Additivity properties for Value-at-Risk under Archimedean dependence and heavy-tailedness

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Abstract

Mainly due to new capital adequacy standards for banking and insurance, an increased interest exists in the aggregation properties of risk measures like Value-at-Risk (VaR). We show how VaR can change from sub- to superadditivity depending on the properties of the underlying model. Mainly, the switch from a finite to an infinite mean model gives a completely different asymptotic behaviour. Our main result proves a conjecture made in Barbe et al. [3].

Keywords: Value-at-Risk, subadditivity, dependence structure, Archimedean copula, aggregation.

1 Introduction

Based on the regulatory framework for banking and insurance supervision, the financial industry has to come up with regulatory capital for several risk categories (market, credit, insurance underwriting, claims reserving, operational, ...). An overview of the underlying issues and further references can be found in McNeil et al. [17]. The quantile based Value-at-Risk (VaR) is the main risk measure used in industry to quantify regulatory capital. For instance, market risk has to be measured at a 99%-VaR with a 10-day holding period, whereas for operational risk this is 99.9% and 1 year. Hence the regulatory capital for operational risk corresponds to a one-in-1000-year event. In several papers the strengths and weaknesses of VaR as a risk measure have been discussed; in Artzner et

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al. [2] and in various subsequent papers VaR is critizised for not being subadditive. See McNeil et al. [17] for more details. However, the subadditivity axiom for risk measures is also critizised in various papers, see for example Rootzén-Klüppelberg [26], Dhaene et al. [7], Daníelsson et al. [5] or Heyde et al. [12]. The latter paper also contains a critical discussion on robustness of risk measures. The authors replace for instance tail conditional expectation (which is a subadditive risk measure) by the more robust tail conditional median which is simply VaR at a higher confidence level.

Though the present paper can be read context free, the main motivation comes from the present discussions around the measurement of operational risk. Current insurance regulation gives no clear guidance for quantification of operational risk. Larger international banks complying with international banking regulation have to come up with a one-in-1000-year VaR estimate under the so-called loss distribution approach (LDA). There is an ongoing intensive discussion between practitioners, regulators and academics concerning the appropriate way to estimate such a 99.9% quantile based capital charge. Influential papers from the regulatory side concerning several quantitative impact studies are Moscadelli [19], de Fontnouvelle et al. [10] and Dutta-Perry [8]. The Moscadelli paper relies heavily on extreme value theory (EVT) based modelling and also uses the tail conditional median as a risk measure. The Dutta-Perry paper introduces the g-and-h distribution as a parametric model for operational risk severity data. These authors claim that EVT methods may occasionally lead to unrealistically high capital charges. This is mainly due to fitted infinite mean models; see Nešlehová et al. [22], Degen et al. [6] and Böcker-Klüppelberg [4] for a discussion in the context of operational risk. Daníelsson et al. [5] discuss this issue from the point of view of multivariate regular variation.

Our paper contributes to this discussion by looking at dependent data using copula methodology. Our main results are Theorems 2.3 and 2.5. Qualitatively they highlight the difficulties switching from a finite to an infinite mean model. Theorem 2.3 solves an open problem posed in Barbe et al. [3].

Organisation. In Section 2 we define the model, state the main convergence results (see Proposition 2.2 and Corollary 2.4), and provide a regime switch picture in Theorem 2.5. These statements are based on our main theorem, Theorem 2.3, which describes the basic behaviour of the limiting constants in Proposition 2.2. In Section 3 we present general properties of asymptotic VaR behaviour. Finally, the proofs of the main results are given in Section 4.

2 A multivariate model and main results

Consider loss random variables $X_1, \ldots, X_n$, $n \geq 2$, defined on some probability space $(\Omega, \mathcal{F}, P)$. Our main focus is on the calculation of $\mathrm{VaR}_p(X_1 + \cdots + X_n)$, where p is a given confidence level typically close to 1. The quantile risk measure $\mathrm{VaR}_p(X)$ for a random variable X and $p \in (0,1)$ is defined as:

$$\mathrm{VaR}_p(X) = \inf\{t \in \mathbb{R} : \; P(X > t) \leq 1 - p\}. \tag{2.1}$$

We do not assume independence between the loss random variables. For modelling dependence in loss data, the theory of Archimedean copulas is particularly useful; see McNeil et al. [17] for details, definitions and further references.

We assume that $\psi : [0,1] \to [0, \infty]$ is the generator of the Archimedean copula

$$C^{\psi}(x_1, \ldots, x_n) = \psi^{-1}\left(\sum_{i=1}^{n} \psi(x_i)\right). \tag{2.2}$$

Note that in order to generate a true copula, ψ has to satisfy certain growth and regulatory conditions, in particular, ψ needs to be completely monotone to generate a copula for any dimension $n \geq 2$; see Kimberling [15], Nelsen [21], Joe [13] and Alink et al. [1]. The condition of complete monotonicity can be relaxed if one only requires that ψ generates a copula for a fixed dimension n; see McNeil-Nešlehová [18].

Given that these conditions are fulfilled, the following formula always defines an n-dimensional distribution with marginals F_i and Archimedean copula C^{ψ}

$$H(x_1, \ldots, x_n) = C^{\psi}(F_1(x_1), \ldots, F_n(x_n)) = \psi^{-1}\left(\sum_{i=1}^{n} \psi(F_i(x_i))\right). \tag{2.3}$$

As has already been discussed in the literature (see for example Daníelsson et al. [5] or Nešlehová et al. [22]), the (sub- or super-)additivity properties of VaR depend on the interplay between the tail behaviour of the marginal loss random variables and their dependence structure. In the present paper, we investigate the properties of VaR in the following specific model.

