following result:

Theorem 2.4 *In the Cramér–Lundberg model with $F_I \in \mathcal{S}$ and safety loading $\vartheta > 0$ one has*

$$\Psi(u) \sim \frac{1}{\vartheta}\, \overline{F}_I(u) \ , \quad u \to \infty \, . \tag{18}$$

□

The main examples of claim size distributions where (18) holds are the Pareto, lognormal and the heavy-tailed Weibull distributions.

2.2 Practical evaluation of $\Psi(u, T)$

In the previous sections we discussed various expressions for the ruin probabilities $\Psi(u) = \Psi(u, \infty)$ in the Cramér–Lundberg model. These results were either exact, see (8), in inequality form (see (14)) or asymptotic for large initial capital u (see (16) and (18)). Alternative techniques may lead to integral-differential equations, see Section 2.3.3 below, Fourier-type representation and, for any of these, specific numerical techniques like the fast-Fourier-transform, simulation, or recursive methods. The most widely used method in the latter category is the so-called Panjer recursion which is based on a discretization of (8); see for instance Panjer and Willmot (1992), p.171. One particular method for estimating $\Psi(u,T)$ for finite T is based on a so-called diffusion approximation. We include a discussion mainly because of its relevance for the general theme of the paper rather than for its practical usefulness which is limited. Often one can imbed a (classical) risk process in a sequence $(U^{(n)})_n$ of risk processes and hope for the existence of a reasonable weak limiting process Z, say. If the risk process $U^{(n)}$ is approximated by the limiting process Z, then, under some regularity conditions, the hitting times (ruin probabilities) of Z should also approximate the hitting times (ruin probabilities) of $U^{(n)}$.

A Cramér–Lundberg risk process U is càdlàg, i.e. it has sample paths which are right continuous with left limits. (The word "*càdlàg*" is an acronym from the French "continu à droite, limites à gauche"). Stone (1963) extends the Skorokhod J_1–metric for càdlàg functions on *compact* intervals to $\mathbb{D} = \mathbb{D}[0,\infty)$, making $\mathbb{D}$ a Polish space. Hence, we can talk of weak convergence in $\mathbb{D}$.

Definition 2.5 *A sequence $(X^{(n)})_n$ of stochastic processes in $\mathbb{D} = \mathbb{D}[0,\infty)$ is said to converge weakly in the Skorokhod J_1–topology to a stochastic process X if for every bounded continuous functional f on $\mathbb{D}$ it follows that*

$$\lim_{n\to\infty} E\big(f(X^{(n)})\big) = E(f(X)) \,.$$

In this case one writes $X^{(n)} \Rightarrow X\,,\ n \to \infty$.

The main ingredients for weak approximations in risk theory are a functional central limit theorem in conjunction with the continuous mapping theorem. Suppose that (X_k) is a sequence of iid rvs with mean μ and finite variance σ^2. The famous Donsker invariance principle then says that, on $[0,1]$,

$$Z_n(\,\cdot\,) := \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{[n\,\cdot\,]} (X_k - \mu) \Rightarrow W_{(\cdot)}\,, \quad n \to \infty\,.$$

, where W denotes standard Brownian motion. The process $1/(\sigma\sqrt{n})\sum_{k=1}^{N(nt)}(X_k-\mu)$ is a random time transformation of Z_n, i.e.

$$\frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{N(nt)} (X_k - \mu) = Z_n\left(\frac{N(nt)}{n}\right) .$$

Moreover, $N(nt)/n \Rightarrow \lambda I$, where I denotes the identity map. The composition mapping is continuous, implying that

$$Z_n\left(\frac{N(n\,\cdot\,)}{n}\right) = \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{N(n\,\cdot\,)} (X_k - \mu) \Rightarrow W_{(\lambda\,\cdot\,)} \stackrel{d}{=} \sqrt{\lambda}\, W_{(\cdot)}\,. \tag{19}$$

The last equality in law follows from the scaling property of Brownian motion. Relation (19) is the key to the diffusion approximation in risk theory, which was first introduced in insurance mathematics by Iglehart (1969), see also Grandell (1991, Appendix A.4) for an extensive discussion of the method. The diffusion approach yields approximations for $\Psi(u,T)$ as well as for $\Psi(u)$, namely

$$\begin{aligned}\Psi(u,T) &\approx P\left(\inf_{0\le s<T}(u+\lambda\mu\vartheta s+\sqrt{\lambda}W_s)<0\right)\\ &=\overline{\mathcal{N}}\left(\frac{\lambda\mu\vartheta T+u}{\sqrt{\lambda T}}\right)+e^{-2\mu\vartheta u}\mathcal{N}\left(\frac{\lambda\mu\vartheta T-u}{\sqrt{\lambda T}}\right),\\ \Psi(u) &\approx P\left(\inf_{s\ge 0}(u+\lambda\mu\vartheta s+\sqrt{\lambda}W_s)<0\right)=e^{-2\mu\vartheta u},\end{aligned}$$

where $\mathcal{N}$ denotes the df of a standard normal rv. The results in the above equalities can for instance be found in Borodin and Salminen (1996). The latter approach is called *diffusion approximation* since Brownian motion is a special diffusion process. One of the advantages of the diffusion approximation is that it is applicable to more general models which derive from the classical risk process. For these more general processes the classical methods from renewal theory usually fail, and the diffusion approach is then one of the few tools that work.

Brownian motion has been studied for a long time and its usefulness in stochastic modeling is well accepted. However, Gaussian processes and variables do not allow for large fluctuations and may sometimes be inadequate for modeling high variability. For instance, the above diffusion approximation does not apply when the observed data give rise to a heavy–tailed claim size distribution such as Pareto with shape parameter $1<\alpha<2$, implying that the variance σ^2 does not exist. This phenomenon very often arises in non-life insurance and in particular in reinsurance; see Embrechts, Klüppelberg and Mikosch (1997). Both stable rv's and stable processes arise naturally as alternative modeling tools. The class of stable laws is defined as follows:

Definition 2.6 *A rv X is said to have a stable distribution, if for any $n\ge 2$ there is a $c_n>0$ and a real number d_n such that*

$$X_1+\cdots+X_n\stackrel{d}{=}c_nX+d_n\,, \tag{20}$$

where the X_i are independent copies of X.

It turns out (Feller (1971)) that in (20) we have necessarily $c_n=n^{1/\alpha}$ for some $\alpha\in(0,2]$. The parameter α is called *index of stability*. The case $\alpha=2$ corresponds to the normal distribution. Stable laws share many properties with the Gaussian distribution. In particular we may think of the central limit theorem: only stable laws appear as weak limits of normalized sums of iid rvs. The main difference between the normal distribution and non–Gaussian stable distributions is the tail behaviour. The (upper) tails of the latter decrease like $kx^{-\alpha}$, $x\to\infty$, for some constant k. The smaller the value of α, the slower the decay and the heavier the tails.

We now introduce another class of Lévy processes which contains Brownian motion as a special case.

Definition 2.7 *A càdlàg process Z is said to be an α–stable Lévy motion if the following properties hold:*

(a) $Z_0=0$ a.s.,

(b) Z has independent, stationary increments, and

(*c*) *For every t, Z_t has an α–stable distribution.*

Notice that 2–stable Lévy motion is Brownian motion. A stable Lévy motion with parameter $\alpha < 2$ exhibits jumps whose directions are governed by a so-called skewness parameter $\beta \in [-1, 1]$. If $|\beta| = 1$, the Lévy measure is concentrated on a half line and consequently there are only jumps in one direction. Figure 5 depicts some simulations of α–stable Lévy motion.

The analogous powerful result to the Donsker invariance principle in the regime of heavy–tailedness is a *stable* functional central limit theorem: Suppose that (X_k) is a sequence of iid rvs with finite mean μ and such that

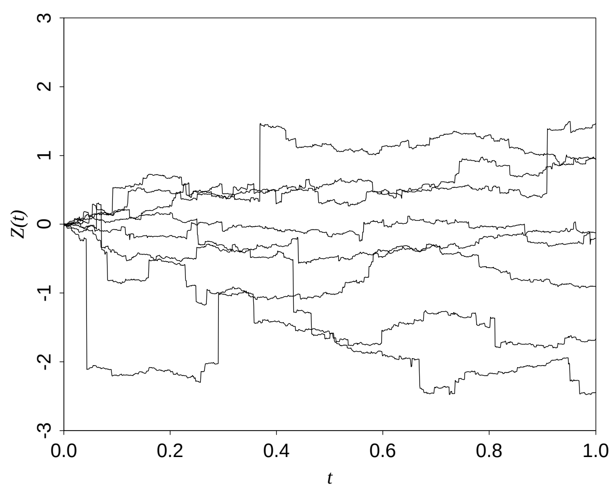

Figure 5: Simulations of 1.2–stable Lévy motion ($\beta = 0$) .

$$\frac{1}{\varphi(n)} \sum_{k=1}^{n} (X_k - \mu) \Rightarrow Y \ , \quad n \to \infty \, ,$$

where $\varphi(n) = n^{1/\alpha} L(n)$ for an appropriate slowly varying function L and Y has a stable distribution with index $1 < \alpha < 2$ and skewness parameter $|\beta| \le 1$. Then, for $0 \le t \le 1$,

$$\frac{1}{\varphi(n)} \sum_{k=1}^{[n\,\cdot]} (X_k - \mu) \Rightarrow Z. \ , \quad n \to \infty \, ,$$

where Z denotes α–stable Lévy motion with index α and skewness parameter β. Moreover, $Z_1 \stackrel{d}{=} Y$. Following the same approach as in the Brownian diffusion approximation, it is suggested to use the following approximations for the ruin probabilities when the variance of the claim size distribution does not exist:

$$\Psi(u, T) \approx P\left(\inf_{0 \le s < T} (u + \lambda\mu\vartheta s + \lambda^{1/\alpha} Z_s) < 0\right) ,$$

$$\Psi(u) \approx P\left(\inf_{s \ge 0} (u + \lambda\mu\vartheta s + \lambda^{1/\alpha} Z_s) < 0\right) = \sum_{n=0}^{\infty} \frac{(-a\mu\vartheta)^n}{\Gamma(1 + \tilde{\alpha} n)} \, u^{\tilde{\alpha} n} \, ,$$

where $\tilde{\alpha} = \alpha - 1$, $a = |\cos(\pi\alpha/2)|$ and $\Gamma(x) = \int_0^\infty e^{-u} u^{x-1}\,du$ denotes the Gamma function, see Furrer, Michna and Weron (1997) and Furrer (1998). In the latter reference an explicit formula for the distribution of the infimum of an α–stable Lévy motion with linear drift is derived in terms of the so–called Mittag–Leffler function $E_\sigma(x) = \sum_{n=0}^\infty x^n/\Gamma(1+\sigma n)$, $\sigma > 0$.

2.3 Generalizations of the claim number process

One can think of various generalizations of the classical risk process in order to obtain a more reasonable description of reality. Note that the homogeneous Poisson process is a stationary process, implying that the *size* of the portfolio can not increase (or decrease). In addition, fire and automobile insurance for instance ask for models allowing for *risk* fluctuations. As already mentioned in Section 1, the simplest way to take size fluctuations into account is to consider inhomogeneous Poisson processes with intensity measure $\Lambda(t)$. The purpose of this section is mainly to discuss the choice of point processes describing such risk fluctuations.

2.3.1 Mixed Poisson processes

Definition 2.8 *Let $\widetilde{N}$ be a homogeneous Poisson process with intensity* 1 *and Λ a random variable with $P(\Lambda > 0) = 1$, independent of $\widetilde{N}$. Then the process*

$$N = \widetilde{N} \circ \Lambda = (\widetilde{N}(\Lambda t))_t$$

is called a mixed Poisson process. The random variable Λ is called structure variable.

A mixed Poisson process has stationary increments, however the independent increments condition is violated. The stochastic variation of the claim number intensity can be interpreted as random changes of the Poisson parameter from its expected value λ. The most common choice for the distribution of the structure variable Λ is certainly the gamma distribution whose density function is given by

$$f_\Lambda(x) = \frac{\delta^\gamma}{\Gamma(\gamma)}\, x^{\gamma-1}\, e^{-\delta x}\,, \quad x \geq 0\,. \tag{21}$$

We use the notation $\Lambda \sim \Gamma(\gamma, \delta)$ to indicate that the random variable Λ has a gamma distribution with density function given in (21).

Definition 2.9 *A mixed Poisson process N is called a negative binomial process or Pólya process if $\Lambda \sim \Gamma(\gamma, \delta)$.*

We then have for a Pólya process N

$$\begin{aligned} P(N(t) = n) &= P\big(\widetilde{N}(\Lambda t) = n\big) \\ &= \int_0^\infty P\big(\widetilde{N}(\Lambda t) = n \mid \Lambda = \lambda\big)\, f_\Lambda(\lambda)\, d\lambda \\ &= \int_0^\infty e^{-\lambda t}\, \frac{(\lambda t)^n}{n!}\, \frac{\delta^\gamma}{\Gamma(\gamma)}\, \lambda^{\gamma-1}\, e^{-\delta\lambda}\, d\lambda \\ &= \binom{\gamma+n-1}{n} \left(\frac{\delta}{\delta+t}\right)^\gamma \left(\frac{t}{\delta+t}\right)^n , \end{aligned}$$

i.e. $N(t)$ has a negative binomial distribution. The corresponding risk model is also known as the Pólya–Eggenberger model. If one compares the total claim amount up to time t for the

Poisson model, then for equal means the variance of the Pólya model is bigger than in the Poisson model. This phenomenon is referred to as over-dispersion and is often encountered in real insurance data, see for instance Seal (1978) .