Model 2.1 (Archimedean survival copula)

Assume that the random vector $(X_1, \ldots, X_n)$ satisfies:

(a) All coordinates X_i have the same continuous marginal distribution function F. The tail distribution $\overline{F} = 1 - F$ is regularly varying at infinity with index $-\beta < 0$, i.e. $\overline{F}(x) = x^{-\beta} L(x)$ for some function L slowly varying at infinity.

(b) $(-X_1, \ldots, -X_n)$ has an Archimedean copula with generator ψ, which is regularly varying at 0 with index $-\alpha < 0$.

For background on regular variation see Embrechts et al. [9]. For $n = 2$, the additivity properties of VaR within this model have been analyzed in Alink et al. [1] and Barbe et al. [3]. Our aim is to discuss the case $n > 2$. The following result yields the basics for our analysis.

Proposition 2.2 (Archimedean survival copula) *Under Model 2.1 we have for all dimensions $n \geq 2$*

$$\lim_{u \to \infty} \frac{1}{\overline{F}(u)} \; P\left(\sum_{i=1}^{n} X_i > u \right) = q_n(\alpha, \beta), \tag{2.4}$$

where the limiting constant $q_n(\alpha, \beta) \in [n^{\beta - 1/\alpha}, \infty)$ is given by the following integral representation

$$q_n(\alpha, \beta) = \int_{\mathbb{R}_+^n} 1_{\{\sum_{i=1}^n x_i^{-1} \geq 1\}} \; \frac{d^n}{dx_1 \cdots dx_n} \left(\sum_{i=1}^{n} x_i^{-\alpha\beta} \right)^{-1/\alpha} dx_1 \cdots dx_n. \tag{2.5}$$

Proof. Consider the transformation $X_i \mapsto -X_i$. Then $-X_i$ satisfies the assumptions of Theorem 3.3 (Fréchet case) in Wüthrich [28]. Hence the existence of the limiting constant $q_n(\alpha, \beta)$ follows from $\lim_{u \to \infty} \overline{F}(u)/\overline{F}(un) = n^{\beta}$ and Theorem 3.3 in Wüthrich [28]. The integral representation (2.5) for $q_n(\alpha, \beta)$ was then proved in Alink et al. [1], Theorem 2.2.

□

Remarks.

- The proof of Proposition 2.2 strongly uses the property of regular variation of the marginal distribution function F and of the generator ψ; see Alink et al. [1]. An alternative proof can be based on Proposition 5.17 of Resnick [24] or Theorem 1 of Resnick [25]. Indeed, this has recently been done in Barbe et al. [3] using the theory of multivariate regular variation: the limiting constant in (2.5) can then be expressed via the Radon exponent measure obtained in Theorem 1 of Resnick [25]. However, the analysis of the limiting constant for general dependence structures turns out to be rather difficult. Barbe et al. [3] derive some basic properties in their Proposition 2.2. In order to obtain specific asymptotic additivity properties one needs to switch to explicit dependence structures; see also Barbe et al. [3], Section 3. Whereas we restrict to Archimedean survival copulas, other approaches are possible that yield results like Proposition 2.2 under alternative dependence structures.

- The case of the two-dimensional Archimedean copula is solved in Alink et al. [1]; as we will see, the case $n \geq 3$ is much more involved.

- Under Model 2.1 assumption (a), we immediately have the following results.

  Independent case. If $X_1, \ldots, X_n$ are i.i.d. then

$$\lim_{u\to\infty} \frac{1}{\overline{F}(u)} P\left(\sum_{i=1}^{n} X_i > u\right) = n. \tag{2.6}$$

  This property is referred to as subexponentiality of $\overline{F}$ and holds for a wider class of distribution functions; see Corollary 1.3.2 in Embrechts et al. [9]. We also would like to note that numerical approximations based on (2.4) and (2.6) are often bad.

  Comonotonic case. If $X_1, \ldots, X_n$ are comonotone then

$$\lim_{u\to\infty} \frac{1}{\overline{F}(u)} P\left(\sum_{i=1}^{n} X_i > u\right) = n^{\beta}. \tag{2.7}$$

  It turns out that $q_n(\alpha, \beta)$ lies between these two extreme cases; see Lemma 3.1 below. Moreover, note that the case $\beta = 1$ plays a special role, since then the limiting constants of the independent and the comonotonic case are the same.

□

The main difficulty turns out to be the determination of properties of the limiting constant $q_n(\alpha, \beta)$. For $n = 2$ the representation for the constant $q_2(\alpha, \beta)$ yields the following properties (see Theorem 2.5 in Alink et al. [1]): For every $\alpha > 0$, $q_2(\alpha, \beta)$ is strictly increasing in β. The behaviour of the limiting constant as a function of α is given by:

(a) For $\beta > 1$, $q_2(\alpha, \beta)$ is strictly increasing in α.

(b) For $\beta = 1$, $q_2(\alpha, \beta) = 2$.

(c) For $\beta < 1$, $q_2(\alpha, \beta)$ is strictly decreasing in α.

Our main theorem extends the above result to the case $n \geq 3$. It is proven in Section 4 and reads as follows:

Theorem 2.3 *Take $n \geq 2$ in the Model 2.1 setup.*

(a) For $\beta > 1$, $q_n(\alpha, \beta)$ is increasing in α.

(b) For $\beta = 1$, $q_n(\alpha, \beta) = n$.

(c) For $\beta < 1$, $q_n(\alpha, \beta)$ is decreasing in α.

Remark. The decreasing/increasing property of the limiting constant is the crucial property for obtaining asymptotic superadditivity or subadditivity for VaR. Figure 1 gives the graph $\alpha \mapsto q_3(\beta, \alpha)$ for different β's in dimension $n = 3$. Note that we cannot prove strict monotonicity for $n \geq 3$ though we believe this to hold. Theorem 2.3 proves the conjecture made by Barbe et al. [3] in Remark 4.2, where it was stated that "a proof for the monotonicity property remains elusive".

□

An immediate consequence of Proposition 2.2 is the following asymptotic VaR behaviour.