From a purely mathematical point of view, ruin calculations in the mixed Poisson case are easily performed. The idea is to first condition on the outcome of Λ and then weight over ruin probabilities computed in the Poisson case. Let $\Psi(u,\lambda)$ be the infinite–time ruin probability when N is a homogeneous Poisson process with intensity λ. Observe that $\Psi(u,\lambda)=1$ when the net profit condition (7) is violated, i.e. when $\lambda \geq c/\mu$. Thus we can write

$$\Psi(u) = \int_0^{c/\mu} \Psi(u,\ell)\, dF_\Lambda(\ell) + 1 - F_\Lambda(c/\mu)\,, \tag{22}$$

where F_Λ denotes the df of the structure variable Λ. It follows from (22) that $\Psi(u) \geq \overline{F}_\Lambda(c/\mu) > 0$ for all u. This implies that any insurer who does not constantly adjust his premium rate c according to the the risk fluctuations runs a large risk of being ruined.

Assume now that there exists $\ell_1 < c/\mu$ such that $F_\Lambda(\ell_1)=1$. It is natural to let ℓ_1 be the right endpoint of F_Λ, i.e. $\ell_1 = \sup\{\ell : F_\Lambda(\ell) < 1\}$. It follows from (8) that

$$1-\Psi(u) = \sum_{n=0}^{\infty} p_n F_I^{n*}(u)\,,$$

where $p_n = \int_0^{\ell_1} (1-\ell\mu/c)(\ell\mu/c)^n\, dF_\Lambda(\ell)$. An extension of the Cramér–Lundberg approximation in the mixed Poisson case in general seems not possible; see Grandell (1997) for a discussion on this. However, the situation is different in the regime of heavy tails, where the following result can be derived, see Grandell (1997):

Theorem 2.10 *Let ℓ_1 be the right endpoint of F_Λ and suppose that $\ell_1 < c/\mu$ and that $F_I \in \mathcal{S}$. Then*

$$\Psi(u) \sim E\left(\frac{1}{\vartheta(\Lambda)}\right) \overline{F}_I(u)\,, \quad u \to \infty\,,$$

where $\vartheta(\lambda) = c/(\lambda\mu) - 1$. □

2.3.2 Cox processes

We shall now consider the case where the occurrence of claims is described by a Cox process N. The first treatment of Cox processes in insurance mathematics originates from Ammeter (1948). Cox processes seem to form a natural class to model risk and size fluctuations.

Definition 2.11 *A stochastic process $\Lambda = (\Lambda(t))_t$ with P–a.s. $\Lambda(0)=0$, $\Lambda(t)<\infty$ for each $t<\infty$ and non–decreasing sample paths is called a random measure. If Λ has P–a.s. continuous realizations, it is called diffuse.*

Definition 2.12 *Let Λ be a random measure and $\widetilde{N}$ a homogeneous Poisson process with intensity $\lambda=1$, independent of Λ. The point process $N = \widetilde{N}\circ\Lambda$ is called Cox process or doubly stochastic Poisson process.*

Definition 2.12 is one of several equivalent definitions. Strictly speaking we only require that N and $\widetilde{N}\circ\Lambda$ are equal in distribution. For this question and related measurability conditions we refer to Grandell (1976).

Now let Λ be a diffuse random measure with $\Lambda(\infty) = \infty$ a.s. and N be the corresponding Cox process. As a generalization of (4) we obtain the risk process

$$U(t) = u + (1+\vartheta)\mu\Lambda(t) - \sum_{k=0}^{N(t)} X_k\,, \quad t \geq 0\,.$$

Assume now that Λ has the representation $\Lambda(t) = \int_0^t \lambda(s)\,ds$, where $(\lambda(t))_t$ is called the *intensity process*. If $(\lambda(t))_t$ has right continuous and Riemann integrable trajectories, then the corresponding Cox process is well defined (Grandell (1976)). The premium rate is then given by $c(t) = (1+\vartheta)\mu\lambda(t)$, i.e. it is a stochastic process. The martingale approach to Cox models, due to Björk and Grandell (1988), is an extension of the basic martingale approach considered in Proposition 1.12. See also Embrechts, Grandell and Schmidli (1993) for a discussion on finite time ruin probabilities in the Cox case. Let N be a Cox process with intensity process $(\lambda(t))_t$. A suitable filtration $\mathbb{F}$ is given by $\mathcal{F}_t = \mathcal{F}_\infty^\Lambda \vee \mathcal{F}_t^U$. It seems natural to try to find an $\mathbb{F}$–martingale as close as possible to the one used in Proposition 1.12. We therefore consider the process

$$M_r(t) = \exp\{-rU(t) - \theta(r,t)\}\,, \quad t \geq 0\,, \tag{23}$$

where $\theta(r,t) = \Lambda(t)(Ee^{rX_1} - 1) - rct$, i.e. we simply replace λt by $\Lambda(t)$. Then the following proposition holds:

Proposition 2.13 *The process $(M_r(t))_t$ given in (23) is an $\mathbb{F}$–martingale, where the filtration $\mathbb{F}$ is given by $\mathcal{F}_t = \mathcal{F}_\infty^\Lambda \vee \mathcal{F}_t^U$.* □

A lower bound for the ruin probability $\Psi(u)$ is easily obtained in the same way as in Section 2.1, namely

$$\begin{aligned} M_r(0) = e^{-ru} &\geq E(M_r(\tau \wedge t); \tau \leq t|\mathcal{F}_0) \\ &= E(M_r(\tau); \tau \leq t|\mathcal{F}_0) \\ &\geq E(\exp -\theta(r,\tau); \tau \leq t|\mathcal{F}_0) \\ &\geq \inf_{0 \leq s \leq t} \exp\{-\theta(r,s)\}\, P(\tau \leq t|\mathcal{F}_0)\,. \end{aligned}$$

Taking expectations on both sides and using monotone convergence yields

$$\begin{aligned} P(\tau < \infty) &\leq e^{-ru} E\left(\sup_{t\geq 0} \exp\{\Lambda(t)(Ee^{rX_1} - 1) - rct\}\right) \\ &= C(r)\, e^{-ru}\,, \end{aligned}$$

say. Like in the Poisson case we would like to choose r as large as possible. This suggests the following definition:

Definition 2.14 *The Lundberg coefficient R_C in the Cox model is defined as*

$$R_C = \sup\left\{ r : E\left(\sup_{t\geq 0} \exp\{\Lambda(t)(Ee^{rX_1} - 1) - rct\}\right) < \infty \right\}.$$

Consider a Cox model where the intensity process is stationary with $E\lambda(t) = \lambda$ and denote by R_P the Lundberg coefficient in a classical risk model where the homogeneous Poisson process N has intensity λ. Let $r > R_P$, implying that $\theta(r) = \lambda(Ee^{rX_1} - 1) - rc > 0$ and therefore

$$\begin{aligned} C(r) &\geq \sup_{t\geq 0} E\left(\exp\{\Lambda(t)(Ee^{rX_1} - 1) - rct\}\right) \\ &\geq \sup_{t\geq 0} \exp\left\{t(\lambda(Ee^{rX_1} - 1) - rc)\right\} = \infty\,, \end{aligned}$$

where the second inequality follows from Jensen's inequality. Hence $R_C \leq R_P$ which means that the stationary Cox case is "more dangerous" than the Poisson case. For a more detailed discussion of this comparison we refer to Theorem 22, p95 and to Section 4.6 of Grandell (1991).

A very special class of Cox models are the Cox processes with an independent jump intensity. Intuitively, an independent jump intensity is a jump process where the jump times form a renewal process and where the value of the intensity between two successive jumps may depend only on the distance between those two jumps. Although Cox processes with an independent jump intensity are a special class of Cox models, they are still general enough to obtain non–trivial models allowing for fairly explicit results in the ruin type setting. Cox processes also appear as limiting processes of certain thinning procedures and therefore seem to be natural point processes for modeling claim arrivals. If we consider claims which are caused by "risk situations" or *incidents*, then each incident becomes a claim with probability p independent of all other incidents. Under these assumptions, the claim number process is the result of a thinning procedure of the incident number process. A rigorous treatment of Cox models is to be found in Grandell (1991). The forthcoming book Rolski, Schmidli, Schmidt, and Teugels (1999) gives a readable introduction to risk theory overall.

2.3.3 Renewal processes

In this section we let the occurrence of claims be described by a renewal process N. Denote by T_k the interarrival times between two successive claims.

Definition 2.15 *A point process on $\mathbb{R}_+$ is called a renewal process if the variables $(T_k)_{k\geq 1}$ are independent and if $T_2, T_3, \ldots$ have the same df G. N is called ordinary renewal process if T_1 also has df G.*
We call N a stationary renewal process if G has finite mean $1/\lambda$ and if the df G_0 of T_1 satisfies $G_0(x) = \lambda \int_0^x \overline{G}(s)\, ds$.

The first treatment of ruin problems when the occurence of claims is modelled by a renewal process is due to Andersen (1957).

Ordinary renewal processes

Let N be an ordinary renewal process and assume that T_k has finite mean $1/\lambda$. N is *not* stationary, and $EN(t) \neq \lambda t$ unless T_k has an exponential distribution. We consider the associated random walk $S_n = \sum_{k=1}^n Y_k$, $(S_0 = 0)$, where $Y_k = -cT_k + X_k$. We assume that $EY_k = -c/\lambda + \mu < 0$, implying that the random walk S_n drifts to $-\infty$. The safety loading ϑ is defined in a natural way as $\vartheta = c/(\lambda\mu) - 1$. Since ruin can occur only at renewal epochs, we have that

$$\Psi(u) = P\left(\max_n S_n > u\right) .$$

Denote by K the df of Y_k and let $\widetilde{\mu}$ be the mean of Y_k, i.e. $\widetilde{\mu} = -\mu\vartheta$. Then

$$Ee^{rS_n} = \left(Ee^{rY_k}\right)^n = \left(Ee^{-rcT_k}Ee^{rX_k}\right)^n$$
$$= \left(\hat{g}(rc)\hat{f}(-r)\right)^n =: \left(\hat{k}(-r)\right)^n .$$

We assume that $K(0) < 1$ since $K(0) = 1$ implies $\Psi(u) \equiv 0$. The function $\hat{k}(-r)$ will be important. Under the assumption that the appropriate exponential moments of X_k exist for some $r > 0$, one can show that $\hat{k}(0) = 1$, $\hat{k}'(0) = \widetilde{\mu} < 0$ and that $\hat{k}$ is convex and continuous on $[0, r_\infty)$, where r_∞ denotes the abscissa of convergence of Ee^{rX_k}. Moreover, $\hat{k}(-r) \to \infty$ as $r \to r_\infty$. From this it follows that there exists a constant $R_R > 0$ such that $\hat{k}(-R_R) = 1$. Again R_R is called the Lundberg coefficient. Indeed, if T_k is exponentially distributed, then R_R coincides with the Lundberg coefficient from the classical model. The process (S_n) is a random walk and therefore has stationary and independent increments. This is exactly the property we used in the classical case to construct a family of martingales and to prove the Lundberg inequality (14). It is therefore not surprising that the derivation goes through in the ordinary renewal setup:

Proposition 2.16 *The discrete time process $(M_r(n))_n$ given by*

$$M_r(n) = \frac{e^{-r(u-S_n)}}{\left(\hat{k}(-r)\right)^n}, \quad n \geq 0,$$

is a martingale with respect to the filtration $\mathbb{F}^S$ given by $\mathcal{F}_n^S = \sigma(S_k : k \leq n)$.

Let N_u be the claim number causing ruin, i.e. $N_u = \min\{n : S_n > u\}$. Then N_u is a stopping time and $\Psi(u) = P(N_u < \infty)$. Again $N_u \wedge n_0$ is a bounded $\mathbb{F}^S$–stopping time for $n_0 < \infty$ and by the stopping theorem for martingales (Theorem 1.10) we obtain as before

$$\Psi(u) \leq e^{-ru} \sup_{n \geq 0} \left(\hat{k}(-r)\right)^n .$$

The best choice of r is the Lundberg exponent R_R, yielding

$$\Psi(u) \leq e^{-R_R u}, \quad u \geq 0 . \tag{24}$$

Asymptotic estimates for $\Psi(u)$ as in the Cramér–Lundberg approximation can be derived by means of renewal and random walk theory. Consider the rv $A_1 = S_{N_0}$ on $\{N_0 < \infty\}$, where $N_0 = \min\{k : S_k > 0\}$. Define $A(y) = P(A_1 \leq y, N_0 < \infty)$ and note that $A(\infty) = P(N_0 < \infty) = \Psi(0)$. Thus A has a defective distribution. The defect $1 - A(\infty)$ is the probability that the random walk never becomes positive starting from 0. By separating the cases $A_1 > u$ and $A_1 \leq u$ we obtain

$$\Psi(u) = A(\infty) - A(u) + \int_0^u \Psi(u - y)\, dA(y), \quad u \geq 0, \tag{25}$$

which is a defective renewal equation. By the so-called Esscher transform defined below one can remove the defect provided the appropriate exponential moments exist. Assume that there is a constant κ such that

$$\int_0^\infty e^{\kappa y}\, dA(y) = 1 .$$

Then we get, by multiplying (25) with $e^{\kappa u}$, a proper renewal equation and Smith's key renewal theorem again yields, for nonarithemtic A,

$$\lim_{u\to\infty} e^{\kappa u}\Psi(u) = \frac{1-A(\infty)}{\kappa\int_0^\infty y e^{\kappa y}\,dA(y)}\,.$$

Using random walk theory (see for instance Feller (1971)) one can show that $\kappa = R_R$, and we obtain

$$\begin{aligned}\Psi(u) &\sim \frac{1-A(\infty)}{R_R\displaystyle\int_0^\infty y e^{R_R y}\,dA(y)}\, e^{-R_R u}\,, \quad u\to\infty\,, \\ &= C_0 e^{-R_R u}\,,\end{aligned} \tag{26}$$

say. Since A is in general unknown, the constant C_0 cannot be calculated explicitly. However, it follows that R_R is the "right" exponent in (24) and this is undoubtedly the most important consequence of (26). If the claim size distribution is such that $F_I \in \mathcal{S}$, then the following proposition holds, see Embrechts and Veraverbeke (1982).