Corollary 2.4 *For $n \geq 2$ and assuming Model 2.1,*

$$\lim_{p \to 1} \frac{\mathrm{VaR}_p\left(\sum_{i=1}^{n} X_i\right)}{\mathrm{VaR}_p\left(X_1\right)} = q_n(\alpha, \beta)^{1/\beta}, \tag{2.8}$$

where $q_n(\alpha, \beta)$ is defined in (2.5).

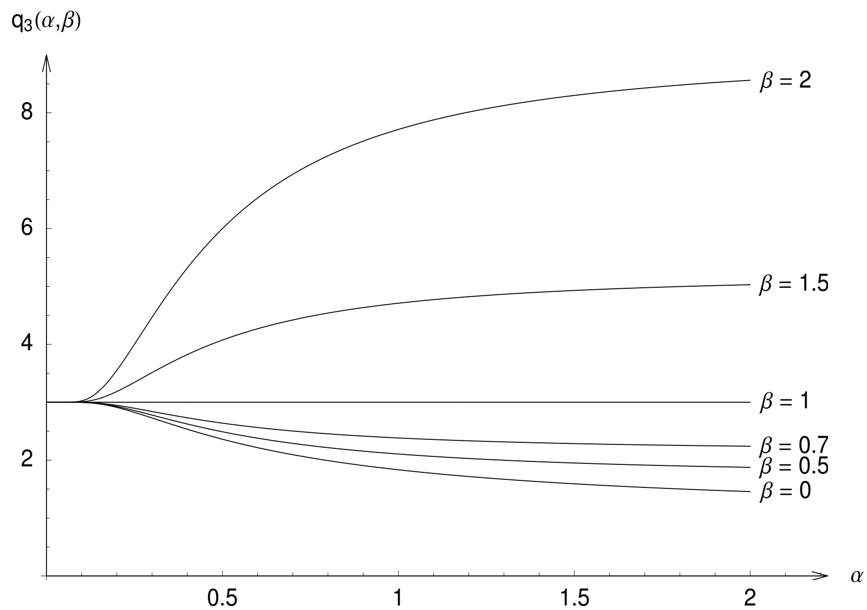

Figure 1: Function $\alpha \mapsto q_3(\beta, \alpha)$ for different $\beta \geq 0$

Combining Corollary 2.4 and Theorem 2.3 leads to the following result:

Theorem 2.5 (Additivity properties of VaR) *For $n \geq 2$ and assuming Model 2.1 we have that:*

(a) For all $\beta > 1$ and $\alpha > 0$ there exists $p_0 > 0$ such that for all $p_0 < p < 1$

$$\mathrm{VaR}_p\left(\sum_{i=1}^{n} X_i\right) < \sum_{i=1}^{n} \mathrm{VaR}_p\left(X_i\right). \tag{2.9}$$

(b) For all $\beta < 1$ and $\alpha > 0$ there exists $p_1 > 0$ such that for all $p_1 < p < 1$

$$\mathrm{VaR}_p\left(\sum_{i=1}^{n} X_i\right) > \sum_{i=1}^{n} \mathrm{VaR}_p\left(X_i\right). \tag{2.10}$$

The proofs of Corollary 2.4 and Theorem 2.5 are given in Section 4. Theorem 2.5 gives the basic qualitative picture for the asymptotic VaR behaviour. It clearly highlights the switch from subadditivity to superadditivity when moving from a finite mean model ($\beta > 1$) to an infinite mean model ($\beta < 1$). The crucial property for these results to hold is whether the limiting constant $q_n(\alpha, \beta)$ is an increasing or decreasing function in α. Note that this happens regardless the explicit choice of the Archimedean dependence structure as long as the condition of Model 2.1 (b) holds.

3 Some properties of $q_n(\alpha,\beta)$ for $n \geq 3$

It follows from Corollary 2.4 that the asymptotic behaviour of VaR is determined by the limiting constant $q_n(\alpha,\beta)$. This section is devoted to the discussion of additional properties of the latter constant; in particular, we address its monotonicity as a function of β and its limiting behaviour in α and β, respectively. Our results complement those of Barbe et al. [3].

Following Section 2.3 in Barbe et al. [3], we first reformulate the problem in polar coordinates. Define $\mathbf{x} = (x_1, \ldots, x_n)'$ and

$$g_\alpha(\mathbf{x}) = \frac{d^n}{dx_1 \cdots dx_n} \left(\sum_{i=1}^n x_i^\alpha \right)^{-1/\alpha} = \prod_{i=0}^{n-1} \left(\frac{1}{\alpha} + i \right) \left(\sum_{i=1}^n x_i^\alpha \right)^{-1/\alpha - n} \prod_{i=1}^n \alpha x_i^{\alpha-1}. \qquad (3.1)$$

Use the transformation $x_i \mapsto x_i^{-1/\beta}$ to obtain

$$q_n(\alpha,\beta) = \int_{\mathbb{R}_+^n} 1_{\left\{\sum_{i=1}^n x_i^{1/\beta} \geq 1\right\}} \; g_\alpha(\mathbf{x}) \; d\mathbf{x}. \qquad (3.2)$$

The function $g_\alpha(\cdot)$ is well-known and corresponds to the class of logistic distributions; consider for instance the densities $h_{m,c}(\mathbf{w})$ in Kotz-Nadarajah [16] with $\psi_{k,c} = 1$.