Proposition 2.17 *If the claim size distribution is such that $F_I \in \mathcal{S}$, then the ruin probability in the ordinary renewal model satisfies*

$$\Psi(u) \sim \frac{1}{\vartheta}\overline{F}_I(u)\,, \quad u\to\infty\,.$$

$\square$

Stationary renewal processes

Ruin type estimates for the stationary renewal model basically derive from the ordinary situation. Indeed, by conditioning on the first claim epoch T_1 (with df G_0) the process starts anew with iid interarrival times (T_k), hence we are then in the situation of the ordinary renewal model. To make the above heuristic reasoning mathematically precise, denote by $\Psi^S(u)$ the ruin probability of a stationary renewal model and by $\Psi(u)$ the ruin probability of the ordinary model. Then a renewal argument yields

$$\Psi^S(u) = \frac{\lambda}{c}\int_u^\infty \overline{F}(y)\,dy + \frac{\lambda}{c}\int_0^u \Psi(u-y)\overline{F}(y)\,dy\,. \tag{27}$$

Here, as before, F denotes the claim size df. For a detailed derivation of (27) see Section 3.2 of Grandell (1991). From the ordinary renewal model we know that $\Psi(u) \le e^{-R_R u}$, yielding

$$\begin{aligned}\Psi^S(u) &\le \frac{\lambda}{c}\int_0^\infty e^{-R_R(u-y)}\overline{F}(y)\,dy \\ &= \frac{\lambda}{R_R c}\left(\hat{f}(-R_R)-1\right) e^{-R_R u}\,,\end{aligned}$$

hence Lundberg's inequality holds, but the constant may be greater than one. A Cramér–Lundberg approximation follows from (27) by multiplying $\Psi^S(u)$ by $e^{R_R u}$ and taking the limit as $u \to \infty$:

$$\begin{aligned}
\lim_{u\to\infty} e^{R_R u}\Psi^S(u) &= \lim_{u\to\infty} \frac{\lambda}{c}\int_0^u e^{R_R(u-y)}\Psi(u-y)e^{R_R y}\overline{F}(y)\,dy \\
&= \frac{\lambda C_0}{c}\int_0^\infty e^{R_R y}\overline{F}(y)\,dy \\
&= \frac{\lambda C_0}{R_R c}\left(\hat{f}(-R_R)-1\right) =: C\,,
\end{aligned}$$

where $0 < C < \infty$; a result due to Thorin (1975) .

2.4 A general insurance risk model

To stress further why martingales play an important role in risk theory we consider the general structure of a risk process

$$u + P(t) - S(t)\,,$$

where u denotes the initial capital, P the premium income up to time t and S the liabilities (claims). If for the moment we forget about the initial capital u and assume that $S(t)$ is a general stochastic process, then a natural way to construct the process P is to make the difference

$$M(t) = P(t) - S(t)$$

a "fair game" (i.e. a martingale) between the insurer and the insured. Delbaen and Haezendonck (1987) use this direct martingale approach for the construction of fairly general risk models allowing for economic factors such as interest and inflation to be incorporated to the classical Cramér–Lundberg model. Paulsen (1993) goes one step further and allows the economic factors to be stochastic. Semimartingales coupled with integro–differential equations lead in some cases to exact probabilities of ruin and in others to inequalities. Economic factors and their influence on ruin probabilities for the Brownian diffusion approximation of a classical risk process are discussed by Sørensen (1996) or Norberg (1997). In this section we present a martingale approach based on the theory of piecewise deterministic Markov processes (PDMPs). The class of PDMPs was introduced by Davis (1984) and further discussed in Davis (1993) . To motivate the usage of PDMPs in risk theory consider the basic Cramér–Lundberg model. Note that the state space of a classical risk process $(U(t))_t$ is $\mathbb{R}$ and that the sample path behaviour of U has a deterministic inter–jump evolution along linear trajectories with rate $c > 0$. In the language of Davis (1984) one reformulates the latter as "$(U(t))_t$ follows the integral curves of the vector field $\chi = c\,\partial/\partial x$". Moreover, the hazard rate along integral curves is $\lambda(x) = \lambda$ and the Markov measure governing the stochastic evolution of the process equals $Q(dy, x) = dF(x - y)$. Dassios and Embrechts (1989) employ the PDMP framework for solving insurance risk problems where borrowing money below a certain surplus barrier is allowed. All these processes share the property that they are PDMPs for which $\phi(t, y)$ will denote the integral curve of a vector field χ starting at $y \in \mathbb{R}$. The efficiency of PDMPs in risk theory is strongly based on martingale methodology. For a general Markov process, the martingale construction can effectively be obtained via the integration of the infinitesimal generator along sample paths of the process.

Definition 2.18 *Let $\mathcal{D}(\mathcal{A})$ be the set of all measurable real functions on $\mathbb{R}$ with the property that an operator $\mathcal{A}$ exists such that $\mathcal{A}f$ is almost surely Lebesgue integrable and*

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)\,ds$$

is a local $\mathcal{F}_t$–martingale. We call $\mathcal{A}$ the extended generator of (X_t) and $\mathcal{D}(\mathcal{A})$ the domain of the generator $\mathcal{A}$.

When $\mathcal{A}$ corresponds to the infinitesimal generator of a PDMP (X_t), then Davis (1984) gives necessary and sufficient conditions for a function f to belong to $\mathcal{D}(\mathcal{A})$. However, for applications in risk theory, it turns out that the following condition from Dassios and Embrechts (1989) suffices. Denote by S_i the time of the ith claim.

Lemma 2.19 *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying*

(*i*) *for all $x \in \mathbb{R}$, the mapping $t \mapsto f(\phi(t,x))$ from $[0,\infty)$ to $\mathbb{R}$ is absolutely continuous,*

(*ii*) *for all $t \geq 0$, $E\left(\sum_{S_i \leq t} |f(X_{S_i}) - f(X_{S_i-})|\right) < \infty$.*

Then $f \in \mathcal{D}(\mathcal{A})$ and the generator of the PDMP (X_t) is given by

$$\mathcal{A}f(x) = \chi f(x) + \lambda \int_0^\infty (f(x-y) - f(x))\,dF(y)\,.$$

Furthermore, $(M_t^f)_t$ is a martingale. □

The idea now is to construct martingales via functions $f_0 \in \mathcal{D}(\mathcal{A})$ satisfying $\mathcal{A}f_0 = 0$, implying that $f_0(X_t) - f_0(X_0)$ is a martingale for bounded f_0.

A RISK MODEL WITH INTEREST STRUCTURE

As a generalization of the classical model, assume that a company can borrow money if needed (i.e. for a negative or low surplus) and gets interest for capital above a certain level Δ, say, the amount of capital the company retains as a liquid reserve. The interest rates are assumed to be constant and denoted by β_1 for invested money and β_2 for borrowed money. The associated vector field becomes

$$\chi = \begin{cases} (\beta_1(x-\Delta) + c)\frac{\partial}{\partial x} & \Delta \leq x\,, \\ c\,\frac{\partial}{\partial x} & 0 \leq x < \Delta\,, \\ (\beta_2 x + c)\frac{\partial}{\partial x} & x < 0\,. \end{cases}$$

The integral curve corresponding to χ is decreasing for $x \leq -c/\beta_2$. Whenever the process hits the boundary $-c/\beta_2$, the company will a.s. not be able ro repay its debts. So $\tau := \inf\{t > 0 : X_t \leq -c/\beta_2\}$ will be called the ruin time. Above the liquid reserve level Δ the paths are exponentially increasing. Between 0 and Δ their behaviour is as in the classical case and below 0 the slopes of the paths are smaller. The model where $\Delta = \infty$ was studied in Dassios and Embrechts (1989). Using PDMP theory, one can show that for $\Delta \in [0,\infty]$ one has

$$P(\tau < \infty) = 1 - \frac{f(0)}{f(\infty)}\,,$$

where f is the solution of complicated integral–differential equations. Moreover, $P(\tau < \infty) = 1$ if and only if $\Delta = \infty$ and $c \leq \lambda\mu$. As a special case, consider exponentially distributed claims with mean μ. Then the function f above becomes

$$f(x) = f_1(x)\mathbb{I}_{[\Delta,\infty)}(x) + f_2(x)\mathbb{I}_{[0,\Delta)}(x) + f_3(x)\mathbb{I}_{(-\infty,0)}(x)\,,$$

where

$$\begin{aligned}
f_3(x) &= K\int_0^{x+c/\beta_2} s^{(\lambda/\beta_2)-1}e^{-s/\mu}\,ds\,,\\
f_2(x) &= f_3(0) + \frac{f_3'(0)}{R}\left(1-e^{-Rx}\right)\,,\\
f_1(x) &= f_2(\Delta) + \left(\frac{\beta_1}{c}\right)^{(\lambda/\beta_1)-1} e^{c/(\beta_1\mu)} f_2'(\Delta)\int_{c/\beta_1}^{x+c/\beta_1-\Delta} s^{(\lambda/\beta_1)-1}e^{-s/\mu}\,ds\,,
\end{aligned}$$

for some constant K which can be calculated explicitly. Here $R = 1/\mu - \lambda/c$ denotes the Lundberg coefficient for exponentially distributed claims in the Cramér–Lundberg model. As a consequence of this result one obtains the following adjustement coefficient estimate:

$$\lim_{u\to\infty} P(\tau < \infty)\,e^{ru} = \begin{cases} 0 & r < 1/\mu \ \text{ or } (r = 1/\mu \ \text{ and } \ \lambda < \beta_1)\,,\\ c & r = 1/\mu \ \text{ and } \ \lambda = \beta_1\,,\\ \infty & \text{otherwise}\,, \end{cases}$$

where

$$c = e^{\Delta\mu}(\beta_1/c)^{(\lambda/\beta_1)-1}\,\mu f_2'(\Delta)/f_1(\infty)\,.$$

An "extended" PDMP framework also allows to consider ruin type problems of the following model

$$U_W(t) = u + ct - \sum_{k=1}^{N(t)} Y_k + W_t\,,$$

where u and c are constants, N is a claim number process and W is standard Brownian motion describing small perturbations around the risk process U, see Furrer and Schmidli (1994). Finally, an interesting application of the PDMP-methodology to a health-insurance problem is to be found in Davis (1993), p.107.

2.5 Remarks on the use of stochastic processes in insurance

The above sections have only highlighted some (definitely from a historical perspective the most important) ways in which stochastic processes enter as key building blocks in the stochastic modeling of insurance. Although it was not stated explicitly, it should be clear to the reader that the models that have been treated so far refer mainly to non-life and re-insurance. A very important field of applications, which increasingly sees stochastic modeling being used, is life-insurance. One of the main reasons for this is the increasing convergence of insurance and finance, both structually, i.e. at the company level, and at the level of products being offered. Think for instance of the so-called equity-linked life products where the payment due at the end of the policy is partly contingent on the returns (performance) of an equity portfolio. On the other hand even standard life products are increasingly being modeled by finite-state Markov-processes. It is impossible for us to enter into some of the models in this area in the course of this paper. We refer the interested reader to Wolthuis (1994) and Norberg (1991, 1995), for a start. The forthcoming monographs Koller (1999) and Milbrodt and Helbig (1999) give an excellent overview of stochastic processes in life-insurance mathematics.

3 Stochastic processes in finance

3.1 Pricing and hedging of derivatives: standard theory

3.1.1 Introduction

We start our discussion of stochastic processes in finance by a review of the standard approach for the pricing of derivative securities such as options. Our exposition is based on Föllmer (1991) and Frey (1997) .

Modern derivative asset analysis has its origins in the seminal papers Black and Scholes (1973) and Merton (1973). A few years later it was given an almost definitive conceptual structure by Harrison and Kreps (1979) and Harrison and Pliska (1981). These papers show that the natural mathematical framework for the analysis of derivative securities is provided by the theory of martingales and stochastic integrals. The theory of stochastic integration had been developed by probabilists long before its applicability to Finance was discovered, starting with the fundamental work of Itô and culminating in the "general theory" of the French School. A brief history of stochastic integration theory is provided in Protter (1992).

For our exposition we consider a market with two traded assets: a riskless asset B representing some bond or money market account and a risky asset which will be called the stock. The price fluctuations of stock and bond will be described by some stochastic process $S_t(\omega)$ respectively $B_t(\omega)$ on our underlying probability space $(\Omega, \mathcal{F}, P)$. For simplicity we assume that $B_t = 1$ for all $t \geq 0$. This assumption does not exclude nonzero interest rates from our analysis, if we interprete S as forward price process of the stock, i.e. if we choose the bond as *numeraire.*

To complete the description of our setup we have to specify the information that is available to our financial decision makers at a particular point in time. As in Section 1.2 this is done via a filtration $(\mathcal{F}_t)_t$; it is understood that at time t agents have access to the information contained in $\mathcal{F}_t$. We will always assume that our stock price is adapted and that its trajectories follow càdlàg sample paths.

Now imagine an investor such as an investment bank who considers selling a contingent claim, i.e. a $\mathcal{F}_T$-measurable random variable H. In this context H is interpreted as payoff of some financial contract which occurs at the maturity date T. Typically H is a derivative asset, i.e. the value of H is determined by the realization of the price path of S. The most popular examples are European call and put options with maturity date T and exercise price K, where $H = (S_T - K)^+$ or $H = (K - S_T)^+$, respectively. More complicated contracts are also traded nowadays; as an example we mention the so-called average option where $H = (1/T \int_0^T S_s ds - K)^+$.