On $\mathbb{R}_+^n$ we now consider the polar coordinate transformation

$$\mathbf{x} \mapsto \left(|\mathbf{x}|, \frac{\mathbf{x}}{|\mathbf{x}|} \right), \qquad \text{with} \quad |\mathbf{x}| = \sum_{i=1}^n x_i.$$

Using this transformation we rewrite (3.2) as follows

$$\begin{aligned} q_n(\alpha,\beta) &= \int_{[0,\infty)\times\mathcal{N}_+} 1_{\left\{\left(\sum_{i=1}^n w_i^{1/\beta}\right)^{-\beta} \leq r\right\}} \; r^{-2} \; g_\alpha(\mathbf{w}) \; dr \; d\mathbf{w} \\ &= \int_{\mathcal{N}_+} \left(\sum_{i=1}^n w_i^{1/\beta} \right)^\beta \; g_\alpha(\mathbf{w}) \; d\mathbf{w}, \end{aligned} \qquad (3.3)$$

where $\mathcal{N}_+ = \{\mathbf{w} \in \mathbb{R}_+^n \; : \; |\mathbf{w}| = 1\}$ is the unit simplex. This is exactly the integral representation (20) in Barbe et al. [3]. The following lemmas contain some results which are essentially known. We add short proofs where necessary.

Lemma 3.1 *For all $n \geq 2$ and $\alpha > 0$ the following properties hold:*

(a) $g_\alpha(\cdot)/n$ is a probability density on $\mathcal{N}_+$;

(b) $q_n(\alpha, \beta)$ *is strictly increasing in* β*;*

(c) $q_n(\alpha, 1) = n$*, and*

(d) $\min\{n^\beta, n\} \le q_n(\alpha, \beta) \le \max\{n^\beta, n\}$.

Proof.

(a) See for instance Kotz-Nadarajah [16], formula (3.7).

(b) This was proved in Proposition 2.2 of Barbe et al. [3].

(c) This was proved in Proposition 4.1 of Barbe et al. [3].

(d) From (b) and (c) it is clear that $q_n(\alpha, 1)$ is an upper bound on $q_n(\alpha, \beta)$ for $\beta < 1$ and that $q_n(\alpha, 1)$ is a lower bound on $q_n(\alpha, \beta)$ for $\beta > 1$. The maximum and minimum of $n\left(\sum_{i=1}^n w_i^{1/\beta}\right)^\beta$ on $\mathcal{N}_+$ are given by $\max(n^\beta, n)$ and $\min(n^\beta, n)$, respectively. Hence, since $g_\alpha(\cdot)/n$ is a probability density on $\mathcal{N}_+$, the proof of (d) is straightforward.

□

Lemma 3.2 *For all* $n \ge 2$ *and* $\alpha > 0$ *we have*

$$\lim_{\beta \to 0} q_n(\alpha, \beta) = \int_{\mathcal{N}_+} \max\{w_1, \ldots, w_n\} \; g_\alpha(\mathbf{w}) \; d\mathbf{w} = \sum_{k=1}^n \binom{n}{k} \; (-1)^{k-1} \; k^{-1/\alpha}, \tag{3.4}$$

and

$$\lim_{\beta \to \infty} \frac{q_n(\alpha, \beta)}{n^\beta} = \int_{\mathcal{N}_+} \prod_{i=1}^n w_i^{1/n} \; g_\alpha(\mathbf{w}) \; d\mathbf{w}. \tag{3.5}$$

Proof. Claim (3.5) as well as the first part of (3.4) follow from Proposition 2.2 in Barbe et al. [3]. Define $|\mathbf{w}|_\infty = \max\{w_1, \ldots, w_n\}$. Hence it remains to show that

$$\int_{\mathcal{N}_+} |\mathbf{w}|_\infty \; g_\alpha(\mathbf{w}) \; d\mathbf{w} = \sum_{k=1}^n \binom{n}{k} \; (-1)^{k-1} \; k^{-1/\alpha}.$$

For this, note that

$$\begin{aligned} \int_{\mathcal{N}_+} |\mathbf{w}|_\infty \; g_\alpha(\mathbf{w}) \; d\mathbf{w} &= \int_{[0,\infty) \times \mathcal{N}_+} 1_{\{1/|\mathbf{w}|_\infty \le r\}} r^{-2} \; g_\alpha(\mathbf{w}) \; dr \; d\mathbf{w} \\ &= \int_{\mathbb{R}_+^n} 1_{\{|\mathbf{x}|_\infty \ge 1\}} \; g_\alpha(\mathbf{x}) \; d\mathbf{x} = \int_{B_\infty^c(1)} g_\alpha(\mathbf{x}) \; d\mathbf{x}, \end{aligned}$$

where $B^c_\infty(1) = \{\mathbf{x} \in \mathbb{R}_n^+ \; : \; |\mathbf{x}|_\infty \geq 1\}$. Choose $\mathbf{a} \in \mathbb{R}^n_+$ and define the set $A_n = [a_1, \infty) \times \ldots \times [a_n, \infty) \subset \mathbb{R}^n_+$. Then we have

$$\int\limits_{A_n} g_\alpha(\mathbf{x}) \; d\mathbf{x} = \left(\sum_{i=1}^n a_i^\alpha \right)^{-1/\alpha},$$

completing the proof.

□

Lemma 3.3 *For all $n \geq 2$ and $\beta > 0$ we have*

$$\lim_{\alpha \to \infty} q_n(\alpha, \beta) = n^\beta, \tag{3.6}$$

and

$$\lim_{\alpha \to 0} q_n(\alpha, \beta) = n. \tag{3.7}$$

Proof. Claim (3.6) is proved in Proposition 4.1 of Barbe et al. [3]. The proof of (3.7) is more involved; Barbe et al. [3] give a hint in their Section 3.1. We decided to give the explicit arguments since one needs to distinguish the two cases $\beta \leq 1$ and $\beta > 1$.

For $\beta \leq 1$ we know that $q_n(\alpha, \beta) \leq q_n(\alpha, 1) = n$; see Lemma 3.1. Moreover, from Lemmas 3.1 and 3.2 we know that for all $\beta > 0$ and $\alpha > 0$ we have

$$q_n(\alpha, \beta) \geq \sum_{k=1}^n \binom{n}{k} \; (-1)^{k-1} \; k^{-1/\alpha}.$$

This implies that

$$\liminf_{\alpha \to 0} \; q_n(\alpha, \beta) \;\; \geq \;\; \lim_{\alpha \to 0} \;\; \sum_{k=1}^n \binom{n}{k} \; (-1)^{k-1} \; k^{-1/\alpha} \;\; = \;\; n.$$

Hence for $\beta \leq 1$ we have

$$\lim_{\alpha \to 0} q_n(\alpha, \beta) = n \;\; \lim_{\alpha \to 0} \int\limits_{\mathcal{N}_+} \left(\sum_{i=1}^n w_i^{1/\beta} \right)^\beta g_\alpha(\mathbf{w})/n \; d\mathbf{w} = n. \tag{3.8}$$

Observe that $g_\alpha(\mathbf{w})/n$ is a probability density on $\mathcal{N}_+$ for all $\alpha > 0$ and that the right-hand side of (3.8) is independent of $\beta \leq 1$. Moreover, $\left(\sum_{i=1}^n w_i^{1/\beta} \right)^\beta$ takes its maximas in the corners of $\mathcal{N}_+$ for $\beta < 1$ (i.e. in $\mathcal{C} = \{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\}$) and

$$\left(\sum_{i=1}^n w_i^{1/\beta} \right)^\beta = 1 \qquad \text{for } \mathbf{w} \in \mathcal{C}.$$

This immediately implies that $g_\alpha(\mathbf{w})/n$ converges to a point measure on $\mathcal{C}$ for $\alpha \to 0$, giving weight $1/n$ to each corner of $\mathcal{N}_+$ using a symmetry argument; this corresponds to the independent case. Having a uniform distribution in the corners of $\mathcal{N}_+$ also proves the claim for $\beta > 1$. This completes the proof.

□

4 Proofs of the results from Section 2

In order to prove our main result, Theorem 2.3, we need to introduce a specific stochastic ordering and review some results given in Kemperman [14], Müller [20] and Wei-Hu [27].

4.1 Stochastic Ordering

Definition 4.1 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is supermodular if for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$*

$$f(\mathbf{x} \wedge \mathbf{y}) + f(\mathbf{x} \vee \mathbf{y}) \geq f(\mathbf{x}) + f(\mathbf{y}). \tag{4.1}$$

Here $\wedge$ and $\vee$ denote the componentwise minimum and maximum, respectively. In statistical physics condition (4.1) is often referred to as the FKG lattice condition; see Newman [23] and Fortuin et al. [11].

From Kemperman [14], it follows that a function $f : \mathbb{R}^n \to \mathbb{R}$ is supermodular if and only if $f(\ldots, x_i, \ldots, x_j, \ldots)$ is supermodular in all pairs (x_i, x_j); see also Wei-Hu [27]. Theorem 2.5 in Müller [20] implies that, if f is twice differentiable, then it is supermodular if and only if

$$\frac{\partial^2}{\partial x_i\, \partial x_j} f(\mathbf{x}) \geq 0 \qquad \text{for all } \mathbf{x} = (x_1, \ldots, x_n)' \in \mathbb{R}^n \text{ and } 1 \leq i < j \leq n. \tag{4.2}$$

Definition 4.2 *A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is said to be smaller than a random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ in supermodular ordering if for all supermodular functions $f : \mathbb{R}^n \to \mathbb{R}$ we have*

$$Ef(\mathbf{X}) \leq Ef(\mathbf{Y}).$$

Write $\mathbf{X} \leq_{sm} \mathbf{Y}$ and $F_{\mathbf{X}} \leq_{sm} F_{\mathbf{Y}}$ (where $F_{\mathbf{X}}$ and $F_{\mathbf{Y}}$ denote the distribution functions of $\mathbf{X}$ and $\mathbf{Y}$, respectively).

From Theorem 3.2 in Müller [20] it follows that the supermodular ordering for random vectors is preserved under coordinatewise increasing transformations. This immediately implies that the supermodular property $\mathbf{X} \leq_{sm} \mathbf{Y}$ only depends on the supermodular ordering of the copulas of $\mathbf{X}$ and $\mathbf{Y}$. This means that

$$\mathbf{X} \leq_{sm} \mathbf{Y} \quad \Longleftrightarrow \quad C_{\mathbf{X}} \leq_{sm} C_{\mathbf{Y}}, \tag{4.3}$$

where $C_{\mathbf{X}}$ and $C_{\mathbf{Y}}$ are the copulas of $\mathbf{X}$ and $\mathbf{Y}$, respectively. We now choose as an example a specific Archimedean copula $\psi_\alpha(t) = t^{-\alpha} - 1$ for $\alpha > 0$. Then

$$C_\alpha(u_1, \ldots, u_n) = \left(\sum_{i=1}^{n} u_i^{-\alpha} - (n-1) \right)^{-1/\alpha}$$

is the so-called Clayton copula with parameter α. Observe that the generator $\psi_\alpha(t)$ of the Clayton copula is regularly varying in 0 with index $-\alpha < 0$.

Lemma 4.3 *Assume that* $\mathbf{X}$ *and* $\mathbf{Y}$ *have a Clayton copula with respective parameters* $\alpha_X > 0$ *and* $\alpha_Y > 0$. *Then*

$$\alpha_X < \alpha_Y \quad \Longrightarrow \quad \mathbf{X} \leq_{sm} \mathbf{Y}.$$

Proof. From (4.3) it follows that it remains to prove that the Clayton copulas C_α are ordered with respect to α. From Theorem 3.1 in Wei-Hu [27] it suffices to prove that

$$(-1)^{k-1} \frac{\partial^k}{\partial t^k} \psi_{\alpha_X} \circ \psi_{\alpha_Y}^{-1}(t) \geq 0 \qquad \text{for all } k \geq 1 \text{ and } t > 0, \tag{4.4}$$

and $\psi_{\alpha_X} \circ \psi_{\alpha_Y}^{-1}(0) = 0$ and $\psi_{\alpha_X} \circ \psi_{\alpha_Y}^{-1}(\infty) = \infty$. Note that

$$\psi_{\alpha_X} \circ \psi_{\alpha_Y}^{-1}(t) = (1+t)^{\alpha_X/\alpha_Y} - 1.$$

Hence (4.4) is easily verified for $\alpha_X < \alpha_Y$. This completes the proof.