The common feature of all these contracts is that the payoff H is unknown at $t = 0$ and therefore constitutes a risk for the seller. Hence two questions arise for our investor: How should he price the claim and how should he deal with the risk incurred by selling the contract? The "modern" answer to these questions dates back to the seminal papers Black and Scholes (1973) and Merton (1973), where it was shown for the first time that under certain assumptions the payoff of a derivative security can be replicated by a dynamic trading strategy in the underlying asset, such that its risk can be eliminated. This concept of dynamic hedging and not some particular pricing formula is actually the major contribution of these papers.

3.1.2 A two-period example

We start by explaining this idea in a very simple two-period setting which represents for instance one time step in the binomial model of Cox, Ross and Rubinstein (1979). Suppose that the current price of S is given by $S_0 = 150$ and that there are two possible "scenarios" for the future stock price: the price of S at the terminal time T could be $S_T = 180$ (with probability $p > 0$) or be equal to $S_T = 120$ (with probability $1-p > 0$). Consider a European call option with payoff $K = 140$. We claim that a fair price of this option is given by $C_0 = 20$, and that this price is moreover independent of the probability p.

To justify this claim we construct a portfolio in stock and bond whose value at T equals the price of our option: At $t = 0$ we buy $(2/3)$ units of the stock and sell 80 bonds. At $t = T$ there are two possibilities for the value V_T of our portfolio.

- $S_T = 180$: In that case V_T equals $V_T = (2/3)180 - 80 = 40$.
- $S_T = 120$: In that case the option is worthless; moreover we have $V_T = 0$.

In either case the value of our portfolio at T equals the payoff of the option. Hence the fair price of the option should also equal the value of our portfolio at $t = 0$ which is given by $V_0 = (2/3)150 - 80 = 20$. Otherwise either the buyer or the seller could make some riskless profit. To construct the hedge portfolio in this simple two-period setting we have to consider two linear equations: Denote by ξ and η the number of stocks and bonds in our portfolio at $t = 0$. For our portfolio to replicate the option we must have

$$\xi 180 + \eta = 40 \quad \text{and} \quad \xi 120 + \eta = 0\,, \tag{28}$$

which leads to the above values of $\xi = 2/3$ and $\eta = -80$.

Note that the probability p did not enter our argument; this probability mattered only in so far as the requirements $P(S_T = 180) > 0$ and $P(S_T = 120) = 1 - p > 0$ determine the set of possible scenarios at $t = T$. Nonetheless it is still possible to compute the fair price of the option as expected value of the terminal payoff under some "artificial" probability measure Q which turns the investment in the stock into a fair game (a martingale). In our case such a probability measure is unique and given by $Q(S_T = 180) = Q(S_T = 120) = 0.5$. If we now compute the expected terminal value (under Q) of the terminal payoff of our option we get

$$E^Q(S_T - 140)^+ = (1/2)40 + (1/2)0 = 20\,. \tag{29}$$

This is of course not a lucky coincidence and the general argument justifying (29) will be given in the next section.

3.1.3 The general argument

We now extend the argument from the previous two-period example to a more realistic continuous-time setting. Our basic assumption is that the process S admits an equivalent local martingale measure Q, i.e. a probability measure $Q \sim P$ such that S is a Q-local martingale. We will comment on the economic meaning of this assumption below. From a mathematical viewpoint this assumption ensures that S is a *semimartingale* under P such that we may define stochastic integrals with respect to S. Recall that a semimartingale X is an adapted càdlàg process which can be decomposed as $X_t = X_0 + M_t + A_t$, where M is a local martingale and A is a process of finite variation. If A is predictable (e.g. left-continuous) such a decomposition is unique. Semimartingales are natural stochastic integrators; a good treatment of semimartingale theory and in particular of their role as natural stochastic integrators is given in Protter (1992).

To replicate the payoff of a contingent claim we use a dynamic trading strategy (ξ, η) where ξ_t gives the amount held in the risky asset at time t and η_t gives the position in the bond. Of course our position at t should depend only on information available up to time t, that is we require ξ to be *predictable* and η to be *adapted* with respect to our filtration; ξ should moreover be locally bounded. We refer the reader to Chapter 4 of Protter (1992) for a formal definition of predictable processes and mention only that every adapted and left-continuous process is locally bounded and predictable. At time t the value of our hedge portfolio equals

$$V_t = \xi_t S_t + \eta_t \,. \tag{30}$$

As $B_t \equiv 1$ the cumulated *gains from trade* of following this strategy up to time t are measured by the stochastic integral $\int_0^t \xi_s dS_s$. This is obvious for so-called simple predictable strategies ξ of the form

$$\xi_t = \sum_{i=1}^{n} \xi_i(\omega) 1_{(T_i, T_{i+1}]}(t) \,,$$

where $0 = T_0 < T_1 < \ldots < T_{n+1} < \infty$ is a finite sequence of stopping times and where each ξ_i is $\mathcal{F}_{T_i}$-measurable and bounded. If we follow such a strategy the gains (or losses) from trade up to time t are given by

$$\sum_{i=1}^{n} \xi_i (S_{T_{i+1} \wedge t} - S_{T_i \wedge t}) = \int_0^t \xi_s dS_s \,,$$

by definition of the stochastic integral for simple predictable processes. For general strategies the modeling of the gains from trade as a stochastic integral can be justified by limit arguments. The *cumulative cost* C_t from following this strategy up to time t is given by

$$C_t = V_t - V_0 - \int_0^t \xi_s dS_s \,. \tag{31}$$

It measures the cumulative in- or outflows to our strategy. The strategy will be called *selffinancing* if the cumulative cost is zero, i.e. if

$$V_t = V_0 + \int_0^t \xi_s dS_s \quad \text{for all } 0 \leq t \leq T \,. \tag{32}$$

Suppose now that our contingent claim can be represented as a stochastic integral with respect to S, i.e. $H = H_0 + \int_0^T \xi_s^H dS_s$. Then we may construct a dynamic hedging strategy for H as follows. Define

$$\xi_t = \xi_t^H \text{ and } \eta_t := H_0 + \int_0^t \xi_s^H dS_s - \xi_t^H S_t \,. \tag{33}$$

This strategy is selffinancing with value process $V_t^H = H_0 + \int_0^t \xi_s^H dS_s$. In particular $V_T^H = H$. Therefore, at any time $t \le T$ we can replicate the claim by starting with an investment of V_t^H and following the above strategy. There are no further payments and hence no further risk. This implies that at time t the *fair price* of the claim should be equal to V_t^H.

Harrison and Pliska (1981) showed how the fair price of the claim can be computed using the concept of martingales. The stochastic integral $\int_0^t \xi_s^H dS_s$ is a Q-local martingale and a martingale under some uniform integrability assumptions. Hence

$$E^Q\left(\int_t^T \xi_s^H dS_s \mid \mathcal{F}_t\right) = 0 \quad \text{for all } t\,.$$

This yields the so-called *risk-neutral pricing rule* for the claim H

$$H_t := V_t^H = E^Q(H \mid \mathcal{F}_t)\,; \tag{34}$$

in particular the fair price process $H = (H_t)_{0\le t\le T}$ is a Q-martingale. Harrison and Pliska (1983) moreover showed that the market is *complete*, i.e. every Q-integrable claim admits a representation as stochastic integral with respect to S, if and only if there is only one equivalent martingale measure for S.

The assumption that S admits an equivalent (local) martingale measure needs of course some economic justification, which is provided by the so-called "First Fundamental Theorem of Asset Pricing". This theorem, whose origins go back to the work of Harrison and Kreps (1979), states that the existence of an equivalent martingale measure is "essentially equivalent" to the absence of arbitrage opportunities. As a precise mathematical statement of this theorem is relatively cumbersome, we refer the reader to Dalang, Morton and Willinger (1990) for an analysis in discrete time and to the fundamental paper Delbaen and Schachermayer (1994) for definitive results in continuous-time models.

3.1.4 Diffusion Models

Now we want to apply this general approach to cases where the stock price process S is given by a diffusion. More precisely we assume that S is given by the solution to the following SDE

$$dS_t = \mu(t, S_t) S_t\, dt + \sigma(t, S_t) S_t\, dW_t\,, \quad S_0 = x\,, \tag{35}$$

where W is a standard Brownian motion as in Definition 1.1 and μ and σ are sufficiently smooth such that there is a unique solution to (35); σ is moreover strictly positive. The model (35) has the following intuitive interpretation: at a given point in time $\mu(t, S_t)$ describes the instantaneous growth rate of the asset, while the *volatility* $\sigma(t, S_t)$ measures the instantaneous variance of the process $\log S$. Hence $\sigma(t, S_t)$ can be interpreted as (local) measure of the risk incurred by investing one unit of the money market account into the stock. In case that σ is a constant independent of S_t the SDE (35) can be solved explicitely; the solution is given by the exponential martingale (13) from Proposition 1.13(b) with $\beta = 1$. In that case the stock price process is referred to as *classical Black-Scholes model* or as *geometric Brownian motion.* This model was first proposed by Samuelson (1965), who replaced Bachelier's arithmetic Brownian motion by geometric Brownian motion, the main argument in favour of this change being that real stock prices cannot be negative because of the limited liability of shareholders.

Fix some $T > 0$. To determine an equivalent martingale measure for the stock price process (35) we define

$$G_T := \exp\left(-\int_0^T [\mu(t,S_t)/\sigma(t,S_t)]dW_t - \frac{1}{2}\int_0^T [\mu(t,S_t)/\sigma(t,S_t)]^2 dt \right).$$

Under some integrability conditions we have $E(G_T) = 1$. In that case we may define a new probability measure Q on $\mathcal{F}_T$ by putting $dQ/dP := G_T$. According to Girsanov's theorem the process $W_t^Q := W_t + \int_0^t [\mu(s,S_s)/\sigma(s,S_s)]ds$ is a Brownian motion under Q, see e.g. Section 3.5 of Karatzas and Shreve (1988) . Hence S solves under Q the SDE $dS_t = \sigma(t,S_t)S_t dW_t^Q$ and is therefore a local Q-martingale and a martingale under some integrability assumptions.

As the volatility function $\sigma(t,x)$ is strictly positive, market completeness follows from the martingale representation theorem for Brownian motion, see e.g. Section 3.4 D of Karatzas and Shreve (1988). This theorem ensures that for any Q-integrable $\mathcal{F}_T$ measurable random variable H the martingale $H_t = E^Q(H \mid \mathcal{F}_t)$, $0 \le t \le T$, can be represented as a stochastic integral, i.e. there is a predictable process ψ^H such that $H_t = H_0 + \int_0^t \psi_s^H dW_s^Q$. If we now define $\xi_s^H := \psi_s^H/(\sigma(s,S_s)\,S_s)$ we immediately get $H = H_0 + \int_0^T \xi_s^H dS_s$.

Now there remains of course the task of computing price and hedging strategy. For the purposes of this paper it is enough to consider claims whose payoff has the form $H = g(S_T)$, so-called *terminal value claims*. For the pricing of path-dependent options in the framework of the classical Black-Scholes model see for instance Chapter 9 of Musiela and Rutkowski (1997) and the references given therein. For path-independent derivatives the price and the hedge portfolio can be computed by means of a parabolic partial differential equation. Denote by $h(t,x)$ the solution of the terminal value problem

$$\frac{\partial}{\partial t}h(t,x) + \frac{1}{2}\sigma^2(t,x)x^2\frac{\partial^2}{\partial x^2}h(t,x) = 0, \quad h(T,x) = g(x)\,. \tag{36}$$

By Itô's formula (see e.g. Karatzas and Shreve (1988)) we obtain from (36)

$$g(S_T) = h(T,S_T) = h(t,S_t) + \int_t^T \frac{\partial}{\partial x}h(s,S_s)dS_s\,.$$

Hence $\xi_t^H = \frac{\partial}{\partial x}h(t,S_t)$ and the fair price of the derivative is given by $H_t := h(t,S_t)$. In the classical Black-Scholes model with constant volatility σ the terminal value problem (36) can be solved explicitely for $g = (x-K)^+$. This yields the famous Black-Scholes formula for the price $C_{BS}(t,x,\sigma)$ of a European call option.

$$C_{BS}(t,x) = x\mathcal{N}(d_t^1) - K\mathcal{N}(d_t^2)\,,$$

where

$$d_t^1 = \frac{\ln(x/K) + (T-t)\sigma^2/2}{\sqrt{(T-t)\sigma^2}}\,,\ d_t^2 = d_t^1 - \sqrt{(T-t)\sigma^2}\,,$$

and where $\mathcal{N}$ denotes the distribution function of the one-dimensional standard normal distribution. Alternatively one could derive the Black-Scholes formula using probabilistic methods to compute the conditional expectation in (34). For an application of this approach in a more general setting see for instance Musiela and Rutkowski (1997) or Frey and Sommer (1996).

Of course up to now we have only been able to present the very basics of modern derivative pricing theory and had to omit many interesting topics. In particular we have to refer Björk

(1997) or Musiela and Rutkowski (1997) for a treatment of models for interest rate derivatives and to Myeni (1992) for a discussion of American-type derivatives. Excellent textbooks on derivative pricing theory with a focus on continuous-time modeling include Duffie (1992), Lamberton and Lapeyre (1996), Musiela and Rutkowski (1997) and the advanced Karatzas (1997) or Karatzas and Shreve (1998). Moreover, we strongly recommend the excellent essays in Runggaldier (1997). Taleb (1996) finally gives a trader's account of dynamic hedging.

3.1.5 Discussion

Over the last 20 years this approach to pricing and hedging derivative securities turned out very successful from a theoretical and from an applied point of view. One should bear in mind however, that this elegant theory hinges on several crucial assumptions. Obviously, if our hedging argument is to work for all claims the market must be complete. Moreover, in our definition of the gains from trade we implicitely assumed that there are no market frictions like taxes and transaction costs or constraints on the stockholdings ξ. The definition of the gains from trade is reasonable only if our hedger is small relative to the size of the market, meaning that the implementation of his hedging strategy does not affect the price process of the stock.

This is of course a very stylized picture of real markets, which is why much of the recent research in Finance has concentrated on relaxing these assumptions. The hedging of derivatives under market frictions has mainly been studied in the framework of the classical Black-Scholes model. Cvitanic (1997) gives an excellent and detailed introduction to the theory of hedging under portfolio constraints. Davis, Panas, and Zariphopoulou (1993), Barles and Soner (1998) or Cvitanic, Pham, and Touzi (1999) are representative examples of recent work on option pricing with transaction costs. The pricing and hedging of options in markets with a large trader is for instance studied by Jarrow (1994) or Frey and Stremme (1997) and Frey (1998). Many of these papers employ techniques from stochastic control theory and from the theory of nonlinear PDE's. In particular the pricing PDE (36) is often replaced by a nonlinear PDE where the volatility depends on the derivatives of the option price, see e.g. Barles and Soner (1998) or Avellaneda, Levy and Paras (1995) .

Typically we enter the realm of incomplete markets whenever we want to use models for asset price dynamics which are more "realistic" than the simple model (35). For instance the simple model from Section 3.1.2 is incomplete if we allow for a third possible value for the stock price at the terminal time T. Perhaps more importantly, markets are incomplete if we consider asset price processes with random volatility or with jumps of varying size. There is in fact a lot of statistical support for such models, as most empirical evidence suggests that the classical Black-Scholes model does not describe the statistical properties of financial time series very well. According to this model log-returns, i.e. differences of the form $\log S_{t+h} - \log S_t$, are independent and identically normally distributed. The following Figure 6 shows daily log-returns of the American S&P 500 stock index and simulated iid normal variates with variance equal to the sample variance of the S&P 500 log-returns.

This picture makes two stylized facts immediately apparent, which are typical for most financial time series.

- We see that large asset prize movements occur more frequently than in a model with normally distributed increments. This feature is often referred to as *excess curtosis* or *fat tails*; it is the main reason for considering asset price processes with jumps.
- There is evidence for *volatility clusters*, i.e. there seems to be a succession of periods

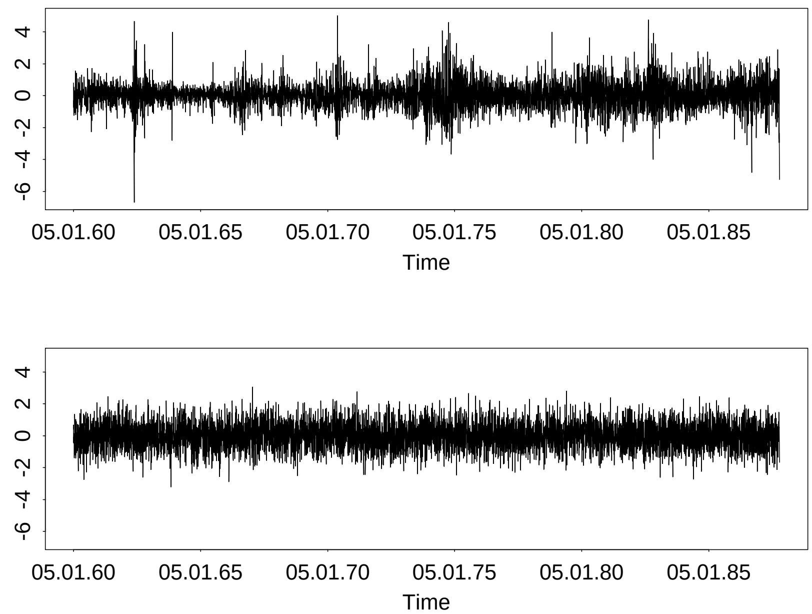

Figure 6: Daily log-returns of the S&P 500 index (top picture) and simulated normal variates with mean and variance equal to the sample mean and sample variance of the S&P 500 log-returns.

with high return variance and with low return variance. This observation motivates the introduction of diffusion models for asset prices where volatility is itself stochastic.

Of course these findings have been confirmed by many rigorous statistical tests, see e.g. Pagan (1996) for an extensive survey. In the remainder of this paper we will discuss some recent work on derivative asset analysis in models with jumps and/or stochastic volatility; this will allow us also to make contact with some approaches to derivative pricing in incomplete markets.

3.2 Some new models for asset prices

3.2.1 Stochastic volatility models

Most of the diffusion models that have been proposed in recent years as an extension to the classical Black-Scholes model belong to the class of stochastic volatility models (SV-models). In this class of models the volatility is modeled as a stochastic process whose innovations are only imperfectly correlated to the asset price process. Our definition of a SV-model is as follows.

Assumption 3.1 *S follows a general stochastic volatility model, if it solves the SDE*

$$dS_t = S_t(\sigma_t dW_t + \mu_t dt) \tag{37}$$

for predictable processes σ_t and μ_t. We assume that $\sigma_t > 0$, $\int_0^t \sigma_s^2 ds < \infty$ and that σ_t is not adapted to the filtration generated by W.

In economic terms this last assumption simply means that besides W there is a second source of randomness, influencing the system. In most papers from the financial literature it is assumed that the instantaneous variance $v_t = \sigma_t^2$ follows a one-dimensional diffusion:

Assumption 3.2 *S and v satisfy the SDE*

$$dS_t = S_t(v_t^{1/2} dW_t^{(1)} + \mu(v_t)dt)\,, \tag{38}$$

$$dv_t = a(v_t)dt + \eta_1(v_t)dW_t^{(1)} + \eta_2(v_t)dW_t^{(2)}\,, \tag{39}$$

for $W_t = (W_t^{(1)}, W_t^{(2)})$ a standard twodimensional Wiener process. We assume that the coefficients are such that the vector SDE (38), (39) has a non-exploding and strictly positive solution. Moreover, there is some $0 \le a < b \le \infty$ such that $\eta_2(v) > 0$ for all $v \in (a, b)$.

The above class of volatility models contains among others the SV-models considered by Wiggins (1987) , Hull and White (1987) or Heston (1993) as special cases. The function η_1 models the instantaneous correlation of $\log S$ and v. Most empirical studies have found that at least on equity markets η_1 is significantly negative, an observation which is termed the *leverage effect* since Black (1976) .

SV-models can be obtained as diffusion limits of certain popular GARCH-models. This has potentially important implications for parameter estimation and for derivative asset analysis in these models. For a detailed analysis of "ARCH-models as diffusion approximation" and related topics see Nelson (1990), Duan (1997) or the surveys Frey (1997) and Ghysels, Harvey, and Renault (1996). Duffie and Protter (1992) give an in-depth discussion of results on weak convergence of asset price processes and implications in Finance.

SV models are typically incomplete, meaning that there are derivatives which cannot be replicated by dynamic hedging. As explained in Section 3.1 this is equivalent to the fact that there are now many probability measures $Q \sim P$ such that the stock price process is a (local) Q-martingale. The next proposition characterizes the set of all equivalent local martingale measures for the stock price process defined in Assumption 3.2. For similar results and a proof see e.g. Hofmann, Platen and Schweizer (1992) and the references given therein.

Proposition 3.3 *a) Under Assumption 3.2 a probability measure Q equivalent to P on $\mathcal{F}_T$ is a local martingale measure for S on $\mathcal{F}_T$ if and only if there is a progressively measurable process $\nu = (\nu_t)_{0 \le t \le T}$ with $\int_0^T \nu_s^2 ds < \infty$ P- a.s. such that the following holds: The local martingale $(G_t)_{0 \le t \le T}$ with*

$$G_t := \exp\left(\int_0^t (-\mu(v_s)/\sqrt{v_s})dW_s^{(1)} + \int_0^t \nu_s dW_s^{(2)} - \frac{1}{2}\int_0^t (\mu(v_s)/\sqrt{(v_s)})^2 + \nu_s^2\, ds\right) \tag{40}$$

satisfies $E(G_T) = 1$ and $G_T = dQ/dP$ on $\mathcal{F}_T$.

b) Suppose that Q is an equivalent local martingale measure corresponding to some process ν. Then S and v solve the following SDE under Q

$$dS_t = S_t|v_t|^{1/2} d\widetilde{W}_t^{(1)}\,, \tag{41}$$

$$dv_t = a(v_t) - \eta_1(v_t)\mu(v_t)/\sqrt{v_t} + \eta_2(v_t)\nu_t dt + \eta_1(v_t)d\widetilde{W}_t^{(1)} + \eta_2(v_t)d\widetilde{W}_t^{(2)}\,, \tag{42}$$

where $\widetilde{W}$ is a two-dimensional standard Brownian motion under Q. □

In the financial literature the process ν is usually referred to as *market price of volatility risk process.* Proposition 3.3 shows that there is a one to one correspondance between market price of volatility risk processes ν satisfying some regularity conditions and equivalent (local) martingale measures. In particular market incompleteness is equivalent to nonuniqueness of the market price of risk process.

3.2.2 Models with discontinuous price paths

Real markets exhibit from time to time very large price movements over short time periods. Even if we allow for stochastic volatility these price movements are only very difficult to reconcile with the assumption that asset prices follow diffusion models with continuous trajectories. Moreover, in an interesting empirical study Bakshi, Cao and Chen (1997) have shown that in order to explain observed option prices one should allow for both, stochastic volatility and the possibility of occasional jumps.

A rather general jump-diffusion model has been proposed by Colwell and Elliot (1993). They assume the following dynamics for the stock price S:

$$dS_t = \mu(t, S_{t-})dt + \sigma(t, S_{t-})dW_t + \int_{\mathbb{R}} \gamma(t, S_{t-}, y)(\mu(dt, dy) - H(dy)dt)\,. \tag{43}$$

Here W is standard Brownian motion and μ is a random measure with deterministic compensator $\nu = H(dy)dt$, which is assumed to be independent of W. We can alternatively write model (43) as follows.

$$S_t = S_0 + \int_0^t \mu(s, S_{s-})ds + \int_0^t \sigma(s, S_{s-})dW_s \tag{44}$$

$$+ \sum_{i=1}^{N_t} \gamma(\tau_i, S_{\tau_i -}, Y_i) - \int_0^t \int_{\mathbb{R}} \gamma(s, S_{s-}, y)\widetilde{H}(dy)ds\,. \tag{45}$$

As the compensator ν is deterministic, $Z_t = \sum_{i=1}^{N_t} Y_i$ is a compound Poisson process with intensity $\lambda = \int_{\mathbb{R}} 1H(dy)$. The stopping times τ_i denote the successive jump-times of N. The distribution of the Y_i is given by $\widetilde{H} = \lambda^{-1}dH$. This notation makes the similarities to the models studied in Chapter 2 apparent. It follows from general results on SDE's driven by random measures that S is a Markov process.

Most jump-diffusion models from the financial literature are special cases of (44). If we take $\mu(t,x) = \mu(t)x$, $\sigma(t,x) = \sigma(t)x$ and $\gamma(t,x,y) = \gamma(t)x$ for deterministic functions μ, σ and γ with $\gamma(t) > -1$ for all t we obtain the models of Merton (1976) or Mercurio and Runggaldier (1993). Bakshi, Cao and Cheng (1997) consider a model where $\gamma(t,x,y) = xy$, and where $1+Y$ is lognormally distributed; they allow moreover for stochastic volatility. In all these models the assumption that $\gamma(t,x,y) > -1$ a.s. is made to ensure that the asset price process is strictly positive.

Jump-diffusion models of the form (43) are typically incomplete as there are many different equivalent martingale measures. Intuitively speaking this is due to the fact that by an equivalent change of measure we may change the drift, the jump-size distribution and the jump-intensity of the process; there are typically many different combinations of these parameters and hence many different equivalent probability measures that turn S into a (local) martingale. Colwell and Elliot (1993) determine the class of equivalent martingale measures for the model (43) that preserve the Markov property.

Eberlein and Keller (1995) introduce another class of discontinuous stochastic processes for asset prices. Their analysis is motivated by statistical considerations which show that the hyperbolic distribution (see e.g. Barndorff-Nielsen and Halgreen (1977)) yields an excellent fit to the distribution of log-returns for various stocks. The hyperbolic distribution is infinitely divisible and generates therefore a Lévy process, the so-called hyperbolic Lévy motion. The Lévy-Khinthine representation of this process shows that the hyperbolic Lévy motion is a quadratic pure jump process, i.e. orthogonal (in the sense of quadratic variation) to all continuous semimartingales. We refer the interested reader to Eberlein and Keller (1995) for further information.

3.3 Pricing and hedging of derivatives in incomplete markets

As we have just seen, if we move on from the classical Black-Scholes models to more realistic models with jumps and stochastic volatility we usually end up with an incomplete market where perfect hedging strategies for derivatives do not exist. Hence a conceptual problem arises: how should we value contingent claims, and how should we manage the risk we incur by selling the claim? Of course, there is now no longer a unique answer to these questions. However, in recent years a number of interesting concepts for the risk-managements of derivatives in incomplete markets have been developed, and we are now going to survey two such approaches.

3.3.1 Superreplication

If the precise duplication of a contingent claim is not feasible one might try to find a *superreplicating strategy*, i.e. the "cheapest" selffinancing strategy with terminal value no smaller than the payoff of the contingent claim. This concept has been developed first by El Karoui and Quenez (1995). To explain their results we have to give some definitions first.