□

4.2 Proof of Theorem 2.3

Proof of Theorem 2.3 (b), case $\beta = 1$. This was proved in Proposition 4.1 of Barbe et al. [3].

Proof of Theorem 2.3 (a), case $\beta > 1$. Because of Proposition 2.2 we may obtain a proof on any model that satisfies the assumptions of Model 2.1. We choose two random vectors $\mathbf{X}$ and $\mathbf{Y}$ as follows: all marginals are Pareto distributed with fixed parameter $\beta > 1$ and threshold $\theta = 1$, and $-\mathbf{X}$ and $-\mathbf{Y}$ have a Clayton copula with parameters $0 < \alpha_X < \alpha_Y$. Proposition 2.2 implies that for this choice

$$q_n(\alpha_X, \beta) = \lim_{u\to\infty} \frac{1}{\overline{F}(u)}\ P\left(\sum_{i=1}^n X_i > u\right)$$

and

$$q_n(\alpha_Y, \beta) = \lim_{u\to\infty} \frac{1}{\overline{F}(u)}\ P\left(\sum_{i=1}^n Y_i > u\right),$$

so that we have to show that $q_n(\alpha_X, \beta) \leq q_n(\alpha_Y, \beta)$.

The proof goes by contradiction. Define $\widetilde{\mathbf{X}} = -\mathbf{X}$ and $\widetilde{\mathbf{Y}} = -\mathbf{Y}$ and note that

$$\widetilde{\mathbf{X}} \leq_{sm} \widetilde{\mathbf{Y}} \tag{4.5}$$

by our choice $\alpha_X < \alpha_Y$; see Lemma 4.3. Assume that $q_n(\alpha_Y, \beta) < q_n(\alpha_X, \beta)$. Then there exist $\varepsilon \in (0,\ (q_n(\alpha_X, \beta) - q_n(\alpha_Y, \beta))/3\ )$ and u_0 such that for all $u > u_0$ we have

$$\begin{aligned} P\left(\sum_{i=1}^n -\widetilde{Y}_i > u\right) &\leq \overline{F}(u)\,(q_n(\alpha_Y, \beta) + \varepsilon) \\ &\leq \overline{F}(u)\,(q_n(\alpha_X, \beta) - 2\varepsilon) \\ &\leq P\left(\sum_{i=1}^n -\widetilde{X}_i > u\right) - \varepsilon\overline{F}(u). \end{aligned} \tag{4.6}$$

Choose $\beta_1 \in (1, \beta)$ and define for $\mathbf{x} \in \mathbb{R}^n_-$ the functions

$$\begin{aligned} f_1(\mathbf{x}) &= \left(-\sum_{i=1}^n x_i\right)^{\beta_1}, \\ f_2(\mathbf{x}) &= \left(\left(-\sum_{i=1}^n x_i\right)^{\beta_1} - u_0^{\beta_1}\right)_+. \end{aligned} \tag{4.7}$$

Observe that for $i \neq j$ and $\beta_1 > 1$

$$\frac{\partial^2}{\partial x_i\,\partial x_j} f_1(\mathbf{x}) = \beta_1(\beta_1 - 1)\left(-\sum_{i=1}^n x_i\right)^{\beta_1 - 2} \geq 0. \tag{4.8}$$

Hence because of (4.2), f_1 is a supermodular function on $\mathbb{R}^n_-$ (the domain of the random vectors $\widetilde{\mathbf{X}}$ and $\widetilde{\mathbf{Y}}$). Also note that f_2 is a supermodular function on $\mathbb{R}^n_-$ for all $u_0 > 0$, this follows from the fact that the stop-loss transformation of a convex increasing function is still convex. Hence we have by the supermodular ordering (4.5)

$$Ef_2\left(\widetilde{\mathbf{X}}\right) \leq Ef_2\left(\widetilde{\mathbf{Y}}\right) < \infty.$$

This implies that

$$\int_0^\infty P\left(\left(\left(-\sum_{i=1}^n \widetilde{X}_i\right)^{\beta_1} - u_0^{\beta_1}\right)_+ > x\right) dx \leq \int_0^\infty P\left(\left(\left(-\sum_{i=1}^n \widetilde{Y}_i\right)^{\beta_1} - u_0^{\beta_1}\right)_+ > x\right) dx,$$

which is equivalent to

$$\int_0^\infty P\left(\sum_{i=1}^n -\widetilde{X}_i > \left(x + u_0^{\beta_1}\right)^{1/\beta_1}\right) dx \leq \int_0^\infty P\left(\sum_{i=1}^n -\widetilde{Y}_i > \left(x + u_0^{\beta_1}\right)^{1/\beta_1}\right) dx. \quad (4.9)$$

This contradicts (4.6), hence $q_n(\alpha_Y, \beta) \geq q_n(\alpha_X, \beta)$. This completes the proof of case (a) of Theorem 2.3.