Definition 3.4 *Consider a contingent claim H with nonnegative payoff. An adapted, nonnegative càdlàg process $\widetilde{H}$ with $\widetilde{H}_T = H$ is called an admissible price for sellers, if $\widetilde{H}$ is the value process of some trading strategy with nonincreasing cost process C. An admissible price process for sellers H^* will be called the ask price for H, if $H_t^* \leq \widetilde{H}_t$ for any other admissible price for sellers H and for all $t \in [0,T]$.*

This definition deserves a comment. Suppose that an investor sells at time $t < T$ the claim H at an admissible selling price $\widetilde{H}_t$. By following the corresponding portfolio strategy he can then completely eliminate the risk incurred by selling the claim and moreover he earns the nonnegative amount $-(C_T - C_t)$. Hence he will certainly agree to sell the claim for the price $\widetilde{H}_t$. The following is an example for an admissible price process for sellers in the case of a European call option. Define

$$\widetilde{H}_t = S_t\,,\ \xi_t = 1 \ \text{ for } 0 \leq t < T \ \text{ and } \widetilde{H}_T = (S_T - K)^+\,,\ \xi_T = 0\,. \tag{46}$$

The cost process is then given by $C_t = 0$ for $t < T$ and $C_T = (S_T - K)^+ - S_T$.

It is a priori not clear that an ask-price for a contingent claim exists. Here we have the following result, which was proved in increasing generality by Delbaen (1992), El Karoui and Quenez (1995), Kramkov (1996), and Föllmer and Kabanov (1998).

Theorem 3.5 *Assume that the set $\mathcal{Q}$ of equivalent local martingale measures for the asset price process S is nonempty. Then the ask price exists for every contingent claim H with* *nonnegative payoff; it is given by*

$$H_t^* = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}} E^Q(H|\mathcal{F}_t)\,. \tag{47}$$

□

It is easily seen that the ask price cannot be smaller than H^*. In fact, we have for every admissible price process for sellers $\widetilde{H}$

$$H = \widetilde{H}_T = \widetilde{H}_t + \int_t^T \xi_s dS_s - (C_T - C_t) \leq \widetilde{H}_t + \int_t^T \xi_s dS_s\,. \tag{48}$$

Fix some $Q \in \mathcal{Q}$. The stochastic integral $\int_0^t \xi_s dS_s$ is a nonnegative local martingale and hence a supermartingale. Taking expectations on both sides of (48) we get

$$E^Q(H|\mathcal{F}_t) \leq \widetilde{H}_t + E^Q\left(\int_t^T \xi_s dS_s|\mathcal{F}_t\right) \leq \widetilde{H}_t\,.$$

Hence we must have $\widetilde{H}_t \geq \operatorname{ess\,sup}\{E^Q(\bar{H}|\mathcal{F}_t)\,,\ Q \in \mathcal{Q}\}$. The difficult part in the proof of Theorem 3.5 is to show that the process H^* can be represented as the sum of a stochastic integral w.r.t. S and an adapted nonincreasing process.

At a first glance superreplication seems to be a very attractive concept for the pricing and the hedging of derivatives in incomplete markets. Unfortunately, in applications it often leads to results which are not very satisfactory. Consider for instance the SV-model which was introduced in Assumption 3.2, and assume that — as in most models from the financial literature — $\eta_2(v) > 0$ for all $v > 0$. By well-known results on one-dimensional diffusions this implies that the range of v_t is unbounded. For this class of models Frey and Sin (1999) have shown that under some minor technical conditions we have

$$\operatorname*{ess\,sup}_{Q \in \mathcal{Q}} E^Q((S_T - K)^+|\mathcal{F}_t) = S_t \text{ for all } t < T, K > 0\,;$$

see also Cvitanic, Pham, and Touzi (1997) for related results. In light of Theorem 3.5 we can therefore conclude that the ask price process and the corresponding hedge portfolio are given by (46); in other words the cheapest superreplicating strategy for a call option is to buy the stock. Similar results have been obtained for the other new model classes introduced in Section 3.2; see Bellamy and Jeanblanc (1997) for an analysis of superhedging in jump-diffusion models and Eberlein and Jacod (1997) for results in the context of discontinuous Lévy processes.

In spite of these disappointing results there are good financial reasons to study superhedging strategies. For instance, these strategies appear as building blocks in the *quantile hedging* approach of Föllmer and Leukert (1998). These authors relax the condition that the terminal value of the hedging strategy should almost surely be no smaller than the payoff of the claim under consideration; instead they focus on the cheapest hedging strategy with nonnegative value process which superreplicates the claim with a given success probability. We refer the reader to their paper for further details.

There are other situations where the superhedging approach yields very interesting and relevant results. Several authors have applied the concept of superhedging to the problem of hedging a derivative in the Black-Scholes model but with certain constraints on the hedging portfolio, see for instance Cvitanic (1997). Many papers address the problem of superhedging in stochastic volatility models with known a-priori bounds on the volatility. These bounds are usually interpreted as confidence interval for the range of future volatility. In this situation the ask-price of a call-option is given by the Black-Scholes price of the option corresponding to the upper volatility bound. For details on this work see the papers by El Karoui, Jeanblanc and Shreve (1998), Avellaneda, Levy and Paras (1995), Lyons (1995) or Frey (1998).

3.3.2 Mean-variance hedging

In the theory of *mean-variance hedging* which subsumes the so-called (local) *risk-minimization* and *variance-minimization* approaches one wants to find a trading strategy that reduces the actual risk of a derivative position to some "intrinsic component." While the computation of the strategy usually involves the computation of "prices" for contingent claims, the emphasis of this theory is not on the valuation of derivatives but on the reduction of risk.

We now explain these approaches in more detail. We restrict ourselves to trading strategies with square-integrable cost- and value processes. In the theory of (local) risk-minimization the conditional variance of C under the "real-world" probability measure P is used as a measure for the risk of a strategy. For a given claim H one tries to determine a strategy (ξ^R, η^R) with terminal value equal to H that minimizes at each time t the remaining risk

$$R_t := E^P((C_T - C_t)^2 | \mathcal{F}_t). \tag{49}$$

Here the minimization is over all admissible continuations of (ξ^*, η^*) after t with terminal value equal to H. Föllmer and Sondermann (1986) have studied existence and uniqueness of such a strategy if the stock price process is a P-martingale. In that case existence and uniqueness of such a strategy follows from the well-known *Kunita-Watanabe decomposition* of the P-martingale $H_t = E^P(H|\mathcal{F}_t)$ with respect to the P-martingale S. This decomposition result implies that the martingale H_t can be decomposed as

$$H_t = H_0 + \int_0^t \xi_s^H dS_s + L_t^H\,, \tag{50}$$

where L^H is a martingale orthogonal to S, i.e. the product SL^H is again a martingale. A proof of this result can be found in all major textbooks on stochastic analysis. The risk-minimizing strategy (ξ^R, η^R) is then given by

$$\xi^R := \xi^H\,,\ \eta^R := H_t - \xi_t^R S_t\,,\ \text{and hence } C_t = L_t^H\,.$$

Note that the risk-minimizing strategy is no longer selffinancing as the cost process does not necessarily vanish; however, the strategy is *mean selffinancing*, i.e. the cost process is a P-martingale with $E(C_T) = 0$.

In the variance-minimization approach one seeks to determine a *selffinancing* strategy (ξ^V, η^V) which minimizes the L^2-norm of the hedging error, i.e. the expression

$$E\left(H - (V_0 + \int_0^T \xi_s^V dS_S)\right)^2 .$$

If S is a P-martingale a unique solution solution exists; it can again be described in terms of the Kunita-Watanabe decomposition (50). We now put $\xi^V := \xi^H$, $V_0 := H_0$ and $\eta^V := H_0 + \int_0^t \xi_s^V dS_S - \xi_t S_t$, which is typically not equal to η^R.

Let us now turn to the general situation where S is only a semimartingale under P. Here the risk-minimization approach and the variance-minimization approach lead to different solutions also for the stockholdings ξ of the optimal strategy.

As shown by Schweizer (1991) for semimartingales a globally risk-minimizing strategy does not always exist. He therefore introduces a criterion of local risk-minimization. Roughly speaking a strategy (ξ^R, η^R) is locally risk-minimizing if it minimizes the remaining risk over all strategies that "deviate" from (ξ^R, η^R) only over a sufficiently short time period. Schweizer (1991) shows that under some technical conditions a strategy is locally risk-minimizing if and only if the associated cost process is a martingale orthogonal to the martingale part of S. To compute such a strategy we have to find a decomposition of our claim H of the following form

$$H = H_0 + \int_0^T \xi_s^H dS_s + L_T^H \,, \tag{51}$$

where L^H is a P-martingale orthogonal to the martingale part of S under P. The local risk-minimizing strategy is then defined via $\xi^R := \xi^H$ and $C^R := L^H$. In particular the strategy is still mean-selffinancing. In case that S is a P-martingale the decomposition (51) reduces to the Kunita-Watanabe decomposition of the P-martingale H with respect to S. If S is only a semimartingale the decomposition (51) is usually referred to as *Föllmer-Schweizer decomposition.*

The main tool for the computation of the Föllmer-Schweizer decomposition is the *minimal martingale measure* Q^* introduced in Föllmer and Schweizer (1991). In particular, Föllmer and Schweizer show that for *continuous* asset price processes the decomposition (51) is uniquely determined. It exists under some integrability assumptions and is then given by the Kunita-Watanabe decomposition of the Q^* martingale $H_t = E^{Q^*}(H|\mathcal{F}_t)$ with respect to the Q^*-martingale S. Using this approach locally risk-minimizing strategies in various kinds of stochastic volatility models have been computed; see e.g. Föllmer and Schweizer (1991), Hofmann, Platen and Schweizer (1992), Di Masi, Kabanov and Runggaldier (1994) or Frey (1997). Colwell and Elliott (1993) apply the concept of local risk-minimization to the jump-diffusion model introduced in Section 3.2.2.

The key point in ensuring existence of a variance-minimizing hedging strategy is the closedness in $L^2(P)$ of the following set of random variables

$$G := \left\{ \int_0^T \xi_s dS_s \,, \quad \xi \text{ ``admissible''} \right\} .$$

If this set is closed a variance-minimizing strategy for a contingent claim H can — at least theoretically — be computed as orthogonal projection of H onto G. Unfortunately, the analysis of the closedness of G is rather technical and we refer the reader to Delbaen et.al. (1997) for details on this issue. For continuous processes some easier proofs and more concrete examples are given in Pham, Rheinländer and Schweizer (1998).

4 On the interplay between finance and insurance

Historically the fields of finance and insurance have developed separately, unified mainly by the common use of the theory of stochastic processes as principal tool of analysis. However, caused by developments in the financial sector such as the inreasing collaboration between insurance companies and banks (all-finance) or the emergence of finance–related insurance products, the interplay between finance and insurance has recently become a "hot topic," and we believe that a lot of important future research in finance and insurance will combine ideas from both fields. It seems therefore a good idea to conclude this survey with a brief discussion of some recent developments in this area. For a related discussion see also the paper Embrechts (1996).

4.1 Methodological differences

To prepare the ground for our discussion we now summarize the preceding chapters and point out the main differences between the classical actuarial and financial approaches to dealing with financial risk as presented in the preceding parts of the paper.

In modern derivative asset analysis one aims at "hedging away" financial risks by dynamic trading. Prices are determined by the funds needed to finance this hedge. Consequently, the distribution under the real world probability measure of some financial risk (e.g. the payoff of a derivative) is not used for pricing this risk; instead prices are computed using some "artificial" martingale measure whose existence is intimately related to the economic notion of no-arbitrage.

The standard actuarial approach to dealing with financial risks is fundamentally different. Insurance companies are ready to bear some of the financial risks (claims) of an insured in exchange for a premium that equals the expected value of the claim plus some risk premium or loading. This loading is computed via actuarial premium principles; see e.g. Goovaerts, De Vylder, and Haezendonck (1984) for a detailed discussion. While the insurance company might pass on a part of this risk to a reinsurer, it can typically not "hedge away" the risks in its portfolio by dynamic trading. Consequently, the computation of insurance premiums, ruin probabilities or necessary reserves is done using the real distribution of the claims; martingales enter the analysis only as an — albeit very important — technical tool.

The difference between the actuarial and the financial approach to financial risk management is also highlighted by the following quote from Jensen and Nielsen (1996).

> Theories and models dealing with price formation in financal markets are divided into (at least) two markedly different types. One type of models is attempting to explain levels of asset prices, risk premiums etc. in an *absolute* manner in terms of the so-called *fundamentals.* A crucial model of this type includes the well-known rational expectation model equating stock prices to the discounted value of expected future dividends. Another type of models has a more modest scope, namely to explain in a *relative manner* some asset prices in terms of other, given and observable prices.

It is clear from the preceding discussion that derivative pricing theory adheres to the latter approach, whereas actuarial models come closer to an absolute pricing theory.

A second difference between the standard models in the two fields concerns the class of stochastic processes used. Insurance risk-processes like the Cramér-Lundberg-model have discontinuous sample paths which are of finite variation; whereas most standard finance models use diffusion processes with continuous trajectories to describe asset price fluctuations. However, we have seen in Section 3.2.2 that certain "new" models for asset prices resemble closely actuarial risk processes.

In summary, from a methodological viewpoint the two fields seem to be relatively far apart. However, if we look at recent developments it is very likely that in the future the gap between both disciplines will become much smaller than it appears to be now.

4.2 Financial pricing of insurance

The fundamental papers on this topic are due to Sondermann (1991) and in particular to Delbaen and Haezendonck (1989b). We now explain the "martingale approach to premium calculation in an arbitrage-free-market" proposed in the latter paper.

Delbaen and Haezendonck start from the underlying risk process X_t that represents the total claim amount of a fixed portfolio of insurance contracts that has been paid out up to time t. X_t is modeled as a compound Poisson process as in Section 1.1, i.e. we have $X_t = \sum_{i=1}^{N_t} Y_i$ for iid random variables $(Y_i)_{i\in\mathbb{N}}$, and N is a standard Poisson process independent of the Y_i. Delbaen and Haezendonck assume that at every point in time t the insurance company can sell the remaining claim payments $X_T - X_t$ of this portfolio over the period $(t, T]$ for some premium p_t. Necessarily such a premium must be a predictable process. Hence the underlying price process S_t (the value of the portfolio of claims at time t) has the form

$$S_t = p_t + X_t\,.$$

Now comes the crucial point that marks the departure from usual insurance pricing principles. Delbaen and Haezendonck argue that

> The possibility of buying and selling at time t represents the possibility of "take–over" of this policy. This liquidity of the market should imply that there are no arbitrage opportunities and hence by the Harrison-Kreps theory (Harrison and Kreps (1979)) there should be a risk neutral probability distribution Q such that $\{S_t : 0 \le t \le T\}$ is a Q–martingale.

The next step in the pricing of insurance contracts by no–arbitrage arguments is the selection of an appropriate measure Q. Delbaen and Haezendonck are interested in all those measures Q that lead to linear premiums of the form $p_t = p^Q(T-t)$ for the underlying risk-process X itself and for all excess–of–loss reinsurance contracts with payoff

$$C_K = \sum_{i=1}^{N_T} (Y_i - K)^+\ .$$

The number p^Q — which depends of course on the particular excess–of–loss contract under consideration — is then called a *premium density*. It can be shown that this implies that under Q the process S must again be a compound Poisson process, possibly with different loss-distribution μ^Q and loss–intensity λ^Q. A premium density p^Q then takes on the form

$$p^Q = E^Q(S_1) = E^Q(N_1)\,E^Q(Y) = \lambda^Q \int_0^\infty y\,\mu^Q(dy)\,.$$

Delbaen and Haezendonck show that we may obtain any claim-size distribution μ^Q which is equivalent to the original claim size distribution μ and every intensity $\lambda^Q > 0$ in this way. In particular, they show how certain well-known premium principles can be obtained by an appropriate choice of λ^Q and μ^Q. Here a word of warning is in order: while we may justify a particular premium principle for X by choosing Q appropriately (say $Q = Q^*$), our no–arbitrage pricing approach will not necessarily yield the same premium principle simultaneously for all insurance derivatives like our excess of loss contracts C_K: the expected value

$$E^{Q^*}\left(\sum_{i=1}^{N_1} (Y_i - K)^+\right) = \lambda^{Q^*} \int_0^\infty (y-K)^+ \mu^{Q^*}(dy)$$

need not correspond to the same premium principle. On the other hand there are typically several measures leading to the same premium density for X. A critical statement concerning actuarial premium principles is to be found in Venter (1991).

Delbaen and Haezendonck derive their results directly; see also Embrechts and Meister (1997). Alternatively one might use Girsanov-type theorems on equivalent change-of-measure for marked point processes as presented among others in Brémaud (1981).

4.3 Insurance derivatives

An area closely related to the pricing of insurance contracts by no-arbitrage arguments is the valuation of insurance derivatives. The payoff of such derivatives is (partially) linked to the losses of some predetermined insurance portfolio or to some standardized loss index. Examples include the PCS–options traded on the Chicago Board of Trade or certain so–called CAT–bonds (catastrophe bonds) issued by individual (re–)insurance companies. Insurance companies use these instruments in order to pass on some of their risk to the capital markets; for certain investors on the other hand these derivatives might be interesting tools to further diversify their investment risks. For more institutional details about these derivatives see e.g. Canter, Cole, and Sandor (1996).

A stylized mathematical description of an insurance derivative could be as follows. Let X be a risk-process of the form $X_t = \sum_{i=1}^{N_t} Y_i$ representing the underlying loss index. Then the payoff of a typical insurance derivative is given by some function $F(X_T)$; for instance we have in the case of a PCS-option

$$F(X_T) = (X_T - K_1)^+ - (X_T - K_2)^+ \quad \text{for some } 0 < K_1 < K_2 \,.$$

To explain the main problem arising in the pricing of such contracts let us assume as in Section 4.2 that X is a compound Poisson process, and that at every point in time t the remaining risk $X_T - X_t$ can be bought or sold for the price $p^*(T-t)$. Arbitrage pricing theory now only tells us that – after discounting – every viable price process for our derivative must be of the form

$$H_t = E^Q\left(F\left(X_T\right) \mid \mathcal{F}_t\right) \,,$$

where $Q \sim P$ and $E^Q\left(X_T \mid \mathcal{F}_t\right) = X_t + p^*(T-t)$ for all t. As soon as the claim sizes Y are variable — certainly the relevant case if we are talking about insurance against catastrophic events — there are many measures with this property, even if we stick to the assumption that X is compound Poisson under Q. In fact, under some technical conditions every new intensity $\lambda^Q > 0$ and every claim–size distribution μ^Q equivalent to the distribution μ of the Y_i under P would be in order, provided that

$$\lambda^Q \int_0^\infty y\, \mu^Q(dy) = p^* \,. \tag{52}$$

Equation (52) leaves plenty of choice as soon as the support of μ has at least two elements. Hence the pricing of insurance derivatives leads to a pricing problem in incomplete markets, and one might apply one of the concepts introduced in Section 3.3; we think that the risk–minimization approach is particularly well suited here.

We refer the reader to Embrechts and Meister (1997) for a detailed discussion of the methodological questions related to the pricing of insurance derivatives and for a more complete list of the relevant literature. Schmock (1998) contains an interesting discussion of some statistical issues arising in the area.

4.4 Actuarial methods in Finance

So far we have dealt mainly with the application of financial pricing techniques to insurance problems. However, actuarial concepts are also of increasing relevance for finance problems. We have seen that realistic models for asset price processes are typically incomplete. In addition, the results mentioned in Section 3.3.1 have shown that in many incomplete market models the concept of superhedging does not lead to satisfactory answers for the risk-management of derivatives. Consequently, interesting approaches to this problem must involve some sort of risk-sharing between buyer and seller; in particular the seller has to bear a part of the "remaining risk." Moreover, participants in derivative markets are faced with a large amount of credit risk, and it would be illusory to believe that all this risk can be hedged away. We refer the reader to the survey Lando (1997) for more information on financial models for credit risky securities.

Actuarial concepts for risk-management might prove helpful in dealing with these "unhedgeable" risks. To mention an example where such concepts are already applied, the RAC-(risk adjusted capital) approach in insurance has become popular among investment banks as a tool for the determination of risk capital and capital allocations. It is no coincidence that Swiss Bank Cooperation (now UBS) called its new credit risk management system ACRA which stands for Actuarial Credit Risk Accounting.

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References

Ammeter, H. (1948): "A Generalization of the Collective Theory of Risk in Regard to Fluctuating Basic Probabilities," *Skand. Aktuar Tidskr.*, pp. 171–198.

Andersen, E. (1957): "On the collective theory of risk in the case of contagion between the claims," in *Transactions XVth International Congress of Actuaries, II New York*, pp. 219–22.

Avellaneda, M., A. Levy, and A. Paras (1995): "Pricing and Hedging Derivative Securities in Markets with Uncertain Volatilities," *Applied Mathematical Finance*, 2, 73–88.

Bachelier, L. (1900): "Théorie de la spéculation," in *The Random Character of Stock Market Prices*, ed. by P. Cootner, pp. 17–78. MIT Press, Cambridge, Mass. (1964).

Bakshi, G., C. Cao, and Z. Chen (1997): "Empirical Performance of Alternative Option Pricing Models," *Journal of Finance*, 52, 2003–2049.

Barles, G., and M. Soner (1998): "Option Pricing with Transaction Costs and a Nonlinear Black-Scholes Equation," *Finance and Stochastics*, 2, 369–397.

Barndorff-Nielsen, O., and O. Halgreen (1977): "Infinite Divisibility of the Hyperbolic and Generalized Inverse Gaussian Distribution," *Zeitschrift für Wahrscheinlichkeitstheorie*, 38, 309–312.

BELLAMY, N., and M. JEANBLANC (1997): "Incompleteness of Markets Driven by a Mixed Diffusion," Preprint, Université d'Evry, Paris, France.

BERTOIN, J. (1996): *Lévy Processes.* Cambridge University Press, Cambridge.

BJÖRK, T. (1997): "Interest Rate Theory," in *Financial Mathematics, Springer Lecture Notes in Mathematics 1656*, pp. 53–122.

BJÖRK, T., and J. GRANDELL (1988): "Exponential inequalities for ruin probabilities in the Cox case," *Scandinavian Actuarial Journal*, pp. 77–111.

BLACK, F. (1976): "Studies in Stock Price Volatility Change," in *Proceedings of the 1976 Business Meeting of the Business and Economics Statistics Selection, American Statistical Association*, pp. 177–181.

BLACK, F., and M. SCHOLES (1973): "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81(3), 637–654.

BORODIN, A., and P. SALMINEN (1996): *Handbook of Brownian Motion – Facts and Formulae.* Birkhäuser, Basel.

BRÉMAUD, P. (1981): *Point Processes and Queues: Martingale Dynamics.* Springer, New York.

BÜHLMANN, H. (1970): *Mathematical Methods in Risk Theory.* Springer, Berlin.

CANTER, M., J. COLE, and R. SANDOR (1996): "Insurance Derivatives: A New Asset Class for the Capital Markets and a New Hedging Tool for the Insurance Industry," *Journal of Derivatives*, Winter 1996, 89–104.

CLARK, P. (1973): "A subordinated stochastic process model with finite variance for speculative prices," *Econometrica*, 41, 135–156.

COLWELL, D., and R. ELLIOTT (1993): "Discontinuous Asset Prices and Non-attainable Contingent Claims," *Mathematical Finance*, 3, 295–308.

COX, J., S. ROSS, and M. RUBINSTEIN (1979): "Option Pricing: A Simplified Approach," *Journal of Financial Economics*, 7, 229–263.

CVITANIC, J. (1997): "Optimal Trading under Constraints," in *Financial Mathematics, Springer Lecture Notes in Mathematics 1656*, pp. 123–190.

CVITANIC, J., H. PHAM, and N. TOUZI (1997): "Superreplication in Stochastic Volatility Models Under Portfolio Constraints," preprint, CREST, Paris, forthcoming in Journal of Applied Probability 36 (1999).

CVITANIC, J., H. PHAM, and N. TOUZI (1999): "A Closed-Form Solution to the Problem of Superreplication under Transaction Costs," *Finance and Stochastics*, 3, 35–54.

DALANG, R., A. MORTON, and W. WILLINGER (1990): "Equivalent Martingale Measures and No-arbitrage in Stochastic Security Market Models," *Stochastics Stochastics Reports*, 29, 185–201.

DASSIOS, A., and P. EMBRECHTS (1989): "Martingales and insurance risk," *Communications in Statistical and Stochastic Models*, 5, 181–217.

DAVIS, M. (1984): "Piecewise-deterministic Markov Processes: A General Class of Non-diffusion Stochastic Models," *Journal of the Royal Statistical Society B*, 46, 353–388.

DAVIS, M. (1993): *Markov Models and Optimization*. Chapman & Hall, London.

DAVIS, M., V. PANAS, and T. ZARIPHOPOULOU (1993): "European Option Pricing with Transaction Costs," *SIAM Journal of Optimal Control*, 31, 470–493.

DELBAEN, F. (1992): "Representing Martingale Measures when Asset Prices are Continuous and Bounded," *Mathematical Finance*, 2, 107–130.

DELBAEN, F., and J. HAEZENDONCK (1989a): "Classical Risk Theorie in an Economic Environment," *Insurance: Mathematics and Economics*, 6, 85–116.

DELBAEN, F., and J. HAEZENDONCK (1989b): "A Martingale Approach to Premium Calculation Principles in an Arbitrage-free Market," *Insurance: Mathematics and Economics*, 8, 269–277.

DELBAEN, F., P. MONAT, W. SCHACHERMAYER, M. SCHWEIZER, and C. STRICKER (1997): "Weighted Norm Inequalities and Hedging in Incomplete Markets," *Finance and Stochastics*, 1, 181–228.

DELBAEN, F., and W. SCHACHERMAYER (1994): "A General Version of the Fundamental Theorem of Asset Pricing," *Mathematische Annalen*, 300, 463–520.

DI MASI, G., Y. KABANOV, and W. RUNGGALDIER (1994): "Mean-variance Hedging of Options on Stocks with Stochastic Volatilities," *Theory of Probability and Applications*, 39, 172–182.

DOOB, J. (1953): *Stochastic Processes*. Wiley, New York.

DUAN, J. (1997): "Augmented GARCH (p, q) Process and its Diffusion Limit," *Journal of Econometrics*, 79, 97–127.

DUFFIE, D. (1992): *Dynamic Asset Pricing Theory*. Princeton University Press, Princeton, New Jersey.

DUFFIE, D., and P. PROTTER (1992): "From Discrete to Continuous Time Finance: Weak Convergence of the Financial Gain Process," *Mathematical Finance*, 2, 1–15.

EBERLEIN, E., and J. JACOD (1997): "On the Range of Option Prices," *Finance and Stochastics*, 1(2), 131–140.

EBERLEIN, E., and U. KELLER (1995): "Hyperbolic Distributions in Finance," *Bernoulli*, 1, 281–299.