□

Proof of Theorem 2.3 (c), case $\beta < 1$. We use the same notations as in the proof of part (a) and prove the assertion also by contradiction. Choose $0 < \alpha_X < \alpha_Y$ and assume that $q_n(\alpha_Y, \beta) > q_n(\alpha_X, \beta)$. Similarly as in (4.6) there exists $\varepsilon \in (0,\ (q_n(\alpha_Y, \beta) - q_n(\alpha_X, \beta))/3\ )$ and u_1 such that for all $u \geq u_1$ we have

$$P\left(\sum_{i=1}^n -\widetilde{X}_i > u\right) \leq P\left(\sum_{i=1}^n -\widetilde{Y}_i > u\right) - \varepsilon \overline{F}(u). \quad (4.10)$$

Choose $\beta_1 \in (\beta, 1)$ and consider for $\mathbf{x} \in \mathbb{R}_-^n$ the function $f_1(\mathbf{x})$ defined in (4.7). Observe that the second derivative in (4.8) now becomes negative for $\beta_1 < 1$, hence because of (4.2), $-f_1$ is a supermodular function on $\mathbb{R}_-^n$. Now define for $u_2 > 0$

$$f_3(\mathbf{x}) = \min\{f_1(\mathbf{x}), u_2\}. \quad (4.11)$$

The function $-f_3$ is also supermodular on $\mathbb{R}_-^n$ (this has the fact that concave increasing functions stay concave under transformation (4.11)). The supermodular ordering (4.5) implies that for all $u_2 > 0$

$$Ef_3\left(\widetilde{\mathbf{Y}}\right) \leq Ef_3\left(\widetilde{\mathbf{X}}\right) \leq u_2 < \infty. \quad (4.12)$$

Choose $u_3 = u_1^{\beta_1}$ fixed (see (4.10)). For the choice $u_2 = u_3$ in definition (4.11) we define (see also (4.12))

$$\begin{aligned}\delta_1 &= Ef_3\left(\widetilde{\mathbf{X}}\right) - Ef_3\left(\widetilde{\mathbf{Y}}\right) \\ &= E\left[\min\{f_1\left(\widetilde{\mathbf{X}}\right), u_3\}\right] - E\left[\min\{f_1\left(\widetilde{\mathbf{Y}}\right), u_3\}\right] \in [0,\infty).\end{aligned} \tag{4.13}$$

For all $u_2 > u_3$ we obtain the following decompositions

$$\begin{aligned}E\left[\min\{f_1\left(\widetilde{\mathbf{X}}\right), u_2\}\right] &= E\left[\min\{f_1\left(\widetilde{\mathbf{X}}\right), u_3\}\right] + E\left[\left(f_1\left(\widetilde{\mathbf{X}}\right) - u_3\right) 1_{\{u_3 \le f_1(\widetilde{\mathbf{X}}) \le u_2\}}\right] \\ &\qquad + (u_2 - u_3)\, P\left[f_1\left(\widetilde{\mathbf{X}}\right) \ge u_2\right] \\ &= E\left[\min\{f_1\left(\widetilde{\mathbf{X}}\right), u_3\}\right] + E\left[\left(\min\{f_1\left(\widetilde{\mathbf{X}}\right), u_2\} - u_3\right)_+\right],\end{aligned}$$

and analogously

$$E\left[\min\{f_1\left(\widetilde{\mathbf{Y}}\right), u_2\}\right] = E\left[\min\{f_1\left(\widetilde{\mathbf{Y}}\right), u_3\}\right] + E\left[\left(\min\{f_1\left(\widetilde{\mathbf{Y}}\right), u_2\} - u_3\right)_+\right].$$

With (4.13) we find that for all $u_2 > u_3$

$$\begin{aligned}&E\left[\min\{f_1\left(\widetilde{\mathbf{X}}\right), u_2\}\right] - E\left[\min\{f_1\left(\widetilde{\mathbf{Y}}\right), u_2\}\right] \\ &\quad = \delta_1 + E\left[\left(\min\{f_1\left(\widetilde{\mathbf{X}}\right), u_2\} - u_3\right)_+\right] - E\left[\left(\min\{f_1\left(\widetilde{\mathbf{Y}}\right), u_2\} - u_3\right)_+\right] \\ &\quad = \delta_1 + \int_0^{u_2-u_3} P\left(f_1\left(\widetilde{\mathbf{X}}\right) > x + u_3\right) dx - \int_0^{u_2-u_3} P\left(f_1\left(\widetilde{\mathbf{Y}}\right) > x + u_3\right) dx.\end{aligned}$$

Using (4.10) we obtain

$$\begin{aligned}&E\left[\min\{f_1\left(\widetilde{\mathbf{X}}\right), u_2\}\right] - E\left[\min\{f_1\left(\widetilde{\mathbf{Y}}\right), u_2\}\right] \\ &\quad = \delta_1 + \int_0^{u_2-u_3} \left(P\left(\sum_{i=1}^n -\widetilde{X}_i > (x+u_3)^{1/\beta_1}\right) - P\left(\sum_{i=1}^n -\widetilde{Y}_i > (x+u_3)^{1/\beta_1}\right)\right) dx \\ &\quad \le \delta_1 - \varepsilon \int_0^{u_2-u_3} \overline{F}\left((x+u_1^{\beta_1})^{1/\beta_1}\right) dx \\ &\quad = \delta_1 - \varepsilon \int_0^{\infty} P\left(\left(\min\{X^{\beta_1}, u_2\} - u_1^{\beta_1}\right)_+ > x\right) dx.\end{aligned}$$

This implies, for all $u_2 > u_3 = u_1^{\beta_1}$, that

$$E\left[\min\{f_1\left(\widetilde{\mathbf{X}}\right), u_2\}\right] - E\left[\min\{f_1\left(\widetilde{\mathbf{Y}}\right), u_2\}\right] \le \delta_1 - \varepsilon E\left[\left(\min\{X^{\beta_1}, u_2\} - u_1^{\beta_1}\right)_+\right]. \tag{4.14}$$

Observe that X^{β_1} has infinite mean for $\beta_1 > \beta$, hence the right-hand side of (4.14) converges to $-\infty$ for $u_2 \to \infty$ which contradicts (4.12). This completes the proof.

□

4.3 Proofs of Corollary 2.4 and Theorem 2.5

Proof of Corollary 2.4. This follows easily from the regular variation property of the marginals and from Proposition 2.2.