EL KAROUI, N., M. JEANBLANC-PICQUÉ, and S. SHREVE (1998): "Robustness of the Black and Scholes Formula," *Mathematical Finance*, 8, 93–126.

EL KAROUI, N., and M.-C. QUENEZ (1995): "Dynamic Programming and Pricing of Contingent Claims in an Incomplete Market," *SIAM Journal on Control and Optimization*, 33(1), 27–66.

EMBRECHTS, P. (1983): "A property of the generalized inverse Gaussian distribution with some applications," *Journal of Applied Probability*, 20, 537–544.

EMBRECHTS, P. (1996): "Actuarial versus Financial Pricing of Insurance," Working Paper no 96-17, Wharton Financial Institutions Center.

EMBRECHTS, P., C. GOLDIE, and N. VERAVERBEKE (1979): "Subexponentiality and infinite divisibility.," *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 49, 335–347.

EMBRECHTS, P., J. GRANDELL, and H. SCHMIDLI (1993): "Finite–time Lundberg inequalities in the Cox case," *Scandinavian Actuarial Journal*, pp. 17–41.

EMBRECHTS, P., C. KLÜPPELBERG, and T. MIKOSCH (1997): *Modelling Extremal Events for Insurance and Finance*. Springer, Berlin.

EMBRECHTS, P., and S. MEISTER (1997): "Pricing insurance derivatives, the case of CAT–futures," in *Proceedings of the 1995 Bowles Symposium on Securitization of Insurance Risk, Gorgia State University Atlanta*, ed. by S. Cox, pp. 15–26. Society of Actuaries, Monograph M–FI97-1.

EMBRECHTS, P., and N. VERAVERBEKE (1992): "Estimates for the Probability of Ruin with Special Emphasis on the Possibility of Large Claims," *Insurance: Mathematics and Economics*, 1, 55–72.

FELLER, W. (1971): *An Introduction to Probability Theory and Its Applications: Volume II*. Wiley, New York.

FÖLLMER, H. (1991): "Probabilistic Aspects of Options," SFB 303 Discussion Paper B-202, University of Bonn.

FÖLLMER, H., and Y. KABANOV (1998): "Optional Decomposition and Lagrange Multipliers," *Finance and Stochastics*, 2, 69–81.

FÖLLMER, H., and P. LEUKERT (1998): "Quantile Hedging," preprint, Humboldt-Universität Berlin, to appear in Finance and Stochastics 3 (3) (1999).

FÖLLMER, H., and M. SCHWEIZER (1993): "A Microeconomic Approach to Diffusion Models for Stock Prices," *Mathematical Finance*, 3(1), 1–23.

FÖLLMER, H., and D. SONDERMANN (1986): "Hedging of non-redundant Contingent-Claims," in *Contributions to Mathematical Economics*, ed. by W. Hildenbrand, and A. Mas-Colell, pp. 147–160. North Holland.

FREY, R. (1997): "Derivative Asset Analysis in Models with Level-Dependent and Stochastic Volatility," *CWI Quarterly, Amsterdam*, 10, 1–34.

FREY, R. (1998a): "Perfect Option Replication for a Large Trader," *Finance and Stochastics*, 2, 115–148.

FREY, R. (1998b): "Superhedging in Stochastic Volatility Models and Optimal Stopping," Discussion paper, ETH Zürich.

FREY, R., and C. SIN (1999): "Bounds on European Option Prices under Stochastic Volatility," *Mathematical Finance*, 9, 97–116.

FREY, R., and D. SOMMER (1996): "A Systematic Approach to Pricing and Hedging of International Derivatives with Interest Rate Risk," *Applied Mathematical Finance*, 3, 295–317.

FREY, R., and A. STREMME (1997): "Market Volatility and Feedback Effects from Dynamic Hedging," *Mathematical Finance*, 7(4), 351–374.

FURRER, H. (1998): "Risk processes perturbed by α-stable Lévy motion," *Scandinavian Actuarial Journal, to appear.*

FURRER, H., Z. MICHNA, and A. WERON (1997): "Stable Lévy motion approximation in collective risk theory," *Insurance: Mathematics and Economics*, 20, 23–36.

FURRER, H., and H. SCHMIDLI (1994): "Exponential inequalities for ruin probabilities of risk processes perturbed by diffusion," *Insurance: Mathematics and Economics*, 15, 23–36.

GEMAN, H., and T. ANÉ (1996): "Stochastic Subordination," *RISK*, 9(9), 146–149.

GERBER, H. (1979): *An Introduction to Mathematical Risk Theory.* Huebner Foundation Monographs 8, distributed by Richard D. Irwin Inc., Homewood Illinois.

GHYSELS, E., A. HARVEY, and E. RENAULT (1996): "Stochastic Volatility," in *Handbook of Statistics*, ed. by G. Maddala, and C. Rao, vol. 14, Statistical Methods in Finance, pp. 119–191. North Holland.

GOOVAERTS, M., F. DE VYLDER, and J. HAEZENDONCK (1984): *Insurance Premiums.* North Holland, Amsterdam.

GRANDELL, J. (1976): *Doubly Stochastic Poisson Processes*, Lecture Notes in Mathematics 529. Springer, Berlin.

GRANDELL, J. (1991): *Aspects of Risk Theory.* Springer, Berlin.

GRANDELL, J. (1997): *Mixed Poisson Processes.* Chapman and Hall, London.

GUILLAUME, D., M. DACOROGNA, R. DAVÉ, U. MÜLLER, R. OLSEN, and P. PICTET (1997): "From the bird's eye to the microscope: A survey of new stylized facts of the intra-daily foreign exchange markets," *Finance and Stochastics*, 1, 95–129.

HARRISON, J. (1985): *Brownian Motion and Stochastic Flow Systems.* Wiley, New York.

HARRISON, J., and D. KREPS (1979): "Martingales and Arbitrage in Multiperiod Securities Markets," *Journal of Economic Theory*, 20, 381–408.

HARRISON, J., and S. PLISKA (1981): "Martingales and Stochastic Integrals in the Theory of Continuous Trading," *Stochastic Processes and Applications*, 11, 215–260.

HARRISON, J., and S. PLISKA (1983): "A Stochastic Calculus Model of Continuous Trading: Complete Markets," *Stochastic Processes and Applications*, 15, 313–316.

HESTON, S. (1993): "A Closed Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options," *Review of Financial Studies*, 6, 327 – 343.

HOFMANN, N., E. PLATEN, and M. SCHWEIZER (1992): "Option Pricing under Incompleteness and Stochastic Volatility," *Mathematical Finance*, 2, 153–187.

HULL, D., and A. WHITE (1987): "The Pricing of Options on Assets with Stochastic Volatilities," *Journal of Finance*, 42(2), 281–300.

IGLEHART, D. (1969): "Diffusion Approximations in Collective Risk Theory," *Journal of Applied Probability*, 6, 285–292.

JARROW, R. (1994): "Derivative Securities Markets, Market Manipulation and Option Pricing Theory," *Journal of Financial and Quantitative Analysis*, 29, 241–261.

JENSEN, B., and J. NIELSEN (1996): "Pricing by No–Arbitrage," in *Time Series Models in Econometrics, Finance and other Fields*, ed. by D. Cox, D. Hinkley, and O. Barndorff-Nielsen. Chapman and Hall, London.

KARATZAS, I. (1997): *Lectures in Mathematical Finance*. American Mathematical Society, Providence.

KARATZAS, I., and S. SHREVE (1988): *Brownian Motion and Stochastic Calculus*. Springer, Berlin.

KARATZAS, I., and S. SHREVE (1998): *Methods of Mathematical Finance*. Springer, Berlin.

KLUGMAN, S., H. PANJER, and G. WILLMOT (1988): *Loss Models*. Wiley, New York.

KOLLER, M. (1999): *Lebensversicherungsmathematik mit Markovketten und stochastischen Zinsmodellen*. Springer, Berlin, to appear.

KOPP, P. (1984): *Martingales and Stochastic Integrals*. Cambridge University Press, Cambridge.

KRAMKOV, D. (1996): "Optional Decomposition of Supermartingales and Hedging Contingent Claims in Incomplete Security Markets," *Probability Theory and Related Fields*, 105, 459–479.

LAMBERTON, D., and B. LAPEYRE (1996): *Introduction to Stochastic Calculus Applied to Finance*. Chapman and Hall, London.

LANDO, D. (1997): "Modelling bonds and derivatives with credit risk," in *Mathematics of Financial Derivatives*, ed. by M. Dempster, and S. Pliska, pp. 369–393. Cambridge University Press, Cambridge.

LUNDBERG, F. (1903): *Approximerad framställning av sannolikhetsfunktionen. Återförsäkring av kollektivrisker*. Akad. Afhandling. Almqvist och Wiksell, Uppsala.

LYONS, T. (1995): "Uncertain Volatility and the Risk-free Synthesis of Derivatives," *Applied Mathematical Finance*, 2, 117–133.

MERCURIO, F., and W. RUNGGALDIER (1993): "Option Pricing for Jump-Diffusions: Approximations and their Interpretation," *Mathematical Finance*, 3, 191–200.

MERTON, R. (1973): "The Theory of Rational Option Pricing," *Bell Journal of Economics and Management*, 7, 141–183.

MERTON, R. (1976): "Option Pricing when Underlying Stock Returns are Discontinuous," *Journal of Financial Economics*, pp. 125–144.

MILBRODT, H., and M. HELBIG (1999): *Grundzüge der Personenversicherungsmathematik.* Buchmanuskript, Universität Köln.

MUSIELA, M., and M. RUTKOWSKI (1997): *Martingale Methods in Financial Modelling*, Applications of Mathematics. Springer, Berlin.

MYENI, R. (1992): "The Pricing of the American Option," *The Annals of Applied Probability*, 2, 1–23.

NELSON, D. (1990): "ARCH Models as Diffusion Approximations," *Journal of Econometrics*, 45, 7–38.

NORBERG, R. (1991): "Reserves in life and pension insurance," *Scandinavian Actuarial Journal*, pp. 3–24.

NORBERG, R. (1995): "A time-continuous Markov chain interest model with applications to insurance," *Journal of Applied Stochastic Models and Data Analysis*, 11, 245–256.

NORBERG, R. (1997): "Ruin problems with assets and liabilities of diffusion type," *Stochastic Processes and Applications, to appear.*

PAGAN, A. (1996): "The Econometrics of Financial Markets," *Journal of Empirical Finance*, 3, 15–102.

PANJER, H., and G. WILLMOT (1992): *Insurance Risk Models.* Society of Actuaries, Schaumburg, Illinois.

PAULSEN, J. (1993): "Risk theory in a stochastic economic environment," *Stochastic Processes and their Applications*, 46, 327–361.

PHAM, H., T. RHEINLÄNDER, and M. SCHWEIZER (1998): "Mean-variance Hedging for Continuous Processes: New Proofs and Examples," *Finance and Stochastics*, 2, 173–198.

PROTTER, P. (1992): *Stochastic Integration and Differential Equations: A New Approach*, Applications of Mathematics. Springer, Berlin.

RESNICK, S. (1992): *Adventures in Stochastic Processes.* Birkhäuser, Boston.

REVUZ, D., and M. YOR (1994): *Continuous Martingales and Brownian Motion.* Springer, Berlin.

ROGERS, L., and D. WILLIAMS (1987): *Diffusions, Markov Processes, and Martingales Volume 2 Itô Calculus.* Wiley, New York.

ROGERS, L., and D. WILLIAMS (1994): *Diffusions, Markov Processes, and Martingales Volume 1 Foundations, 2nd edition.* Wiley, New York.

ROLSKI, T., H. SCHMIDLI, V. SCHMIDT, and J. TEUGELS (1999): *Stochastic Processes for Insurance and Finance.* Wiley, Chichester, to appear.

RUNGGALDIER, W. (ed.) (1997): *Financial Mathematics* Lecture Notes in Mathematics No 1656. C.I.M.E., Springer, Berlin.

Samuelson, P. (1965): "Rational Theory of Warrant Pricing," *Industrial Management Review*, 6, 13–31.

Schmock, U. (1998): "Estimating the value of the Wincat coupons of the Winterthur Insurance convertible bond: A study of the model risk," Discussion paper, ETH Zürich, forthcoming in ASTIN Bulletin.

Schweizer, M. (1991): "Option Hedging for Semimartingales," *Stochastic Processes and Applications*, 37, 339–363.

Seal, H. (1978): *Survival Probabilities. The Goal of Risk Theory.* Wiley, Chichester.

Sondermann, D. (1991): "Reinsurance in arbitrage free markets," *Insurance: Mathematics and Economics*, 10, 191–202.

Sørensen, M. (1996): "A Semimartingale Approach to some Problems in Risk Theory," *ASTIN Bulletin*, 26, 15–23.

Stone, C. (1963): "Weak convergence of stochastic processes defined on semi–infinite time intervals," *Proceedings of the American Mathematical Society*, 14, 694–700.

Taleb, N. (1996): *Dynamic Hedging.* Wiley, New York.

Thorin, O. (1975): "Stationarity Aspects of the Sparre Andersen Risk Process and the Corresponding Ruin Probabilities," *Scandinavian Actuarial Journal*, pp. 81–99.

Venter, G. (1991): "Premium calculation implications of reinsurance without arbitrage," *ASTIN Bulletin*, 21, 223–230.

Wiggins, J. B. (1987): "Option Valuation under Stochastic Volatility, Theory and Empirical Estimates," *Journal of Financial Economics*, 19, 351–372.

Williams, D. (1991): *Probability with Matingales.* Cambridge University Press, Cambridge.

Wolthuis, H. (1994): *Life Insurance Mathematics (The Markovian Model)*, CAIRE Education Series 2. CAIRE, Brussels.

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