□

Proof of Theorem 2.5. For $\beta > 1$ and $\alpha > 0$ we know from the proofs of Proposition 4.1 in Barbe et al. [3], Lemma 3.3 and Theorem 2.3 that $q_n(\alpha, \beta) < n^\beta$. Hence choose $\varepsilon \in (0, n^\beta/q_n(\alpha, \beta) - 1)$. From Corollary 2.4 we know that there exists $p_0 = p_0(\alpha, \beta) > 0$ such that for all $p > p_0$ we have

$$\begin{aligned} \mathrm{VaR}_p\left(\sum_{i=1}^n X_i\right) &\leq ((1+\varepsilon)q_n(\alpha,\beta))^{1/\beta}\, \mathrm{VaR}_p\left(X_1\right) \\ &< n\mathrm{VaR}_p\left(X_1\right) = \sum_{i=1}^n \mathrm{VaR}_p\left(X_i\right). \end{aligned}$$

The claim for $\beta < 1$ follows analogously from the fact that $q_n(\alpha, \beta) > n^\beta$; see proofs of Proposition 4.1 in Barbe et al. [3], Lemma 3.3 and Theorem 2.3. This finishes the proof.

□

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References

[1] Alink, S., Löwe, M., Wüthrich, M.V. (2004). Diversification of aggregate dependent risks. Insurance: Math. Econom. 35, 77–95.

[2] Artzner, P., Delbaen, F., Eber, J.M., Heath, D. (1999). Coherent risk measures. Math. Finance 9, 203-228.

[3] Barbe, P., Fougères, A.L., Genest, C. (2007). On the tail behavior of sums of dependent risks. ASTIN Bulletin 36/2, 361-374.

[4] Böcker, K., Klüppelberg, C. (2006). Multivariate models for operational risk. Preprint, TU Munich.

[5] Daníelsson, J., Jorgensen, B.N., Samorodnitsky, G., Sarma, M., de Vries, C.G. (2005). Subadditivity re-examined: the case for Value-at-Risk. Preprint, London School of Economics.

[6] Degen, M., Embrechts, P., Lambrigger (2007). The quantitative modeling of operational risk: between g-and-h and EVT. ASTIN Bulletin, to appear.

[7] Dhaene, J., Laeven, R., Vanduffel, S., Darkiewicz, C., Goovaerts, M. (2007). Can a coherent risk measure be too subadditive? Journal of Risk and Insurance, to appear.

[8] Dutta, K., Perry, J. (2007). A tale of tails: an empirical analysis of loss distribution models for estimating operational risk capital. Working paper, Federal Reserve Bank of Boston, No. 06-13.

[9] Embrechts, P., Klüppelberg, C., Mikosch, T. (1997). Modelling Extremal Events for Insurance and Finance. Springer, Berlin.

[10] de Fontnouvelle, P., Rosengren, E., Jordan, J. (2006). Implications of alternative operational risk modeling techniques. In: M. Carey and R. Stulz, eds., The Risks of Financial Institutions, NBER/University of Chicago Press.

[11] Fortuin, C.M., Kasteleyn, P.W., Ginibre, J. (1971). Correlation inequalities on some partially ordered sets. Comm. Math. Phys. 22, 89–103.

[12] Heyde, C.C., Kou, S.G., Peng, X.H. (2006). What is a good risk measure: bridging the gaps between data, coherent risk measures, and insurance risk measures. Preprint, Columbia University.

[13] Joe, H. (1997). Multivariate Models and Dependence Concepts. Chapman and Hall, London.

[14] Kemperman, J.H.B. (1977). On the FKG-inequalities for measures on a partially ordered space. Proc. Akad. Wetenschappen. Ser. A 80, 313-331.

[15] Kimberling, C. (1974). A probabilistic interpretation of complete monotonicity. Aequationes Mathematica 10, 152-164.

[16] Kotz, S., Nadarajah, S. (2000). Extreme Value Distributions: Theory and Applications. Imperial College Press, London.

[17] McNeil, A.J., Frey, R., Embrechts, P. (2005). Quantitative Risk Management: Concepts, Techniques and Tools. Princeton University Press, Princeton.

[18] McNeil, A.J., Nešlehová, J. (2007). Multivariate Archimedean copulas, d-monotone functions and ℓ_1-norm symmetric distributions. FIM Preprint, ETH Zurich.

[19] Moscadelli, M. (2004). The modelling of operational risk: experience with the analysis of the data collected by the Basel Committee. Technical Report 517, Banca d'Italia.

[20] Müller, H. (1997). Stop-loss order for portfolios of dependent risks. Insurance: Math. Econom. 21, 219-223.

[21] Nelsen, R.B. (1999). An Introduction to Copulas. Springer, New York.

[22] Nešlehová, J., Embrechts, P., Chavez-Demoulin, V. (2006). Infinite mean models and the LDA for operational risk. Journal of Operational Risk 1/1, 3-25.

[23] Newman, C.M. (1997). Topics in Disordered Systems. Lectures in Mathematics, ETH Zürich. Birkhäuser Verlag, Basel.

[24] Resnick, S.I. (1987). Extreme Values, Regular Variation and Point Processes. Springer, New York.

[25] Resnick, S.I. (2004). The extremal dependence measure and asymptotic independence. Stoch. Models 20/2, 205-227.

[26] Rootzén, H., Klüppelberg, C. (1999). A single number can't hedge against economic catastrophes. Ambio-Royal Swedish Academy of Sciences 28, 550-555.

[27] Wei, G., Hu, T. (2002). Supermodular dependence ordering on a class of multivariate copulas. Stat. Prob. Letters 57, 375-385.

[28] Wüthrich, M.V. (2003). Asymptotic Value-at-Risk estimates for sums of dependent random variables. ASTIN Bulletin 33/1, 75–92.